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Variational Inequalities with General Multivalued Lower Order Terms Given by Integrals

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Abstract

This paper is about the existence and some properties of solutions of variational inequalities associated with the 2nd order inclusion

$$\mathrm{div}[A(x, \nabla u)] + L \in f(x, u) \ \text{ in } \ \Omega,$$

where the lower order term $f(x, u)$ is a general multivalued function. Both coercive and noncoercive cases are considered. In the noncoercive case, we use a sub-supersolution approach to study the existence, comparison, and other properties of the solution set such as its compactness, directedness, and the existence of extremal solutions.

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1 Introduction

We are concerned in this paper with the existence and some properties of the solutions to the following inclusion

$$\begin{cases} \mathrm{div}[A(x, \nabla u)] + L \in f(x, u) \ \text{ in } \ \Omega \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.1}$$

or the more general variational inequality

$$\begin{cases} \int_\Omega A(x, \nabla u)(\nabla v - \nabla u)dx + \int_\Omega f(x, u)(v - u)dx - \langle L, v - u\rangle \geq 0, \forall v \in K \\ u \in K, \end{cases} \tag{1.2}$$

where Ω is an open bounded region in $\mathbb{R}^N$ ($N \geq 1$) with Lipschitz boundary $\partial\Omega$, $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is an operator of Leray–Lions type, L is a bounded linear functional, and K is a closed convex subset of a function space of admissible functions. A new feature in (1.1) and (1.2) that we would like to concentrate here is that $f : \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is a general multivalued function. The lower order term in (1.2) is thus an integral operator of multivalued functions. The inequality (1.2) is described in detail in the sequel.

Note that first order ordinary differential inclusions of the form

$$x'(t) \in f(t, x(t)), t \in [0, T],$$

and related problems in control or viability theory have been studied extensively (cf. e.g. [1, 2, 14, 18] and the rich references therein). Here, we are interested in second order partial differential inclusions or inequalities. Compared to inclusions based on first order ordinary differential equations such as above, general inclusions based on second order partial differential equations or inequalities have been investigated to a lesser extent. On the other hand, problems of the form (1.1) or (1.2) with $f(x, u) = \partial_u j(x, u)$ being Clarke's generalized gradient of a locally Lipschitz function $j(x, u)$ (with respect to the second variable) have attracted much attention after the pioneering works of Clarke and Chang ([12, 13]), see e.g. the monographs [6, 10, 13, 15, 22, 23, 24] and their references. In this paper, we are concerned with problems such as (1.1) or more generally (1.2) with f being a general multivalued function without such variational structure (that is, being the derivative of some other smooth or nonsmooth function even in a certain generalized sense).

We consider problems (1.1) and (1.2) in both coercive and noncoercive cases. Even in the coercive case, the problem seems interesting and has not been studied before from the point of view here. Moreover, our problem here also serves as an interesting and relevant example for Browder–Hess' abstract theory of multivalued pseudomonotone operators ([4]), applied to general multivalued integral operators not necessarily given by Clarke's generalized gradients (or closely related functions) as in [24] or [10], which seem so far the only type of examples and applications for Browder–Hess' theory to boundary value problems. In the noncoercive case, we follow a sub-supersolution approach to get the existence of solutions and also some qualitative properties of the solution sets between sub- and supersolutions. As shown in the sequel, although the general ideas of regularization and truncation in the sub-supersolution method are followed, many new arguments and techniques are needed in the proofs and calculations in our case of inequalities with general multivalued lower order terms. Since Clarke's generalized gradients are upper semicontinuous multivalued functions with closed, convex values, the existence and comparison theorems and other properties of solutions in the noncoercive case considered here improve and extend several nonsmooth existence and enclosure results related to hemivariational and variational-hemivariational inequalities to the case of general multivalued lower order terms without nonsmooth potential functionals, therefore complement several of our results established previously in [7, 9, 8, 10] etc.

The paper is organized as follows. In Section 2, we present a precise formulation of the problem together with the necessary assumptions on the involved sets and mappings. Our main theorems and their preparation are presented in Sections 3 and 4. The coercive case, including the case where the multivalued lower order term has a "sublinear" growth, is studied in Section 3 where abstract existence theorems (Theorems 3.1 and 3.3) together with an illustrating example of a unilateral problem with a mixed Neumann–Dirichlet boundary condition (Theorem 3.7) are considered. Section 4 is devoted to the noncoercive case. We introduce the concepts of sub- and supersolutions for our problem and prove a general existence/enclosure result (Theorem 4.2); some further properties of the solution set such as its compactness and directedness and the existence of extremal solutions are derived in Corollary 4.3.

2 Assumptions - problem setting

Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying the following conditions:

(A1) There exists $p \in (1, \infty)$ such that

$$|A(x, \xi)| \le b_1|\xi|^{p-1} + a_1(x), \tag{2.1}$$

$$A(x, \xi)\xi \ge b_2|\xi|^p - a_2(x), \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N, \tag{2.2}$$

where $b_1, b_2 > 0$, $a_1 \in L^{p'}(\Omega)$ (p' is the Hölder conjugate of p), and $a_2 \in L^1(\Omega)$,

(A2) A is monotone in the following sense:

$$[A(x, \xi_1) - A(x, \xi_2)](\xi_1 - \xi_2) \ge 0, \quad \text{for a.e. } x \in \Omega, \text{ all } \xi_1, \xi_2 \in \mathbb{R}^N. \tag{2.3}$$

Due to the growth condition of A, an appropriate choice of our function space is the usual Sobolev space $W^{1,p}(\Omega)$. Suppose K is a closed convex (nonempty) subset of $W^{1,p}(\Omega)$.

It is easy to see from (A1)–(A2) that the operator $\mathcal{A} : W^{1,p}(\Omega) \to [W^{1,p}(\Omega)]^*$,

$$\langle \mathcal{A}(u), v\rangle = \int_\Omega A(x, \nabla u(x))\nabla v(x)dx, \; u, v \in W^{1,p}(\Omega),$$

is well defined, continuous, bounded, and monotone ($\langle \cdot, \cdot \rangle_{X^*,X}$ denotes the dual pairing between X and its dual X^* and $\langle \cdot, \cdot\rangle = \langle \cdot, \cdot\rangle_{[W^{1,p}(\Omega)]^*, W^{1,p}(\Omega)}$).

Remark 2.1 (a) The variational (weak) formulation of (1.1) is the inclusion: Find $u \in W_0^1(\Omega)$ such that

$$\int_\Omega A(x, \nabla u)\nabla v dx + \int_\Omega f(x, u)v dx \ni \langle L, v\rangle, \tag{2.4}$$

for all $v \in W_0^1(\Omega)$. (The integral containing the multivalued lower order term $f(x, u)$ and the inclusion in (2.4) will be defined in a precise way later.) Hence, (1.2) reduces to (1.1) in the particular case when $K = W_0^{1,p}(\Omega)$.

If $K = h + W_0^{1,p}(\Omega)$ is a linear manifold in $W^{1,p}(\Omega)$ ($h \in W^{1,p}(\Omega)$) then (1.2) is the nonhomogeneous Dirichlet problem of the inclusion in (1.1) with the boundary condition $u = h$ on $\partial\Omega$. In the case $K = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Gamma\}$ (Γ is a measurable subset of $\partial\Omega$), (1.2) becomes a mixed Neumann-Dirichlet boundary value problem which reduces to a Neumann problem when $\Gamma = \emptyset$. If

$K = \{u \in W^{1,p}(\Omega) : u = \text{ constant on } \partial\Omega\}$, then we obtain a no-flux boundary problem, which is a multidimensional generalization of the periodic boundary condition for ordinary differential equations. In general, K is used to describe various obstacle or other unilateral constraints on Ω or its boundary.

(b) The results hereafter seem new even in the case of Dirichlet or Neumann boundary value problems, i.e., when the inequality (1.2) is an equation, due to the presence of the multivalued lower order term.

(c) If a boundary integral term such as $\int_{\partial\Omega} g(x,u)(v-u)dS$ where g is a multivalued function from $\partial\Omega \times \mathbb{R}$ to $2^{\mathbb{R}}$ is included in the right hand side of (1.2), then the variational inequality can be used to formulate other boundary conditions, such as nonhomogeneous Neumann or Robin condition, or a Steklov type problem. In such cases, the obtained results extend those in [21] or [5]. Since the adding of such terms could be done by combining the arguments and calculations in the sequel with those in [21], it will not be considered here for the sake of simplicity of presentation.

Assume L is an element of the dual $[W^{1,p}(\Omega)]^*$. Concerning f, we use the notation

$$\mathcal{K}(X) = \{A \subset X : A \neq \emptyset, A \text{ is closed and convex}\},$$

where X is a normed vector space. Let f be a function from $\Omega \times \mathbb{R}$ to $\mathcal{K}(\mathbb{R})$ such that:

(F1) f is graph measurable on $\Omega\times\mathbb{R}$, that is, $\mathrm{Gr}(f) = \{(x,u,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R} : \xi \in f(x,u)\}$ belongs to $[\mathcal{L}(\Omega)\times\mathcal{B}(\mathbb{R})]\times\mathcal{B}(\mathbb{R})$, where $\mathcal{L}(\Omega)$ is the family of Lebesgue measurable subsets of Ω and $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets in $\mathbb{R}$.

Note that if f is measurable from $\Omega \times \mathbb{R}$ to $\mathcal{K}(\mathbb{R})$ in the usual sense, that is $f^-(W) := \{(x,u) \in \Omega\times\mathbb{R} : f(x,u) \cap W \neq \emptyset\} \in \mathcal{L}(\Omega)\times\mathcal{B}(\mathbb{R})$ for all $W \subset \mathbb{R}$ open, then f is graph measurable on $\Omega\times\mathbb{R}$.

(F2) For a.e. $x \in \Omega$, the function $f(x,\cdot) : \mathbb{R} \to \mathcal{K}(\mathbb{R})$ is upper semicontinuous, that is, for each $u \in \mathbb{R}$ and each open $U \subset \mathbb{R}$ such that $f(x,u) \subset U$, there exists $\delta > 0$ such that if $|v-u| < \delta$ then $f(x,v) \subset U$.

In many places in the sequel, we also need the following growth condition on f:

(F3) There exist $q \in [1,p^*)$ (p^* is the Sobolev conjugate of p) and $a_3 \in L^{q'}(\Omega)$ (q' is the Hölder conjugate of q), $b_3 \geq 0$ such that

$$\sup\{|\xi| : \xi \in f(x,u)\} \leq a_3(x) + b_3|u|^{q-1}, \tag{2.5}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$. Note that if (2.5) is assumed then $f(x,u)$ is a compact interval in $\mathbb{R}$, hence (F2) is equivalent to the Hausdorff upper semicontinuity (h-u.s.c.) of $f(x,\cdot)$ for a.e. $x \in \Omega$ (cf. Theorem 2.68, Chap. 1, [17]).

We are now ready for a precise formulation of (1.2).

Definition 2.2 A function $u \in K$ is a *solution of (1.2)* if there exist $q \in [1,p^*)$ and $\eta \in L^{q'}(\Omega)$ (q' is the Hölder conjugate of q) such that

$$\eta(x) \in f(x,u(x)), \quad \text{for a.e. } x \in \Omega, \tag{2.6}$$

and

$$\int_\Omega A(x,\nabla u)(\nabla v - \nabla u)dx + \int_\Omega \eta(v-u)dx - \langle L, v-u\rangle \geq 0, \ \forall v \in K. \tag{2.7}$$

For $1 \le q \le p^*$, we denote by i_q the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. If $1 \le q < p^*$ then i_q is compact. Therefore its adjoint i_q^*, which is the projection from $L^{q'}(\Omega) \equiv [L^q(\Omega)]^*$ to $[W^{1,p}(\Omega)]^*$, is also compact. Note that $i_q(u) = u$ for $u \in W^{1,p}(\Omega)$, that is, $i_q(u)(x) = u(x)$ for a.e. $x \in \Omega$. Thus, to simplify the notation in the sequel, we shall use in many places u instead of $i_q(u)$. Similarly, i_q^* is the restriction of elements in $L^{q'}(\Omega) \equiv [L^q(\Omega)]^*$ on the functions in $W^{1,p}(\Omega)$, i.e., for $\eta \in L^{q'}(\Omega)$, $i_q^*(\eta) = \eta|_{W^{1,p}(\Omega)}$,

$$\langle i_q^*(\eta), v\rangle_{[W^{1,p}(\Omega)]^*, W^{1,p}(\Omega)} = \langle \eta, i_q(v)\rangle_{L^{q'}(\Omega), L^q(\Omega)} = \int_\Omega \eta v dx, \forall v \in W^{1,q}(\Omega). \tag{2.8}$$

Inequality (1.2) can also be stated as an inclusion as follows. Let I_K be the indicator functional of K, $I_K : W^{1,p}(\Omega) \to [0, \infty]$,

$$I_K(u) = \begin{cases} 0 & \text{if} \quad u \in K \\ \infty & \text{if} \quad u \notin K. \end{cases}$$

I_K is a proper, convex, and lower semicontinuous functional on $W^{1,p}(\Omega)$ with effective domain $D(I_K) = K$. Let ∂I_K be the subdifferential of I_K (in the sense of Convex Analysis), then (1.2) could be formulated as the following inclusion: Find $u \in K$ such that

$$\mathcal{A}(u) + (i_q^* \tilde{f} i_q)(u) + \partial I_K(u) - L \ni 0. \tag{2.9}$$

With the above assumptions, we see that u is a solution of (2.9) if and only if it satisfies (2.6)–(2.7). In fact, assume $u \in K$ satisfies (2.6) and (2.7). From (2.6), we have $i_q^*\eta \in i_q^*\tilde{f}(u) = i_q^*\tilde{f} i_q(u)$. Inequality (2.7) is equivalent to

$$\langle \mathcal{A}(u) + i_q^*\eta - L, v - u\rangle + I_K(v) - I_K(u) \ge 0, \forall v \in W^{1,p}(\Omega).$$

This means that $-[A(u) + i_q^*\eta - L] \in \partial I_K(u)$, i.e., u satisfies (2.9). On the other hand, if u satisfies (2.9) then there are $\eta \in \tilde{f}(u)$ and $D \in \partial I_K(u)$ such that $\mathcal{A}(u) + i_q^*\eta + D - L = 0$. The condition on η implies (2.6). Since $D = -[A(u) + i_q^*\eta - L]$, and $I_K(v) - I_K(u) \ge \langle D, v - u\rangle$, $\forall v \in W^{1,p}(\Omega)$, we have $0 \ge \langle -[A(u) + i_q^*\eta - L], v - u\rangle$, $\forall v \in K$. Together with (2.8), we see that (2.7) is satisfied.

Let u be any measurable function on Ω. From (F1), the function $f(\cdot, u(\cdot))$, $x \mapsto f(x, u(x))$, is also a measurable function from Ω to $\mathcal{K}(\mathbb{R})$. Let $\tilde{f}(u)$ be the set of all measurable selections of $f(\cdot, u(\cdot))$, that is,

$$\tilde{f}(u) = \{\eta : \Omega \to \mathbb{R} : \eta \text{ is measurable on } \Omega \text{ and } \eta(x) \in f(x, u(x)) \text{ for a.e. } x \in \Omega\}. \tag{2.10}$$

We know that $\tilde{f}(u) \neq \emptyset$ whenever u is measurable on Ω since $f(\cdot, u(\cdot))$ is measurable. Moreover, if the growth condition (2.5) is fulfilled then $\tilde{f}(u) \subset L^{q'}(\Omega)$ whenever $u \in L^q(\Omega)$. Some further properties of $\tilde{f}$ are given in the next section.

In the following sections, we study the existence and some properties of solutions of (2.9), i.e. of (2.6)-(2.7).

3 Coercive case

To study the existence of solutions of (2.9) under some coercivity conditions on A, f, and K, we need the following abstract result, which is a variant of Theorem 4.1 and Proposition 4.1 in [19].

Theorem 3.1 *Let $(X, \|\cdot\|)$ be a reflexive Banach space and $T : X \to 2^X$ be a multivalued mapping such that:*

(T1) (Condition (pm$_1$), [19] or Condition (a), Definition 1, [4]) For each $x \in X$, $T(x)$ is nonempty, convex, and closed in X^*.

(T2) (Condition (pm$_2$), [19] or Condition (c), Definition 1, [4]) If $\{x_n\} \subset X$, $\{x_n^*\} \subset X^*$ are sequences such that $x_n^* \in T(x_n)$, $\forall n \in \mathbb{N}$, $x_n \rightharpoonup x$ (weakly) in X, and

$$\limsup \langle x_n^*, x_n - x\rangle \le 0,$$

then to each $y \in X$, there exists $x^*(y) \in T(x)$ such that

$$\liminf \langle x_n^*, x_n - y\rangle \ge \langle x^*(y), x - y\rangle.$$

(T3) (Condition (pm$_4$), [19]) For each $x_0 \in K$, each bounded subset B of X, there exists a constant $N(B, x_0) \in \mathbb{R}$ such that $\langle x^*, x - x_0\rangle \ge N(B, x_0)$ for all $x \in B$, all $x^* \in T(x)$.

Assume K is a nonempty closed convex subset of X and $\phi : X \to \mathbb{R} \cup \{\infty\}$ is a proper convex, lower semicontinuous functional such that $D(\phi) \cap K \neq \emptyset$. Let $f \in X^*$.

If (T, K, ϕ, f) has the following coercivity condition: There exists $a \in D(\phi) \cap K$ such that

$$\lim_{x\in K, \|x\|\to\infty} \left(\inf_{x^*\in T(x)} [\langle x^* - f, x - a\rangle + \phi(x)] \right) = \infty, \tag{3.1}$$

then there exist $x_0 \in K$ and $x_0^* \in T(x_0)$ such that

$$\langle x_0^* - f, x - x_0\rangle + \phi(x) - \phi(x_0) \ge 0, \ \forall x \in K. \tag{3.2}$$

Proof. Let $R > \|a\|$ and $\overline{B}_R = \overline{B_R(0)} = \{x \in X : \|x\| \le R\}$ be the closed ball with radius R centered at 0. From Proposition 4.1 of [19], the variational inequality (3.2), restricted to $K \cap \overline{B}_R$, has a solution x_R, i.e., there exist $x_R \in K \cap \overline{B}_R$ and $x_R^* \in T(x_R)$ such that

$$\langle x_R^* - f, x - x_R\rangle + \phi(x) - \phi(x_R) \ge 0, \ \forall x \in K \cap \overline{B}_R. \tag{3.3}$$

Next, we prove that there exists $R > \|a\|$ such that

$$\|x_R\| < R, \tag{3.4}$$

where x_R is any solution of (3.3). In fact, assume otherwise that $\|x_R\| = R$ for all solutions x_R of (3.3), all $R > \|a\|$. Letting $a \in K \cap \overline{B}_R$ into (3.3) yields

$$\langle x_R^* - f, a - x_R\rangle + \phi(a) - \phi(x_R) \ge 0,$$

i.e.,

$$\langle x_R^* - f, x_R - a\rangle + \phi(x_R) \le \phi(a),$$

and thus

$$\limsup_{R\to\infty} [\inf_{x^*\in T(x_R)} \langle x^* - f, x_R - a\rangle + \phi(x_R)] \le \phi(a),$$

contradicting (3.1) since $\|x_R\| = R \to \infty$ as $R \to \infty$.

Let x_R be a solution of (3.3) that satisfies (3.4). Let us show that x_R is also a solution of (3.2). In fact, let $x \in K$. For $t > 0$ sufficiently small, we have $v = x_R + t(x - x_R) \in \overline{B}_R$. Moreover, $v = (1-t)x_R + tx \in K$ since $x, x_R \in K$. Letting $v \in \overline{B}_R \cap K$ in (3.3) gives

$$\langle x_R^* - f, t(x - x_R)\rangle + \phi((1-t)x_R + tx) - \phi(x_R) \geq 0.$$

However,

$$\phi((1-t)x_R + tx) - \phi(x_R) \leq t[\phi(x) - \phi(x_R)],$$

and thus $t[\langle x_R^* - f, x - x_R\rangle + \phi(x) - \phi(x_R)] \geq 0$. Since $t > 0$, this gives us (3.2) with $x_0 = x_R$. □

Remark 3.2 (a) If

$$\lim_{\|x\|\to\infty, x\in K} \left(\inf_{x^*\in T(x)} \frac{\langle x^*, x - a\rangle + \phi(x)}{\|x\|} \right) = \infty,$$

then (3.1) is satisfied for all $f \in X^*$. Hence, Theorem 3.1 implies Theorem 4.1, [19].

(b) Since $\phi + I_K$ is convex, the subdifferential $\partial(\phi + I_K) : X \to 2^{X^*}$ is maximal monotone. The variational inequality above can be written equivalently as the following inclusion for multivalued operators: Find $x_0 \in X$ such that

$$T(x_0) + \partial(\phi + I_K)(x_0) \ni f.$$

Theorem 3.1 does not follow from existence theorems for multivalued pseudomonotone perturbations of maximal monotone operators such as Theorem 7 in [4], Theorem 3.2 in [19], or Theorems 2.11 and 2.12 in [24]. In those cited theorems, the coercivity conditions are solely on the pseudomonotone operators and not on the combinations of the pseudomonotone operators and maximal monotone operators (or convex functionals) as considered in Theorem 3.1. Such combined coercivity conditions for variational inequalities or equalities are particularly relevant for boundary value problems with principal terms given by Leray–Lions type operators and lower order terms given by Niemytskii operators associated with some functions.

As an example, the p-Laplace equation with

$$\langle T(u), v\rangle = \int_\Omega [|\nabla u|^{p-2}\nabla u \nabla v + f(x,u)v]dx,$$

with sublinear lower order term $f(x,u)$ with $|f(x,u)| \leq a_1 + b_1|u|^\alpha$, $0 \leq \alpha < p-1$, is coercive in the sense of (3.1) on the linear manifold $K = h + W_0^{1,p}(\Omega)$, T is a pseudomonotone operator on $X = W^{1,p}(\Omega)$, but is not coercive on $X = W^{1,p}(\Omega)$.

(c) As noted in [19], if T satisfied (T1), (T2), and (T3) (i.e, (pm_1), (pm_2), and (pm_4) in [19]) then T is pseudomonotone (the definition of multivalued pseudomonotonicity in [4] consists of (T1)-(T2) and the weak upper semicontinuity of T on finite dimensional subspaces of X). On the other hand, if T is monotone with $D(T) = X$ or if T is bounded then T satisfies (T3). Therefore, for bounded operators, the combination (T1)-(T2)-(T3) above is equivalent to the pseudomonotonicity of T.

We have the following existence result for (2.9), or equivalently (2.6)-(2.7), under certain coercivity condition.

Theorem 3.3 *Assume f satisfies (F1)-(F2)-(F3) and there exists $u_0 \in K$ such that*

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty, u\in K} \left\{ \inf_{\eta\in\tilde{f}(u)} \int_\Omega [A(x,\nabla u)(\nabla u - \nabla u_0) + \eta(u-u_0)]dx - \langle L, u-u_0\rangle \right\} = \infty, \tag{3.5}$$

or equivalently,

$$\int_\Omega [A(x,\nabla u)(\nabla u - \nabla u_0) + \eta(u-u_0)]dx - \langle L, u-u_0\rangle \geq c(\|u\|_{W^{1,p}(\Omega)}), \tag{3.6}$$

for all $u \in K$, all $\eta \in \tilde{f}(u)$, where $c : [0,\infty) \to \mathbb{R}$, $c(r) \to \infty$ as $r \to \infty$. Then, (2.6)-(2.7) has a solution.

Some properties of $\tilde{f}$ are need for the proof of Theorem 3.3.

Lemma 3.4 *Under assumptions (F1)-(F2)-(F3), if $u \in L^q(\Omega)$ then, $\tilde{f}(u)$ is a bounded, closed, and convex subset of $L^{q'}(\Omega)$; in particular, $\tilde{f}(u) \in \mathcal{K}(L^{q'}(\Omega))$. Moreover, $\tilde{f}$ is a bounded mapping from $L^q(\Omega)$ to $\mathcal{K}(L^{q'}(\Omega))$.*

Proof. The convexity of $\tilde{f}(u)$ follows from the fact that $f(x,u)$ is a closed interval in $\mathbb{R}$. Let $\eta \in \tilde{f}(u)$. From (2.5),

$$|\eta(x)| \leq a_3(x) + b_3|u(x)|^{q-1}, \quad \text{a.e. } x \in \Omega. \tag{3.7}$$

Since $|u|^{q-1} \in L^{q'}(\Omega)$ due to $u \in L^q(\Omega)$, we have the boundedness of $\tilde{f}(u)$ in $L^{q'}(\Omega)$. Inequality (3.7) also proves that if W is a bounded set in $L^q(\Omega)$ then $\tilde{f}(W) = \bigcup_{u\in W} \tilde{f}(u)$ is a bounded set in $L^{q'}(\Omega)$, that is, $\tilde{f}$ is a bounded operator from $L^q(\Omega)$ to $2^{L^{q'}(\Omega)}$.

To verify that $\tilde{f}(u)$ is closed in $L^{q'}(\Omega)$, let $\{\eta_n\}$ be a sequence in $\tilde{f}(u)$ such that $\eta_n \to \eta$ in $L^{q'}(\Omega)$. By passing to a subsequence, we can assume without loss of generality that $\eta_n(x) \to \eta(x)$ for a.e. $x \in \Omega$. Since $\eta_n(x) \in f(x,u(x))$ for a.e. $x \in \Omega$, all $n \in \mathbb{N}$, and $f(x,u(x))$ is closed in $\mathbb{R}$, we have $\eta(x) \in f(x,u(x))$. Since this holds for a.e. $x \in \Omega$, we have $\eta \in \tilde{f}(u)$, which proves the closedness of $\tilde{f}(u)$ in $L^{q'}(\Omega)$. □

Another property of $\tilde{f}(u)$ is given in the following lemma.

Lemma 3.5 *Under assumptions (F1)-(F2)-(F3), $\tilde{f}$ is Hausdorff upper semicontinuous (h-u.s.c.) from $L^q(\Omega)$ to $\mathcal{K}(L^{q'}(\Omega))$, that is, for each $u_0 \in L^q(\Omega)$, the function*

$$u \mapsto h^*_{L^{q'}(\Omega)}(\tilde{f}(u), \tilde{f}(u_0)) \tag{3.8}$$

is continuous at u_0, where

$$h^*_{L^{q'}(\Omega)}(A,B) = \sup_{u\in A} \left(\inf_{v\in B} \|u-v\|_{L^{q'}(\Omega)} \right), \tag{3.9}$$

for $A, B \subset L^{q'}(\Omega)$.

This property of $\tilde{f}$ is given in Theorem 7.26, [17]. We only note a small misprint in the condition related to $a(\cdot)$ in the statement of that theorem.

The following lemma is crucial for the proof of Theorem 3.3 as well as later developments.

Lemma 3.6 *If f satisfies condition (F3) then the operator $i_q^* \tilde{f} i_q$ is pseudomonotone and bounded from $W^{1,p}(\Omega)$ to $\mathcal{K}([W^{1,p}(\Omega)]^*)$.*

Proof. We first check that $i_q^* \tilde{f} i_q$ is weakly closed from $W^{1,p}(\Omega)$ into $2^{W^{1,p}(\Omega)} \setminus \{\emptyset\}$, that is, if $\{u_n\}$ and $\{\eta_n\}$ are sequences in $W^{1,p}(\Omega)$ and $[W^{1,p}(\Omega)]^*$ respectively such that

$$u_n \rightharpoonup u \ \text{(weakly) in} \ W^{1,p}(\Omega), \tag{3.10}$$

$$\eta_n \rightharpoonup \eta \ \text{(weakly) in} \ [W^{1,p}(\Omega)]^*, \tag{3.11}$$

and

$$\eta_n \in i_q^* \tilde{f} i_q(u_n), \ \forall n \in \mathbb{N}, \tag{3.12}$$

then,

$$\eta \in i_q^* \tilde{f} i_q(u). \tag{3.13}$$

In fact, assume (3.10)-(3.12). As noted in (2.8), $i_q(u_n) = u_n$ and $i_q^*(\eta_n) = \eta_n|_{W^{1,p}(\Omega)}$. From (3.12), for each n, there exists $\tilde{\eta}_n \in \tilde{f} i(u_n) = \tilde{f}(u_n)$ such that $\eta_n = i_q^*(\tilde{\eta}_n) = \tilde{\eta}_n|_{W^{1,p}(\Omega)}$. From (3.10) and the compactness of i_q, we have

$$u_n = i_q(u_n) \to i_q(u) = u \ \text{(strongly) in} \ L^q(\Omega). \tag{3.14}$$

Hence, from the h-upper semicontinuity of $\tilde{f}$ from $L^q(\Omega)$ to $\mathcal{K}(L^{q'}(\Omega))$ (cf. Lemma 3.5), we have

$$h^*(\tilde{f}(u_n), \tilde{f}(u)) \to 0, \tag{3.15}$$

where h^* is given in (3.9). Since $\tilde{\eta}_n \in \tilde{f}(u_n)$,

$$\inf_{v \in \tilde{f}(u)} \|\tilde{\eta}_n - v\|_{L^{q'}(\Omega)} \le h^*(\tilde{f}(u_n), \tilde{f}(u)).$$

Hence, $\inf_{v \in \tilde{f}(u)} \|\tilde{\eta}_n - v\|_{L^{q'}(\Omega)} \to 0$ as $n \to \infty$, and there exists a sequence $\{\eta_n^*\} \subset \tilde{f}(u)$ such that

$$\lim_{n \to \infty} \|\tilde{\eta}_n - \eta_n^*\|_{L^{q'}(\Omega)} = 0. \tag{3.16}$$

Since $\{\eta_n^*\} \subset \tilde{f}(u)$ and $\tilde{f}(u)$ is a bounded subset of $L^{q'}(\Omega)$, by passing to a subsequence if necessary, we can assume that

$$\eta_n^* \rightharpoonup \eta_0 \ \text{(weakly) in} \ L^{q'}(\Omega) \tag{3.17}$$

for some $\eta_0 \in L^{q'}(\Omega)$. As $\tilde{f}(u)$ is closed and convex in $L^{q'}(\Omega)$, it is weakly closed there; thus $\eta_0 \in \tilde{f}(u)$ as a consequence of (3.17). Hence, (3.16) and (3.17) imply that

$$\tilde{\eta}_n \rightharpoonup \eta_0 \ \text{(weakly) in} \ L^{q'}(\Omega). \tag{3.18}$$

Since $i_q^* : L^{q'}(\Omega) \to [W^{1,q}(\Omega)]^*$ is continuous (in the strong topologies), it is also continuous in the weak topologies of both $L^{q'}(\Omega)$ and $[W^{1,q}(\Omega)]^*$. From (3.18),

$$\eta_n = i_q^*(\tilde{\eta}_n) = \tilde{\eta}_n|_{W^{1,p}(\Omega)} \rightharpoonup i_q^*(\eta_0) = \eta_0|_{W^{1,p}(\Omega)} \tag{3.19}$$

weakly in $[W^{1,p}(\Omega)]^*$. From (3.11) and (3.19), we have $\eta = i_q^*(\eta_0) \in i_q^* \tilde{f}(u)$ since $\eta_n \rightharpoonup \eta$ and $\eta_n \rightharpoonup i_q^*(\eta_0)$ both in the sense of distribution. (3.13) is thus proved, which completes our proof of the weakly closed property of $i_q^*\tilde{f}i_q$. As a direct consequence of this closedness, we see that $(i_q^*\tilde{f}i_q)(u)$ is closed in $[W^{1,p}(\Omega)]^*$.

The above arguments show that for each $u \in W^{1,p}(\Omega)$, $(i_q^*\tilde{f}i_q)(u) \in \mathcal{K}([W^{1,p}(\Omega)]^*)$. Moreover, since i_q^* is linear and bounded, $i_q^*\tilde{f}i_q$ is a bounded operator from $W^{1,p}(\Omega)$ to $\mathcal{K}([W^{1,p}(\Omega)]^*)$ from the corresponding property of $\tilde{f}$ from $L^q(\Omega)$ to $\mathcal{K}(L^{q'}(\Omega))$. In particular, $i_q^*\tilde{f}i_q(u)$ is a bounded, closed, and nonempty convex subset of $[W^{1,p}(\Omega)]^*$.

Next, we check that if $\{u_n\} \subset W^{1,p}(\Omega)$, $\{\eta_n\} \subset [W^{1,p}(\Omega)]^*$ are sequences satisfying (3.10)-(3.12) then

$$\langle \eta_n, u_n\rangle_{[W^{1,p}(\Omega)]^*,W^{1,p}(\Omega)} \to \langle \eta, u\rangle_{[W^{1,p}(\Omega)]^*,W^{1,p}(\Omega)}. \tag{3.20}$$

Let $\{\tilde{\eta}_n\}$ and η_0 be as above. We have

$$\begin{aligned}\langle \eta_n, u_n\rangle_{[W^{1,p}(\Omega)]^*,W^{1,p}(\Omega)} &= \langle \tilde{\eta}_n|_{W^{1,p}(\Omega)}, u_n\rangle_{[W^{1,p}(\Omega)]^*,W^{1,p}(\Omega)}\\ &= \langle i_q^*(\tilde{\eta}_n), u_n\rangle_{[W^{1,p}(\Omega)]^*,W^{1,p}(\Omega)}\\ &= \langle \tilde{\eta}_n, i_q(u_n)\rangle_{L^{q'}(\Omega),L^q(\Omega)} = \langle \tilde{\eta}_n, u_n\rangle_{L^{q'}(\Omega),L^q(\Omega)}.\end{aligned} \tag{3.21}$$

From (3.14) and (3.18),

$$\begin{aligned}\langle \tilde{\eta}_n, u_n\rangle_{L^{q'}(\Omega),L^q(\Omega)} &\to \langle \eta_0, u\rangle_{L^{q'}(\Omega),L^q(\Omega)} = \langle \eta_0, i_q(u)\rangle_{L^{q'}(\Omega),L^q(\Omega)}\\ &= \langle i_q^*(\eta_0), u\rangle_{[W^{1,p}(\Omega)]^*,W^{1,p}(\Omega)}\\ &= \langle \eta, u\rangle_{[W^{1,p}(\Omega)]^*,W^{1,p}(\Omega)}.\end{aligned}$$

This limit, together with (3.21), proves (3.20).

The weakly closed property of $i_q^*\tilde{f}i_q$ and (3.20) show that $i_q^*\tilde{f}i_q$ is generalized pseudomonotone (cf. [4]) which together with its boundedness, implies that $i_q^*\tilde{f}i_q$ is pseudomonotone from $W^{1,p}(\Omega)$ to its dual $[W^{1,p}(\Omega)]^*$ (cf. Proposition 4 and Definitions 1 and 2 in [4]). □

Proof of Theorem 3.3. Let $T : W^{1,p}(\Omega) \to 2^{[W^{1,p}(\Omega)]^*}$, $T(u) = \mathcal{A}(u) + (i_q^*\tilde{f}i_q)(u)$. From (A1)-(A2), we see that $\mathcal{A}$ is a (singlevalued) monotone and bounded operator from $W^{1,p}(\Omega)$ to $[W^{1,p}(\Omega)]^*$.

From Proposition 8 in [4], $\mathcal{A}$ is pseudomonotone (and bounded) on $W^{1,p}(\Omega)$. From Lemma 3.6, $i_q^*\tilde{f}i_q$ is bounded and pseudomonotone. Hence, from Proposition 9 in [4], $T = \mathcal{A} + i_q^*\tilde{f}i_q$ is a multivalued pseudomonotone and bounded operator from $W^{1,p}(\Omega)$ to $\mathcal{K}([W^{1,p}(\Omega)]^*)$. As noted in Remark 3.2 (c), T satisfies conditions (T1)-(T3) in Theorem 3.1 with $X = W^{1,p}(\Omega)$. Letting $\phi = 0$ and $f = L \in X^*$, we see that condition (3.5) (or equivalently (3.6)) is the same as condition (3.1) in our setting here. Also, in this setting (3.2) is the same as inequality (2.7). □

As an illustrating consequence of the above abstract theorem, let us prove an existence result for a mixed nonhomogeneous Dirichlet problem with "sublinear" multivalued lower order term. Let Γ be a nonempty open subset of $\partial\Omega$ and

$$W_\Gamma^{1,p} = \{u \in W^{1,p}(\Omega) : u|_{\partial\Omega}(x) = 0 \text{ for a.e. } x \in \Gamma\}.$$

Also, let $h \in W^{1,p}(\Omega)$ and let K be a nonempty closed convex subset of $h + W_\Gamma^{1,p}$. We have the following existence theorem for (2.7).

Theorem 3.7 *Assume f satisfies (F1)-(F2) and (2.5) with*

$$1 \leq q < p. \tag{3.22}$$

Then there exist u (and η) that satisfy (2.6)-(2.7).

Proof. Poincaré's inequality implies that there exists $C_0 > 0$ such that

$$\| |\nabla u| \|_{L^p(\Omega)} \geq C_0 \|u\|_{L^p(\Omega)}, \ \forall u \in W^{1,p}_\Gamma. \tag{3.23}$$

Therefore, there exists $C_1 > 0$ such that

$$\| |\nabla u| \|_{L^p(\Omega)} \geq C_1 \|u\|_{W^{1,p}(\Omega)}, \ \forall u \in W^{1,p}_\Gamma.$$

Let u_0 be a (fixed) element of K. We have from (2.2) that

$$\int_\Omega A(x, \nabla u)\nabla u dx \geq b_2 \| |\nabla u| \|^p_{L^p(\Omega)} - \|a_2\|_{L^1(\Omega)}. \tag{3.24}$$

Since $u - u_0 \in W^{1,p}_\Gamma$ and $|\nabla u| \geq | |\nabla(u-u_0)| - |\nabla u_0| |$ a.e. on Ω, we have

$$\begin{aligned} \| |\nabla u| \|_{L^p(\Omega)} &\geq \| |\nabla(u-u_0)| - |\nabla u_0| \|_{L^p(\Omega)} \\ &\geq C_1 \|u - u_0\|_{W^{1,p}(\Omega)} - \| |\nabla u_0| \|_{L^p(\Omega)} \\ &\geq C_1 \|u\|_{W^{1,p}(\Omega)} - C_2, \end{aligned}$$

where $C_2 = C_1 \|u_0\|_{W^{1,p}(\Omega)} + \| |\nabla u_0| \|_{L^p(\Omega)}$. If $\|u\|_{W^{1,p}(\Omega)} \geq \dfrac{2C_2}{C_1}$ then $\dfrac{C_1}{2}\|u\|_{W^{1,p}(\Omega)} \geq C_2$ and thus $\| |\nabla u| \|_{L^p(\Omega)} \geq \dfrac{C_1}{2}\|u\|_{W^{1,p}(\Omega)}$ and

$$\int_\Omega A(x, \nabla u)\nabla u dx \geq C_3 \|u\|^p_{W^{1,p}(\Omega)}, \tag{3.25}$$

for all $u \in W^{1,p}_\Gamma$, $\|u\|_{W^{1,p}(\Omega)} \geq \dfrac{2C_2}{C_1}$, where $C_3 = b_2 \left(\dfrac{C_1}{2}\right)^p$. (Here and in the next estimates, C_k stands for a generic constant that does not depend on $u \in W^{1,p}_\Gamma$.) On the other hand, it follows from (2.1) that

$$\begin{aligned} &\int_\Omega A(x, \nabla u)\nabla u_0 dx \\ \leq\ & b_1 \int_\Omega |\nabla u|^{p-1} |\nabla u_0| dx + \int_\Omega a_1(x)|\nabla u_0| dx \\ \leq\ & b_1 \| |\nabla u| \|^{p-1}_{L^p(\Omega)} \| |\nabla u_0| \|_{L^p(\Omega)} + \|a_1\|_{L^{p'}(\Omega)} \| |\nabla u_0| \|_{L^p(\Omega)} \\ \leq\ & C_4 (\|u\|^{p-1}_{W^{1,p}(\Omega)} + 1). \end{aligned} \tag{3.26}$$

Using (2.5), we have for all $u \in W^{1,p}_\Gamma$, all $\eta \in \tilde{f}(u)$,

$$\begin{aligned} \int_\Omega \eta(u - u_0) dx &\leq \int_\Omega (a_3 + b_3|u|^{q-1})(|u| + |u_0|) dx \\ &\leq \|a_3\|_{L^{q'}(\Omega)} \|u\|_{L^q(\Omega)} + b_3 \|u\|^q_{L^q(\Omega)} + b_3 \|u\|^{q-1}_{L^q(\Omega)} \|u_0\|_{L^q(\Omega)} \\ &\quad + \|a_3\|_{L^{q'}(\Omega)} \|u_0\|_{L^q(\Omega)} \\ &\leq C_5 (\|u\|_{W^{1,p}(\Omega)} + \|u\|^q_{W^{1,p}(\Omega)} + \|u\|^{q-1}_{W^{1,p}(\Omega)} + 1). \end{aligned} \tag{3.27}$$

Lastly,

$$\begin{aligned} |\langle L, u-u_0\rangle| &\le \|L\|_{[W^{1,p}(\Omega)]^*}(\|u\|_{W^{1,p}(\Omega)} + \|u_0\|_{W^{1,p}(\Omega)}) \\ &\le C_6(\|u\|_{W^{1,p}(\Omega)} + 1). \end{aligned} \tag{3.28}$$

Combining (3.25)-(3.28), we see that

$$\begin{aligned} &\int_\Omega [A(x,\nabla u)(\nabla u - \nabla u_0) + \eta(u-u_0)]dx - \langle L, u-u_0\rangle \\ \ge\ & C_3\|u\|^p_{W^{1,p}(\Omega)} - C_7\left(\|u\|^{p-1}_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)} + \|u\|^q_{W^{1,p}(\Omega)} + \|u\|^{q-1}_{W^{1,p}(\Omega)} + 1\right), \end{aligned}$$

for all $u \in K$ with $\|u\|_{W^{1,p}(\Omega)} \ge \dfrac{2C_2}{C_1}$.

Since $q < p$ and $p > 1$, this estimate immediately gives us (3.5) (or (3.6)), which in view of Theorem 3.3, ensures the existence of solutions of (2.6)-(2.7). □

4 Noncoercive case - sub-supersolution method

In cases where the coercivity condition (3.5) (or the growth condition (3.22)) is not satisfied then (2.6)-(2.7) may not have solutions. However, if sub- and supersolutions of (2.6)-(2.7), in a certain appropriate sense, exist and f only satisfies some local growth condition, then we still get the solvability of (2.6)-(2.7), together with other qualitative properties of their solutions.

Definition 4.1 A function $\underline{u} \in W^{1,p}(\Omega)$ is a *subsolution* of (1.2), or more precisely of (2.6)-(2.7), if there exists $q \in [1, p^*)$ and $\underline{\eta} \in L^{q'}(\Omega)$ such that

$$\underline{\eta}(x) \in f(x, \underline{u}(x)) \ \text{ for a.e. } \ x \in \Omega, \tag{4.1}$$

(i.e., $\underline{\eta} \in L^{q'}(\Omega) \cap \tilde{f}(\underline{u})$) such that

$$\int_\Omega A(\cdot, \nabla\underline{u})(\nabla v - \nabla\underline{u})dx + \int_\Omega \underline{\eta}(v-\underline{u})dx - \langle L, v-\underline{u}\rangle \ge 0, \tag{4.2}$$

for all $v \in \underline{u} \wedge K := \{\underline{u} \wedge w = \min\{\underline{u}, w\} : w \in K\}$.

Similarly, $\overline{u} \in W^{1,p}(\Omega)$ is a *supersolution* of (2.6)-(2.7) if there is $q \in [1, p^*)$ and

$$\overline{\eta} \in L^{q'}(\Omega) \cap \tilde{f}(\overline{u}) \tag{4.3}$$

such that

$$\int_\Omega A(\cdot, \nabla\overline{u})(\nabla v - \nabla\overline{u})dx + \int_\Omega \overline{\eta}(v-\overline{u})dx - \langle L, v-\overline{u}\rangle \ge 0, \tag{4.4}$$

for all $v \in \overline{u} \vee K := \{\overline{u} \vee w = \max\{\overline{u}, w\} : w \in K\}$.

We have the following general existence and enclosure/comparison theorem for (2.7) when sub- and supersolutions exist.

Theorem 4.2 *Let A and f satisfy (A1)-(A2)-(A3) and (F1)-(F2). Assume there are subsolutions $\underline{u}_i$, $i = 1, \dots, k$, and supersolutions $\overline{u}_j$, $j = 1, \dots, m$, of (2.6)-(2.7) such that*

$$\underline{u} := \max\{\underline{u}_i : 1 \le i \le k\} \le \overline{u} := \min\{\overline{u}_j : 1 \le j \le m\}, \tag{4.5}$$

and

$$\underline{u}_i \vee K \subset K, \overline{u}_j \wedge K \subset K, \forall i \in \{1, \dots, k\}, j \in \{1, \dots, m\}. \tag{4.6}$$

Suppose f satisfies the following local growth condition: There exist $q \in [1, p^*)$ and $a_4 \in L^{q'}(\Omega)$ such that

$$\sup\{|\xi| : \xi \in f(x, u)\} \le a_4(x), \tag{4.7}$$

for a.e. $x \in \Omega$, all $u \in [\underline{u}(x), \overline{u}(x)]$.

Then, there exists a solution u of (2.6)-(2.7) such that

$$\underline{u} \le u \le \overline{u} \text{ a.e. on } \Omega. \tag{4.8}$$

Proof. Note that the numbers $q \in [1, p^*)$ in the definition of $\underline{u}_i$ and $\overline{u}_j$ and in the growth condition (4.7) may be different. However, the conditions in the definition of sub- and supersolutions in Definition 4.1 and in (4.7) still hold if q is replaced by q_0 with $1 \le q \le q_0 < p^*$ (thus $\infty \ge q' \ge q_0'$). Hence, by replacing such numbers q's in the definitions of $\underline{u}_i$ and $\overline{u}_j$ and in (4.7) by their greatest value (which still is in the interval $[1, p^*)$), we can assume without loss of generality and for simplicity of notation that we have the same number q in (4.7) and in the conditions of $\underline{\eta}_i$ and $\overline{\eta}_j$, where $\underline{\eta}_i$ and $\overline{\eta}_j$ are the measurable selections of $f(\cdot, \underline{u}_i)$ and $f(\cdot, \overline{u}_j)$ in Definition 4.1.

We use the usual truncation-regularization method with some essential modifications and extensions for multivalued functions, as seen below. As noted above, for each $i \in \{1, \dots, k\}$ (resp. $j \in \{1, \dots, m\}$), $\underline{\eta}_i$ (resp. $\overline{\eta}_j$) is a function in $L^{q'}(\Omega)$ satisfying (4.1) and (4.2) with $\underline{\eta}_i$ instead of $\underline{\eta}$ (resp. (4.3) and (4.4) with $\overline{\eta}_j$ instead of $\overline{\eta}$). We construct families $\{\Omega_i : 1 \le i \le k\}$ and $\{\Omega^j : 1 \le j \le m\}$ of subsets of Ω inductively as follows. Let $\Omega_1 = \{x \in \Omega : \underline{u}(x) = \underline{u}_1(x)\}$, and $\Omega_i = \left\{x \in \Omega \setminus \bigcap_{l=1}^{i-1} \Omega_l : \underline{u}(x) = \underline{u}_i(x)\right\}$ for $i = 2, \dots, k$. Similarly, let $\Omega^1 = \{x \in \Omega : \overline{u}(x) = \overline{u}_1(x)\}$, and $\Omega^j = \left\{x \in \Omega \setminus \bigcap_{l=1}^{j-1} \Omega^l : \overline{u}(x) = \overline{u}^j(x)\right\}$ for $j = 2, \dots, m$. It is clear that $\Omega_i (1 \le i \le k)$ (resp. $\Omega^j (1 \le j \le m)$) are disjoint measurable subsets of Ω and

$$\Omega = \bigcup_{i=1}^{k} \Omega_i = \bigcup_{j=1}^{m} \Omega^j.$$

Let us define

$$\underline{\eta} = \sum_{i=1}^{k} \underline{\eta}_i \chi_{\Omega_i} \text{ and } \overline{\eta} = \sum_{j=1}^{m} \overline{\eta}_j \chi_{\Omega^j},$$

where χ_A $(A \subset \Omega)$ is the characteristic function of A. From their definitions, we see that $\underline{\eta}, \overline{\eta} \in L^{q'}(\Omega)$. Moreover, since $\underline{\eta}(x) = \underline{\eta}_i(x)$ and $\underline{u}(x) = \underline{u}_i(x)$ for a.e. $x \in \Omega_i$ $(1 \le i \le k)$, we have

$$\underline{\eta}(x) \in f(x, \underline{u}(x)) \text{ for a.e. } x \in \Omega. \tag{4.9}$$

Similarly, $\overline{\eta}(x) \in f(x, \overline{u}(x))$ for a.e. $x \in \Omega$.

Next, we define the truncated function for $f(x, u)$. Let $f_0 : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ be given by

$$f_0(x, u) = \begin{cases} \{\underline{\eta}(x)\} & \text{if} \quad u < \underline{u}(x) \\ f(x, u) & \text{if} \quad \underline{u}(x) \le u \le \overline{u}(x) \\ \{\overline{\eta}(x)\} & \text{if} \quad u > \overline{u}(x). \end{cases} \tag{4.10}$$

Then, f_0 satisfies (F_1) and (F_2). In fact, we first note that

$$\begin{aligned} \mathrm{Gr}(f_0) &= \{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : \xi \in f_0(x, u)\} \\ &= [\{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : u > \overline{u}(x)\} \cap \{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : \xi = \overline{\eta}(x)\}] \\ &\quad \cup [\{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : u < \underline{u}(x)\} \cap \{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : \xi = \underline{\eta}(x)\}] \\ &\quad \cup [\{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : u \ge \underline{u}(x)\} \cap \{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : u \le \overline{u}(x)\} \\ &\quad \cap \{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : \xi \in f(x, u)\}]. \end{aligned}$$

All sets in the right hand side belong to $\mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ (the last set does because f satisfies condition (F1), the other sets are in $\mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ thanks to the fact that Carathéodory functions are jointly measurable). Therefore $\mathrm{Gr}(f_0) \in \mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, i.e. f_0 satisfies $(F1)$.

To verify that $(F2)$ holds for f_0, let x be any point in Ω such that $\overline{\eta}(x) \in f(x, \overline{u}(x))$ and $\underline{\eta}(x) \in f(x, \underline{u}(x))$ (note that the set of all $x \in \Omega$ not satisfying these inclusions has measure 0 by (4.9)). We show that the function $f_0(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is u.s.c. at each $u \in \mathbb{R}$. Assume W is an open set in $\mathbb{R}$ such that $f_0(x, u) \subset W$.

If $u > \overline{u}(x)$ then $f_0(x, u) = \overline{\eta}(x) \in W$. Let $\delta \in (0, u - \overline{u}(x))$. If $v \in \mathbb{R}$, $|v - u| < \delta$ then $v > \overline{u}(x)$ and thus $f_0(x, v) = \overline{\eta}(x) \in W$. Hence, $f_0(x, \cdot)$ is u.s.c. at u. Similar arguments show the upper semicontinuity of $f_0(x, \cdot)$ at every $u \in (-\infty, \underline{u}(x))$. Assume now that $u \in [\underline{u}(x), \overline{u}(x)]$. By the upper semicontinuity of $f(x, \cdot)$ at u assumed in (F2), there exists $\delta > 0$ such that $f(x, v) \subset W$ for any $v \in (u - \delta, u + \delta)$. For such δ and v, we have three possibilities: (i) $v \in [\underline{u}(x), \overline{u}(x)]$, (ii) $v > \overline{u}(x)$, and (iii) $v < \underline{u}(x)$. In case (i), we have $f_0(x, v) = f(x, v) \subset W$. In case (ii), we have $f_0(x, v) = \{\overline{\eta}(x)\}$. Since $u \le \overline{u}(x) < v$ and $|u - v| = v - u < \delta$, one has $|\overline{u}(x) - u| = \overline{u}(x) - u < v - u < \delta$, and thus from the choice of δ, $f(x, \overline{u}(x)) \subset W$. However, since $\overline{\eta}(x) \in f(x, \overline{u}(x))$, we have $f_0(x, v) = \{\overline{\eta}(x)\} \subset f(x, \overline{u}(x)) \subset W$. Similar proofs show that in case (iii), $f_0(x, v) = \{\underline{\eta}(x)\} \subset f(x, \underline{u}(x)) \subset W$ if $|v - u| < \delta$ and $v < \underline{u}(x)$. We have shown that in all cases, $f_0(x, v) \subset W$ whenever $|v - u| < \delta$. The upper semicontinuity of $f_0(x, \cdot)$ at u when $\underline{u}(x) \le u \le \overline{u}(x)$ and therefore in all cases is proved. We have checked that f_0 satisfies (F2).

Next, it follows from (4.7) and (4.10) that

$$\sup\{|\xi| : \xi \in f_0(x, u)\} \le a_4(x) + |\overline{\eta}(x)| + |\underline{\eta}(x)|, \tag{4.11}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, where $a_4 + |\overline{\eta}| + |\underline{\eta}| \in L^{q'}(\Omega)$. Hence, f_0 satisfies (2.5) in (F3) with $a_3 = a_4 + |\overline{\eta}| + |\underline{\eta}|$ and $b_3 = 0$.

As in the singlevalued case, we also need the following regularization function $b : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$b(x,u) = \begin{cases} [u - \overline{u}(x)]^{p-1} & \text{if} \quad u > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \le u \le \overline{u}(x) \\ -[\underline{u}(x) - u]^{p-1} & \text{if} \quad u < \underline{u}(x), \quad \text{for } x \in \Omega, u \in \mathbb{R}. \end{cases} \tag{4.12}$$

We see that b is a Carathéodory function and since $\underline{u}, \overline{u} \in L^p(\Omega)$,

$$|b(x,u)| \le a_5(x) + b_5|u|^{p-1}, \ (x \in \Omega, u \in \mathbb{R}), \tag{4.13}$$

with $a_5 \in L^{p'}(\Omega)$, $b_5 > 0$, and

$$\int_\Omega b(x,u)u dx \ge b_6 \|u\|^p_{L^p(\Omega)} - a_6, \ \forall u \in L^p(\Omega), \tag{4.14}$$

$a_6, b_6 > 0$ (see e.g. [20, 10]). Let $\mathcal{B} : L^p(\Omega) \to L^{p'}(\Omega) = [L^p(\Omega)]^*$ be given by

$$\langle \mathcal{B}(u), v\rangle_{L^{p'}(\Omega),L^p(\Omega)} = \int_\Omega b(\cdot,u) v dx, \ \forall u, v \in L^q(\Omega).$$

Estimate (4.13) shows that $\mathcal{B}$ is well defined and is a bounded continuous operator from $L^p(\Omega)$ to $L^{p'}(\Omega)$. Let us verify that $i_p^* \mathcal{B} i_p$ is a (single-valued) pseudomonotone operator from $W^{1,p}(\Omega)$ to its dual. In fact, assume $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. We have $u_n = i_q(u_n) \to i_q(u) = u$ strongly in $L^p(\Omega)$. Estimate (4.13), together with usual convergence arguments based on the Lebesgue dominated convergence theorem, implies that $\mathcal{B}(u_n) \to \mathcal{B}(u)$ in $L^{p'}(\Omega)$. In particular, $\langle \mathcal{B}(u_n), u_n - v\rangle_{L^{p'}(\Omega),L^p(\Omega)} \to \langle \mathcal{B}(u), u - v\rangle_{L^{p'}(\Omega),L^p(\Omega)}$, $\forall v \in L^p(\Omega)$, and thus, for all $v \in W^{1,p}(\Omega)$,

$$\begin{aligned} \langle i_p^* \mathcal{B} i_p(u_n), u_n - v\rangle_{[W^{1,p}(\Omega)]^*, W^{1,p}(\Omega)} &= \langle \mathcal{B}(u_n), u_n - v\rangle_{L^{p'}(\Omega),L^p(\Omega)} \\ &\to \langle \mathcal{B}(u), u - v\rangle_{L^{p'}(\Omega),L^p(\Omega)} \\ &= \langle i_p^* \mathcal{B} i_p(u), u - v\rangle_{[W^{1,p}(\Omega)]^*, W^{1,p}(\Omega)}. \end{aligned}$$

Next, let us define certain truncation functions needed for the regularization of the involved setvalued mappings. For $i \in \{1, \ldots, k\}$, let $T_i(x,u)$ be a Carathéodory function such that for $x \in \Omega$, $u \in \mathbb{R}$,

$$T_i(x,u) = \begin{cases} |\underline{\eta}_i(x) - \underline{\eta}(x)| & \text{if} \quad u \le \underline{u}_i(x) \\ 0 & \text{if} \quad u \ge \underline{u}(x), \end{cases} \tag{4.15}$$

and

$$0 \le T_i(x,u) \le |\underline{\eta}_i(x) - \underline{\eta}(x)|, \text{ for a.e. } x \in \Omega, \text{ all } u \in \mathbb{R}. \tag{4.16}$$

A simple choice of such function is

$$T_i(x,u) = |\underline{\eta}_i(x) - \underline{\eta}(x)| \sigma\left(\frac{u - \underline{u}_i(x)}{\underline{u}(x) - \underline{u}_i(x)}\right), \tag{4.17}$$

for $x \in \Omega$, $u \in \mathbb{R}$, where $\sigma \in C(\mathbb{R},\mathbb{R})$, $0 \le \sigma(s) \le 1, \forall s \in \mathbb{R}$, $\sigma(s) = 1$ if $s \le 0$, and $\sigma(s) = 0$ if $s \ge 1$. σ can be simply chosen as

$$\sigma(s) = \begin{cases} 1, & s \le 0 \\ 1 - s, & 0 \le s \le 1 \\ 0, & s \ge 1. \end{cases} \tag{4.18}$$

It is clear that T_i given by (4.17)-(4.18) is a Carathéodory function satisfying (4.15) and (4.16). Similarly, for $j = 1, \ldots, m$, we define $T^j : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$T^j(x,u) = |\overline{\eta}_j(x) - \overline{\eta}(x)| \left[1 - \sigma\left(\frac{u - \overline{u}(x)}{\overline{u}_j(x) - \overline{u}(x)} \right) \right]. \tag{4.19}$$

T^j is a Carathéodory function with

$$T^j(x,u) = \begin{cases} |\overline{\eta}_j(x) - \overline{\eta}(x)| & \text{if} \quad u \ge \overline{u}_j(x) \\ 0 & \text{if} \quad u \le \overline{u}(x), \end{cases} \tag{4.20}$$

and similarly to (4.16),

$$0 \le T^j(x,u) \le |\overline{\eta}_j - \overline{\eta}(x)|, \text{ for a.e. } x \in \Omega, \text{ all } u \in \mathbb{R}. \tag{4.21}$$

Consequently,

$$T_i(\cdot,u), T^j(\cdot,u) \in L^{q'}(\Omega), \forall u \in L^q(\Omega), \tag{4.22}$$

and $\mathcal{T}_i : u \mapsto T_i(\cdot,u)$, $\mathcal{T}^j : u \mapsto T^j(\cdot,u)$ are bounded operators from $L^q(\Omega)$ to $L^{q'}(\Omega)$. Using standard convergence arguments based on Lebesgue's dominated convergence theorem and the growths in (4.16) and (4.21), we see that $\mathcal{T}_i, \mathcal{T}_j$ $(1 \le i \le k, 1 \le j \le m)$ are also continuous from $L^q(\Omega)$ to $L^{q'}(\Omega)$. Hence, $i_q^*\mathcal{T}_i i_q$ and $i_q^*\mathcal{T}^j i_q$ are completely continuous and are thus (singlevalued) pseudomonotone operators from $W^{1,p}(\Omega)$ to $[W^{1,p}(\Omega)]^*$.

Let us consider the following auxiliary variational inequality of (2.7): Find $u \in K$, $\eta \in L^q(\Omega)$ such that

$$\eta(x) \in f_0(x, u(x)) \text{ a.e. } x \in \Omega \text{ and} \tag{4.23}$$

$$\begin{aligned} &\int_\Omega A(x,\nabla u)(\nabla v - \nabla u)dx + \int_\Omega \eta(x)(v-u)dx + \int_\Omega b(x,u)(v-u)dx \\ &- \sum_{i=1}^k \int_\Omega T_i(x,u)(v-u)dx + \sum_{j=1}^m \int_\Omega T^j(x,u)(v-u)dx - \langle L, v-u\rangle \\ &\ge 0, \ \forall v \in K. \end{aligned} \tag{4.24}$$

This inequality is equivalent to finding $u \in K$ and $\tilde{\eta} \in (i_q^* \tilde{f}_0 i_q)(u)$ ($\tilde{\eta} = i_q^* \eta i_q$, η as in (4.23)) such that

$$\left\langle \mathcal{A}(u) + \tilde{\eta} + (i_p^*\mathcal{B}i_p)(u) - \sum_{i=1}^k (i_q^*\mathcal{T}_i i_q)(u) + \sum_{j=1}^m (i_q^*\mathcal{T}^j i_q)(u) - L, v - u \right\rangle \ge 0, \forall v \in K. \tag{4.25}$$

This inequality is of the form (3.2) with $f = L$, $\phi = 0$, $X = W^{1,p}(\Omega)$, and $T : X \to 2^{X^*}$,

$$T = \mathcal{A} + i_q^* \tilde{f} i_q + i_p^* \mathcal{B} i_p - \sum_{i=1}^{k} (i_q^* \mathcal{T}_i i_q) + \sum_{j=1}^{m} (i_q^* \mathcal{T}^j i_q). \tag{4.26}$$

Note that among the components of T, only $i_q^* \tilde{f} i_q$ is multivalued. Since all the components of T are pseudomonotone from $W^{1,p}(\Omega)$ to its dual, so is T.

Next, we check that the operators in (4.25) satisfy the coercivity condition (3.1). Letting u_0 be any (fixed) element of K, we have

$$\int_\Omega A(x, \nabla u) \nabla u dx \geq b_2 \| \, |\nabla u| \, \|_{L^p(\Omega)}^p - \|a_2\|_{L^1(\Omega)}, \tag{4.27}$$

and

$$\begin{aligned} \left| \int_\Omega A(x, \nabla u) \nabla u_0 dx \right| &\leq \int_\Omega \left(b_1 |\nabla u|^{p-1} + a_1 \right) |\nabla u_0| dx \\ &\leq b_1 \| \, |\nabla u| \, \|_{L^p(\Omega)}^{p-1} \| \, |\nabla u_0| \, \|_{L^p(\Omega)} + \|a_1\|_{L^{p'}(\Omega)} \| \, |\nabla u_0| \, \|_{L^p(\Omega)} \\ &\leq C_1 \|u\|_{W^{1,p}(\Omega)}^{p-1} + C_2. \end{aligned} \tag{4.28}$$

As above, here and in the next estimates C_k's stand for positive constants that do not depend on $u \in W^{1,p}(\Omega)$. From (4.11), for any $\tilde{\eta} \in (i_q^* \tilde{f}_0 i_q)(u)$,

$$\begin{aligned} |\langle \tilde{\eta}, u - u_0 \rangle| &\leq (\|a_4\|_{L^{q'}(\Omega)} + \|\underline{\eta}\|_{L^{q'}(\Omega)} + \|\overline{\eta}\|_{L^{q'}(\Omega)})(\|u\|_{L^q(\Omega)} + \|u_0\|_{L^q(\Omega)}) \\ &\leq C_3 \|u\|_{W^{1,p}(\Omega)} + C_4. \end{aligned} \tag{4.29}$$

From (4.13) and (4.14),

$$\begin{aligned} \langle (i_p^* \mathcal{B} i_p)(u), u - u_0 \rangle &= \int_\Omega b(x, u)(u - u_0) dx \\ &\geq b_6 \|u\|_{L^p(\Omega)}^p - a_6 - \|a_5\|_{L^{p'}(\Omega)} \|u_0\|_{L^p(\Omega)} - b_5 \|u\|_{L^p(\Omega)}^{p-1} \|u_0\|_{L^p(\Omega)} \\ &\geq b_6 \|u\|_{L^p(\Omega)}^p - C_5 \|u\|_{W^{1,p}(\Omega)}^{p-1} - C_6. \end{aligned} \tag{4.30}$$

From (4.16), for $i \in \{1, \ldots, k\}$,

$$\begin{aligned} |\langle (i_q^* \mathcal{T}_i i_q)(u), u - u_0 \rangle| &= \left| \int_\Omega T_i(\cdot, u)(u - u_0) dx \right| \\ &\leq \|\underline{\eta}_i - \underline{\eta}\|_{L^{q'}(\Omega)} \|u\|_{L^q(\Omega)} + \|\underline{\eta}_i - \underline{\eta}\|_{L^{q'}(\Omega)} \|u_0\|_{L^q(\Omega)}. \end{aligned}$$

Hence,

$$\sum_{i=1}^{k} |\langle (i_q^* \mathcal{T}_i i_q)(u), u - u_0 \rangle| \leq C_7 \|u\|_{W^{1,p}(\Omega)} + C_8. \tag{4.31}$$

Similarly,

$$\sum_{j=1}^{m} |\langle (i_q^* \mathcal{T}^j i_q)(u), u - u_0 \rangle| \leq C_9 \|u\|_{W^{1,p}(\Omega)} + C_{10}. \tag{4.32}$$

Finally,

$$|\langle L, u - u_0\rangle| \leq \|L\|_{[W^{1,p}(\Omega)]^*}\|u\|_{W^{1,p}(\Omega)} + \|L\|_{[W^{1,p}(\Omega)]^*}\|u_0\|_{W^{1,p}(\Omega)}. \tag{4.33}$$

Combining the estimates from (4.27) to (4.33) shows that for any $u \in W^{1,p}(\Omega)$ and $u^* \in T(u)$, that is,

$$u^* = \Big[\mathcal{A} + i_p^*\mathcal{B}i_p - \sum_{i=1}^{k}(i_q^*\mathcal{T}_i i_q) + \sum_{j=1}^{m}(i_q^*\mathcal{T}^j i_q)\Big](u) + \tilde{\eta}$$

with $\tilde{\eta} \in (i_q\tilde{f}i_q)(u)$, we always have

$$\begin{aligned}\langle u^* - L, u - u_0\rangle &\geq b_2\| \,|\nabla u|\,\|^p_{L^p(\Omega)} + b_6\|u\|^p_{L^p(\Omega)} - C_{11}\left(\|u\|^{p-1}_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)} + 1\right)\\ &\geq C_{12}\|u\|^p_{W^{1,p}(\Omega)} - C_{11}\left(\|u\|^{p-1}_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)} + 1\right).\end{aligned}$$

Since $p > 1$, this implies that

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty}\left[\inf_{u^*\in T(u)}\langle u^* - L, u - u_0\rangle\right] = \infty.$$

It follows from Theorem 3.1 that inequality (4.25), or (4.23)-(4.24), has a solution $u \in K$.

Let $u \in K$ be any solution of (4.23)-(4.24). In the next step, we prove that

$$\underline{u} \leq u \leq \overline{u} \ \text{ a.e. on } \Omega. \tag{4.34}$$

To verify the first inequality, we let s be any number in $\{1, \ldots, k\}$ and prove that

$$\underline{u}_s \leq u \ \text{ a.e. on } \Omega. \tag{4.35}$$

From (4.6), we have $\underline{u}_s \vee u \in K$. Letting $v = \underline{u}_s \vee u = u + (\underline{u}_s - u)^+$ in (4.24) yields

$$\begin{aligned}&\int_\Omega A(x, \nabla u)\nabla[(\underline{u}_s - u)^+]dx + \int_\Omega \eta(\underline{u}_s - u)^+dx + \int_\Omega b(x, u)(\underline{u}_s - u)^+dx\\ &-\sum_{i=1}^{k}\int_\Omega T_i(x, u)(\underline{u}_s - u)^+dx + \sum_{j=1}^{m}\int_\Omega T^j(x, u)(\underline{u}_s - u)^+dx - \langle L, (\underline{u}_s - u)^+\rangle\\ &\geq 0.\end{aligned} \tag{4.36}$$

From (4.2) with $\underline{u}_s$ and $\underline{\eta}_s$ instead of $\underline{u}$ and $\underline{\eta}$, and $v = \underline{u}_s - (\underline{u}_s - u)^+ = \underline{u}_s \wedge u \in \underline{u}_s \wedge K$, we obtain

$$-\int_\Omega A(x, \nabla\underline{u}_s)\nabla[(\underline{u}_s - u)^+]dx - \int_\Omega \underline{\eta}_s(\underline{u}_s - u)^+dx + \langle L, (\underline{u}_s - u)^+\rangle \geq 0. \tag{4.37}$$

Adding inequalities (4.36) and (4.37), we get

$$\begin{aligned}&\int_\Omega[A(x, \nabla u) - A(x, \underline{u}_s)]\nabla[(\underline{u}_s - u)^+]dx + \int_\Omega(\eta - \underline{\eta}_s)(\underline{u}_s - u)^+dx\\ &+ \int_\Omega b(x, u)(\underline{u}_s - u)^+dx - \sum_{i=1}^{k}\int_\Omega T_i(x, u)(\underline{u}_s - u)^+dx\\ &+ \sum_{j=1}^{m}\int_\Omega T^j(x, u)(\underline{u}_s - u)^+dx \geq 0.\end{aligned} \tag{4.38}$$

Stampacchia's theorem (cf. e.g. [16]) and the monotonicity of A in (A2) imply that

$$\begin{aligned} &\int_\Omega [A(x,\nabla u) - A(x,\underline{u}_s)]\nabla[(\underline{u}_s - u)^+]dx \\ = &\int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} [A(x,\nabla u) - A(x,\underline{u}_s)]\nabla[(\underline{u}_s - u)^+]dx \\ \geq\ & 0. \end{aligned} \tag{4.39}$$

At $x \in \Omega$ such that $\underline{u}_s(x) > u(x)$, since $\underline{u}_s(x) \leq \underline{u}(x) \leq \overline{u}(x)$, we have from (4.20) that $T^j(x, u(x)) = 0$ and thus

$$\int_\Omega T^j(x,u)(\underline{u}_s - u)^+ dx = \int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} T^j(x,u)(\underline{u}_s - u)dx = 0, \tag{4.40}$$

for all $j \in \{1,\ldots,m\}$. Furthermore, for $x \in \Omega$ such that $\underline{u}_s(x) > u(x)$, we have $u(x) < \underline{u}(x)$ which, together with (4.23) and (4.10), implies that $\eta(x) \in \{\underline{\eta}(x)\}$, i.e.,

$$\eta(x) = \underline{\eta}(x). \tag{4.41}$$

Also, for such x, (4.15) gives

$$T_s(x,u(x)) = |\underline{\eta}_s(x) - \underline{\eta}(x)|. \tag{4.42}$$

As a direct consequence of (4.16),

$$\int_\Omega T_i(x,u)(\underline{u}_s - u)^+ dx \geq 0,\ \forall i \in \{1,\ldots,k\}.$$

Thanks to (4.41) and (4.42), we get

$$\begin{aligned} &\int_\Omega (\eta - \underline{\eta}_s)(\underline{u}_s - u)^+ dx - \sum_{i=1}^k \int_\Omega T_i(x,u)(\underline{u}_s - u)^+ dx \\ \leq\ &\int_\Omega (\eta - \underline{\eta}_s)(\underline{u}_s - u)^+ dx - \int_\Omega T_s(x,u)(\underline{u}_s - u)^+ dx \\ =\ &\int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} \{[\underline{\eta}(x) - \underline{\eta}_s(x)] - |\underline{\eta}(x) - \underline{\eta}_s(x)|\}(\underline{u}_s - u)dx \\ \leq\ & 0. \end{aligned} \tag{4.43}$$

Combining (4.38) with (4.39), (4.40), and (4.43), we obtain

$$0 \leq \int_\Omega b(x,u)(\underline{u}_s - u)^+ dx = \int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} b(x,u)(\underline{u}_s - u)dx.$$

From (4.12), if $\underline{u}_s(x) > u(x)$ then $\underline{u} > u(x)$ and $b(x,u(x)) = -[\underline{u}(x) - u(x)]^{p-1}$. Hence, $0 \leq -\int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} [\underline{u}(x) - u(x)]^{p-1}[\underline{u}_s(x) - u(x)]dx$. Since $\underline{u}(x) - u(x) > 0$ and $\underline{u}_s(x) - u(x) > 0$ on the set $\{x \in \Omega : \underline{u}_s(x) > u(x)\}$, this inequality implies that this set must have measure 0, which means that $u(x) \geq \underline{u}_s(x)$ for a.e. $x \in \Omega$. We have proved (4.35). Since (4.35) holds for all $s \in \{1,\ldots,k\}$, we get the first inequality of (4.34). The second inequality of (4.34) is verified analogously.

From (4.34) and (4.12)-(4.15)-(4.20), we have

$$b(\cdot, u) = T_i(\cdot, u) = T^j(\cdot, u) = 0 \text{ a.e. on } \Omega,$$

for all $i \in \{1, \dots, k\}$, $j \in \{1, \dots, m\}$. Also, from (4.34) and (4.10), together with (4.23), we see that $\eta(x) \in f_0(x, u(x)) = f(x, u(x))$ for a.e. $x \in \Omega$. In view of these observations, (4.23)-(4.24) reduce to (2.6)-(2.7). Our proof of Theorem 4.2 is complete. □

Let $\underline{u}_i$, $1 \le i \le k$ and $\overline{u}_j$, $1 \le j \le m$, be sub- and supersolutions that satisfy conditions (4.5), (4.6), and (4.7), in Theorem 4.2. We have proved that the set $\mathcal{S}$ of solutions of (2.7) between $\underline{u}$ and $\overline{u}$,

$$\mathcal{S} = \{u \in K : u \text{ satisfies (2.6)-(2.7) and } \underline{u} \le u \le \overline{u} \text{ a.e. on } \Omega\},$$

is nonempty. As consequences of Theorem 4.2, some further properties of $\mathcal{S}$ are given in the following theorem. Since the proofs of these properties do not require substantial modifications as in Theorem 4.2 compared to the case of singlevalued lower order terms, they are just outlined here with the necessary changes indicated. We assume in the sequel that A is strictly monotone, that is, strict inequality holds in (2.3) whenever $\xi_1 \ne \xi_2$.

Corollary 4.3 *(a) $\mathcal{S}$ is a compact subset of $W^{1,p}(\Omega)$.*

(b) If

$$\mathcal{S} \wedge K \subset K \text{ (resp. } \mathcal{S} \vee K \subset K), \tag{4.44}$$

then

(i) any $u \in \mathcal{S}$ is a subsolution (resp. supersolution) of (2.7), and

(ii) $\mathcal{S}$ is directed downward (resp. upward), that is, for all $u_1, u_2 \in \mathcal{S}$, there exists $u \in \mathcal{S}$ such that

$$u \le \min\{u_1, u_2\} \text{ (resp. } u \ge \max\{u_1, u_2\}).$$

(c) If both inclusions in (4.44) hold then $\mathcal{S}$ has least and greatest elements, that is, there are $u_*, u^* \in \mathcal{S}$ such that $u_* \le u \le u^*$, $\forall u \in \mathcal{S}$.

Proof. Since $\underline{u}, \overline{u} \in W^{1,p}(\Omega)$, it follows from (4.8) that the set $\{\|u\|_{L^p(\Omega)} : u \in \mathcal{S}\}$ is bounded. Let $\{u_n\}$ be a sequence in $\mathcal{S}$ and $\{\eta_n\}$ be a corresponding sequence in $L^q(\Omega)$ that satisfies (2.6) and (2.7) (for each $u = u_n$ and $\eta = \eta_n$).

From (4.7), $\{\eta_n\}$ is a bounded sequence in $L^q(\Omega)$. Using (2.7) with u_n, η_n, and $v = v_0$, a fixed element of K, we see that $\left\{\int_\Omega A(\cdot, \nabla u_n)\nabla u_n dx\right\}$ is a bounded sequence which thanks to (A1) implies that the set $\{\| |\nabla u_n| \|_{L^p(\Omega)} : n \in \mathbb{N}\}$ is also bounded. Hence, $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$ and there exists a subsequence $\{u_{n_l}\} \subset \{u_n\}$ such that $u_{n_l} \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega)$ for some $u_0 \in K$ (note that K is weakly closed in $W^{1,p}(\Omega)$) and thus $u_{n_l} \to u_0$ in $L^q(\Omega)$.

By passing to a subsequence if necessary, we can also assume that $u_{n_l} \to u_0$ a.e. in Ω and because of the boundedness of $\{\eta_n\}$ in $L^{q'}(\Omega)$, $\eta_{n_l} \rightharpoonup \eta_0$ weakly in $L^{q'}(\Omega)$ for some $\eta_0 \in L^{q'}(\Omega)$. It follows that $i_q^*\eta_{n_l} \rightharpoonup i_q^*\eta_0$ weakly in $[W^{1,p}(\Omega)]^*$ and therefore

$$\int_\Omega \eta_{n_l}(u_{n_l} - u_0)dx \to 0 \text{ and } \langle L, u_{n_l} - u_0\rangle \to 0 \text{ as } l \to \infty. \tag{4.45}$$

From (2.7) with $u = u_{n_l}$ and $v = u_0$, we see that

$$\liminf_{l\to\infty} \langle \mathcal{A}(u_{n_l}), u_0 - u_{n_l}\rangle \ge 0.$$

Since $\mathcal{A}$ is of class $(S)_+$ (cf. e.g. [3]), we must have $u_{n_l} \to u_0$ (strongly) in $W^{1,p}(\Omega)$. Next, we prove that $u_0 \in \mathcal{S}$. It is evident that

$$\underline{u} \le u_0 \le \overline{u} \ \text{ a.e. on } \Omega. \tag{4.46}$$

Let f_0 be defined by (4.10) in the proof of Theorem 4.2. Since $\underline{u} \le u_n \le \overline{u}$ a.e. on Ω, we see that u_n and η_n satisfy (2.6)-(2.7) with f_0 instead of f. From (4.45) and the fact that $i_q^* \tilde{f}_0 i_q$ is pseudomonotone and thus generalized pseudomonotone from $W^{1,p}(\Omega)$ to $[W^{1,p}(\Omega)]^*$ (see the proof of Theorem 4.2) we have $\eta_0 \in (i_q^* \tilde{f}_0 i_q)(u_0)$, i.e.,

$$\eta_0(x) \in f_0(x, u_0(x)) = f(x, u(x)) \text{ for a.e. } x \in \Omega, \tag{4.47}$$

(from (4.46)) and $\int_\Omega \eta_{n_l} u_{n_l} dx \to \int_\Omega \eta_0 u_0 dx$. Therefore, for all $v \in K$,

$$\begin{aligned} &\int_\Omega A(\cdot, \nabla u_{n_l})(\nabla v - \nabla u_{n_l}) dx + \int_\Omega \eta_{n_l}(v - u_{n_l}) - \langle L, v - u_{n_l} \rangle \\ \to \quad &\int_\Omega A(\cdot, \nabla u_0)(\nabla v - \nabla u_0) dx + \int_\Omega \eta_0(v - u_0) - \langle L, v - u_0 \rangle. \end{aligned}$$

Since $u_{n_l} \in \mathcal{S}$, this limit together with (4.47) shows that u_0 and η_0 satisfy (2.6)-(2.7) which in view of (4.46) implies that $u_0 \in \mathcal{S}$. We thus obtain the compactness of $\mathcal{S}$ in $W^{1,p}(\Omega)$.

(b) Assume the first inclusion in (4.44). If $u_0 \in \mathcal{S}$ then $u_0 \wedge K \subset K$ and thus u_0 is a subsolution of (2.7). If $u_1, u_2 \in \mathcal{S}$ then they are subsolutions of (2.7) and Theorem 4.2 thus implies the existence of a solution u of (2.7) such that $\max\{u_1, u_2\} \le u \le \min\{\overline{u}_j : 1 \le j \le m\} = \overline{u}$. It is clear that $u \in \mathcal{S}$.

(c) Since $W^{1,p}(\Omega)$ is separable, so is $\mathcal{S}$ with the metric generated by $\|\cdot\|_{W^{1,p}(\Omega)}$. Let $\{w_n\}$ be a dense sequence in $\mathcal{S}$. Using the directedness of $\mathcal{S}$, we can construct inductively a sequence $\{u_n\}$ in $\mathcal{S}$ such that $w_n \le u_n \le u_{n+1}, \forall n \in \mathbb{N}$. Let $u^*(x) = \sup\{u_n(x) : n \in \mathbb{N}\} = \lim_{n\to\infty} u_n(x),\ x \in \Omega$. As a consequence of the compactness of $\mathcal{S}$, $u_n \to u^*$ in $W^{1,p}(\Omega)$ and $u^* \in \mathcal{S}$. Since $u^* \ge w_n$ a.e. in Ω for all $n \in \mathbb{N}$, from the density of $\{w_n\}$ in $\mathcal{S}$, we see that $u^* \ge u$ a.e. in Ω for all $u \in \mathcal{S}$. The existence of the smallest element u_* of $\mathcal{S}$ is proved in a similar way. □

We conclude our paper with some remarks regarding the consideration of multivalued integral lower order terms in (1.1) and (1.2).

Remark 4.4 (a) If $f = f(u)$ depends only on u then the upper semicontinuity of f in condition (F2) implies its measurability in condition (F1) (cf. e.g. [1]).

(b) Let us consider $f(x, u) = \partial F(x, u)$ where $F(x, u)$ is a Carathéodory function which is locally Lipschitz in u and $\partial F(x, u)$ is the Clarke generalized gradient with respect to u. Since $\partial F(x, u)$ is a closed and bounded interval in $\mathbb{R}$ and the mapping $u \mapsto \partial F(x, u)$ is upper semicontinuous from $\mathbb{R}$ into $\mathcal{K}(\mathbb{R})$ (cf. [13]), our discussions and existence results above and their variants are natural extensions of several nonsmooth existence results containing Clarke's generalized gradient (cf. [7, 9, 8, 10, 5, 11]), to more general multivalued functions without nonsmooth potential functionals.

Furthermore, as seen in the above arguments, we do not need the one-sided conditions, assumed in some works related to Clarke's generalized gradient as in the above references. Therefore, the concepts and results presented here are natural continuation and complements of several results in those works.

(c) Here are some simple examples of multivalued functions $f : \mathbb{R} \to 2^{\mathbb{R}}$ that satisfy the above conditions (F1)-(F2) (and (F3)) but are not generalized gradients of any locally Lipschitz functions (and are not subdifferentials of any convex functions either).

Example 1. (a) $f(u) = \begin{cases} [-1, 1] & \text{if } \quad u = 0 \\ \{0\} & \text{if } \quad u \neq 0. \end{cases}$

(b) $f(u) = \begin{cases} \{-1\} & \text{if } \quad u < 0 \\ [-2, 2] & \text{if } \quad u = 0 \\ \{1\} & \text{if } \quad u > 0. \end{cases}$

In Example 1(a), if $f(u) = \partial F(u)$ for some locally Lipschitz function F (∂F denotes Clarke's generalized gradient) then since $F(u) = C_1$ if $u < 0$, $F(u) = C_2$ for $u > 0$ (C_1, C_2 = constant) and F is continuous at 0, we must have $C_1 = C_2$ and thus $\partial F(0) = 0$. Similarly, in Example 1(b), if $f(u) = \partial F(u)$ then $F(u) = -u + C_1$ for $u < 0$, $F(u) = u + C_2$ for $u > 0$ and again by the continuity of F at 0, $C_1 = C_2$. Therefore, $F(u) = |u|$, $\forall u \in \mathbb{R}$ and $\partial F(0) = [-1, 1]$, not $[-2, 2]$. Note that in both examples, f is upper semicontinuous at every $u \in \mathbb{R}$.

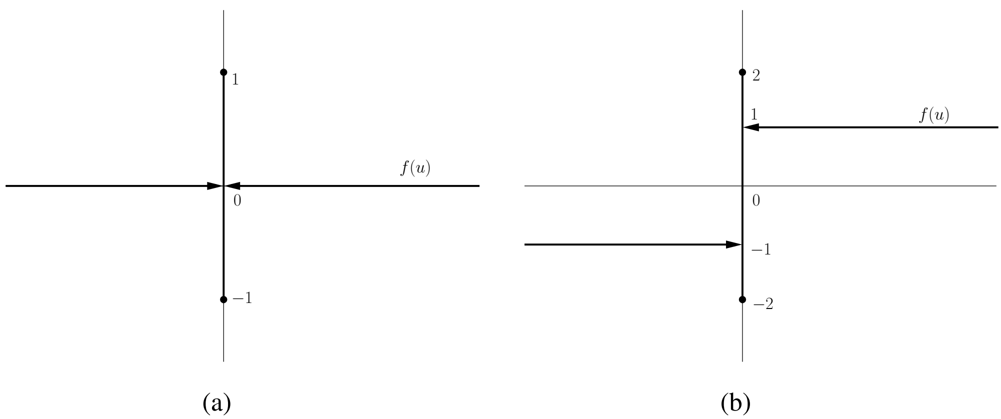

Figure 1: Example 1

Example 2. A more general example in this vein is the following function:

$$f(u) = \begin{cases} \{h_1(u)\} & \text{if } \quad u < a \\ [\alpha_1, \beta_1] & \text{if } \quad u = a \\ [g_1(u), g_2(u)] & \text{if } \quad a < u < b \\ [\alpha_2, \beta_2] & \text{if } \quad u = b \\ \{h_2(u)\} & \text{if } \quad u > b, \end{cases}$$

where $a \leq b, \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2$ ($[\alpha_1, \beta_1] = [\alpha_2, \beta_2]$ if $a = b$), $h_1 \in C((-\infty, a])$, $h_2 \in C([b, \infty))$, with $h_1(a) \in [\alpha_1, \beta_1]$, $h_2(b) \in [\alpha_2, \beta_2]$, $g_1, g_2 \in C([a, b])$ with $g_1(u) \leq g_2(u)$, $\forall u \in [a, b]$, and $[g_1(a), g_2(a)] \subset [\alpha_1, \beta_1]$, $[g_1(b), g_2(b)] \subset [\alpha_2, \beta_2]$. The graph of f is as follows.

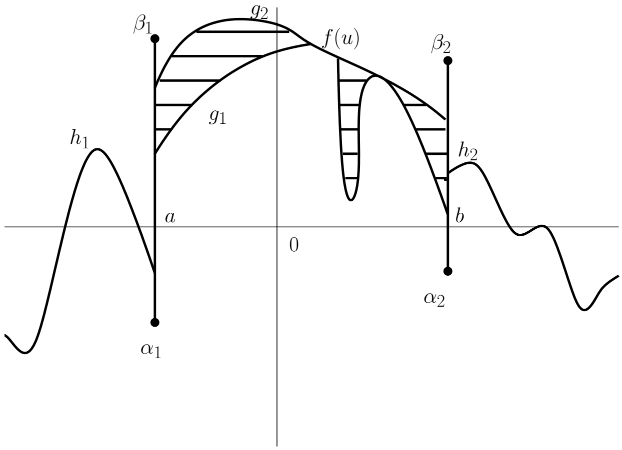

Figure 2: Example 2

Note that such function f satisfies conditions (F1), (F2), and (F3) above, but f is generally not a Clarke's generalized gradient.

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