Advanced Nonlinear Studies **10** (2010), 23–52

Invariant Manifolds for Random and Stochastic Partial Differential Equations

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Received in revised form 10 November 2008
Communicated by Peter Kloeden

[*]The author acknowledges the support Ministerio de Educación y Ciencia (Spain) BFM2002-03068 and MTM2005-01412, and Junta de Andalucía (Spain) P07-FQM-02468.
[†]The author acknowledges the support NSF 0620539
[‡]The author acknowledges the support NSF DMS 0200961 and NSF DMS 0401708
[§]The author acknowledges the support DFG17355596

Abstract

Random invariant manifolds are geometric objects useful for understanding complex dynamics under stochastic influences. Under a nonuniform hyperbolicity or a nonuniform exponential dichotomy condition, the existence of random pseudo-stable and pseudo-unstable manifolds for a class of *random* partial differential equations and *stochastic* partial differential equations is shown. Unlike the invariant manifold theory for stochastic *ordinary* differential equations, random norms are not used. The result is then applied to a nonlinear stochastic partial differential equation with linear multiplicative noise.

2000 Mathematics Subject Classification. Primary: 37L55, 35R60; Secondary: 58B99, 35L20.

Key words. Stochastic PDEs, random PDEs, multiplicative ergodic theorem, random dynamical systems, nonuniform hyperbolicity, invariant manifolds.

1 Introduction

Invariant structures in state spaces are essential for describing and understanding dynamical behavior of nonlinear random systems. For random dynamical systems, these invariant structures are usually random geometric objects. Stable, unstable, center, and inertial manifolds, as special random invariant structures, have been considered in the investigation of stochastic partial differential equations or stochastic evolutionary equations in infinite dimensional spaces. More precisely, an inertial manifold for a stochastic partial differential equation driven by white noise is constructed in [5] . Some inertial manifolds have been used in [15] to construct a stationary solution for such kind of an equation. Backward integration ideas for stochastic equations are used in [11] with more general noise. Invariant manifolds related to a stochastic pitchfork bifurcation are studied in [8]. In [12], a graph transform has been developed, based on a random fixed point theorem, to obtain random invariant manifolds. The existence of *smooth* random invariant manifolds is proved in [13]. In [19] one can find a general theorem about random invariant manifolds for a stochastic partial differential equation with linear diffusion part, while in [27] it is shown that an invariant manifold is asymptotically complete, and use this manifold to study stationary solutions to hyperbolic stochastic partial differential equations. More detailed historical account of this subject may be found in [8, 12].

In this paper, we are concerned with invariant stable or unstable manifolds for infinite dimensional random dynamical systems, especially those systems generated by *stochastic* or *random* partial differential equations (SPDEs or RPDEs), under some weak conditions. Our approach for establishing invariant manifolds for infinite dimensional *random* dynamical systems is based on a nonuniform exponential dichotomy, also called nonuniform pseudo-hyperbolicity, for the linearized random dynamical systems. When a multiplicative ergodic theorem (MET) holds [21, 18], nonuniform pseudo-hyperbolicity also holds. Moreover, unlike the invariant manifolds theory for finite dimensional random dynamical systems [28, 2], we make no use of random norms. To be more precise, the structure of our analysis is the following. Before proving the existence of invariant manifolds for a nonlinear (random or stochastic) partial differential equation (PDE), we analyse the linear system as a first approximation. We prove that the fundamental solutions of our linear PDE generates a random dynamical system that is linear and compact (for every positive time t). The partial differential operator generating this equation is supposed to be uniformly elliptic and random. The long-time behaviour of this linear random dynamical system is analysed under a nonuniform pseudo-hyperbolicity condition, which also implies an exponential dichotomy result. We then use a cut-off procedure to obtain the existence of local (pseudo) invariant stable and unstable manifolds for nonlinear random systems by using the Lyapunov-Perron technique.

The paper is organized as follows. In Section 2, we recall some basic concepts for random dynamical systems. In Section 3, we discuss multiplicative ergodic theorems and exponential dichotomies for linear cocycles. We prove that when a multiplicative ergodic theorem (MET) holds in an infinite dimensional Hilbert space, a *nonuniform* exponential dichotomy (i.e., nonuniform pseudo-hyperbolicity) also holds (Theorem 3.4) in the same Hilbert space. Furthermore, we obtain sufficient conditions under which a stochastic partial differential equation generates a continuous random dynamical system (Theorem 3.5). We then prove pseudo-stable and pseudo-unstable manifold theorems for random and stochastic partial differential equations (Theorems 4.1 and 4.2), under nonuniform pseudo-hyperbolicity (see Definition 4.1), in Section 4. Finally, in Section 5, we demonstrate our invariant manifold theorem for an example of stochastic partial differential equations.

2 Random dynamical systems

We now recall some basic concepts in random dynamical systems. First we introduce an appropriate model for a noise. Such a model is given by a metric dynamical system defined by a quadrupel $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a measurable flow with time set $\mathbb{T}$ being $\mathbb{R}$ or $\mathbb{Z}$:

$$\theta : (\mathbb{T} \times \Omega, \mathcal{B}(\mathbb{T}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F}).$$

For the partial mappings $\theta(t, \cdot)$ we use the symbol θ_t. We then have

$$\theta_t \circ \theta_\tau =: \theta_t \theta_\tau = \theta_{t+\tau} \quad \text{for } t,\, \tau \in \mathbb{T}, \qquad \theta_0 = \mathrm{id}_\Omega.$$

The measure $\mathbb{P}$ is taken to be ergodic with respect to the *shift* operators θ_t; see [6]. The standard example for a metric dynamical system is induced by the Brownian motion. Let V be a separable Hilbert space and let $C_0(\mathbb{R}, V)$ be the set of continuous functions on $\mathbb{R}$ with values in U which are zero at zero equipped with the compact open topology. We denote by $\mathcal{F}$ the associated Borel-σ-algebra. Let $\mathbb{P}$ be the Wiener measure on $\mathcal{F}$ which is given by the distribution of a two-sided Wiener process with trajectories in $C_0(\mathbb{R}, V)$. For the definition of a two-sided Wiener process see Arnold [2] page 547. The flow θ is given by the Wiener shifts

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \qquad \omega \in \Omega = C_0(\mathbb{R}, V).$$

In this case the measure $\mathbb{P}$ is *ergodic* with respect to the flow θ.

For some Polish space (complete separable metric space) H a random dynamical system is given by a mapping

$$\varphi : (\mathbb{T}^+ \times \Omega \times H, \mathcal{B}(\mathbb{T}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H)) \to (H, \mathcal{B}(H))$$

which has the *cocycle* property:

$$\begin{aligned} \varphi(t+\tau,\omega,x) &= \varphi(t,\theta_\tau\omega,\varphi(\tau,\omega,x)), \qquad t,\tau \in \mathbb{T}^+, \quad \omega \in \Omega, \\ \varphi(0,\omega,x) &= x. \end{aligned} \tag{2.1}$$

Cocycles are generalizations of semigroups reflecting some non-autonomous dynamics.

Suppose that for some flow θ the differential equation

$$u' = f(\theta_t\omega, u), \quad u(0) = x \in H$$

possesses a unique solution on any interval $[0,T]$ for $T > 0$. Then, the solution mapping $(t,\omega,x) \to \varphi(t,\omega,x)$ defines a cocycle. If this operator depends measurably on its variables then φ defines a random dynamical system.

In what follows, we have to transform one random dynamical system into another. To do this, we need the following lemma.

Lemma 2.1 *Consider the mapping*

$$T : \Omega \times H \to H,$$

and assume that $T(\omega,\cdot)$ is a homeomorphism for any $\omega \in \Omega$, and $T(\cdot,x)$, $T(\cdot,x)^{-1}$ are measurable for any $x \in H$. If φ is a continuous random dynamical system, then so is ψ defined by

$$\psi(t,\omega,x) := T(\theta_t\omega, \varphi(t,\omega,T(\omega,x)^{-1}).$$

The proof is straightforward. We note that, by the assumptions of the lemma, the mappings T and T^{-1} are measurable from $\Omega \times H$ to H, see Castaing and Valadier [9], Lemma III.14.

For our purpose, a class of random variables will be crucial. A random variable

$$X : (\Omega, \mathcal{F}) \to (\mathbb{R}^+ \setminus \{0\}, \mathcal{B}(\mathbb{R}^+ \setminus \{0\})) \tag{2.2}$$

is called *tempered* if

$$\lim_{t\to\infty} \frac{\log^+ X(\theta_t\omega)}{t} = 0$$

for ω contained in a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant set of full measure. Such a random variable X is called *tempered from below* if X^{-1} is tempered. We note that, in the case of ergodicity, the random variable defined in (2.2) is either tempered or, alternatively, there exists a $\{\theta_t\}_{t\in\mathbb{T}}$ invariant set $\tilde{\Omega}$ of full measure such that

$$\limsup_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{t} = +\infty, \quad \omega \in \tilde{\Omega}.$$

A random variable is tempered if and only if there exists a positive constant Λ and a positive random variable $C_\Lambda(\omega)$ such that

$$X(\theta_t\omega) \le C_\Lambda(\omega)e^{\Lambda t} \quad \text{for} \quad t \in \mathbb{T} \tag{2.3}$$

for ω in some $\{\theta_t\}_{t\in\mathbb{T}}$ invariant set $\tilde{\Omega}$ of full measure.

We need the following definitions and conclusions about the measurability of linear operators.

Let H_1, H_2 be separable Hilbert spaces. A mapping $\omega \to B(\omega) \in L(H_1, H_2)$ is called *strongly measurable* if $\omega \to B(\omega)h$ is a random variable on H_2 for every $h \in H_1$.

Lemma 2.2 *Let H_1, H_2, H_3 be three separable Banach spaces. Let B be a strongly measurable operator in $L(H_1, H_2)$, and let C be a strongly measurable operator in $L(H_2, H_3)$. Then*
(i) $B : \Omega \times H_1 \to B(\omega)h \in H_2$ is measurable.
(ii) $C \circ B$ is strongly measurable in $L(H_1, H_3)$
(iii) $\omega \to \|B(\omega)\|_{L(H_1,H_2)}$ is measurable.
(iv) Let $\dot{H}_1$ a dense set in H_1, and suppose that $\omega \to B(\omega)h$ is measurable for $h \in \dot{H}_1$. Then B is strongly measurable.

Proof. (i) Follows from Castaing and Valadier [9], Lemma III.14, and (ii) is a consequence of (i). (iii) follows because the unit ball in H_1 contains a dense countable set and, for (iv), we note that $B(\cdot)h$, $h \in H_1$, is the pointwise limit for some sequence $(B(\cdot)h_n)$, $h_n \in \dot{H}_1$. □

3 Multiplicative ergodic theorem and exponential dichotomy

In this section we introduce random dynamical systems U consisting of linear continuous operators $U(t,\omega) \in L(H,H)$. In particular, we study linear random dynamical systems generated by random linear evolution equations

$$\frac{du}{dt} + A(\theta_t\omega)u = 0, \qquad u(0) = x \in H. \tag{3.4}$$

Our intention is to show that this equation generates a *random* dynamical system. To describe the properties of the operator A, let $(H, (\cdot,\cdot), \|\cdot\|)$, $(H_1, (\cdot,\cdot)_1, \|\cdot\|_1)$ be two separable Hilbert spaces, where H_1 is densely and compact embedded into H. We assume that A is given by linear operators $A(\omega) \in L(H_1, H)$ such that

$\omega \to A(\omega)$ is strongly measurable. In addition, $-A(\omega)$ are generators of an analytic C_0–semigroups on H denoted by $e^{-\tau A(\omega)}$, $\tau \geq 0$, and the function $t \to A(\theta_t\omega)$ is Hölder continuous with values in $L(H_1, H)$. Namely, the function

$$\mathbb{R} \ni t \to A(\theta_t\omega)$$

is in $C^\rho(\mathbb{R}, L(H_1, H))$ for $\rho \in (0, 1)$. For the definition of this space see Amann [1], page 40f. We also assume that there exists a random variable $k_1(\omega) \geq 0$ so that the resolvent set of $-(k_1(\omega)\mathrm{id} + A(\omega))$ denoted by $\rho(-(k_1(\omega) + A(\omega)))$ contains $\mathbb{R}^+$, and the mapping $t \to k_1(\theta_t\omega)$ is supposed to be Hölder continuous. We define $A_\omega(t) := A(\theta_t\omega)$, $k_{1,\omega}(t) := k_1(\theta_t\omega)$ for $\omega \in \Omega$. According to the above properties, A_ω generates a *fundamental solution* U_ω (or a parabolic evolution operator), see Amann [1]. For our application we need the following parts of the definition of a fundamental solution. Let J denote either the interval $[0, T]$, for $T > 0$, or $\mathbb{R}^+$. Then

$$U_\omega \in C(J_\Delta, L_s(H)), \quad J_\Delta = \{(t, s) \in J^2, t \geq s\}. \tag{3.5}$$

$$t \to U_\omega(t, s)x,\ x \in H,\ t \geq s \quad \text{solves} \tag{3.6}$$

$$\frac{du}{dt} + A_\omega(t)u = 0,\ u(s) = x, \quad \text{where } U_\omega(\cdot, s) \in C^1(J \cap (s, \infty); L(H)).$$

$$U_\omega(t, t) = \mathrm{id}, \qquad U_\omega(t, s) = U_\omega(t, \tau) \circ U_\omega(\tau, s), \quad T > t \geq \tau \geq s. \tag{3.7}$$

$$\sup_{T \geq t \geq s \geq 0} (t - s)\|A_\omega(t)U_\omega(t, s)\| < \infty. \tag{3.8}$$

$L_s(H)$ denotes the strong convergence on the set of continuous linear operators $L(H)$ on H. For the operator norm in $L(H_1, H)$, and in $L(H)$ we simply write $\|\cdot\|$. In addition, we have

$$\|U_\omega(t, s)\| \leq C_\omega e^{\mu_\omega(t-s)} \tag{3.9}$$

for appropriate constants C_ω, μ_ω, see Amann [1], Theorem II.4.4.1.

We consider the following simple transform

$$U_{k_{1,\omega}}(t, s) := e^{-\int_0^t k_{1,\omega}(\tau)d\tau} U_\omega(t, s) e^{\int_0^s k_{1,\omega}(\tau)d\tau}. \tag{3.10}$$

These operators are fundamental solutions of an equation generated by

$$A_{k_{1,\omega}}(t) = k_{1,\omega}(t)\mathrm{id} + A_\omega(t).$$

We have $k_{1,\omega}(t) \leq K_{1,\omega,T}$ on every interval $[0, T]$. Then, we can introduce the operator

$$A_{K_{1,\omega,T}}(t) = K_{1,\omega,T}\mathrm{id} + A_\omega(t).$$

For such a $K_{1,\omega,T}$, condition (II.4.2.1) in Amann [1], page 55, is satisfied on $[0, T]$. This gives us the existence of a unique fundamental solution with generator $A_{K_{1,\omega,T}}$, and hence with generator A_ω; see [1], Corollary II.4.4.2. In particular, for any $T > 0$ there exists $M_{T,\omega}$ such that

$$M_{T,\omega}^{-1}\|x\|_1 \leq \|A_{K_{1,\omega,T}}(t)x\| \leq M_{T,\omega}\|x\|_1.$$

We then can conclude by (3.8) that

$$\sup_{T\geq t>s\geq 0}(t-s)\|U_{K_{1,\omega,T}}(t,s)\|_1\leq M_{T,\omega}\sup_{T\geq t>s\geq 0}(t-s)\|A_{K_{1,\omega,T}}(t)U_{K_{1,\omega,T}}(t,s)\|<\infty$$

such that $U_{K_{1,\omega,T}}(t,s)$, $s<t\in J$ and hence $U_\omega(t,s)$ for $t>s$ are compact linear operators by the compact embedding $H_1\subset H$. For the case $t=0$, see [1].

Our intention is now to derive from the fundamental solution a random dynamical system. We set

$$U(t,\omega):=U_\omega(t,0).$$

By $A_\omega(t)=A_{\theta_s\omega}(t-s)$, $t\geq s$, the cocycle property follows directly from (3.7)

$$U(t+\tau,\omega)=U(t,\theta_\tau\omega)\circ U(\tau,\omega). \tag{3.11}$$

Replacing A_ω by the operator given by (3.10) we can assume that the resolvent set of $-A(\omega)$ contains $\mathbb{R}^+$.

Consider the Yoshida approximations

$$A^\varepsilon(\omega)=A(\omega)(\mathrm{id}+\varepsilon A(\omega))^{-1}\in L(H,H).$$

By our assumptions on the resolvent set, these operators are defined for $\varepsilon>0$. Then, the solution of the equation

$$\frac{du}{dt}+A^\varepsilon(\theta_t\omega)u=0,\quad u(0)=x$$

can be constructed by Picard iterations so that the associated fundamental solution U^ε forms a random dynamical system if A^ε is strongly measurable. In particular, we note that from Amann[1] (II.6.1.9), it follows that

$$t\to\|A^\varepsilon(\theta_t\omega)\|$$

is Hölder continuous, hence locally integrable.

We have to prove that the Yoshida approximations are strongly measurable. Indeed, for $h\in H$, the operator $(\lambda\mathrm{id}+A(\omega))^{-1}$ exists for every $\lambda>0$ as an operator in $L(H,H_1)$. By Skorochod [25], Chapter II.6.3, the random variable $(\lambda\mathrm{id}+A(\omega))^{-1}h$ is measurable with respect to $\mathcal{B}(H)$. But we have

$$\mathcal{B}(H)\cap H_1=\mathcal{B}(H_1),$$

see Vishik and Fursikov[26], Chapter II.2, which gives the strong measurability of A^ε. Then, by the convergence of the Yoshida approximations, we have the pointwise limit

$$\lim_{\varepsilon\to 0}U^\varepsilon(t,\omega)x=U(t,\omega)x$$

for every $x\in H$ (see Amann [1] Theorem II.6.2.4) which shows that U is a random dynamical system. In particular, it holds, by (3.5), that the mapping $t\to U_\omega(t,0)x$ is continuous for $t\geq 0$. Hence, due to Castaing and Valadier [9], Lemma III.14.,

$$(t,\omega)\to U_\omega(t,0)x$$

is measurable. Similarly, since $(t,\omega) \to U_\omega(t,0)x$ is measurable for fixed $x \in H$, and the mapping $x \to U_\omega(t,\omega)x$ is continuous, we have that

$$(t,\omega,x) \to U_\omega(t,0)x$$

is measurable. Together with (3.11), U defines a continuous random dynamical system. If we consider the original random dynamical system by the inverse transform to (3.10), we can conclude that A generates a random dynamical system.

Summarizing the above discussions, we have the following theorem on linear cocycles.

Theorem 3.1 (**Generation of linear cocycle**) *Let $A(\omega) \in L(H_1,H)$ be generators of analytic C_0–semigroups on H. The separable Hilbert space H_1 is compactly and densely embedded in the separable Hilbert space H. In addition, we assume that $t \to A(\theta_t\omega)$ is Hölder continuous in $L(H_1,H)$, and $\Omega \ni \omega \to A(\omega) \in L(H_1,H)$ is strongly measurable, and the resolvent set of $-A(\omega)$ contains the interval $[k_1(\omega),\infty)$ where $k_1 \geq 0$ is a random variable. Then (3.4) generates a random dynamical system of compact linear operators on H.*

We consider the following example. Let A be the following linear differential operator over a bounded domain $\mathcal{O} \in \mathbb{R}^d$ with C^∞-smooth boundary $\partial\mathcal{O}$:

$$A(x,\omega,D)u = \sum_{|\gamma|,|\delta|\leq m} (-1)^{|\gamma|} D^\gamma (a_{\gamma,\delta}(x,\omega) D^\delta)u. \tag{3.12}$$

We suppose that $a_{\gamma,\delta}$ forms a stochastic process

$$(t,\omega) \to a_{\gamma,\delta}(\theta_t\omega,\cdot) \in C^m(\bar{\mathcal{O}})$$

which has Hölder continuous path. The principal part of A,

$$A_0(x,\omega,D)u = \sum_{|\gamma|,|\delta|= m} (-1)^{|\gamma|} D^\gamma (a_{\gamma,\delta}(x,\omega) D^\delta)u,$$

is supposed to be uniformly elliptic, i.e.,

$$\sum_{|\gamma|,|\delta|=m} a_{\gamma,\delta}(x,\omega) z_\gamma z_\delta \geq 2k_0(\omega)|z|^m, \quad z = (\cdots, z_\gamma, \cdots)$$

where the vector z is indexed by the multi–index γ. The random variable $k_0(\omega) \in (0,\infty)$ is supposed to be independent of $x \in \bar{\mathcal{O}}$. We also assume that $t \to k_0^{-1}(\theta_t\omega)$ is Hölder continuous. The differential operator will be augmented by boundary conditions

$$u|_{\partial\mathcal{O}} = \frac{\partial u}{\partial n}|_{\partial\mathcal{O}} = \cdots = \frac{\partial u^{m-1}}{\partial n^{m-1}}|_{\partial\mathcal{O}}, \tag{3.13}$$

where n denotes the outer normal. We set

$$H = L^2(\mathcal{O}), \quad V = H_0^m(\mathcal{O}), \quad H_1 = V \cap H^{2m}(\mathcal{O})$$

where H_0^m and H^{2m} are standard Sobolev spaces. A more specific example is $A = \Delta$ (Laplace operator), under the zero Dirichlet boundary condition.

We introduce the following continuous bilinear form on V,

$$b_\omega(u,v) = \sum_{\gamma,\delta\le m} (a_{\gamma,\delta}(\omega)D^\delta u, D^\gamma v) + k_1(\omega)(u,v),$$

satisfying the Lax–Milgram condition

$$b_\omega(u,v) \ge k_0(\omega)\|u\|_V^2.$$

Then, $A(x,\omega,D)$ generates an analytic C_0–semigroup in H with generator denoted by $A(\omega)$, and $D(A(\omega)) = H_1$ for every $\omega \in \Omega$. We note that

$$\omega \to A(\cdot,\omega,D)h \in H, \qquad h \in C_0^\infty(\mathcal{O})$$

is measurable, so that $\omega \to A(\omega)h, \quad h \in H$, is measurable. In addition, by the remarks about the processes $a_{\gamma,\delta}(\theta_t\omega,\cdot)$, the operators $A(\theta_t\omega)$ are in $L(H_1,H)$ so that we have to ensure that terms like

$$\sup_{h\in C_0^\infty(\bar{\mathcal{O}}),\|h\|_1=1} \|(A(\theta_t\omega) - A(\theta_s\omega))h\|^2$$

are Hölder continuous for $\omega \in \Omega$. Indeed, for appropriate γ, δ we have

$$\begin{aligned}
&\int_{\mathcal{O}} |D^\gamma(a_{\gamma,\delta}(\theta_s\omega,x) - a_{\gamma,\delta}(\theta_t\omega,x))D^\delta h(x)|^2 dx \\
&\le \sup_{x\in\bar{\mathcal{O}}} |D^\gamma a_{\gamma,\delta}(x,\theta_s\omega) - D^\gamma a_{\gamma,\delta}(x,\theta_t\omega)|^2 \int_{\mathcal{O}} |D^\delta h(x)|^2 dx \\
&\le \|a_{\gamma,\delta}(\theta_s\omega) - a_{\gamma,\delta}(\theta_t\omega)\|^2_{C^m(\bar{\mathcal{O}})} \|h\|^2_{W_2^{2m}(\bar{\mathcal{O}})}.
\end{aligned}$$

We note that k_1 can be calculated by an interpolation argument. Then, by the assumptions on $a_{\gamma,\delta}$ and k_0, it follows that $t \to k_1(\theta_t\omega)$ is Hölder continuous.

In the following we describe the stability behavior of linear random dynamical systems with an infinite dimensional state space. To do this we formulate an infinite dimensional version of the multiplicative ergodic theorem; see Ruelle [21]. A version of this theorem for continuous time can be found in Mohammed et al. [19].

Theorem 3.2 *Let U be a linear random dynamical system of compact operators for $t > 0$ on H satisfying the following integrability condition:*

$$\mathbb{E} \sup_{0\le t\le 1} \log^+ \|U(t,\omega)\| + \mathbb{E} \sup_{0\le t\le 1} \log^+ \|U(1-t,\theta_t\omega)\| < \infty.$$

Then, there exist finitely or infinitely many deterministic numbers $\lambda_1 > \lambda_2 > \cdots$ (with $-\infty$ possible) and linear spaces $H = V_1 \supset V_2(\omega) \supset \cdots$, such that

(i) Each linear space $V_i(\omega)$ has a finite co-dimension independent of ω.
(ii) The following limits and invariance conditions hold:

$$\lim_{t\to\infty}\frac{1}{t}\log\|U(t,\omega)x\| = \lambda_i \qquad for \quad x\in V_i(\omega)\setminus V_{i+1}(\omega)$$
$$U(t,\omega)V_i(\omega)\subset V_i(\theta_t\omega) \quad for \quad t\ge 0$$

for $t\ge 0$ and for all ω contained in a set $\tilde\Omega$ of full measure such that $\theta_t\tilde\Omega\subset\tilde\Omega$, $t\ge 0$.

The numbers $\lambda_1, \lambda_2, \cdots$ are called the *Lyapunov exponents* associated to U. The set of these numbers forms the *Lyapunov spectrum.*

By the above theorem we can derive the following exponential dichotomy condition for U; see Mohammed et al. [19].

Theorem 3.3 *Suppose that the following exponential integrability condition is satisfied:*

$$D(\omega) := \log^+ \sup_{t_1,\, t_2\in[0,1]} \|U(t_1,\theta_{t_2}\omega)\|, \qquad \mathbb{E}D<\infty, \tag{3.14}$$

and suppose that $\lambda\in\mathbb{R}$ is not contained in the Lyapunov spectrum. Then, there exists a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant set $\tilde\Omega$ of full measure such that, for $\omega\in\tilde\Omega$, we have the following properties: There exist linear spaces $E^u(\omega)$, $E^s(\omega)$ such that

$$H = E^u(\omega)\oplus E^s(\omega).$$

The space $E^u(\omega)$ has a finite dimension independent of ω;

$$U(t,\omega)E^u(\omega) = E^u(\theta_t\omega)$$
$$U(t,\omega)E^s(\omega)\subset E^s(\theta_t\omega)$$

for $t\ge 0$. The restriction of $U(t,\omega)$ to $E^u(\omega)$ is invertible. There exist measurable projections $\Pi^u(\omega)$, $\Pi^s(\omega)$ onto $E^u(\omega)$, $E^s(\omega)$. In the case that $\lambda_1<\lambda$ we have $E^u=\{0\}$.

Suppose that $\lambda_1>\lambda$ and let λ_+ be the smallest Lyapunov exponent bigger than λ, and let λ_- be the biggest Lyapunov exponent smaller than λ. Then, we have for any $\varepsilon>0$

$$\|U(t,\omega)x\|\ge\|x\|e^{\alpha t} \quad for\ x\in E^u(\omega), \quad t\ge\tau(\omega,\varepsilon,x), \quad \alpha=\lambda_+-\varepsilon,$$
$$\|U(t,\omega)x\|\le\|x\|e^{\beta t} \quad for\ x\in E^s(\omega), \quad t\ge\tau(\omega,\varepsilon,x), \quad \beta=\lambda_-+\varepsilon,$$

where ε is chosen so small that $\alpha>\beta$.

Remark 3.1 i) The integrability condition of Theorem 3.3 ensures the integrability condition of Theorem 3.2.
ii) The space $E^s(\omega)$ is given by $V_i(\omega)$ if $\lambda_-=\lambda_i$.
iii) It follows directly from the invariance of the spaces $E^u(\omega)$ and $E^s(\omega)$, that

$$\Pi^u(\theta_t\omega)U(t,\omega) = U(t,\omega)\Pi^u(\omega),$$
$$\Pi^s(\theta_t\omega)U(t,\omega) = U(t,\omega)\Pi^s(\omega),$$

on a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set of full measure.

We denote the restriction of $U(t,\omega)$ to $E^u(\omega)$, $E^s(\omega)$ by $U^u_\lambda(t,\omega)$, $U^s_\lambda(t,\omega)$:

$$U^{s/u}_\lambda(t,\omega) : E^{s/u}_\lambda(\omega) \to E^{s/u}_\lambda(\theta_t\omega).$$

In the following we need the norm of these operators $U^{s/u}_\lambda(t,\omega)$ which should be denoted by $\|U^{s/u}_\lambda(t,\theta_s\omega)\|_{L(E^{s/u})(\theta_s\omega),E^{s/u}(\theta_{t+s}\omega))}$. But to avoid these long expressions in the norm we simply write $\|\cdot\|$ for the norm. From the context, this is not to be confused with the norm in H.

Lemma 3.1 *Suppose that the integrability condition of Theorem 3.3 is satisfied. Then, there exists a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant set of full $\mathbb{P}$-measure and a constant H^Λ_ω such that*

$$\|U^u_\lambda(t,\omega)\| \le H^\Lambda_\omega e^{\Lambda t} \quad \text{for } t\ge 0,\ \omega\in\tilde\Omega,$$

for all sufficiently large Λ.

Proof. 1) We show that on a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant set of full measure

$$\limsup_{t\to\infty} \|U^u_\lambda(t,\omega)\|_{L(H)} e^{-\Lambda\, t} < \infty.$$

By

$$\|U^u_\lambda(t,\omega)\| \le \|U(t,\omega)\|$$

we have that $\mathbb{E}\log^u \|U^u_\lambda(1,\omega)\| =: \tilde\Lambda < \infty$. By Kingman's theorem (see Ruelle [21]) there exists a set of measure one such that, for any ω in this set, we have that

$$\lim_{i\to\infty} \frac{1}{i}\log\|U^u_\lambda(i,\omega)\| = \tilde\Lambda. \tag{3.15}$$

Hence

$$\Omega^1_n := \{\omega\in\Omega : \limsup_{i\to\infty}\frac{1}{i}\log\|U^u_\lambda(i,\theta_n\omega)\| \le \tilde\Lambda\} \in \mathcal{F},$$

and set

$$\Omega^1 := \bigcap_{n\in\mathbb{Z}} \Omega^1_n.$$

This set is $\{\theta_t\}_{t\in\mathbb{Z}}$–invariant and has probability one. Let Ω^2 be the $\{\theta_t\}_{t\in\mathbb{Z}}$–invariant set so that

$$\lim_{i\to\pm\infty}\frac{1}{i}D(\theta_{i+n}\omega) = 0, \qquad n\in\mathbb{N}.$$

By the integrability condition (3.14) and by

$$D(\theta_t\omega) \le D(\theta_{1+[t]}\omega) + D(\theta_{[t]}\omega)$$

we have that Ω^2 has full measure

$$\lim_{t\to\pm\infty}\frac{1}{t}D(\theta_t\omega) = 0.$$

such that $\{\theta_t\}_{t\in\mathbb{R}}$–invariant.

2) Since $\Omega^1\cap\Omega^2$ is $\{\theta_t\}_{t\in\mathbb{Z}}$ invariant, we can restrict ourselves to the case that $s\in(-1,0)$ for the invariance with respect to continuous time.
We have, for $\omega\in\Omega^1\cap\Omega^2$, that

$$\begin{aligned}\log\|U_\lambda^u(t,\theta_s\omega)\| \le&\log^+\|U_\lambda^u(s+t-[s+t],\theta_{[s+t]}\omega)\|\\ &+\log\|U_\lambda^u([s+t],\omega)\|+\log^+\|U_\lambda^u(-s,\theta_s\omega)\|\\ \le& D(\theta_{[s+t]}\omega)+\log\|U_\lambda^u([s+t],\omega)\|.\end{aligned}\tag{3.16}$$

(Note $s\le 0$). Thus, $\limsup_{t\to\infty}\log\|U_\lambda^u(t,\theta_s\omega)\|/t\le\tilde\Lambda$. The same is true if we replace ω by $\theta_n\omega$, $n\in\mathbb{Z}$. Hence $\theta_s\omega\in\Omega^1$ and, therefore, $\theta_s\omega\in\Omega^1\cap\Omega^2$. On the other hand, for $s=0$ we obtain the conclusion. □

The following lemma states that one can restrict a metric dynamical system to a smaller invariant set of full measure.

Lemma 3.2 *Let $\tilde\Omega$ be defined in Lemma 3.1 and let $\mathcal{F}_{\tilde\Omega}$ be the trace σ algebra of $\mathcal{F}$ with respect to $\tilde\Omega$. Then, θ is*

$$(\mathcal{F}_{\tilde\Omega}\otimes\mathcal{B}(\mathbb{R}),\mathcal{F}_{\tilde\Omega})-\textit{measurable}.$$

Proof. $A'\in\mathcal{F}_{\tilde\Omega}$ if and only if there exists an $A\in\mathcal{F}$ such that $A'=A\cap\tilde\Omega$. Hence

$$\theta^{-1}(A')=\theta^{-1}(A)\cap(\tilde\Omega\times\mathbb{R})\in(\mathcal{F}\cap\mathcal{B}(\mathbb{R}))\cap(\tilde\Omega\times\mathbb{R})$$

by the invariance of $\tilde\Omega$. Let R be the set of measurable rectangle sets of $\Omega\times\mathbb{R}$. It follows from Halmos [16], Section 5, Theorem E, that

$$(\mathcal{F}\cap\mathcal{B}(\mathbb{R}))\cap(\tilde\Omega\times\mathbb{R})=\sigma(R)\cap(\tilde\Omega\times\mathbb{R})=\sigma(R\cap(\tilde\Omega\times\mathbb{R}))=\mathcal{F}_{\tilde\Omega}\otimes\mathcal{B}(\mathbb{R}).$$

This completes the proof. □

Let $\tilde{\mathbb{P}}$ be the restriction of $\mathbb{P}$ to $\mathcal{F}_{\tilde\Omega}$. In the sequel we will denote the new restricted metric dynamical system $(\tilde\Omega,\mathcal{F}_{\tilde\Omega},\tilde{\mathbb{P}},\theta)$ by $(\Omega,\mathcal{F},\mathbb{P},\theta)$.

Our considerations are based crucially on the following theorem, which says that multiplicative ergodic theorem (MET) (i.e., the existence of Lyapunov exponents) implies nonuniform exponential dichotomy in infinite dimensional Hilbert spaces.

Theorem 3.4 (**MET implies nonuniform exponential dichotomy**) *Assume the assumptions of Theorem 3.3. Suppose that $\lambda_1>\lambda$ and let λ_+ be the smallest Lyapunov exponent bigger than λ, and let λ_- be the biggest Lyapunov exponent smaller than λ. Then, there exist a tempered random variable $K_\lambda^s(\omega)$ and a tempered from below random variable $K_\lambda^u(\omega)$ such that, for $t\ge0$, $\omega\in\Omega$ and $\varepsilon>0$,*

$$\begin{aligned}\|U_\lambda^u(t,\omega)\|&\ge K_\lambda^u(\omega)\,e^{(\lambda_+-\varepsilon)t}\\ \|U_\lambda^s(t,\omega)\|&\le K_\lambda^s(\omega)\,e^{(\lambda_-+\varepsilon)t}.\end{aligned}$$

Remark 3.2 This nonuniform exponential dichotomy is also called nonuniform pseudo-hyperbolicity; see Definition 4.1 in the next section.

Proof. We start with K^u_λ. By Lemma 3.1 we can assume that

$$\|U^u_\lambda(t,\omega)\| \le H^\Lambda_\omega e^{\Lambda t} \quad \text{for} \quad t \ge 0, \omega \in \Omega, \tag{3.17}$$

where Λ is chosen bigger than $\tilde{\Lambda}$ (see Lemma 3.1, (3.15)) and $|\lambda_+|$. Sufficient for the conclusion of the first part is to show that

$$\frac{1}{K^u(\omega)} := \sup_{t\ge 0} \frac{e^{(\lambda^u-\varepsilon)t}}{\|U^u_\lambda(t,\omega)\|} = \sup_{t\in\mathbb{Q}} \frac{e^{(\lambda^u-\varepsilon)t}}{\|U^u_\lambda(t,\omega)\|}$$

is a tempered random variable in $(0,\infty)$. Indeed, to see that $1/K^+_\lambda$ is a random variable we note that $t \to \|U^u_\lambda(t,\omega)\|$ is continuous on $(0,\infty)$ by the finite dimensionality of $E^u(\omega)$. In addition, by $U(\cdot,\omega) \in C(\mathbb{R}^+, L_s(H))$ the norm $\|U^u_\lambda(t,\omega)\|$ is bounded away from zero for $t \to 0$. Indeed, we have

$$\|U^u_\lambda(t,\omega)\| \ge \|U^u_\lambda(t,\omega)x\|, \qquad \|x\| = 1,\, x \in E^u(\omega),$$

where the right hand side converges to one as $t \to 0$. Similarly, we have on Ω

$$\limsup_{t\to\infty} \frac{e^{(\lambda_+-\varepsilon)t}}{\|U^u_\lambda(t,\omega)\|} \le \limsup_{t\to\infty} \frac{e^{(\lambda_+-\varepsilon)t}}{\|U^u_\lambda(t,\omega)x\|} < \infty, \qquad (\|x\| = 1)$$

which follows from Theorem 3.3. According to Lemma 3.1

$$e^{-\Lambda t} \le \frac{H^\Lambda_\omega}{\|U^u_\lambda(t,\omega)\|} \quad \text{for any } t \ge 0.$$

We then see that, for $s > 0$,

$$\begin{aligned}
\frac{1}{K^u_\lambda(\theta_s\omega)} e^{-2\Lambda s} &= \sup_{t\ge 0} \frac{e^{(\lambda_u-\varepsilon)t}}{\|U^u_\lambda(t,\theta_s\omega)\|} e^{-2\Lambda s} \\
&\le \sup_{t\ge 0} \frac{e^{(\lambda_+-\varepsilon)t} e^{(\lambda_+-\varepsilon)s}}{\|U^u_\lambda(t,\theta_s\omega)\| \|U^u_\lambda(s,\omega)\|} e^{-\Lambda s} H^\Lambda_\omega e^{-(\lambda_+-\varepsilon)s} \\
&\le \sup_{t\ge 0} \frac{e^{(\lambda_+-\varepsilon)(t+s)}}{\|U^u_\lambda(t+s,\omega)\|} e^{-\Lambda s} H^\Lambda_\omega e^{-(\lambda_+-\varepsilon)s} \\
&\le \frac{H^\Lambda_\omega}{K^u_\lambda(\omega)} e^{-\Lambda s} e^{-(\lambda_+-\varepsilon)s}
\end{aligned}$$

which goes to zero for $s \to \infty$. Thus the condition (2.3) gives the first part of the conclusion.

Now we show the existence and temperedness of $K^s_\lambda(\omega)$ on the stable space. We first show this for discrete time and then extend it to continuous time. Define

$$K^s_\lambda(\omega) := \sup_{t\in\mathbb{R}^u} \frac{\|U^s_\lambda(t,\omega)\|}{e^{(\lambda_-+\varepsilon)t}}.$$

We use the Kingman subadditive ergodic theorem (see Theorem A. 1 in Ruelle [21]).

Define $F_n(\omega) = \log \|U^s_\lambda(n,\omega)\|$. We can check that F_n satisfies the conditions in the Kingman subadditive ergodic theorem. Therefore, together with Ruelle's MET [21], there exists a $\{\theta_t\}_{t\in\mathbb{Z}}-$invariant measurable function $F(\omega)$ such that

$$F(\omega) = \lim_{n\to\infty} \frac{1}{n} F_n(\omega) = \lim_{n\to\infty} \log \frac{1}{n} \|U^s_\lambda(n,\omega)\| \le \lambda_-. \tag{3.18}$$

Now set $G(\omega) = \lambda_-$. Then $G(\omega) \ge F(\omega)$. As a consequence of Kingman's subadditive ergodic theorem (see Corollary A.2 in Ruelle [21]), for every $\varepsilon > 0$, there exists a finite-valued random variable $K_\varepsilon(\omega)$ such that for $n > m$,

$$\log \|U^s_\lambda(n-m,\theta_m\omega)\| \le (n-m)\lambda_- + n\frac{\varepsilon}{2} + K_\varepsilon(\omega), \quad a.s. \tag{3.19}$$

To see the temperedness of K^s_λ is sufficient to show that

$$\lim_{s\to\infty} K^s_\lambda(\theta_s\omega)e^{\Lambda s} = 0,$$

for some sufficiently large Λ. But this follows from the definition of K^s_λ and from

$$\begin{aligned}\|U^s_\lambda(t,\theta_s\omega)\| &\le \|U(1+t-[t]-1-[s]+s,\theta_{[s]+[t]}\omega)\|\times \\ &\quad\times \|U^s_\lambda([t]-1,\theta_{1+[s]}\omega)\|\|U(1-s+[s],\theta_s\omega)\| \\ &\le e^{D(\theta_{[s]+[t]+1}\omega)}e^{D(\theta_{[s]+[t]}\omega)}e^{K_\varepsilon(\omega)+\varepsilon([s]+1)+(\lambda_-+\frac{\varepsilon}{2})([t]-1)}e^{D(\theta_{[s]}\omega)},\end{aligned}$$

for $t > 1$, and similarly for $t \in [0,1]$. □

We now are ready to show that the random partial differential equation

$$\frac{du}{dt} + A(\theta_t\omega)u = F(\theta_t\omega,u), \qquad u(0) = x \in H, \tag{3.20}$$

via its solution mapping, defines a continuous random dynamical system. Here the nonlinear term F does not depend on the gradient of u. A similar result for stochastic partial differential equations can be found in [14]. However, in contrast to our approach, non-random differential operators are studied but these partial differential equations there have the interpretation of an Ito-equation.

Theorem 3.5 ***(Generation of cocycle)*** *Let $F : \Omega \times H \to H$ be a mapping such that $F(\cdot,x)$ is $(\mathcal{F},\mathcal{B}(H))$ measurable for $x \in H$, and $F(\omega,\cdot)$ is Lipschitz continuous for $\omega \in \Omega$ with a Lipschitz constant $L(\omega)$ such that*

$$\int_a^b L(\theta_s\omega)ds < \infty, \qquad \text{for } -\infty < a < b < \infty.$$

Then, (3.20) has a unique (mild) solution on any interval $[0,T]$ for any $\omega \in \Omega$ which generates a continuous random dynamical system.

Proof. We consider the Polish space $C_{T,x} := C([0,T],H)$ of continuous functions u with values in H and $u(0) = x$. This space is equipped with the norm

$$|||u||| = \sup_{t\in[0,T]} e^{-\Lambda t}\|u(t)\| \quad \text{for some } \Lambda > 0.$$

We consider the mapping

$$\mathcal{T}_x(u)[t] := U(t,\omega)u(0) + \int_0^t U(t-s,\theta_s\omega)F(\theta_s\omega,u(s))ds, \quad t\in[0,T], \quad u\in C_{T,x}. \tag{3.21}$$

According to Amann [1], page 46 f, we have that $\mathcal{T}_x(u) \in C_{T,x}$. Due to (3.9) we obtain

$$\|U(t-s,\theta_s\omega)\| = \|U_\omega(t,s)\| \le C_\omega e^{k_0(t-s)}.$$

We now choose Λ sufficiently large such that

$$\max_{t\in[0,T]} \int_0^t e^{-(\Lambda-\mu)(t-s)} CL(\theta_s\omega)ds \le \frac{1}{2}, \qquad \Lambda > k_0, \tag{3.22}$$

where $\mu = \mu(\omega)$, $C = C_\omega$. Indeed $s \to C\,L(\theta_s\omega)$ is an integrable majorant for the integrand in the above integral with respect to Λ. As $\Lambda \to \infty$, the integrand goes to zero for almost all $s \in [0,t]$. By the Lebesgue theorem, the integrals go to zero for $\Lambda \to \infty$ and for any t. Note that, for fixed t, the integrals are monotone in Λ such that, for sufficiently large Λ, we have the inequality (3.22) by Dini's theorem. We then have the contraction condition

$$|||\mathcal{T}_x(u) - \mathcal{T}_x(v)||| \le \max_{t\in[0,T]} \int_0^t e^{-(\Lambda-\mu)(t-s)} CL(\theta_s\omega)ds\, |||u-v||| \le \frac{1}{2}|||u-v|||.$$

The Banach fixed point theorem gives us a solution of (3.20) which is continuous in t for any $\omega \in \Omega$, $T \ge 0$ and $\Lambda = \Lambda(\omega, T)$ sufficiently large.
The solution of (3.20) depends continuously on x, which follows by the Gronwall Lemma from the Lipschitz continuity of F.

The norms $|||\cdot|||$ are equivalent to the standard supremum norm for every $\Lambda > 0$. Hence we can construct the solution of (3.20) by successive iteration of the operator $\mathcal{T}_x$ starting with the *measurable* function $u^0(t,\omega) \equiv x$. We see that the solution is a pointwise limit of measurable functions, hence measurable. Let $\varphi(t,\omega,x)$ be the solution operator for (3.20). The measurable dependence on x, t, ω follows in the same way as for the linear case (see above). The cocycle property follows by

$$\begin{aligned}
\varphi(t+\tau,\omega,x) &= U(t+\tau,\omega)x + \int_0^{t+\tau} U(t+\tau-s,\theta_s\omega)F(\theta_s\omega,\varphi(s,\omega,x))ds \\
&= U(t,\theta_\tau\omega)(U(\tau,\omega)x + \int_0^{\tau} U(\tau-s,\theta_s\omega)F(\theta_s\omega,\varphi(s,\omega,x))ds) \\
&+ \int_0^t U(t,\theta_s\theta_\tau\omega)F(\theta_s\theta_\tau\omega,\varphi(s,\theta_\tau\omega,\varphi(\tau,\omega,x)))ds.
\end{aligned}$$

Hence φ is a continuous random dynamical system. □

4 Invariant manifolds

In this section, we consider a general nonlinear random evolutionary equation in a Hilbert space H

$$\frac{du}{dt} + A(\theta_t\omega)u = F(\theta_t\omega, u), \tag{4.23}$$

with the random linear operator A, and nonlinear part F. We assume that the linear equation

$$\frac{du}{dt} + A(\theta_t\omega)u = 0 \tag{4.24}$$

generates a linear random dynamical system $U(t,\omega)$ on H for $t \geq 0$. We first introduce a weak hyperbolicity condition on the linear dynamics.

Definition 4.1 $U(t,\omega)$ (or $u = 0$) is said to be *nonuniformly pseudo-hyperbolic* if there exists a θ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full measure such that, for each $\omega \in \tilde{\Omega}$, the phase space H splits into a direct sum of closed subspaces

$$H = E^s(\omega) \oplus E^u(\omega)$$

satisfying:

(i) This splitting is invariant under $U(t,\omega)$:

$$U(t,\omega)E^s(\omega) \subset E^s(\theta_t\omega)$$
$$U(t,\omega)E^u(\omega) \subset E^u(\theta_t\omega)$$

and $U(t,\omega)|_{E^u(\omega)}$ is an isomorphism from $E^u(\omega)$ to $E^u(\theta_t\omega)$.

(ii) There are θ-invariant random variables $\alpha(\omega) > \beta(\omega)$, and a tempered random variable $K(\omega) : \tilde{\Omega} \to [1,\infty)$ such that

$$||U(t,\omega)\Pi^s(\omega)|| \leq K(\omega)e^{\beta(\omega)t} \quad \text{for } t \geq 0 \tag{4.25}$$

$$||\big(U(-t,\theta_t\omega)|_{E^u(\theta_t\omega)}\big)^{-1}\Pi^u(\omega)|| \leq K(\omega)e^{\alpha(\omega)t} \quad \text{for } t \leq 0, \tag{4.26}$$

where $\Pi^s(\omega)$ and $\Pi^u(\omega)$ are the measurable projections associated with the splitting. For our special setting of *ergodicity* we can assume that $\alpha > \beta$ are constant on a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set of a full measure.

Let

$$U^u_\lambda(t,\omega) = U(t,\omega)|_{E^u(\omega)} \quad \text{and} \quad U^s_\lambda(t,\omega) = U(t,\omega)|_{E^s(\omega)},$$

where λ generates some splitting of H, see Section 3. Then, condition (i) in the above definition implies that we may extend $U^u_\lambda(t,\omega)$ to be defined for $t < 0$ as

$$U^u_\lambda(t,\omega) = \big(U^u_\lambda(-t,\theta_t\omega)\big)^{-1}.$$

One can easily verify that the cocycle property holds for the extended system $U^u_\lambda(t,\omega)$ with $t \in \mathbb{R}$.

Remark 4.1 As ω varies, $\beta(\omega)$ may be arbitrarily small and $K(\omega)$ may be arbitrarily large. However, along each orbit $\{\theta_t\omega\}$, $\alpha(\omega)$ and $\beta(\omega)$ are constant and $K(\omega)$ can increase only at a subexponential rate. Thus, the linear system $U(t,\omega)$ is nonuniformly hyperbolic in the sense of Pesin. As an example, let $U(t,\omega)$ be an infinite dimensional linear random dynamical system satisfying the conditions of the following multiplicative ergodic theorem. Then, the nonuniform pseudo-hyperbolicity we introduced here automatically follows.

For the remainder of this paper, we assume that

Hypothesis A: *$U(t,\omega)$ is nonuniformly pseudo-hyperbolic.*

For the nonlinear term $F(\omega,x)$ we assume that

Hypothesis B: *There is a ball, $\mathcal{N}(\omega) = B(0,\rho(\omega)) = \{u \in H \mid ||u|| < \rho(\omega)\}$, where $\rho : \Omega \to (0,\infty)$ is tempered from below and $\rho(\theta_t\omega)$ is locally integrable, such that $F(\omega,\cdot) : \mathcal{N}(\omega) \to H$ is Lipschitz continuous and satisfies $F(\omega,0) = 0$ and*

$$||F(\omega,u) - F(\omega,v)|| \leq \tilde{B}_1(\omega)\,(||u||^{\varepsilon} + ||v||^{\varepsilon})\,||u-v||, \quad \omega \in \Omega,\ u,v \in \mathcal{N}(\omega).$$

where $\tilde{B}_1(\omega)$ is a random variable tempered from above, $\tilde{B}_1(\theta_t\omega)$ is locally integrable in t and $\varepsilon \in (0,1]$.

Later we can see that we can *extend* such an F to $\Omega \times H$ such that the assumptions of Theorem 3.5 are satisfied.

Next, we introduce a modified equation by using a cut-off function [2]. Let $\sigma(s)$ be a C^{∞} function from $(-\infty,\infty)$ to $[0,1]$ with

$$\sigma(s) = 1 \text{ for } |s| \leq 1, \quad \sigma(s) = 0 \text{ for } |s| \geq 2,$$

$$\sup_{s\in\mathbb{R}} |\sigma'(s)| \leq 2.$$

Let $\rho : \Omega \to (0,\infty)$ be a random variable tempered from below such that $\rho(\theta_t\omega)$ is locally integrable in t. We consider a modification of $F(\omega,u)$. Let

$$F_\rho(\omega,u) = \sigma\left(\frac{|u|}{\rho(\omega)}\right) F(\omega,u).$$

An elementary calculation gives

Lemma 4.1 *(i) $F_\rho(\omega,u) = F(\omega,u)$, for $|u| \leq \rho(\omega)$ and $||F_\rho(\omega,u)|| \leq B_0(\omega)$, where $B(\omega) > 0$ is a random variable tempered from above and $B(\theta_t\omega)$ is locally integrable in t ;*

(ii) there exists a random variable $B_1(\omega) > 0$ tempered from above, $B_1(\theta_t\omega)$ is locally integrable in t, such that

$$||F_\rho(\omega,u) - F_\rho(\omega,v)|| \leq B_1(\omega)\big(\rho(\omega)\big)^{\varepsilon}||u-v||, \quad \textit{for all } u,v \in H.$$

We now consider the following modified equation

$$\frac{du}{dt} + A(\theta_t\omega)u = F_\rho(\theta_t\omega, u). \tag{4.27}$$

Using Lemma 4.1, this modified equation has a unique global solution for each given initial value $u(0) = u_0$, and thus generates a random dynamical system.

We consider the Banach Space for $\gamma(\omega) = \frac{\alpha(\omega)+\beta(\omega)}{2}$

$$C_\gamma^- = \left\{ u | u : \mathbb{R}^- \to E \text{ is continuous and } \sup_{t\le 0} ||e^{-\gamma(\omega)t}u(t)|| < \infty \right\}$$

with the norm $|u|_\gamma^- = \sup_{t\le 0} ||e^{-\gamma(\omega)t}u(t)||$. Let $u(t,\omega,u^0)$ denote the solution of equation (4.27) and set

$$M^u(\omega) = \left\{ u^0 | u(t,\omega,u^0) \text{ is defined for all } t \le 0 \text{ and } u(\cdot,\omega,u^0) \in C_\gamma^- \right\}.$$

Then the set M^u is called local unstable invariant set. If $M^u(\omega)$ can be defined by a graph of a Lipschitz continuous function then we call $M^u(\omega)$ Lipschitz pseudo-unstable manifold for equation (4.27).

Theorem 4.1 (**Pseudo-unstable manifold theorem**) *Assume that Hypotheses A and B hold and choose the tempered radius $\rho(\omega)$ such that*

$$0 < \rho(\omega) < \left(\frac{\alpha(\omega) - \beta(\omega)}{8K(\omega)B_1(\omega)} \right)^{\varepsilon}. \tag{4.28}$$

Then there exists a Lipschitz pseudo-unstable manifold for equation (4.27) which is given by

$$M^u(\omega) = \{ p + h^u(\omega, p) | p \in E^u(\omega) \}$$

where $h^u(\omega,\cdot) : E^u(\omega) \to E^s(\omega)$ is Lipschitz continuous and satisfies $h^u(\omega, 0) = 0$.

Remark 4.2 When $\alpha(\omega) < 0$, the assumption $F(\omega, 0) = 0$ can be removed. This corresponds to the inertial manifold in deterministic case. If F is continuously differentiable in u, then h^u is continuously differentiable in u. Note that h^u, and thus the local manifold M^u, depend on ρ. The proof below shows the existence of an unstable manifold for the truncated equation (4.27), and as in [8], it can be shown that this is indeed a local unstable manifold for the original equation (4.23).

Proof. We use the Lyapunov and Perron approach to show this theorem. Then $M^u(\omega)$ is nonempty since $u = 0 \in M^u(\omega)$, and invariant for the random dynamical system generated by (4.27). We will prove that $M^u(\omega)$ is given by the graph of a Lipschitz function over $E^u(\omega)$.

We first claim that for $u(\cdot) \in C_\gamma^-(\omega)$, $u(0) \in M^u(\omega)$ if and only if $u(t)$ satisfies

$$\begin{aligned} u(t) = U_\lambda^u(t,\omega)\xi &+ \int_0^t U_\lambda^u(t-\tau, \theta_\tau\omega)\Pi^u F_\rho(\theta_\tau\omega, u)d\tau \\ &+ \int_{-\infty}^t U_\lambda^s(t-\tau, \theta_\tau\omega)\Pi^s F_\rho(\theta_\tau\omega, u)d\tau, \end{aligned} \tag{4.29}$$

where $\xi = \Pi^u u(0)$. To prove this claim, we first let $u(0) = u^0 \in M^u(\omega)$. By using the variation of constants formula, we have

$$\Pi^u u(t) = U^u_\lambda(t,\omega)\Pi^u u^0 + \int_0^t U^u_\lambda(t-\tau,\theta_\tau\omega)\Pi^u F_\rho(\theta_\tau\omega, u)d\tau, \tag{4.30}$$

and for $t_0 \le t$

$$\Pi^s u(t) = U^s_\lambda(t-t_0,\theta_{t_0}\omega)\Pi^s u(t_0) + \int_{t_0}^t U^s_\lambda(t-\tau,\theta_\tau\omega)\Pi^s F_\rho(\theta_\tau\omega, u)d\tau. \tag{4.31}$$

Since $u \in C^-_\gamma$, we have, for $t_0 < t, t_0 < 0$, that

$$\begin{aligned} &||U^s_\lambda(t-t_0,\theta_{t_0}\omega)\Pi^s u(t_0)|| \le K(\theta_{t_0}\omega)e^{\beta(\omega)(t-t_0)}e^{\gamma(\omega)t_0}|u|^-_\gamma \\ &\le e^{\beta(\omega)t}\Big(K(\theta_{t_0}\omega)e^{(\gamma(\omega)-\beta(\omega))t_0}\Big)|u|_\gamma \to 0 \quad \text{as} \quad t_0 \to -\infty, \end{aligned}$$

where we used the facts that $\beta(\omega) < \gamma(\omega)$ and $K(\omega)$ is tempered from above. Taking the limit $t_0 \to -\infty$ in (4.31),

$$\Pi^s u(t) = \int_{-\infty}^t U^s_\lambda(t-\tau,\theta_\tau\omega)\Pi^s F_\rho(\theta_\tau\omega, u)d\tau. \tag{4.32}$$

Combining (4.30) and (4.32), we obtain (4.29). The converse follows from a direct computation.

Let $\mathcal{J}^u(u,p,\omega)$ be the right hand side of equality (4.29). Using (4.25), (4.26), Lemma 4.1, and (4.28), we have for $u, \bar{u} \in C^-_\gamma$

$$\begin{aligned} &|\mathcal{J}^u(u,p,\omega) - \mathcal{J}^u(\bar{u},p,\omega)|^-_\gamma \\ &\le \sup_{t\le 0}\Big\{\int_t^0 e^{\big(\alpha(\omega)-\gamma(\omega)\big)(t-\tau)}K(\theta_\tau\omega)B_1(\theta_\tau\omega)\rho^\varepsilon(\theta_\tau\omega)d\tau \\ &\qquad + \int_{-\infty}^t e^{-\big(\gamma(\omega)-\beta(\omega)\big)(t-\tau)}K(\theta_\tau\omega)B_1(\theta_\tau\omega)\rho^\varepsilon(\theta_\tau\omega)d\tau\Big\}|u-\bar{u}|^-_\gamma \\ &\le \frac{1}{2}|u-\bar{u}|^-_\gamma \end{aligned}$$

and

$$|\mathcal{J}^u(u,p,\omega) - \mathcal{J}^u(u,\bar{p},\omega)|^-_\gamma \le K(\omega)||p-\bar{p}||.$$

Using the uniform contraction mapping principle, we have that for each $p \in E^u(\omega)$ $\mathcal{J}^u$ has a fixed point, thus equation (4.29) has a unique solution $u(\cdot,p,\omega) \in C^-_\gamma$ which is Lipschitz continuous in p and satisfies

$$|u(\cdot,p,\omega) - u(\cdot,\bar{p},\omega)|^-_\gamma \le 2K(\omega)||p-\bar{p}||.$$

Let

$$h^u(\omega,p) = \Pi^s u(0,p,\omega) = \int_{-\infty}^0 U^s_\lambda(-\tau,\theta_\tau\omega)\Pi^s F_\rho(\theta_\tau\omega, u(\tau,p,\omega))d\tau.$$

Then $h^u(\omega, 0) = 0$ and $h^u(\omega, \cdot)$ is Lipschitz continuous.

By the definition of h^u and the fact that $u^0 \in M^u(\omega)$ if and only if (4.29) has a unique solution $u(\cdot)$ in C_γ^- with $u(0) = u^0 = p + h^u(\omega, p)$ for some $p \in E^u(\omega)$, it follows that

$$M^u(\omega) = \{p + h^u(\omega, p) | p \in E^u(\omega)\}.$$

This completes the proof of the pseudo-unstable manifold theorem. □

Theorem 4.2 (**Pseudo-stable manifold theorem**) *Assume that Hypotheses A and B hold and choose the same tempered radius as in Theorem 4.1. Then, there exists a Lipschitz pseudo-stable manifold for equation (4.27) which is given by*

$$M^s(\omega) = \{q + h^s(\omega, q) | q \in E^s(\omega)\},$$

where $h^s(\omega, \cdot) : E^s(\omega) \to E^u(\omega)$ is Lipschitz continuous and satisfies $h^u(\omega, 0) = 0$.

Remark 4.3 Restricting $M^u(\omega)$ and $M^s(\omega)$ to a random ball $\mathcal{N}(\omega)$ with center zero and a random radius tempered from below gives local random pseudo-unstable and pseudo-stable manifolds for equation (4.23), respectively, see Lu and Schmalfuß [17].

Proof. When H is a finite dimensional space, one can simply reverse the time to get the pseudo stable manifold by using the pseudo-unstable manifold theorem. For an infinite dimensional space H, since the random dynamical systems are generally defined only for $t \geq 0$, the pseudo-unstable manifold theorem cannot be applied here as for the finite dimensional systems. Define the following Banach space for $\gamma(\omega) = \frac{\alpha(\omega)+\beta(\omega)}{2}$

$$C_\gamma^u = \left\{ u | u : \mathbb{R}^u \to E \text{ is continuous and } \sup_{t \geq 0} ||e^{\gamma t} u(t)|| < \infty \right\}$$

with the norm $|u|_\gamma^u = \sup_{t \geq 0} ||e^{\gamma t} u(t)||$.

Let

$$M^s(\omega) = \left\{ u^0 \ : \ u(\cdot, \omega, u^0) \in C_\gamma^u \right\}.$$

It is easy to see that $M^s(\omega)$ is nonempty and invariant for the random dynamical system generated by equation (4.27). We will show that $M^s(\omega)$ is the graph of a Lipschitz function over $E^s(\omega)$. First, a similar computation as in the proof of Theorem 4.1 gives that, for $u(\cdot) \in C_\gamma^+$, $u(0) \in M^s(\omega)$ if and only if $u(t)$ satisfies

$$\begin{aligned} u(t) = U_\lambda^s(t, \omega) q + \int_0^t U_\lambda^s(t - \tau, \theta_\tau \omega) \Pi^s F_\rho(\theta_\tau \omega, u(\tau)) \\ + \int_\infty^t U_\lambda^u(t - \tau, \theta_\tau \omega) \Pi^u F_\rho(\theta_\tau \omega, u(\tau)) d\tau, \end{aligned} \tag{4.33}$$

where $q = \Pi^s u(0)$.

We will show that for each $q \in E^s(\omega)$, equation (4.33) has a unique solution in C_γ^+. To see this, let $\mathcal{J}^s(u, q, \omega)$ be the right hand side of (4.33). A simple calculation gives that $\mathcal{J}^s$ is well-defined from C_γ^+ to itself for each fixed $\omega \in \Omega$ and $q \in E^s(\omega)$. For any $u, \bar{u} \in C_\gamma^+$, using (4.25), (4.26), Lemma 4.1, and (4.28), we have

$$|\mathcal{J}^s(u, q, \omega) - \mathcal{J}^s(\bar{u}, q, \omega)|_\gamma^+ \leq \frac{1}{2}|u - \bar{u}|_\gamma^+ \tag{4.34}$$

and

$$|\mathcal{J}^s(u, q, \omega) - \mathcal{J}^s(u, \bar{q}, \omega)|_\gamma^+ \leq K(\omega)||q - \bar{q}||.$$

Using the uniform contraction principle, we have that for each $\omega \in \Omega$ and $q \in E^s(\omega)$ equation (4.33) has a unique solution $u(\cdot, q, \omega) \in C_\gamma^+$ which is Lipschitz continuous in q and satisfies

$$|u(\cdot, q, \omega) - w(\cdot, \bar{q}, \omega)|_\gamma^+ \leq 2K(\omega)||q - \bar{q}||. \tag{4.35}$$

Let $h^s(\omega, q) = \Pi^s u(0, q, \omega)$. Then

$$h^s(q, \omega) = \int_\infty^0 U_\lambda^u(-\tau, \theta_\tau\omega)\Pi^u F_\rho(\theta_\tau\omega, u(\tau, q, \omega)d\tau,$$

$h^s(\omega, 0) = 0$, $h^s(\omega, q)$ is Lipschitz in q. Using (4.33) and the definitions of $M^s(\omega)$ and h^s, we have

$$M^s(\omega) = \{q + h^s(\omega, q) \ : \ q \in E^s(\omega)\}.$$

This proves the pseudo-stable manifold theorem. □

Remark 4.4 It is possible to solve the above problem when the nonlinearity F contains in an addition derivatives of appropriate order. But methods to find invariant manifolds are qualitatively different, which are worth to be studied in an additional paper.

5 An application

In this section we will illustrate the above random invariant manifold theory by applying it to an example of stochastic partial differential equations.

Let H be a separable Hilbert spaces with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. Consider an (unbounded) operator $A : D(A) =: H_1 \to H$, where it is assumed that $-A$ is the generator of an analytic C_0–semigroup $\{S_A(t)\}_{t\geq 0}$ on H, such that $S_A(t)$ is compact for all $t > 0$, and that $-A$ possesses infinitely many eigenvalues

$$\mu_1 \geq \cdots \geq \mu_j \geq \mu_{j+1} \geq \mu_{j+2} \geq \cdots \quad (\text{with} \ \ \mu_j \to -\infty \ \ \text{as} \ \ j \to \infty),$$

so that their associated eigenvalues $\{e_j\}_{j\geq 1}$ form a complete orthonormal basis of H.

For instance, we can consider as operator A the one given in (3.12) which satisfies the homogeneous Dirichlet boundary conditions (3.13), assuming that is symmetric and has a compact resolvent. Then the above assumptions are satisfied with $H = L^2(\mathcal{O})$.

On the other hand, assume that f is a Lipschitz continuous operator from H to H, i.e.

$$\|f(u_1) - f(u_2)\| \leq L_f \|u_1 - u_2\|, \qquad \text{for all} \;\; u_1, u_2 \in H,$$

$w_1, \cdots, w_N$ are one-dimensional mutually independent standard Wiener processes over the same probability space, and $D_i \in \mathcal{L}(H)$ for $i = 1, \cdots, N$. Then, we consider the following semilinear stochastic partial differential equation with multiplicative Stratonovich linear noise

$$dX + AX\,dt = f(X)dt + \sum_{i=1}^{N} D_i X \circ dw_i. \tag{5.36}$$

The operators D_i generate C_0-groups which we will denote by S_{D_i}. If, in addition, we suppose the operators A, $D_1, \cdots, D_N$ mutually commute (what implies that these groups and the semigroup $S_A(t)$ generated by A are also mutually commuting), then this stochastic equation will generate a random dynamical system by performing a suitable transformation (see Lemma 2.1).

We consider the one-dimensional stochastic differential equation

$$dz = -\nu\, z\, dt + dw(t) \tag{5.37}$$

for some $\nu > 0$. This equation has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process.

Lemma 5.1 ([7]) *Let ν be a positive number and consider the probability space as in Section 2. There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\bar{\Omega} \in \mathcal{F}$ of $\Omega = C_0(\mathbb{R}, \mathbb{R})$ of full measure such that*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} = 0, \tag{5.38}$$

and, for such ω, the random variable given by

$$z^*(\omega) := -\nu \int_{-\infty}^{0} e^{\nu\tau} \omega(\tau) d\tau$$

is well defined. Moreover, for $\omega \in \bar{\Omega}$, the mapping

$$\begin{aligned}(t, \omega) \to z^*(\theta_t \omega) &= -\nu \int_{-\infty}^{0} e^{\nu\tau} \theta_t \omega(\tau) d\tau \\ &= -\nu \int_{-\infty}^{0} e^{\nu\tau} \omega(t + \tau) d\tau + \omega(t)\end{aligned}$$

is a stationary solution of (5.37) with continuous trajectories. In addition, for $\omega \in \bar{\Omega}$

$$\lim_{t\to\pm\infty} \frac{|z^*(\theta_t\omega)|}{|t|} = 0, \qquad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z^*(\theta_\tau\omega)d\tau = 0,$$
$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |z^*(\theta_\tau\omega)|d\tau = \mathbb{E}|z^*| < \infty.$$

Let $\nu_1, \cdots, \nu_N$ be a set of positive numbers. For any pair ν_j, w_j we have a stationary Ornstein-Uhlenbeck process generated by a random variable $z_j^*(\omega)$ on $\bar{\Omega}_j$ with properties formulated in Lemma 5.1 defined on the metric dynamical system $(\bar{\Omega}_j, \mathcal{F}_j, \mathbb{P}_j, \theta)$. We set

$$(\Omega, \mathcal{F}, \mathbb{P}, \theta), \tag{5.39}$$

where

$$\Omega = \bar{\Omega}_1 \times \cdots \times \bar{\Omega}_N, \quad \mathcal{F} = \bigotimes_{i=1}^N \mathcal{F}_i, \quad \mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \cdots \times \mathbb{P}_N,$$

and θ is the flow of Wiener shifts.

To find random fixed points for (5.36) we will transform this equation into an evolution equation with random coefficients but without white noise. Let

$$T(\omega) := S_{D_1}(z_1^*(\omega)) \circ \cdots \circ S_{D_N}(z_N^*(\omega))$$

be a family of random linear homeomorphisms on H. The inverse operator is well defined by

$$T^{-1}(\omega) := S_{D_N}(-z_N^*(\omega)) \circ \cdots \circ S_{D_1}(-z_1^*(\omega))$$

Because of the estimate

$$\|T^{-1}(\omega)\| \le e^{\|D_1\||z_1^*(\omega)|} \cdot \ldots \cdot e^{\|D_N\||z_N^*(\omega)|}$$

and the properties of the Ornstein-Uhlenbeck processes, it follows that $\|T(\theta_t\omega)\|$, $\|T^{-1}(\theta_t\omega)\|$ has sub-exponential growth as $t \to \pm\infty$ for any $\omega \in \Omega$. Hence $\|T\|$, $\|T^{-1}\|$ are tempered. On the other hand, since z_j^*, $j = 1, \cdots, N$ are independent Gaussian random variables, we have that

$$\prod_{j=1}^N \mathbb{E}(\|S_{D_j}(-z_j^*)\|\|S_{D_j}(z_j^*)\|) < \infty.$$

Hence by the ergodic theorem we still have a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\bar{\Omega} \in \mathcal{F}$ of full measure such that

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \|T(\theta_\tau\omega)\|\|T^{-1}(\theta_\tau\omega)\|d\tau = \mathbb{E}\|T\|\|T^{-1}\|$$
$$\le \prod_{j=1}^N \mathbb{E}(\|S_{D_j}(-z_j^*)\|\|S_{D_j}(z_j^*)\|).$$

We can change our metric dynamical system with respect to $\bar{\Omega}$. However the new metric dynamical system will be denoted by the old symbols $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.

We formulate an evolution equation with random coefficients but without white noise

$$\frac{d\psi}{dt} + \left(A - \sum_{j=1}^{N} \nu_j z_j^*(\theta_t\omega) D_j \right) \psi = T^{-1}(\theta_t\omega) f(T(\theta_t\omega)\psi), \tag{5.40}$$

and initial condition $\psi(0) = x \in H$.

Lemma 5.2 *Suppose that A, $D_1, \cdots, D_N$ satisfy the preceding assumptions. Then i) the random evolution equation (5.40) possesses a unique solution, and this solution generates a random dynamical system.*
ii) if ψ is the random dynamical system in i),

$$\varphi(t, \omega, x) = T(\theta_t\omega)\psi(t, \omega, T^{-1}(\omega)x) \tag{5.41}$$

is another random dynamical system for which the process

$$(\omega, t) \to \varphi(t, \omega, x)$$

solves (5.36) for any initial condition $x \in H$.

From now on, we work with the random partial differential equation (5.40) which has been obtained (by conjugation) from our original stochastic PDE. To set our problem in the framework previously developed, we denote

$$C(\omega) = \sum_{j=1}^{N} \nu_j z_j^*(\omega) D_j, \qquad A(\omega) = A - C(\omega), \qquad F(\omega, \cdot) = T^{-1}(\omega) f(T(\omega)\cdot).$$

Note that $F(\omega, \cdot)$ is also Lipschitz continuous. The Lipschitz constant L is locally integrable in the sense of Theorem 3.5.

In order to prove the existence of invariant (stable and unstable) manifolds, we need to check that assumptions in Theorems 4.1 and 4.2 are fulfilled. To this end, we first need to work with the linear part of the RPDE and prove that the solution operator $U(t, \omega)$ generated by $A(\theta_t\omega)$ is nonuniformly pseudo-hyperbolic, what is immediately implied by the MET (Theorem 3.3). So, it is sufficient to prove the integrability condition (3.14) in that theorem.

Indeed, we define $U(t, \omega)$ by

$$U(t, \omega) = S_A(t) \exp\left\{ \int_0^t C(\theta_s\omega) \, \mathrm{d}s \right\}.$$

Then, defining, for fixed $\omega \in \Omega$, $u \in H$, the function

$$v(t) = U(t, \omega)u,$$

and thanks to the commutativity properties of the operators, we have

$$\begin{aligned}\frac{d}{dt}v(t) &= -AS_A(t)\exp\left\{\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\}u \\ &+ S_A(t)\exp\left\{\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\}C(\theta_t\omega)u \\ &= -AU(t,\omega)u + C(\theta_t\omega)S_A(t)\exp\left\{\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\}u \\ &= -A(\theta_t\omega)v(t).\end{aligned}$$

Therefore, $U(t,\omega)$ is the fundamental solution for the linear problem

$$\frac{d}{dt}v(t) + A(\theta_t\omega)v(t) = 0.$$

Observe that the compactness of $S_A(t)$ and the commutativity property implies that $U(t,\omega)$ is also compact.

Let us now prove that assumption (3.14)is satisfied. Indeed, take $t_1, t_2 \in [0,1]$, then

$$\begin{aligned}||U(t_1,\theta_{t_2}\omega)|| &\le ||S_A(t_1)||\left\|\exp\left\{\int_0^{t_1} C(\theta_{t_2+s}\omega)\ \mathrm{d}s\right\}\right\| \\ &\le ||S_A(t_1)||\exp\left\{\int_0^{t_1} \|C(\theta_{t_2+s}\omega)\|\ \mathrm{d}s\right\},\end{aligned}$$

and

$$\begin{aligned}\log^+||U(t_1,\theta_{t_2}\omega)|| &\le \log^+||S_A(t_1)|| + \int_0^{t_1}\|C(\theta_{t_2+s}\omega)\|\ \mathrm{d}s \\ &\le \mu_1 t_1 + \int_{t_2}^{t_1+t_2}\|C(\theta_s\omega)\|\ \mathrm{d}s \\ &\le |\mu_1| + \int_0^2 \|C(\theta_s\omega)\|\ \mathrm{d}s.\end{aligned}$$

Therefore,

$$E\left(\sup_{t_1,t_2\in[0,1]}\log^+||U(t_1,\theta_{t_2}\omega)||\right) \le |\mu_1| + \int_0^2 E\,\|C(\theta_s\omega)\|\ \mathrm{d}s < +\infty$$

thanks to the properties of the Ornstein-Uhlenbeck processes. Hence we can apply Theorem 3.4 to find the existence of tempered random variables K_λ^s, $1/(K_\lambda^u)$ such that $K = K_\lambda^s + 1/(K_\lambda^u)$. For the sake of completeness, we will explicitly determine the Lyapunov exponents of U as well as $\alpha(\omega), \beta(\omega)$, $K(\omega)$ in (4.25)–(4.26). First, we will prove that

$$\lim_{t\to+\infty}\frac{1}{t}\log||U(t,\omega)u|| \ne -\infty, \qquad \text{for all} \ \ u \in H.$$

This fact implies that there exist infinitely many Lyapunov exponents.

Choose an eigenvector e_j of the operator A associated to the eigenvalue μ_j. Then,

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{t}\log||U(t,\omega)e_j|| &= \lim_{t\to+\infty}\frac{1}{t}\log \mathrm{e}^{\mu_j t}\left\|\exp\left\{\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\}e_j\right\| \\ &= \mu_j + \lim_{t\to+\infty}\frac{1}{t}\log\left\|\exp\left\{\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\}e_j\right\| \\ &= \mu_j,\end{aligned}$$

since, by the ergodic theorem, and the following inequalities

$$\begin{aligned} e^{-\left\|\int_0^t C(\theta_s)ds\right\|} \le \left\|e^{\int_0^t C(\theta_s)ds}\right\| &\le e^{\left\|\int_0^t C(\theta_s)ds\right\|} \\ \frac{1}{\left\|e^{-\int_0^t C(\theta_s)ds}\right\|} &\le \left\|e^{\int_0^t C(\theta_s)ds}e_j\right\| \end{aligned}$$

it easily follows that

$$\lim_{t\to\infty}\frac{1}{t}\log\left\|\exp\left(\int_0^t C(\theta_s\omega)ds\right)\right\| = 0,$$

and, as a consequence,

$$\begin{aligned}0 = \lim_{t\to\infty}\frac{1}{t}\log\left\|\exp\left(-\int_0^t C(\theta_s\omega)ds\right)\right\| &\le \lim_{t\to\infty}\frac{1}{t}\log\left\|\exp\left(\int_0^t C(\theta_s\omega)ds\right)e_j\right\| \\ \le \lim_{t\to\infty}\frac{1}{t}\log\left\|\exp\left(\int_0^t C(\theta_s\omega)ds\right)\right\| = 0,\end{aligned}$$

so that the Lyapunov exponents λ_j for the random dynamical system U are equal to the eigenvalues μ_j. As for the associated space V_j it is easy to check that

$$V_j = \bigoplus_{i=j}^{\infty} F_i$$

where F_i are the eigenspaces associated to μ_j.

Let us now determine $\alpha(\omega), \beta(\omega)$ and $k(\omega)$ satisfying relations (4.25)–(4.26). To this end, let us denote by μ_s and μ_u the consecutive eigenvalues which satisfy

$$\mu_s = \mu_{j+1} < 0 < \mu_u = \mu_j.$$

Then,

$$\left\|\exp\left\{\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\}S_A(t)\Pi^s\right\| \le \mathrm{e}^{\mu_s t}\exp\left\|\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\|$$

and we observe that

$$\exp\left\|\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\| = \exp\left\|\sum_{j=1}^N \nu_j \int_0^t z_j^*(\theta_s\omega) D_j\ \mathrm{d}s\right\|$$

$$\leq \exp\left\{\sum_{j=1}^N |\nu_j|\,\|D_j\|\left|\int_0^t z_j^*(\theta_s\omega)\ \mathrm{d}s\right|\right\}.$$

As

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t z_j^*(\theta_s\omega)\ \mathrm{d}s = 0,$$

then for a given $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that

$$\left|\int_0^t z_j^*(\theta_s\omega)\ \mathrm{d}s\right| \leq \frac{\varepsilon}{\delta}t, \quad \text{for all } \ t \geq T(\varepsilon), \quad \text{and all} \quad j = 1, 2, \cdots, N,$$

where $\delta = \sum_{j=1}^N |\nu_j|\,\|D_j\|$. Thus,

$$\exp\left\{\sum_{j=1}^N |\nu_j|\,\|D_j\|\left|\int_0^t z_j^*(\theta_s\omega)\ \mathrm{d}s\right|\right\} \leq \mathrm{e}^{\varepsilon t}, \quad \text{for all } \ t \geq T(\varepsilon).$$

On the other hand, for $t \in [0, T(\varepsilon))$ we have

$$\exp\left\{\sum_{j=1}^N |\nu_j|\,\|D_j\|\left|\int_0^t z_j^*(\theta_s\omega)\ \mathrm{d}s\right|\right\}$$

$$\leq \exp\left\{\sum_{j=1}^N |\nu_j|\,\|D_j\| \max_{r\in[0,T(\varepsilon)]}\left|\int_0^r z_j^*(\theta_s\omega)\ \mathrm{d}s\right|\right\},$$

whence

$$\exp\left\{\sum_{j=1}^N |\nu_j|\,\|D_j\|\left|\int_0^t z_j^*(\theta_s\omega)\ \mathrm{d}s\right|\right\}$$

$$\leq \exp\left\{\sum_{j=1}^N |\nu_j|\,\|D_j\| \max_{r\in[0,T(\varepsilon)]}\left|\int_0^r z_j^*(\theta_s\omega)\ \mathrm{d}s\right|\right\}\mathrm{e}^{\varepsilon t},$$

for all $t \geq 0$, and, finally,

$$\left\|\exp\left\{\int_0^t C(\theta_s\omega)\ \mathrm{d}s\right\} S_A(t)\Pi^s\right\| \leq \mathrm{e}^{(\mu_s+\varepsilon)t}K(\omega),$$

where

$$K(\omega) = \prod_{j=1}^{N} \underbrace{\exp\left\{|\nu_j|\, \|D_j\| \max_{r\in[0,T(\varepsilon)]} \left| \int_0^r z_j^*(\theta_s\omega)\, \mathrm{d}s \right| \right\}}_{=K_j(\omega)}.$$

It is clear that $\beta(\omega) = \mu_s + \varepsilon$, and we need to prove that $K(\omega)$ is tempered. For this, it is enough to prove that each $K_j(\omega)$ is tempered. Indeed, observe that

$$\begin{aligned}
0 &\le \frac{1}{t}\log^+ K_j(\theta_t\omega) \\
&= \frac{1}{t}|\nu_j|\, \|D_j\| \max_{r\in[0,T(\varepsilon)]} \left| \int_0^r z_j^*(\theta_{s+t}\omega)\, \mathrm{d}s \right| \\
&\le |\nu_j|\, \|D_j\| \frac{1}{t} \max_{r\in[0,T(\varepsilon)]} \int_0^r \left| z_j^*(\theta_{s+t}\omega)\right|\, \mathrm{d}s \\
&\le |\nu_j|\, \|D_j\| \frac{1}{t} \int_0^{T(\varepsilon)} \left| z_j^*(\theta_{s+t}\omega)\right|\, \mathrm{d}s \\
&\le |\nu_j|\, \|D_j\| \frac{1}{t} \int_t^{t+T(\varepsilon)} \left| z_j^*(\theta_{s}\omega)\right|\, \mathrm{d}s \\
&\le |\nu_j|\, \|D_j\| \left(\underbrace{\frac{t+T(\varepsilon)}{t}}_{\to 1} \cdot \underbrace{\frac{1}{t+T(\varepsilon)} \int_0^{t+T(\varepsilon)} \left| z_j^*(\theta_{s}\omega)\right|\, \mathrm{d}s}_{\to E|z_j^*|} - \underbrace{\frac{1}{t}\int_0^t \left| z_j^*(\theta_{s}\omega)\right|\, \mathrm{d}s}_{\to E|z_j^*|} \right) \to 0
\end{aligned}$$

as $t \to \infty$. So, $K(\omega)$ is tempered. A similar analysis can be carried out to determine that $\alpha(\omega) = \mu_u - \varepsilon$. Therefore, as the nonlinear term F is globally Lipschitz we can take $B_1 = L_f$ and assumptions in theorems 4.1 and 4.2 are fulfilled. We thus have existence of pseudo-unstable and pseudo-stable manifolds.

Acknowledgements. This work was started in the summer 2003 when the authors participated in a Research in Teams Program, supported by the Banff International Research Station (Banff, Alberta, Canada).

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