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Multiple Nontrivial Solutions for Neumann Problems Involving the p-Laplacian: a Morse Theoretical Approach

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Abstract

We consider nonlinear elliptic Neumann problems driven by the p-Laplacian. Using variational techniques together with Morse theory (in particular, critical groups and the Poincaré-Hopf formula), we prove some multiplicity results: either three or four distinct nontrivial solutions are guaranteed, depending on the geometry and smoothness of the nonlinear term.

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1 Introduction

Let $Z \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary ∂Z and consider the following nonlinear Neumann problem driven by the p-Laplacian differential operator

$$\begin{cases} -\triangle_p x(z) = f(z, x(z)) & \text{on} \quad Z; \\ \frac{\partial x}{\partial \nu} = 0 & \text{on} \quad \partial Z, \end{cases} \tag{P}$$

where $f : Z \times \mathbb{R} \to \mathbb{R}$ is a Carathéordory function, $\triangle_p(\cdot) = \mathrm{div}(\|D(\cdot)\|_{\mathbb{R}^N}^{p-2} D(\cdot))$ is the p-Laplacian, $\frac{\partial x}{\partial \nu} = \|Dx\|_{\mathbb{R}^N}^{p-2}(Dx, \nu)_{\mathbb{R}^N}$, ν being the outward unit normal on ∂Z and $2 \le p < \infty$. The purpose of this paper is to prove multiplicity results for problem (P).

While for the Dirichlet problem with the p-Laplacian there have been several multiplicity results, the case of the Neumann problem – in some sense – is lagging behind. Within the Neumann context, we mention the works of Anello [1], Cammaroto-Chinnì-Di Bella [6], Faraci-Kristály [10], Marano-Motreanu [18], Ricceri [22], where the existence of infinitely many solutions for certain nonlinear elliptic problems are established by imposing certain oscillatory assumption on the nonlinear term. In the aforementioned papers the inequality $p > N$ was crucial, exploiting the fact that the space $W^{1,p}(Z)$ is compactly embedded into $C(\overline{Z})$. Nonlinear eigenvalue problems subjected to Neumann boundary conditions were studied by Bonanno-Candito [4] who prove a three solutions theorem using an abstract result of Ricceri [23]. Binding-Drabek-Huang [3] consider a right hand side nonlinearity of the form $\lambda a(z)|x|^{p-2}x + b(z)|x|^{p^*-2}x + h(z)$ with $a, b \in L^\infty(Z)_+$, $h \in L^\infty(Z)$ and they prove the existence of one or two positive solutions. Filippakis-Gasiński-Papageorgiou [11] use the second deformation lemma to establish the existence of at least two nontrivial smooth solutions for (P). Finally, Motreanu-Papageorgiou [20] prove a multiplicity result for Neumann problems with a nonsmooth potential (hemivariational inequalities) using the nonsmooth local linking theorem (see Gasiński-Papageorgiou [13]) to obtain two nontrivial smooth solutions.

The hypotheses and methods used in the aforementioned works are completely different from ours. More precisely, by applying Morse theoretical arguments (critical groups, Poincaré-Hopf formula), we are able to prove the existence of either three or four distinct nontrivial solutions for problem (P), depending on the geometrical behaviour as well as on the smoothness of the nonlinear term f. We emphasize that no symmetry assumption is required on the nonlinearity f and significant technical difficulties should be overcome in order to treat the quasilinear problem (P). In order to illustrate the type of results we obtain, we consider here the z-independent (autonomous) case. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function such that $f(0) = 0$ and

(f_1) there exist $C_1, C_2 > 0$ such that

$$|u| \le C_1 + C_2|x|^{p-2} \quad \text{for all } u \in \partial f(x) \text{ and } x \in \mathbb{R};$$

(Here, $\partial f(x)$ denotes the generalized gradient of f at $x \in \mathbb{R}$, see §3.)

(f_2) there exist $C_3, C_4 < 0$ such that

$$C_3 \leq \liminf_{|x|\to\infty} \frac{f(x)}{|x|^{p-2}x} \leq \limsup_{|x|\to\infty} \frac{f(x)}{|x|^{p-2}x} \leq C_4;$$

(f_3) there exist $\delta > 0$ and $C_5 < 0$ such that

$$F(x) \leq \frac{C_5}{p}|x|^p \text{ for all } x \in [-\delta, \delta], \text{ where } F(x) = \int_0^x f(r)dr;$$

(f_4) there exist $C_6 < 0 < C_7$ such that $F(C_6) > 0$ and $F(C_7) > 0$;

(f_5) there exists $C_8 > 0$ such that $-C_8|x|^p \leq f(x)x$ for all $x \in \mathbb{R}$.

As a simple consequence of Theorems 3.2 and 4.1, we obtain the following

Theorem 1.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function which satisfies $(f_1) - (f_5)$ and $2 \leq p < \infty$. Then, problem (P) has at least three distinct nontrivial smooth solutions. Moreover, if $f : \mathbb{R} \to \mathbb{R}$ is of class C^1 and $p = 2$ (semilinear case), problem (P) has at least four distinct nontrivial smooth solutions.*

Example 1.1 a) Let

$$F(x) = g(x) \cdot \begin{cases} \frac{c}{p}|x|^p - \ln(1 + |x|^p), & \text{if } |x| \leq 1 \\ -\frac{c}{p}|x|^p - \frac{\beta}{q}|x|^q + k, & \text{if } |x| > 1, \end{cases}$$

where $2 \leq q < p$, $p \ln 2 < c < p$, $\beta = p/2 - 2c$, $k = 2c/p - \ln 2 + \beta/q$, and $g(x) \equiv 1$ or $g(x) = 2 + \text{sgn}(x)$. Then the locally Lipschitz function $f(x) = F'(x)$ verifies the assumptions $(f_1) - (f_5)$, and one has at least three distinct nontrivial solutions for (P). When $g(x) = 2 + \text{sgn}(x)$, the nonlinear term f has no symmetrical property.

b) Let $F(x) = -\frac{1}{2}x^2 + c \ln(g(x)x^4 + 1)$ with $c > 1$ and the function(s) g from a). Then the C^1-function $f(x) = F'(x)$ verifies $(f_1) - (f_5)$ with $p = 2$, and one has at least four distinct nontrivial solutions for (P).

In the next section we recall various notions (mainly from Morse theory) and prove two results which are of independent interest. The first one gives a Brézis-Nirenberg [5] type result, comparing minimizers of the energy functional in different spaces related to our problem (P). Next, a recent result of Perera-Schechter [21] is extended to Banach spaces; to the best of our knowledge, Theorem 1.1 (and Theorems 3.2 and 4.1) is the first powerful application of [21]. In Section 3 we prove two independent multiplicity results for (P); both of them guarantee the existence of at least three distinct nontrivial solutions of (P). Finally, in the last section, within the semilinear context ($p = 2$), we guarantee the existence of at least four distinct nontrivial solutions of (P).

2 Mathematical background

First, let us recall some basic notions and results from Morse theory which we will need in the sequel.

Let X be a Banach space and $\varphi \in C^1(X)$. For every $c \in \mathbb{R}$, we set

$$\varphi^c = \{x \in X : \varphi(x) \le c\} \text{ (the sublevel set at } c \text{ of } \varphi),$$

$$K = \{x \in X : \varphi'(x) = 0\} \text{ (the set of critical points of } \varphi),$$

$$K_c = \{x \in K : \varphi(x) = c\} \text{ (the set of critical points of } \varphi \text{ at level } c).$$

If (Y_1, Y_2) is a topological pair with $Y_2 \subset Y_1 \subseteq X$, for every integer $n \ge 0$, we denote by $H_n(Y_1, Y_2)$ the n^{th}-relative singular homology group of the pair (Y_1, Y_2) with integer coefficients. The *critical groups of* φ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$, are defined by

$$C_n(\varphi, x_0) = H_n(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}) \text{ for all } n \ge 0,$$

where U is a neighborhood of x_0 such that $K \cap \varphi^c \cap U = \{x_0\}$. By the excision property of singular homology theory, we see that the above definition of critical groups is independent of U (see Chang [8], and Mawhin-Willem [19]).

In the sequel, we assume that φ satisfies the standard PS-condition. Suppose $-\infty < \inf \varphi(K)$, and choose $c < \inf \varphi(K)$. The *critical groups of* φ *at infinity* are defined by

$$C_n(\varphi, \infty) = H_n(X, \varphi^c) \text{ for all } n \ge 0,$$

see Bartsch-Li [2]. If $\varphi \in C^1(X)$ and $K = \{x_0\}$, then Morse theory tells us that

$$C_n(\varphi, \infty) = C_n(\varphi, x_0) \text{ for all } n \ge 0.$$

In particular, if x_0 is an isolated critical point of φ and $C_n(\varphi, \infty) \ne C_n(\varphi, x_0)$ for some $n \ge 0$, then φ must have one more critical point distinct from x_0. Moreover, if K is finite, then the *Morse-type numbers of* φ are defined by

$$M_n = \sum_{x \in K} \text{rank} C_n(\varphi, x) \text{ for all } n \ge 0.$$

The *Betti-type numbers of* φ are defined by

$$\beta_n = \text{rank} C_n(\varphi, \infty) \text{ for all } n \ge 0.$$

By Morse theory (see Bartsch-Li [2], Chang [8] and Mawhin-Willem [19]), we have the *Poincaré-Hopf formula*

$$\sum_{n \ge 0} (-1)^n M_n = \sum_{n \ge 0} (-1)^n \beta_n. \tag{2.1}$$

In the analysis of problem (P), we will use the following two spaces:

$$W^{1,p}_\nu(Z) = \{x \in W^{1,p}(Z) : x_n \to x \text{ in } W^{1,p}(Z), x_n \in C^1(\overline{Z}), \frac{\partial x_n}{\partial \nu} = 0 \text{ on } \partial Z\};$$

$$C^1_\nu(\overline{Z}) = \{x \in C^1(\overline{Z}) : \frac{\partial x}{\partial \nu} = 0 \text{ on } \partial Z\}.$$

We know that both spaces are ordered Banach spaces with order cones given by

$$W_+ = \{x \in W^{1,p}_\nu(Z) : x(z) \geq 0 \text{ a.e. on } Z\}$$

and

$$C_+ = \{x \in C^1_\nu(\overline{Z}) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\}.$$

Moreover, we know that $\text{int}C_+ \neq \emptyset$ and

$$\text{int}C_+ = \{x \in C_+ : x(z) > 0 \text{ for all } z \in \overline{Z}\}.$$

In the sequel, we denote by $\|\cdot\|_p$ and $\|\cdot\|$ the usual norms of the spaces $L^p(Z)$ and $W^{1,p}(Z)$, respectively. The norm of the space $W^{1,p}_\nu(Z)$ is also $\|\cdot\|$.

Lemma 2.1 *If $u \in W^{1,p}_\nu(Z)$, then $u^+, u^-, |u| \in W^{1,p}_\nu(Z)$. (We use the notation $u^\pm = \max\{\pm u, 0\}$.)*

Proof. We do the proof for u^+; the proof for u^- is done similarly, while $|u| = u^+ + u^-$. Consider the sequence of functions $\xi_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_n(t) = \begin{cases} 0, & \text{if } \ t < 0, \\ \frac{n}{n+1} t^{\frac{n+1}{n}}, & \text{if } \ t \in [0,1], \\ t - \frac{1}{n+1}, & \text{if } \ t > 1. \end{cases}$$

It is clear that $\xi_n \in C^1(\mathbb{R})$ and we have

$$\xi_n'(t) = \begin{cases} 0, & \text{if } \ t < 0, \\ t^{\frac{1}{n}}, & \text{if } \ t \in [0,1], \\ 1, & \text{if } \ t > 1. \end{cases}$$

Moreover, $\xi_n' \in C_b(\mathbb{R})$ (= the space of bounded continuous functions on $\mathbb{R}$) and $\xi_n' \uparrow \chi_{(0,\infty)}$ as $n \to \infty$.

Since $u \in W^{1,p}_\nu(Z)$, by definition, we can find a sequence $\{u_n\} \subset C^1(\overline{Z})$ such that $u_n \to u$ in $W^{1,p}(Z)$ as $n \to \infty$ and $\frac{\partial u_n}{\partial \nu} = 0$ on ∂Z for all $n \geq 1$. We have $\xi_n \circ u_n \in C^1(\overline{Z})$ for all $n \geq 1$. Also, we know that $\xi_n \circ u \in W^{1,p}(Z)$ and $D(\xi_n \circ u)(z) = \xi_n'(u(z))Du(z)$ a.e. on Z for all $n \geq 1$ (see for example Gasiński-Papageorgiou [14, p. 194]). Note that $(\xi_n \circ u)(z) \uparrow u^+(z)$ a.e. on Z and $D(\xi_n \circ u)(z) = \xi_n'(u(z))Du(z) \to \chi_{\{u>0\}}Du(z) = Du^+(z)$ a.e. on Z, as $n \to \infty$. Since $\|D(\xi_n \circ u)(z)\|_{\mathbb{R}^n} \leq \|Du(z)\|_{\mathbb{R}^n}$ a.e. on Z for all $n \geq 1$, from the Lebesgue dominated convergence theorem we have

$$D(\xi_n \circ u) \to Du^+ \ \text{ in } L^p(Z, \mathbb{R}^N) \text{ as } n \to \infty, \tag{2.2}$$

while from the monotone convergence theorem, we obtain

$$\xi_n \circ u \to u^+ \ \text{ in } L^p(Z) \text{ as } n \to \infty. \tag{2.3}$$

From (2.2) and (2.3) it follows that

$$\xi_n \circ u \to u^+ \text{ in } W^{1,p}(Z) \text{ as } n \to \infty.$$

Moreover, from the definition of ξ_n and Stampacchia's theorem we infer that

$$\|\xi_n \circ u_n - \xi_n \circ u\|_p \to 0 \text{ and } \|\xi_n'(u_n)Du_n - \xi_n'(u)Du\|_p \to 0 \text{ as } n \to \infty.$$

Consequently, we have $\|\xi_n \circ u_n - \xi_n \circ u\| \to 0$ as $n \to \infty$, thus, $\xi_n \circ u_n \to u^+$ in $W^{1,p}(Z)$ as $n \to \infty$. Since $\xi_n \circ u_n \in C^1(\overline{Z})$ and $\frac{\partial(\xi_n \circ u_n)}{\partial \nu} = \xi_n'(u_n)\frac{\partial u_n}{\partial \nu} = 0$ on ∂Z, we conclude that $u^+ \in W^{1,p}_\nu(Z)$. □

The next result compares $C^1_\nu(\overline{Z})$ and $W^{1,p}_\nu(Z)$-local minimizers of the energy functional associated with problem (P). It extends to Neumann problems earlier results of Brézis-Nirenberg [5] ($p = 2$) and Garcia Azorero-Manfredi-Peral Alonso [12] ($p \neq 2$), who had Dirichlet boundary conditions. In order to prove the result we spoke about, we introduce a generic nonlinearity $g : Z \times \mathbb{R} \to \mathbb{R}$ satisfying the following hypotheses:

(H_g): (i) for all $x \in \mathbb{R}$ the function $z \mapsto g(z,x)$ is measurable;

(ii) for almost all $z \in Z$, the function $x \mapsto g(z,x)$ is continuous;

(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$|g(z,x)| \leq a_g(z) + c_g|x|^{r-1}$$

with $a_g \in L^\infty(Z)_+$, $c_g > 0$ and $1 < r < p^*$. As usual, $p^* = Np/(N-p)$ if $N > p$, and $p^* = \infty$ if $N \leq p$.

Let $G(z,x) = \int_0^x g(x,r)dr$ and consider the functional $\gamma : W^{1,p}_\nu(Z) \to \mathbb{R}$ defined by

$$\gamma(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z G(z,x(z))dz \text{ for all } x \in W^{1,p}_\nu(Z).$$

We know that γ is well-defined and it belongs to $C^1(W^{1,p}_\nu(Z))$.

Proposition 2.1 *If $x_0 \in W^{1,p}_\nu(Z)$ is a local $C^1_\nu(\overline{Z})$-minimizer of γ, i.e., there exists $r_1 > 0$ such that*

$$\gamma(x_0) \leq \gamma(x_0 + h) \text{ for all } h \in C^1_\nu(\overline{Z}),\ \|h\|_{C^1_\nu(\overline{Z})} \leq r_1,$$

then $x_0 \in C^1_\nu(\overline{Z})$ and it is a local $W^{1,p}_\nu(Z)$-minimizer of γ, i.e., there exists $r_2 > 0$ such that

$$\gamma(x_0) \leq \gamma(x_0 + h) \text{ for all } h \in W^{1,p}_\nu(Z),\ \|h\| \leq r_2.$$

Proof. Let $h \in C^1_\nu(\overline{Z})$. If $\lambda \in \mathbb{R}$ with $|\lambda|$ small, then $\gamma(x_0) \leq \gamma(x_0 + \lambda h)$. Consequently,

$$0 \leq \langle \gamma(x_0), h \rangle \text{ for all } h \in C^1_\nu(\overline{Z}). \tag{2.4}$$

Hereafter, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}_\nu(Z), W^{1,p}_\nu(Z)^*)$. Since $C^1_\nu(\overline{Z})$ is dense in $W^{1,p}_\nu(Z)$, it follows that (2.4) holds for all $h \in W^{1,p}_\nu(Z)$. Hence we have

$$\gamma'(x_0) = 0 \ \text{ in } W^{1,p}_\nu(Z)^*.$$

Let $A : W^{1,p}_\nu(Z) \to W^{1,p}_\nu(Z)^*$ be the nonlinear maximal monotone operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}_{\mathbb{R}^N}(Dx, Dy)_{\mathbb{R}^N} dz \ \text{ for all } x, y \in W^{1,p}_\nu(Z).$$

Also let $N_g : W^{1,p}_\nu(Z) \to L^{r'}(Z)$ $(1/r + 1/r' = 1)$ be the bounded continuous nonlinear Nemitsky operator corresponding to the nonlinearity g, defined by

$$N_g(x)(\cdot) = g(\cdot, x(\cdot)) \ \text{ for all } x \in W^{1,p}_\nu(Z).$$

We know that

$$0 = \gamma'(x_0) = A(x_0) - N_g(x_0). \tag{2.5}$$

From the representation theorem for the elements of $W^{-1,p'}(Z) = W^{1,p}_0(Z)^*$ $(1/p + 1/p' = 1)$ (see for example Gasiński-Papageorgiou [14, p.212]), we have

$$-\mathrm{div}(\|Dx_0\|^{p-2}_{\mathbb{R}^N} Dx_0) \in W^{-1,p'}(Z). \tag{2.6}$$

We take duality brackets of (2.5) with $v \in C^1_c(Z)$ $(= C^1(Z)$ functions with compact support in $Z)$. Then

$$\langle A(x_0), v\rangle = \int_Z \|Dx_0\|^{p-2}_{\mathbb{R}^N}(Dx_0, Dv)_{\mathbb{R}^N} dz = \int_Z N_g(x_0) v dz. \tag{2.7}$$

From the definition of the distributional derivative, (2.6) and (2.7), we have

$$\langle -\mathrm{div}(\|Dx_0\|^{p-2}_{\mathbb{R}^N} Dx_0), v\rangle_0 = \langle A(x_0), v\rangle = \int_Z N_g(x_0) v dz = \langle N_g(x_0), v\rangle_0, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_0$ denotes the duality brackets for the pair $(W^{1,p}_0(Z), W^{-1,p'}(Z))$. Since $C^1_c(Z)$ is dense in $W^{1,p}_0(Z)$, from (2.8), it follows that

$$-\mathrm{div}(\|Dx_0(z)\|^{p-2}_{\mathbb{R}^N} Dx_0(z)) = g(z, x_0(z)) \ \text{ a.e. on } Z. \tag{2.9}$$

Invoking the nonlinear Green's identity of Casas-Fernandez [7] and Kenmochi [15], we have for all $v \in W^{1,p}(Z)$ that

$$\int_Z [\mathrm{div}(\|Dx_0\|^{p-2}_{\mathbb{R}^N} Dx_0)] v dz + \int_Z \|Dx_0\|^{p-2}_{\mathbb{R}^N}(Dx_0, Dv)_{\mathbb{R}^N} dz = \left\langle \frac{\partial x_0}{\partial \nu}, tr(v)\right\rangle_{\partial Z}. \tag{2.10}$$

Here, $\langle\cdot,\cdot\rangle_{\partial Z}$ denotes the duality brackets for the pair $(W^{\frac{1}{p'},p}(\partial Z), W^{-\frac{1}{p'},p'}(\partial Z))$ and tr is the trace map on $W^{1,p}(Z)$. Using (2.9) and (2.10), for every $v \in W^{1,p}(Z)$ we obtain

$$-\int_Z g(z, x_0(z)) v dz + \int_Z \|Dx_0\|^{p-2}_{\mathbb{R}^N}(Dx_0, Dv)_{\mathbb{R}^N} dz = \left\langle \frac{\partial x_0}{\partial \nu}, tr(v)\right\rangle_{\partial Z}. \tag{2.11}$$

Combining (2.7) and (2.11), we have that

$$\left\langle \frac{\partial x_0}{\partial \nu}, tr(v) \right\rangle_{\partial Z} = 0 \ \text{ for all } v \in W^{1,p}(Z). \tag{2.12}$$

But $tr(W^{1,p}(Z)) = W^{\frac{1}{p'},p}(\partial Z)$. So, (2.12) implies that

$$\frac{\partial x_0}{\partial \nu} = 0 \ \text{ in } W^{-\frac{1}{p'},p'}(\partial Z). \tag{2.13}$$

Because of (2.9) and Ladyzhenskaya-Uraltseva [16, Theorem 7.1, p. 286], we have that $x_0 \in L^\infty(Z)$. Then, the regularity result of Lieberman [17] applies and gives $x_0 \in C^1(\overline{Z})$. This fact together with (2.13) shows that $x_0 \in C^1_\nu(\overline{Z})$.

Now, suppose that x_0 is not a local $W^{1,p}_\nu(Z)$-minimizer of γ. Let $\overline{B}_\varepsilon = \{h \in W^{1,p}_\nu(Z) : \|h\| \leq \varepsilon\}$, $\varepsilon > 0$. This set is w-compact. Moreover, it is easily seen that γ is weakly lower semicontinuous. Therefore, by the Weierstrass theorem, we can find $h_\varepsilon \in \overline{B}_\varepsilon$ such that

$$\gamma(x_0 + h_\varepsilon) = \min\{\gamma(x_0 + h) : h \in \overline{B}_\varepsilon\} < \gamma(x_0). \tag{2.14}$$

From the Lagrange multiplier rule, we know that we can find $\lambda_\varepsilon > 0$ such that

$$\gamma'(x_0 + h_\varepsilon) + \lambda_\varepsilon \xi'_\varepsilon(h_\varepsilon) = 0, \tag{2.15}$$

where $\xi_\varepsilon(h) = \frac{1}{p}(\|h\|^p - \varepsilon^p)$. Then, from (2.15), we obtain

$$A(x_0 + h_\varepsilon) - N_g(x_0 + h_\varepsilon) + \lambda_\varepsilon A(h_\varepsilon) + \lambda_\varepsilon K_p(h_\varepsilon) = 0, \tag{2.16}$$

with $K_p : L^p(Z) \to L^{p'}(Z)$ being the bounded continuous nonlinear map defined by $K_p(h)(\cdot) = |h(\cdot)|^{p-2}h(\cdot)$. Due to the compact embedding of $W^{1,p}_\nu(Z)$ into $L^p(Z)$, the function $K_p|_{W^{1,p}_\nu(Z)}$ is completely continuous from $W^{1,p}_\nu(Z)$ to $L^{p'}(Z) \subset W^{1,p}_\nu(Z)^*$. Combining (2.16) and (2.5), we have

$$A(x_0 + h_\varepsilon) - A(x_0) + \lambda_\varepsilon A(h_\varepsilon) = N_g(x_0 + h_\varepsilon) - N_g(x_0) - \lambda_\varepsilon K_p(h_\varepsilon). \tag{2.17}$$

From (2.17), we obtain

$$\begin{aligned} & -\triangle_p(x_0 + h_\varepsilon)(z) + \triangle_p x_0(z) - \lambda_\varepsilon \triangle_p(h_\varepsilon)(z) = \\ = \ & g(z, (x_0 + h_\varepsilon)(z)) - g(z, x_0(z)) - \lambda_\varepsilon |h_\varepsilon(z)|^{p-2} h_\varepsilon(z) \ \text{ a.e. on } Z. \end{aligned} \tag{2.18}$$

We introduce the function $S : Z \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$S(z, y) = \|Dx_0(z) + y\|^{p-2}_{\mathbb{R}^N}(Dx_0(z) + y) - \|Dx_0(z)\|^{p-2}_{\mathbb{R}^N} Dx_0(z) + \lambda_\varepsilon \|y\|^{p-2}_{\mathbb{R}^N} y.$$

Clearly, the function S is Carathéodory, hence it is jointly measurable on $Z \times \mathbb{R}^N$ and also has a $(p-1)$-polynomial growth in the $y \in \mathbb{R}^N$ variable. We rewrite (2.18) in terms of the map S and we have

$$-\text{div} S(z, Dh_\varepsilon(z)) =$$

$$= g(z, (x_0 + h_\varepsilon)(z)) - g(z, x_0(z)) - \lambda_\varepsilon |h_\varepsilon(z)|^{p-2} h_\varepsilon(z) \text{ a.e. on } Z. \tag{2.19}$$

By (2.16), (2.18) and (2.19), and using as before the nonlinear Green's formula of Casas-Fernandez [7] and Kenmochi [15], we obtain

$$\frac{\partial h_\varepsilon}{\partial \nu} = 0 \text{ in } W^{-\frac{1}{p'},p'}(\partial Z). \tag{2.20}$$

In particular, $h_\varepsilon \in C^1_\nu(Z)$.

Because $2 \le p < \infty$ and $\lambda_\varepsilon > 0$, we see that there exists $\hat{c} > 0$ such that

$$(S(z,y), y)_{\mathbb{R}^N} \ge \hat{c} \|y\|^p_{\mathbb{R}^N} \text{ for a.e. } z \in Z, \text{ all } y \in \mathbb{R}^N.$$

Then (2.19), (2.20), [16, Theorem 7.1, p. 286], [17, Theorem 2], and the continuity of the Lagrange multipliers λ_ε on the parameter $\varepsilon \in (0,1]$ imply the existence of $\alpha \in (0,1)$ and $\beta > 0$, both independent of $\varepsilon \in (0,1]$ and $\lambda_\varepsilon > 0$ such that

$$h_\varepsilon \in C^{1,\alpha}_\nu(\overline{Z}) \text{ and } \|h_\varepsilon\|_{C^{1,\alpha}_\nu(\overline{Z})} \le \beta. \tag{2.21}$$

We consider a sequence $\{\varepsilon_n\} \subset (0,\infty)$ such that $\varepsilon_n \to 0$ and set $h_n = h_{\varepsilon_n}$. Since the space $C^{1,\alpha}_\nu(\overline{Z})$ is embedded compactly into $C^1_\nu(\overline{Z})$, see [24, Theorem A.4, p. 213], due to (2.21), and by passing to a suitable subsequence if necessary, we may assume that $h_n \to \hat{h}$ in $C^1_\nu(\overline{Z})$ as $n \to \infty$. But recall that $\|h_n\| \le \varepsilon_n$. Therefore, $h_n \to 0$ in $W^{1,p}_\nu(Z)$ as $n \to \infty$. Consequently, we have $\hat{h} = 0$. Then, for large $n \ge 1$, we have $\|h_n\|_{C^1_\nu(\overline{Z})} \le r_1$. Due to our hypothesis, we have

$$\gamma(x_0) \le \gamma(x_0 + h_n),$$

which contradicts (2.14). This proves that $x_0 \in C^1_\nu(\overline{Z})$ is also a local $W^{1,p}_\nu(Z)$-minimizer of γ. □

The next result will be useful in computing critical groups at infinity. It extends a corresponding result of Perera-Schechter [21, Lemma 2.4] from Hilbert spaces to Banach spaces.

Lemma 2.2 *If $(X, \|\cdot\|)$ is a Banach space, $(t,x) \mapsto \varphi_t(x)$ is a function belonging to $C^1([0,1] \times X)$ such that $x \mapsto \partial_t \varphi_t(x)$ and $x \mapsto \varphi'_t(x)$ are both locally Lipschitz functions, and there exists $R > 0$ such that*

$$\inf\{\|\varphi'_t(x)\|_* : t \in [0,1], \|x\| > R\} > 0, \tag{2.22}$$

and

$$\inf\{\varphi_t(x) : t \in [0,1], \|x\| \le R\} > -\infty, \tag{2.23}$$

then

$$C_n(\varphi_0, \infty) = C_n(\varphi_1, \infty) \text{ for all } n \ge 0.$$

Proof. First, note that because of (2.22), for every $t \in [0,1]$,

$$K_t = \{x \in X : \varphi_t'(x) = 0\} \subset \overline{B}_R. \tag{2.24}$$

Due to (2.24), the function $\varphi \in C^1([0,1] \times X)$ admits a pseudogradient vector field $\hat{v} = (v_0, v) : [0,1] \times (X \setminus \overline{B}_R) \to [0,1] \times X$. Taking into account the construction of the pseudogradient vector field, one can assume that $v_0(t,x) = \partial_t \varphi_t(x)$. By definition, the map $(t,x) \mapsto v_t(x)$ is locally Lipschitz and in fact, for every $t \in [0,1]$, v_t is a pseudogradient vector field for the function φ_t, see Chang [8, p. 19]. In particular, for every $(t,x) \in [0,1] \times (X \setminus \overline{B}_R)$, we have

$$\langle \varphi_t'(x), v_t(x) \rangle \geq \|\varphi_t'(x)\|_*^2. \tag{2.25}$$

The map

$$X \setminus \overline{B}_R \ni x \mapsto -\frac{|\partial_t \varphi_t(x)|}{\|\varphi_t'(x)\|_*^2} v_t(x) =: w_t(x) \in X \tag{2.26}$$

is well-defined and locally Lipschitz. Due to (2.23), one can fix

$$\alpha < \inf\{\varphi_t(x) : t \in [0,1], \|x\| \leq R\} \tag{2.27}$$

such that $\varphi_0^\alpha \neq \emptyset$ or $\varphi_1^\alpha \neq \emptyset$. (If no such α can be found, then $C_n(\varphi_0, \infty) = C_n(\varphi_1, \infty) = \delta_{n,0}\mathbb{Z}$, and we are done.) To fix our ideas, we assume that $\varphi_0^\alpha \neq \emptyset$, the argument being similar for $\varphi_1^\alpha \neq \emptyset$. Let $u \in \varphi_0^\alpha$ and consider the Cauchy problem

$$\frac{d}{dt}\eta = w_t(\eta) \text{ for all } t \in [0,1], \ \eta(0,u) = u. \tag{2.28}$$

The local existence theorem (see for instance Gasiński-Papageorgiou [14, p. 618]) implies that there exists a local flow η. Then, we have

$$\begin{aligned} \frac{d}{dt}\varphi_t(\eta) &= \left\langle \varphi_t'(\eta), \frac{d}{dt}\eta \right\rangle + \partial_t \varphi_t(\eta) \\ &= \langle \varphi_t'(\eta), w_t(\eta) \rangle + \partial_t \varphi_t(\eta) \quad \text{(see (2.28))} \\ &\leq -|\partial_t \varphi_t(\eta)| + \partial_t \varphi_t(\eta) \quad \text{(see (2.26), (2.25))} \\ &\leq 0. \end{aligned}$$

Consequently, the function $t \mapsto \varphi_t(\eta)$ is non-increasing and we have

$$\varphi_t(\eta(t,u)) \leq \varphi_0(\eta(0,u)) = \varphi_0(u) \leq \alpha.$$

On account of (2.27), we have $\|\eta(t,u)\| > R$ for every $t \in [0,1]$. Hence, $\varphi_t'(\eta) \neq 0$, see (2.22), and the solution is global. In particular, $\eta(1,\cdot)$ is a homeomorphism between φ_0^α and a subset of φ_1^α. Now, by reversing the time-dependence from t to $1-t$ in φ_t, due to the uniqueness of the solution of the new Cauchy problem corresponding to (2.28), we obtain another flow $\tilde{\eta}$ instead of η, with $\tilde{\eta} \circ \eta = \eta \circ \tilde{\eta} = \text{id}$ and $\tilde{\eta}(1,\cdot)$ is a homeomorphism between φ_1^α and a subset of φ_0^α. Consequently, $\eta(1,\cdot)$ defines a homeomorphism between φ_0^α and φ_1^α, which implies that $H_n(X, \varphi_0^\alpha) = H_n(X, \varphi_1^\alpha)$ for every $n \geq 0$. This proves the lemma. □

3 Three nontrivial solutions for problem (P)

In this section we establish two multiplicity results where the nonlinearities have different geometry. Both of them guarantee the existence of at least three nontrivial solutions for problem (P). In the first case, our hypotheses on the nonlinear term $f(z,x)$ are the following:

$\underline{H(f)_1}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$ the function $z \mapsto f(z,x)$ is measurable;

(ii) for almost all $z \in Z$, the function $x \mapsto f(z,x)$ is continuous;

(iii) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that

$$|f(z,x)| \le a_r(z) \quad \text{for a.a. } z \in Z \text{ and all } |x| \le r;$$

(iv) there exist functions $\hat{\theta}, \theta \in L^\infty(Z)$ such that $\hat{\theta}(z) \le \theta(z) \le 0$ a.e. on Z, $\theta \ne 0$, and uniformly for a.a. $z \in Z$

$$\hat{\theta}(z) \le \liminf_{|x|\to\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \limsup_{|x|\to\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \theta(z);$$

(v) there exists $\delta > 0$ such that

$$f(z,x)x > 0 \quad \text{for a.a. } z \in Z, \text{ all } 0 < |x| \le \delta;$$

(vi) if $F(z,x) = \int_0^x f(z,r)dr$, there exist $\mu \in (1,p)$, $q \in (p,p^*)$ and some constants $c_1, c_2 > 0$ such that

$$\mu F(z,x) - f(z,x)x \ge c_1|x|^p - c_2|x|^q \quad \text{for a.a. } z \in Z, \text{ all } x \in \mathbb{R};$$

(vii) there exists $c_3 > 0$ such that

$$-c_3|x|^p \le f(z,x)x \quad \text{for a.a. } z \in Z, \text{ all } x \in \mathbb{R}.$$

Theorem 3.1 *If hypotheses $H(f)_1$ hold, then problem (P) has at least three distinct nontrivial solutions $x_0 \in \mathrm{int}C_+$, $v_0 \in -\mathrm{int}C_+$, and $y_0 \in C^1_\nu(\overline{Z})$.*

A simple verification shows that the function $f : \mathbb{R} \to \mathbb{R}$ (we drop for simplicity the z-dependence), defined by

$$f(x) = g(x) \cdot \begin{cases} |x|^{\mu-2}x - |x|^{p-2}x + |x|^{q-2}x, & \text{if } |x| \le 1 \\ -|x|^{p-2}x + \frac{2}{|x|x}, & \text{if } |x| > 1 \end{cases}$$

satisfies hypotheses $H(f)_1$ whenever $1 < \mu < p < q < p^*$, $p \ge 2$, and g is from Example 1.1 (see §1).

In order to state the next result which guarantees again three nontrivial solutions for (P) within a new framework (modifying the behavior of the nonlinearity f near the origin), we recall some basic notions from the nonsmooth calculus.

Let X be a Banach space, X^* its topological dual and we denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). If $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function, the *generalized directional derivative* $\varphi^0(x; h)$ *of* φ *at* $x \in X$ *in the direction* $h \in X$, is defined by

$$\varphi^0(x; h) = \limsup_{x' \to x; \lambda \to 0^+} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to see that $h \mapsto \varphi^0(x; h)$ is sublinear continuous and so it is the support function of a nonempty, convex and w^*-compact set $\partial \varphi(x) \subseteq X^*$ defined by

$$\partial \varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \le \varphi^0(x; h) \ \text{ for all } h \in X\}.$$

The multifunction $x \mapsto \partial \varphi(x)$ is called the *generalized* (*or Clarke*) *subdifferential of* φ. Note that, if $\varphi : X \to \mathbb{R}$ is continuous convex, then φ is locally Lipschitz and the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, given by

$$\partial_c \varphi(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \le \varphi(y) - \varphi(x) \ \text{ for all } y \in X\}.$$

Also, if $\varphi \in C^1(X)$, then clearly φ is locally Lipschitz and $\partial \varphi(x) = \{\varphi'(x)\}$. The generalized subdifferential has a rich calculus, which generalizes the classical calculus and the subdifferential calculus of continuous convex functionals. For details we refer the reader to Clarke [9].

The new hypotheses on the nonlinearity $f(z, x)$ are the following:

$\underline{H(f)_2}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z and

(*i*) for all $x \in \mathbb{R}$ the function $z \mapsto f(z, x)$ is measurable;

(*ii*) for almost all $z \in Z$, the function $x \mapsto f(z, x)$ is locally Lipschitz;

(*iii*) there exist $a \in L^\infty(Z)_+$ and $c > 0$ such that

$$|u| \le a(z) + c|x|^{p-2} \ \text{ for a.a. } z \in Z, \ \text{all } x \in \mathbb{R} \text{ and all } u \in \partial f(z, x);$$

(*iv*) there exist functions $\hat{\theta}, \theta \in L^\infty(Z)$ such that $\hat{\theta}(z) \le \theta(z) \le 0$ a.e. on Z, $\theta \ne 0$, and uniformly for a.a. $z \in Z$

$$\hat{\theta}(z) \le \liminf_{|x| \to \infty} \frac{f(z, x)}{|x|^{p-2} x} \le \limsup_{|x| \to \infty} \frac{f(z, x)}{|x|^{p-2} x} \le \theta(z);$$

(*v*) there exist $\delta > 0$ and a function $\beta \in L^\infty(Z)$ with $\beta(z) \le 0$ a.e. on Z, $\beta \ne 0$ such that

$$F(z, x) \le \frac{\beta(z)}{p} |x|^p \ \text{ for a.a. } z \in Z, \ \text{all } |x| \le \delta;$$

(*vi*) there exist $c_- < 0 < c_+$ such that $\int_Z F(z, c_\pm)dx > 0$;

(*vii*) there exists $c_3 > 0$ such that

$$-c_3|x|^p \le f(z,x)x \quad \text{for a.a. } z \in Z, \text{ all } x \in \mathbb{R}.$$

The first part of Theorem 1.1 follows at once from the following

Theorem 3.2 *If hypotheses $H(f)_2$ hold, then problem (P) has at least three distinct nontrivial solutions $x_0 \in \mathrm{int}C_+$, $v_0 \in -\mathrm{int}C_+$, and $y_0 \in C^1_\nu(\overline{Z})$.*

In the sequel, we are dealing with the proof of Theorems 3.1 and 3.2, respectively. To do these, we prove certain intermediate results.

Lemma 3.1 *If $\theta \in L^\infty(Z)$, $\theta(z) \le 0$ a.e. on Z and the inequality is strict on a set of positive measure, then there exists $\xi_0 > 0$ such that*

$$\psi(x) = \|Dx\|_p^p - \int_Z \theta(z)|x(z)|^p dz \ge \xi_0 \|x\|^p \quad \text{for all } x \in W^{1,p}(Z).$$

Proof. It is clear that $\psi \ge 0$. Suppose that the lemma is false. Since ψ is p-homogeneous, we can find a sequence $\{x_n\} \subset W^{1,p}(Z)$ such that $\|x_n\| = 1$ and $\psi(x_n) \downarrow 0$ as $n \to \infty$. By passing to a suitable subsequence if necessarily, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W^{1,p}(Z),\ x_n \to x \text{ in } L^p(Z),\ x_n(z) \to x(z) \text{ a.e. on } Z \text{ and}$$

$$|x_n(z)| \le k(z) \text{ a.e. on } Z, \text{ for all } n \ge 1, \text{ with } k \in L^p(Z)_+.$$

We have that $\|Dx\|_p^p \le \liminf_{n\to\infty} \|Dx_n\|_p^p$, while from the dominated convergence theorem, it follows that $\int_Z \theta|x_n|^p dz \to \int_Z \theta|x|^p dz$. So, $\psi(x) \le \liminf_{n\to\infty} \psi(x_n) = 0$. Consequently,

$$\|Dx\|_p^p \le \int_Z \theta|x|^p dz \le 0. \tag{3.29}$$

In particular, $x \equiv c \in \mathbb{R}$.

If $c = 0$, then $\|Dx_n\|_p \to 0$ as $n \to \infty$, so $x_n \to 0$ in $W^{1,p}(Z)$, a contradiction to the fact that $\|x_n\| = 1$ for all $n \ge 1$.

If $c \ne 0$, then from (3.29) and the assumption on θ, we have

$$0 \le |c|^p \int_Z \theta dz < 0,$$

a contradiction. This proves the lemma. □

Let $\tau_\pm : \mathbb{R} \to \mathbb{R}$ be the truncation functions defined by

$$\tau_+(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } x > 0 \end{cases} \quad \text{and} \quad \tau_-(x) = \begin{cases} x, & \text{if } x < 0 \\ 0, & \text{if } x \ge 0. \end{cases}$$

Using these functions, we introduce the following truncations of f

$$f_+(z,x) = f(z, \tau_+(z)) \quad \text{and} \quad f_-(z,x) = f(z, \tau_-(z)).$$

Note that both f_+ and f_- are Carathéodory functions and

$$f_+(z,x)=0 \text{ for a.a. } z\in Z, \text{ all } x\le 0, \quad \text{and} \quad f_-(z,x)=0 \text{ for a.a. } z\in Z, \text{ all } x\ge 0.$$

We set $F_\pm(z,x)=\int_0^x f_\pm(z,r)dr$. We introduce the functionals $\varphi,\varphi_\pm : W^{1,p}_\nu(Z)\to\mathbb{R}$ defined by

$$\varphi(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z F(z,x(z))dz$$

and

$$\varphi_\pm(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z F_\pm(z,x(z))dz \text{ for all } x\in W^{1,p}_\nu(Z).$$

Clearly, $\varphi,\varphi_\pm\in C^1(W^{1,p}_\nu(Z))$.

By means of these functions, we will produce two nontrivial smooth solutions of constant sign for (P).

Proposition 3.1 *If hypotheses $H(f)_1$ hold, then problem (P) has two solutions $x_0\in \text{int}C_+$, $v_0\in -\text{int}C_+$ which are local minimizers of φ.*

Proof. By virtue of hypotheses $H(f)_1$ (iii) and (iv), given $\varepsilon>0$, we can find $a_\varepsilon\in L^\infty(Z)_+$ such that

$$f(z,x)=f_+(z,x)\le(\theta(z)+\varepsilon)x^{p-1}+a_\varepsilon(z) \text{ for a.a. } z\in Z, \text{ all } x\ge 0.$$

Therefore,

$$F_+(z,x)\le\frac{1}{p}(\theta(z)+\varepsilon)x^p+a_\varepsilon(z)x \text{ for a.a. } z\in Z, \text{ all } x\ge 0. \tag{3.30}$$

Then, for every $x\in W_+$, we have

$$\begin{aligned}\varphi_+(x) &= \frac{1}{p}\|Dx\|_p^p-\int_Z F_+(z,x(z))dz\\ &\ge \frac{1}{p}\|Dx\|_p^p-\frac{1}{p}\int_Z\theta|x|^pdz-\frac{\varepsilon}{p}\|x\|^p-c_4\|x\| \quad (\text{for some } c_4>0, \text{ see } (3.30))\\ &\ge \frac{\xi_0-\varepsilon}{p}\|x\|^p-c_4\|x\| \quad (\text{see Lemma 3.1}).\end{aligned} \tag{3.31}$$

If we choose $\varepsilon<\xi_0$, then from (3.31) we infer that φ_+ is coercive on W_+. Moreover, exploiting the compact embedding of $W^{1,p}_\nu(Z)$ into $L^p(Z)$, we can easily verify that φ_+ is weakly lower semicontinuous on W_+. Consequently, we can find $x_0\in W_+$ such that

$$\varphi_+(x_0)=\inf\{\varphi_+(x): x\in W_+\}.$$

From Clarke [9, p. 52], we have $-\varphi_+'(x_0)\in N_{W_+}(x_0)$, where $N_{W_+}(x_0)$ is the normal cone to W_+ at x_0. By definition,

$$N_{W_+}(x_0)=\{v^*\in W^{1,p}_\nu(Z)^*:\langle v^*,v-x_0\rangle\le 0 \text{ for all } v\in W_+\}.$$

Hence

$$0\le\langle\varphi_+'(x_0),v-x_0\rangle \text{ for all } v\in W_+. \tag{3.32}$$

Given $\varepsilon > 0$ and $h \in W^{1,p}_{\nu}(Z)$, set $v = (x_0 + \varepsilon h)^+ = (x_0 + \varepsilon h) + (x_0 + \varepsilon h)^- \in W_+$. We use this as a test function in (3.32) and we obtain

$$-\langle \varphi'_+(x_0), (x_0 + \varepsilon h)^- \rangle \leq \varepsilon \langle \varphi'_+(x_0), h \rangle. \tag{3.33}$$

Note that $\varphi'_+(x_0) = A(x_0) - N_{f_+}(x_0)$ and let $Z^-_\varepsilon = \{z \in Z : (x_0 + \varepsilon h)(z) < 0\}$. Then

$$-\langle \varphi'_+(x_0), (x_0 + \varepsilon h)^- \rangle =$$

$$= \int_{Z^-_\varepsilon} \|Dx_0\|^{p-2}_{\mathbb{R}^N} (Dx_0, D(x_0 + \varepsilon h))_{\mathbb{R}^N} dz - \int_{Z^-_\varepsilon \cap \{x_0 > 0\}} f_+(z, x_0)(x_0 + \varepsilon h) dz \geq$$

$$\geq \varepsilon \int_{Z^-_\varepsilon} \|Dx_0\|^{p-2}_{\mathbb{R}^N} (Dx_0, Dh)_{\mathbb{R}^N} dz + \varepsilon \int_{Z^-_\varepsilon \cap \{x_0 > 0\}} |f_+(z, x_0(z))| h dz.$$

Combining this inequality with (3.33), we obtain

$$\langle \varphi'_+(x_0), h \rangle \geq \int_{Z^-_\varepsilon} \|Dx_0\|^{p-2}_{\mathbb{R}^N} (Dx_0, Dh)_{\mathbb{R}^N} dz + \int_{Z^-_\varepsilon \cap \{x_0 > 0\}} |f_+(z, x_0(z))| h dz. \tag{3.34}$$

But note that $|Z^-_\varepsilon \cap \{x_0 > 0\}|_N \to 0$ as $\varepsilon \to 0$. Also, by Stampacchia's theorem, $Dx_0(z) = 0$ a.e. on $\{x_0 = 0\}$. Hence, if in (3.34) we pass to the limit as $\varepsilon \to 0$, we obtain

$$\langle \varphi'_+(x_0), h \rangle \geq 0.$$

Since h is arbitrarily, we have $\varphi'_+(x_0) = 0$, i.e., $A(x_0) = N_{f_+}(x_0)$. Reasoning as in Proposition 2.1, we have that $-\triangle_p x_0(z) = f_+(z, x_0(z))$ a.e. on Z, and $\frac{\partial x_0}{\partial \nu} = 0$ on ∂Z. Since $x_0 \in W_+$, then $f_+(z, x_0(z)) = f(z, x_0(z))$, so, x_0 is a solution of (P).

Note that $x_0 \neq 0$. Indeed, due to hypothesis $H(f)_1$ (v), for every $0 < s \leq \delta$, we have $F(z, s) > 0$ a.e. on Z. Thus, $\varphi_+(s) = -\int_Z F(z, s) < 0$. Since the constant function s belongs to W_+, then $\varphi_+(x_0) \leq \varphi_+(s) < 0 = \varphi_+(0)$, i.e., $x_0 \neq 0$.

The nonlinear regularity theory implies that $x_0 \in C_+$. Moreover, from $H(f)_1$ (vii) we have that

$$\triangle_p x_0(z) \leq c_3 x_0^{p-1}(z) \;\text{ a.e. on } Z. \tag{3.35}$$

Invoking the nonlinear strong maximum principle of Vázquez [25], from (3.35) we infer that $x_0 \in \text{int} C_+$. Since $\varphi_+|_{C_+} = \varphi|_{C_+}$, we can find $r > 0$ small enough such that

$$\varphi(x_0) \leq \varphi(x_0 + h) \;\text{ for all } h \in C^1_\nu(\overline{Z}),\; \|h\|_{C^1_\nu(\overline{Z})} \leq r.$$

In other words, x_0 is a local $C^1_\nu(\overline{Z})$-minimizer of φ. By virtue of Proposition 2.1, we have that $x_0 \in \text{int} C_+$ is a local $W^{1,p}_\nu(Z)$-minimizer of φ.

In a similar fashion, working with φ_- and $-W_+$ instead of φ_+ and W_+, respectively, we find a solution $v_0 \in -\text{int} C_+$ of (P) which is also a local $W^{1,p}_\nu(Z)$-minimizer of φ. □

Note that $x = 0$ is critical point of φ, hence a solution of problem (P). We may assume that it is an isolated critical point of φ; otherwise, we already have a whole sequence of distinct nontrivial solutions of (P) and so we are done. Next, we calculate the critical groups of φ at $x = 0$.

Proposition 3.2 *If hypotheses $H(f)_1$ hold, then $C_n(\varphi, 0) = 0$ for all $n \geq 0$.*

Proof. Because of hypothesis $H(f)_1$ (vi) and since $p < q$, we can find $\delta_1 < \min\{\delta, 1\}$ ($\delta > 0$ as in hypothesis $H(f)_1$ (v)) such that

$$\mu F(z,x) - f(z,x)x \geq 0 \ \text{ for a.a. } z \in Z, \ \text{all } |x| \leq \delta_1. \tag{3.36}$$

By (3.36) and hypothesis $H(f)_1$ (v), after integration (as in the case of the Ambrosetti - Rabinowitz condition, see for instance Gasiński-Papageorgiou [14, p. 298]), we obtain

$$F(z,x) \geq c_5 |x|^\mu \ \text{ for a.a. } z \in Z, \ \text{all } |x| \leq \delta_1 \ \text{ and with } c_5 > 0. \tag{3.37}$$

Combining (3.37) with hypotheses $H(f)_1$ (vi) (vii), we have

$$F(z,x) \geq c_5|x|^\mu + c_6|x|^p - c_2|x|^q \ \text{ for a.a. } z \in Z, \ \text{all } x \in \mathbb{R}, \tag{3.38}$$

with $c_6 = c_1 - c_3$, and $c_2 > 0$ from $H(f)_1$ (vi).

Let $x \in W^{1,p}_\nu(Z) \setminus \{0\}$ and $t > 0$. Then

$$\begin{aligned}
\varphi(tx) &= \frac{t^p}{p}\|Dx\|_p^p - \int_Z F(z, tx(z))dz \\
&\leq \frac{t^p}{p}\|x\|^p - \int_Z (c_5 t^\mu |x|^\mu + c_6 t^p |x|^p - c_2 t^q |x|^q)dz \quad \text{(see (3.38))} \\
&\leq \frac{t^p}{p}\|x\|^p - c_5 t^\mu \|x\|_\mu^\mu - c_6 t^p \|x\|_p^p + c_2 t^q \|x\|_q^q. \qquad (3.39)
\end{aligned}$$

Since $\mu < p < q$ and $c_5 > 0$, from (3.39) we see that we can find $t_0 = t_0(x) \in (0,1)$ small such that

$$\varphi(tx) < 0 \ \text{ for all } t \in (0, t_0). \tag{3.40}$$

Now, consider $x \in W^{1,p}_\nu(Z)$ such that $\varphi(x) = 0$. Then

$$\begin{aligned}
\frac{d}{dt}\varphi(tx)\Big|_{t=1} &= \langle \varphi'(tx), x\rangle\Big|_{t=1} \\
&= \|Dx\|_p^p - \int_Z f(z,x)x dz \\
&= \left(1 - \frac{\mu}{p}\right)\|Dx\|_p^p + \int_Z (\mu F(z,x) - f(z,x)x)dz \quad (\text{since } \varphi(x) = 0) \\
&\geq \left(1 - \frac{\mu}{p}\right)\|Dx\|_p^p + c_1\|x\|_p^p - c_2\|x\|_q^q \quad (\text{see hypothesis } H(f)_1(vi)) \\
&\geq c_7\|x\|^p - c_8\|x\|^q \quad (\text{for some } c_7, c_8 > 0, \ \text{since } \mu < p). \qquad (3.41)
\end{aligned}$$

Because $p < q$, from (3.41) we see that we can find $r \in (0,1)$ small enough such that

$$\frac{d}{dt}\varphi(tx)\Big|_{t=1} > 0 \ \text{ for all } x \in W^{1,p}_\nu(Z) \ \text{ such that } \varphi(x) = 0, \ 0 < \|x\| \leq r. \tag{3.42}$$

Consider $x \in W^{1,p}_\nu(Z)$ such that $\|x\| \le r$ and $\varphi(x) \le 0$. We claim that

$$\varphi(tx) \le 0 \quad \text{for all } t \in [0,1]. \tag{3.43}$$

We argue indirectly. So, we suppose that we can find $t_0 \in (0,1)$ such that $\varphi(t_0 x) > 0$. Since φ is continuous, there exists $t_1 \in (t_0, 1]$ such that $\varphi(t_1 x) = 0$. We set $t_2 = \min\{t \in [t_0,1] : \varphi(tx) = 0\}$. Then

$$\varphi(tx) > 0 \quad \text{for all } t \in [t_0, t_2). \tag{3.44}$$

Let $u = t_2 x$. Since $\varphi(u) = 0$ and $0 < \|u\| \le r$, on account of (3.42), we have

$$\frac{d}{dt}\varphi(tu)\big|_{t=1} > 0. \tag{3.45}$$

On the other hand, due to (3.44), for every $t \in [t_0, t_2)$, we have

$$0 = \varphi(u) = \varphi(t_2 x) < \varphi(tx).$$

Consequently,

$$\frac{d}{dt}\varphi(tu)\big|_{t=1} = \frac{d}{dt}\varphi(tx)\big|_{t=t_2} = \lim_{t \uparrow t_2} \frac{\varphi(tx) - \varphi(t_2 x)}{t - t_2} \le 0. \tag{3.46}$$

But (3.45) contradicts (3.46). This means that (3.43) is true.

Now, we fix $r \in (0,1)$ so that the origin is the unique critical point of φ in $\overline{B}_r$. Let $h : [0,1] \times (\overline{B}_r \cap \varphi^0) \to \overline{B}_r \cap \varphi^0$ be defined by $h(t,x) = (1-t)x$. Due to (3.43), h is well-defined. Moreover, it is a continuous deformation and we have that $\overline{B}_r \cap \varphi^0$ is contractible in itself.

Fix $x \in \overline{B}_r$ such that $\varphi(x) > 0$. We claim that there exists a unique $t_x \in (0,1)$ such that

$$\varphi(t_x x) = 0. \tag{3.47}$$

The existence is clear. Indeed, relation (3.40), $\varphi(x) > 0$, and the continuity of $t \mapsto \varphi(tx)$ imply the existence of $t_x \in (0,1)$ as in (3.47). Now, we assume that there are $0 < t_x^1 < t_x^2 < 1$ such that $\varphi(t_x^1 x) = \varphi(t_x^2 x) = 0$. Due to (3.43), $\varphi(t t_x^2 x) \le 0$ for every $t \in [0,1]$. In particular, $t_x^1 \in (0,1)$ is a local maximum of the function $t \mapsto \varphi(tx)$, $t \in [0,1]$. Thus,

$$\frac{d}{dt}\varphi(t t_x^1 x)\big|_{t=1} = 0,$$

which contradicts (3.42). Consequently, (3.47) is true, and for every $x \in \overline{B}_r$ with $\varphi(x) > 0$ we have

$$\varphi(tx) < 0 \quad \text{for all } t \in (0, t_x), \quad \text{and} \quad \varphi(tx) > 0 \quad \text{for all } t \in (t_x, 1]. \tag{3.48}$$

Next, let $\xi : \overline{B}_r \setminus \{0\} \to (0,1]$ be defined by

$$\xi(x) = \begin{cases} 1, & \text{if} \quad x \in \overline{B}_r \setminus \{0\} \text{ and } \varphi(x) \le 0; \\ t_x, & \text{if} \quad x \in \overline{B}_r \text{ and } \varphi(x) > 0 \quad (t_x \text{ being from (3.47)}). \end{cases}$$

Due to the above reasons, the function ξ is well-defined. Moreover, due to (3.42), (3.48) and the implicit function theorem, we also have that ξ is continuous. Then we define the map $\beta : \overline{B}_r \setminus \{0\} \to (\overline{B}_r \cap \varphi^0) \setminus \{0\}$ by

$$\beta(x) = \begin{cases} \xi(x)x, & \text{if} \quad x \in \overline{B}_r \setminus \{0\} \text{ and } \varphi(x) \geq 0; \\ x, & \text{if} \quad x \in \overline{B}_r \setminus \{0\} \text{ and } \varphi(x) < 0. \end{cases}$$

The definition of ξ implies that $\xi(x) = 1$ whenever $\varphi(x) = 0$. Therefore, β is well-defined and since ξ is continuous, the same is true for β. Moreover, we have

$$\beta(x) = x \text{ for all } x \in \overline{B}_r \setminus \{0\}, \ \varphi(x) \leq 0.$$

Consequently, β is a retraction of $\overline{B}_r \setminus \{0\}$ onto $(\overline{B}_r \cap \varphi^0) \setminus \{0\}$. But since $W^{1,p}_\nu(Z)$ is an infinite dimensional Banach space, $\overline{B}_r \setminus \{0\}$ is contractible in itself. Recall that retracts of contractible spaces are contractible too. Therefore, $(\overline{B}_r \cap \varphi^0) \setminus \{0\}$ is contractible in itself. It follows that

$$C_n(\varphi, 0) = H_n(\overline{B}_r \cap \varphi^0, (\overline{B}_r \cap \varphi^0) \setminus \{0\}) = 0 \text{ for all } n \geq 0,$$

which proves our assertion. □

Proof of Theorem 3.1. First of all, the hypotheses $H(f)_1$ imply in a standard manner that φ satisfies the PS-condition. Next, Proposition 3.1 implies the existence of two solutions $x_0 \in \text{int}C_+$, $v_0 \in -\text{int}C_+$ for (P) which are local minimizers of φ. As before, we may assume that both x_0, v_0 are isolated critical points of φ. Thus, we can find $r > 0$ small enough such that

$$\varphi(x_0) < \inf\{\varphi(x) : \|x - x_0\| = r\} \text{ and } \varphi(v_0) < \inf\{\varphi(v) : \|v - v_0\| = r\}.$$

Without any loss of generality, we may assume that $\varphi(v_0) \leq \varphi(x_0)$. Consider the sets

$$D = \partial B_r(x_0), \ E = [v_0, x_0], \text{ and } E_0 = \{v_0, x_0\}.$$

It is clear that $\{E, E_0\}$ and D are linking in $W^{1,p}_\nu(Z)$. Then, by the abstract linking theorem (see for instance Gasiński-Papageorgiou [14, p. 644]), we can find a critical point $y_0 \in W^{1,p}_\nu(Z)$ of φ, which is of mountain pass-type and

$$\varphi(v_0) \leq \varphi(x_0) < \varphi(y_0),$$

i.e., $y_0 \neq x_0$ and $y_0 \neq v_0$. Moreover, since y_0 is of mountain pass-type, we have

$$\text{rank}C_1(\varphi, y_0) \geq 1, \tag{3.49}$$

see Mawhin-Willem [19, p. 195]. Comparing (3.49) with Proposition 3.2, we see that $y_0 \neq 0$. Since y_0 is a critical point of φ, as before, we verify that it is a solution of (P) and the nonlinear regularity theory implies that $y_0 \in C^1_\nu(\overline{Z})$. □

Now, we are going to prove Theorem 3.2. The crucial result here is

Proposition 3.3 *If $H(f)_2$ hold, then $C_n(\varphi,\infty)=\delta_{n,0}\mathbb{Z}$ for all $n\geq 0$.*

Proof. A standard argument shows that φ satisfies the PS-condition. Consider the one parameter family of functions $\varphi_t : W^{1,p}_\nu(Z)\to\mathbb{R}$, $t\in[0,1]$, defined by

$$\varphi_t(x)=\frac{1}{p}\|Dx\|_p^p - t\int_Z F(z,x(z))dz-\frac{1-t}{p}\int_Z \theta(z)|x(z)|^p dz \text{ for all } x\in W^{1,p}_\nu(Z).$$

We have

$$\partial_t\varphi_t(x)=-\int_Z F(z,x(z))dz+\frac{1}{p}\int_Z\theta(z)|x(z)|^p dz \tag{3.50}$$

and

$$\varphi_t'(x)=A(x)-tN_f(x)-(1-t)\theta K_p(x), \tag{3.51}$$

with $A: W^{1,p}_\nu(Z)\to W^{1,p}_\nu(Z)^*$ as before, $N_f: W^{1,p}_\nu(Z)\to L^{p'}(Z)$ is the Nemitsky operator corresponding to f, i.e., $N_f(x)(\cdot)=f(\cdot,x(\cdot))$ for all $x\in W^{1,p}_\nu(Z)$ and $K_p: L^p(Z)\to L^{p'}(Z)$ is defined by $K_p(x)(\cdot)=|x(\cdot)|^{p-2}x(\cdot)$ for all $x\in W^{1,p}_\nu(Z)$. Because of the compact embedding of $W^{1,p}_\nu(Z)$ into $L^p(Z)$, we can easily verify that both maps N_f and K_p are completely continuous.

From (3.50) and Clarke [9, p.83], we have that the map

$$x\mapsto \partial_t\varphi_t(x) \text{ is locally Lipschitz.} \tag{3.52}$$

Since $2\leq p$, the maps $x\mapsto A(x)$ and $x\mapsto K_p(x)$ are both locally Lipschitz. Moreover, due to hypotheses $H(f)_2$ $(ii),(iii)$, as in the proof of Clarke [9, Theorem 2.7.5, p. 83], using the nonsmooth mean value theorem (by Lebourg), we can show that $x\mapsto N_f(x)$ is locally Lipschitz. On account of (3.51), we have that the map

$$x\mapsto \varphi_t'(x) \text{ is locally Lipschitz.} \tag{3.53}$$

We claim that there exists $R>0$ such that

$$\inf\{\|\varphi_t'(x)\|_* : t\in[0,1], \|x\|>R\}>0. \tag{3.54}$$

We proceed by contradiction. So suppose that the claim is not true. We can find $\{t_n\}\subset[0,1]$ and $\{x_n\}\subset W^{1,p}_\nu(Z)$ such that

$$t_n\to t,\ \|x_n\|\to\infty \text{ and } \varphi_{t_n}'(x_n)\to 0 \text{ in } W^{1,p}_\nu(Z)^* \text{ as } n\to\infty. \tag{3.55}$$

Let $y_n=\frac{x_n}{\|x_n\|}$, $n\geq 1$. Then $\|y_n\|=1$ for all $n\geq 1$ and so we may assume that

$y_n\xrightarrow{w} y$ in $W^{1,p}_\nu(Z)$, $y_n\to y$ in $L^p(Z)$, $y_n(z)\to y(z)$ a.e. on Z and

$|y_n(z)|\leq k(z)$ a.e. on Z, for all $n\geq 1$, with $k\in L^p(Z)_+$.

Due to (3.55), we have

$$|\langle\varphi_{t_n}'(x_n),u\rangle|\leq\varepsilon_n\|u\| \text{ for all } u\in W^{1,p}_\nu(Z) \text{ with } \varepsilon_n\downarrow 0. \tag{3.56}$$

Let $u=x_n$. We obtain

$$\left|\|Dx_n\|_p^p - t_n\int_Z f(z,x_n)x_n dz-(1-t_n)\int_Z\theta|x_n|^p dz\right|\leq\varepsilon_n\|x_n\|.$$

Consequently,

$$\left| \|Dy_n\|_p^p - t_n \int_Z \frac{f(z,x_n)}{\|x_n\|^{p-1}} y_n dz - (1-t_n) \int_Z \theta |y_n|^p dz \right| \le \frac{\varepsilon_n}{\|x_n\|^{p-1}} = \varepsilon_n', \tag{3.57}$$

with $\varepsilon_n' \to 0$ as $n \to \infty$. From $H(f)_2$ (iii) after integration, we have

$$|f(z,x)| \le a_1(z)|x| + c_1|x|^{p-1} \tag{3.58}$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$ and with $a_1 \in L^\infty(Z)_+$, $c_1 > 0$. From (3.58) it follows that the sequence

$$\left\{ \eta_n(\cdot) = \frac{f(\cdot, x_n(\cdot))}{\|x_n\|^{p-1}} \right\} \subset L^{p'}(Z) \text{ is bounded } (1/p + 1/p' = 1).$$

By passing to a suitable subsequence if necessarily, we may assume that

$$\eta_n \xrightarrow{w} \eta \text{ in } L^{p'}(Z) \text{ as } n \to \infty. \tag{3.59}$$

For every $\varepsilon > 0$ and $n \ge 1$, we introduce the following two sets

$$C_{\varepsilon,n}^+ = \left\{ z \in Z : x_n(z) > 0,\ \hat{\theta}(z) - \varepsilon \le \frac{f(z, x_n(z))}{x_n^{p-1}(z)} \le \theta(z) + \varepsilon \right\}$$

and

$$C_{\varepsilon,n}^- = \left\{ z \in Z : x_n(z) < 0,\ \hat{\theta}(z) - \varepsilon \le \frac{f(z, x_n(z))}{|x_n(z)|^{p-2} x_n(z)} \le \theta(z) + \varepsilon \right\}.$$

Note that $x_n(z) \to +\infty$ a.e. on $\{y > 0\}$ and $x_n(z) \to -\infty$ a.e. on $\{y < 0\}$ as $n \to \infty$. Then hypothesis $H(f)_2$ (iv) implies

$$\chi_{C_{\varepsilon,n}^+}(z) \to 1 \text{ a.e. on } \{y > 0\} \text{ and } \chi_{C_{\varepsilon,n}^-}(z) \to 1 \text{ a.e. on } \{y < 0\}.$$

The dominated convergence theorem and (3.59) imply

$$\chi_{C_{\varepsilon,n}^+} \eta_n \xrightarrow{w} \eta \text{ in } L^{p'}(\{y > 0\}) \text{ as } n \to \infty, \tag{3.60}$$

and

$$\chi_{C_{\varepsilon,n}^-} \eta_n \xrightarrow{w} \eta \text{ in } L^{p'}(\{y < 0\}) \text{ as } n \to \infty. \tag{3.61}$$

We have

$$\chi_{C_{\varepsilon,n}^+}(z)(\hat{\theta}(z) - \varepsilon) y_n^{p-1}(z) \le \chi_{C_{\varepsilon,n}^+}(z) \eta_n(z) \le \chi_{C_{\varepsilon,n}^+}(z)(\theta(z) + \varepsilon) y_n^{p-1}(z) \tag{3.62}$$

a.e. on $\{y > 0\}$, and

$$\begin{aligned} \chi_{C_{\varepsilon,n}^-}(z)(\theta(z) + \varepsilon)|y_n(z)|^{p-2} y_n(z) &\le \chi_{C_{\varepsilon,n}^-}(z) \eta_n(z) \\ &\le \chi_{C_{\varepsilon,n}^-}(z)(\hat{\theta}(z) - \varepsilon)|y_n(z)|^{p-2} y_n(z) \end{aligned} \tag{3.63}$$

a.e. on $\{y < 0\}$.

We pass to the limit as $n \to \infty$ in (3.62) and (3.63) and we use (3.60) and (3.61), Mazur's lemma and let $\varepsilon \downarrow 0$. So we obtain

$$\hat{\theta}(z)y^{p-1}(z) \leq \eta(z) \leq \theta(z)y^{p-1}(z) \text{ a.e. on } \{y > 0\} \tag{3.64}$$

and

$$\theta(z)|y(z)|^{p-2}y(z) \leq \eta(z) \leq \hat{\theta}(z)|y(z)|^{p-2}y(z) \text{ a.e. on } \{y < 0\}. \tag{3.65}$$

Moreover, from (3.59) it is clear that

$$\eta(z) = 0 \text{ a.e. on } \{y = 0\}. \tag{3.66}$$

From (3.64), (3.65) and (3.66), it follows that

$$\eta(z) = g(z)|y(z)|^{p-2}y(z) \text{ a.e. on } Z \tag{3.67}$$

with $g \in L^\infty(Z)$ such that $\hat{\theta}(z) \leq g(z) \leq \theta(z)$ a.e. on Z. So passing to the limit as $n \to \infty$ in (3.57) and using (3.67), we obtain

$$\|Dy\|_p^p \leq \int_Z (tg(z) + (1-t)\theta(z))|y(z)|^p dz \leq 0. \tag{3.68}$$

Thus, $y = c \in \mathbb{R}$. We know from (3.56) that

$$\left|\langle A(y_n), y_n - y\rangle - t_n \int_Z \eta_n(y_n - y)dz - (1-t_n)\int_Z \theta|y_n|^{p-2}y_n(y_n - y)dz\right| \leq$$
$$\leq \frac{\varepsilon_n}{\|x_n\|^{p-1}}\|y_n - y\|. \tag{3.69}$$

Clearly,

$$\int_Z \eta_n(y_n - y)dz \to 0 \text{ and } \int_Z \theta|y_n|^{p-2}y_n(y_n - y)dz \to 0 \text{ as } n \to \infty.$$

Thus, from (3.69), it follows that

$$\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0. \tag{3.70}$$

But A is maximal monotone, it is generalized pseudomonotone and so from (3.70) we have $\lim_{n\to\infty}\langle A(y_n), y_n\rangle = \langle A(y), y\rangle$, see Gasiński-Papageorgiou [14, p. 330]. Consequently, $\|Dy_n\|_p \to \|Dy\|_p$ as $n \to \infty$. We also have $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, \mathbb{R}^N)$ and $L^p(Z, \mathbb{R}^N)$ being uniformly convex, it has the Kadec-Klee property. Therefore, $Dy_n \to Dy$ in $L^p(Z, \mathbb{R}^N)$. Thus, $y_n \to y$ in $W^{1,p}_\nu(Z)$. In particular, $\|y\| = 1$, i.e., $y \equiv c \neq 0$. Due to (3.68),

$$\|Dy\|_p^p \leq |c|^p \int_Z (tg(z) + (1-t)\theta(z))dz < 0,$$

a contradiction. This proves that there exists $R > 0$ such that (3.54) is true.

It is clear that

$$\inf\{\varphi_t(x) : t \in [0,1], \|x\| \le R\} > -\infty. \tag{3.71}$$

Then (3.52), (3.53), (3.54) and (3.71) permit the use of Lemma 2.2; thus

$$C_n(\varphi_0, \infty) = C_n(\varphi_1, \infty) \ \text{ for all } n \ge 0. \tag{3.72}$$

Note that

$$\varphi_0(x) = \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta(z)|x(z)|^p dz \equiv \frac{1}{p}\psi(x) \ \text{ for all } x \in W_\nu^{1,p}(Z),$$

see Lemma 3.1, and

$$\varphi_1(x) = \varphi(x) \ \text{ for all } x \in W_\nu^{1,p}(Z).$$

From Lemma 3.1, we know that $x = 0$ is a minimizer of $\varphi_0 = \psi/p$ and its only critical point. So,

$$C_n(\varphi_0, \infty) = C_n(\varphi_0, 0) = \delta_{n,0}\mathbb{Z} \ \text{ for all } n \ge 0. \tag{3.73}$$

From (3.72) and (3.73) it follows that

$$C_n(\varphi, \infty) = C_n(\varphi_1, \infty) = \delta_{n,0}\mathbb{Z} \ \text{ for all } n \ge 0.$$

The proof is complete. □

Next, we compute the critical groups of φ at zero. Without loosing the generality, we can assume that 0 is an isolated critical point; otherwise, we already have a sequence of distinct nontrivial solutions of (P). We have

Proposition 3.4 *If $H(f)_2$ holds, then $C_n(\varphi, 0) = \delta_{n,0}\mathbb{Z}$ for all $n \ge 0$.*

Proof. Let $x \in C_\nu^1(\overline{Z})$ be such that $|x(z) \le \delta|$ for all $z \in \overline{Z}$. ($\delta > 0$ is from $H(f)_2$ (v).) Then

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z F(z, x(z))dz \\ &\ge \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \beta|x|^p dz \quad \text{(see hypothesis } H(f)_2 \ (v)) \\ &\ge \frac{\xi_0}{p}\|x\|^p \quad \text{(see Lemma 3.1).} \end{aligned}$$

Therefore, $x = 0$ is a local $C_\nu^1(\overline{Z})$-minimizer of φ. Invoking Proposition 2.1, we infer that $x = 0$ is a local $W_\nu^{1,p}(Z)$-minimizer of φ. Then $C_n(\varphi, 0) = \delta_{n,0}\mathbb{Z}$ for all $n \ge 0$. □

Proof of Theorem 3.2. In a standard way, hypotheses $H(f)_2$ imply that φ satisfies the PS-condition. Note that $H(f)_2$ (iii) implies $H(f)_1$ (iii), while $H(f)_2$ (iv) and $H(f)_1$ (iv) are the same. Thus, the conclusion of Proposition 3.1 remains valid also within the context of $H(f)_2$. Therefore, we obtain $x_0 \in W_+$ and $v_0 \in -W_+$ two local minimizers of φ. This time, the nontriviality of x_0 and v_0 is a consequence of $H(f)_2$

(vi). As before, the hypothesis $H(f)_2$ (vii) permits the use of the strong maximum principle of Vázquez [25], which implies that $x_0 \in \mathrm{int}C_+$, $v_0 \in -\mathrm{int}C_+$ and they are solutions of (P). From a well-known characterization of the critical groups at a local minimizer of a C^1-functional (see Chang [8, p. 33] and Mawhin-Willem [19, p. 175]), we have that

$$C_n(\varphi, x_0) = C_n(\varphi, v_0) = \delta_{n,0}\mathbb{Z} \text{ for all } n \geq 0. \tag{3.74}$$

Suppose that $\{0, x_0, v_0\}$ are the only critical points of φ. Using relation (3.74), Propositions 3.3 and 3.4, and the Poincaré-Hopf formula, we obtain that

$$(-1)^0 + (-1)^0 + (-1)^0 = (-1)^0,$$

a contradiction. So, there exists $y_0 \in W_\nu^{1,p}(Z)$ one more critical point of φ distinct from $0, x_0, v_0$. Again, this is a solution of (P) and by nonlinear regularity theory, we have $y_0 \in C_\nu^1(\overline{Z})$. This completes our proof. □

4 Four nontrivial solutions for problem (P): the semilinear case

In this section we assume that

$\underline{H(f)_3}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z, hypotheses $H(f)_3$ $(i), (iii), (iv), (v), (vi), (vii)$ are the same as hypotheses $H(f)_2$ (i), (iii), (iv), (v), (vi), (vii) with $p = 2$, and

(ii) for a.a. $z \in Z$, the function $x \mapsto f(z, x)$ is a C^1-function.

Theorem 4.1 *If hypotheses $H(f)_3$ hold, then problem (P) has at least four distinct nontrivial solutions $x_0 \in \mathrm{int}C_+$, $v_0 \in -\mathrm{int}C_+$, and $y_0, w_0 \in C_\nu^1(\overline{Z})$.*

Proof. Note that in this case $\varphi \in C^2(W_\nu^{1,2}(Z))$. As before, we obtain $x_0 \in \mathrm{int}C_+$ and $v_0 \in -\mathrm{int}C_+$ two solutions which are local minimizers of φ. So (3.74) is valid. Since x_0, v_0 are local minimizers of φ, by the linking theorem (see Gasiński-Papageorgiou [14, p. 644]), we obtain another critical point $y_0 \in W_\nu^{1,2}(Z)$ which is of mountain pass-type. Therefore,

$$C_n(\varphi, y_0) = \delta_{n,1}\mathbb{Z} \text{ for all } n \geq 0. \tag{4.75}$$

Comparing (4.75) with Proposition 3.4, we infer that $y_0 \neq 0$. Now, let us suppose that $\{0, x_0, v_0, y_0\}$ are the only critical points of φ. Then, from relations (3.74), (4.75), Propositions 3.3 and 3.4, and the Poincaré-Hopf formula, we have that

$$(-1)^0 + (-1)^0 + (-1)^0 + (-1)^1 = (-1)^0,$$

a contradiction. This means that there exists a fourth critical point $w_0 \in W_\nu^{1,2}(Z)$ of φ which is distinct from $0, x_0, v_0, y_0$. As before, y_0, w_0 are both solution of (P) and standard regularity theory implies that $y_0, w_0 \in C_\nu^1(\overline{Z})$. □

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