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Existence of Solution for a Class of Quasilinear Systems

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Abstract

This paper proves the existence of nontrivial solution for a class of quasilinear systems on bounded domains in $\mathbb{R}^N$, $N \geq 2$, whose nonlinearity has a double criticality. The proof is based on a linking theorem without the Palais-Smale condition.

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1 Introduction

In the present paper, we consider the existence of nontrivial solution for a class of quasilinear systems of the type

$$(S_{pq}) \qquad \begin{cases} -\Delta_p u = \quad H_u(x,u,v), & \text{in } \Omega, \\ -\Delta_q v = -H_v(x,u,v), & \text{in } \Omega, \\ \qquad u = v = 0, & \text{on } \partial\Omega. \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, p and q belong to $(1,N]$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $\Delta_q v = div(|\nabla v|^{q-2}\nabla v)$ is the q-Laplacian of v and $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function.

For the case where $p = q = 2$, this class of systems is called noncooperative and many recent studies have focused on it. By using variational methods, the existence and multiplicity of solutions for different classes of nonlinearities $H(x,u,v)$ have been intensively studied by various authors, see for example, [2, 4, 5, 7, 10, 16, 19, 22] and references therein. In [22], Zuo considered the multiple existence of solutions to (S_{22}) in the case that the function H has an asymptotically linear growth. In [11], Hirano established the existence of infinitely many solutions to systems like (S_{22}) which are perturbed from a noncooperative odd elliptic systems. In both articles only subcritical systems have been considered. In [7], Ding and Figueiredo considered (S_{22}) allowing some supercritical growth. More precisely, the function $H(x,u,v)$ can assume a supercritical and subcritical growth on v and u respectively. They established the existence of infinitely many solutions to (S_{22}) provided the nonlinear term H is even in (u,v). In [4], Clapp, Ding and Hernández showed that multiple existence of solutions to (S_{22}) with some supercritical growth can be established without the symmetry assumption. In all these papers, the existence results are obtained as an application of an abstract critical point theorem for strongly indefinite functionals.

Motivated by some results found in [4] and [7], a natural question arises whether existence of nontrivial solutions continues to hold for (S_{pq}) when p and q are different from 2 and H has a *supercritical* and *critical* growth on the variables v and u respectively for $N \geq 2$. Here, for example, we will considerer two cases. The first one, we can assume that $H(x,u,v)$ has a supercritical growth on variable v and has a critical growth at infinity on variable u of the type $|u|^{p^*}$ with $p^* = pN/(N-p)$, the critical exponent of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. In this case the concentration–compactness principle due to Lions [13] is crucial to overcome the lack of compactness of the energy functional. In the second case, we assume that $p = q = N$ and H has critical exponential growth on the variables u and v. The variational formulation to this class of systems is given by the Trundiger and Moser inequality (see [14]). We would like to emphasize that in the literature rather less attention has been paid to noncooperative systems involving exponential critical growth to the case $N \geq 2$. Nevertheless, we should also mention the article [6], where a class of Hamiltonian systems with exponential critical growth has been considered.

The main difficulty in the cases above-mentioned is the lack of compactness of the energy functional associated to system. To overcome this difficulty, we make carefully estimates and prove that there is a Palais-Smale sequence that has a strongly convergent subsequence. The method employed here is based on a Galerkin type approximation developed by Bartsch and Clapp in [2] together with a linking theorem due to Rabinowitz [15], however in the version proved here is not assumed the Palais-Smale condition.

The main results in the present paper have concentrated on existence of nontrivial solutions to (S_{pq}) and can be seen as a complement of the studies developed in [4] and [7] for multiple existence of solutions. We will pursue our investigation of multiple existence of solutions to (S_{pq}) in the future.

This paper is organized as follows. In Section 2 we state and prove the linking theorem that we will use in this work. In Section 3, we apply the linking theorem to get a nontrivial solution to (S_{pq}) assuming that the nonlinear term $H(x,u,v)$ has double critical growth on u and v and $N \geq 3$. Finally, in Section 4, we consider the case $p = q = N$ and we prove the existence of nontrivial solution for the corresponding system (S_{NN}) with double critical exponential growth on bounded domain of $\mathbb{R}^N$ for $N \geq 2$.

2 The linking theorem: a review

This section is devoted to establish a version of the linking theorem of Rabinowitz [15] without Palais-Smale condition. The proof of this theorem is very similar to the one found in [15] with few modifications; however, for convenience of the reader we will show it here by adapting some arguments due to Kryszewski and Szulkin [12].

Theorem 2.1 *Let X be a real Banach space with $X = Y \oplus Z$, where Y is finite dimensional. Suppose $\Phi \in C^1(X, \mathbb{R})$ satisfies:*

(I_1) *There is $\sigma > 0$ such that if $\mathcal{N} = \{u \in Z : \|u\| = \sigma\}$, then $b \doteq \inf_{\mathcal{N}} \Phi > 0$.*

(I_2) *There are $z_* \in Z \cap \partial B_1$ and $\rho > \sigma > 0$ such that*

$$0 = \sup_{\partial \mathcal{M}} \Phi < d \doteq \sup_{\mathcal{M}} \Phi,$$

where $\mathcal{M} = \{u = \lambda z_* + y : \|u\| \leq \rho,\ \lambda \geq 0,\ y \in Y\}$.

Then, there is a sequence $(u_n) \subset X$, such that

$$\Phi(u_n) \to c \in [b, d] \qquad \textit{and} \qquad \Phi'(u_n) \to 0.$$

Proof. Arguing by contradiction, we suppose that the thesis is false. Then there exist $\epsilon > 0$ and $a > 0$ such that

$$\|\Phi'(u)\| \geq a \quad \text{for all } u \in \Phi^{-1}([b - 2\epsilon, d + 2\epsilon]).$$

Setting
$$A = \Phi^{-1}([b-2\epsilon, d+2\epsilon]) \quad \text{and} \quad B = \Phi^{-1}([b-\epsilon, d+\epsilon]),$$
we have that $B \subset A \subset \widetilde{X} \doteq \{u \in X; \Phi'(u) \neq 0\}$. Let V be a pseudo-gradient vector field for Φ on $\widetilde{X}$, that is, for all $u \in \widetilde{X}$,
$$\|V(u)\| \leq 2\|\Phi'(u)\|, \tag{2.1}$$
$$(\Phi'(u), V(u)) \geq \|\Phi'(u)\|^2. \tag{2.2}$$

Set
$$\phi(u) = \frac{\operatorname{dist}(u, X \setminus A)}{\operatorname{dist}(u, B) + \operatorname{dist}(u, X \setminus A)}$$
and
$$\mathcal{G}(u) = \begin{cases} \dfrac{\phi(u)V(u)}{\|V(u)\|}, & u \in A, \\ 0, & u \in X \setminus A. \end{cases}$$
Then by construction, $\mathcal{G}$ is locally Lipschitz continuous on X. Thus, for each $u \in X$, the Cauchy problem
$$\frac{d}{dt}\eta(t,u) = -\mathcal{G}(\eta(t,u)), \qquad \eta(0,u) = u,$$
has a unique solution $\eta(\cdot, u)$ defined on $\mathbb{R}$. We claim that
$$\sup_{u \in \mathcal{M}} \Phi(\eta(T,u)) < b,$$
where $T = 2(d-b+\epsilon)/a$. In fact, if not, $\sup\limits_{u \in \mathcal{M}} \Phi(\eta(T,u)) \geq b$, which implies that there exists $u \in \mathcal{M}$ with $\Phi(\eta(T,u)) > b - \epsilon$. Thus,
$$\begin{aligned} b-\epsilon &< \Phi(\eta(T,u)) = \Phi(u) + \int_0^T \frac{d}{dt}\Phi(\eta(t,u))\,dt \\ &= \Phi(u) - \int_0^T (\Phi'(\eta(t,u)), \mathcal{G}(\eta(t,u)))\,dt \\ &\leq d - \frac{Ta}{2}, \end{aligned}$$
where in the last inequality we used (2.1), (2.2) and that $\eta(t,u) \in B$. But this contradicts the choice of T.

Now we claim that there exists $\overline{u} \in \mathcal{M}$ such that $\eta(T, \overline{u}) \in \mathcal{N}$. In fact, consider the function $G : \overline{\mathcal{M}} \times [0,T] \to Y \oplus \langle z_* \rangle$ given by
$$G(u,t) = P(\eta(t,u)) + (\|Q(\eta(t,u)\| - \sigma)z_*,$$
where $P : X \to Y$ and $Q : X \to Z$ denote the projections. From the definition of G, we can observe that $G^{-1}\{0\} \cap \partial\mathcal{M} = \emptyset$. Then, applying the Brouwer topological degree, we derive
$$d(G(\cdot,T), \mathcal{M}, 0) = d(G(\cdot,0), \mathcal{M}, 0).$$

That is,

$$d(G(\cdot,T),\mathcal{M},0)=d(Id,\mathcal{M},\sigma z_*)=1.$$

Therefore, there exists $\overline{u}\in\mathcal{M}$ such that $G(\overline{u},T)=0$, which concludes the verification of the claim. Finally, from the above considerations,

$$b\le I(\eta(T,\overline{u}))\le\sup_{u\in\mathcal{M}}I(\eta(T,u))<b,$$

which is impossible. This completes the proof of Theorem 2.1.

3 Systems with critical growth for $N>\max\{p,q\}$

In this section, we study the existence of solution for the following class of quasilinear systems:

$$(S_{pq})\qquad\begin{cases}-\Delta_p u=H_u(x,u,v), \text{ in } \Omega\\ -\Delta_q v=-H_v(x,u,v), \text{ in } \Omega\\ u=v=0, \text{ on } \partial\Omega\end{cases}$$

where Δ_p and Δ_q denote the p and q Laplacian operator, respectively, $p,q>1$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $N>\max\{p,q\}$ and $H:\overline{\Omega}\times\mathbb{R}^2\to\mathbb{R}$ is given by

$$H(x,u,v)=\frac{1}{p^*}|u|^{p^*}+G(v)+R(x,u,v),$$

with $p^*=Np/(N-p)$. The assumptions on the functions G and R are the following:

(G_1) $G\in C^1(\mathbb{R},\mathbb{R})$ and there exist constants $C>0$ and $r\ge q$ such that

$$(i)\qquad |G(s)|\le C|s|^r, \text{ for all } s\in\mathbb{R}.$$

If $r\ge q^*=Nq/(N-q)$, we add that

$$(ii)\qquad (g(t)-g(s))(t-s)\ge C|t-s|^r, \text{ for all } t,\ s\in\mathbb{R}.$$

where $g(s)=G'(s)$.

(G_2) There exists $\nu\in(0,q)$ such that

$$0\le\nu G(s)\le g(s)s, \text{ for all } s\in\mathbb{R}.$$

(R_1) $R\in C^1(\overline{\Omega}\times\mathbb{R}^2)$, $R_u(x,0,0)=0$, $R_v(x,0,0)=0$, $R(x,u,v)\ge0$ and $R_u(x,u,v)u\ge0$, for all $(x,u,v)\in\overline{\Omega}\times\mathbb{R}^2$.

(R_2) There exist $p_i \in (p, p^*)$, $q_i \in (q, q^*)$, $i = 1, 2$, with $\max\{p_2, q_1\} < \min\{p^*, q^*\}$ such that

$$|R_u(x, u, v)| \leq C(|u|^{p_1 - 1} + |v|^{q_1 - 1}),$$

$$|R_v(x, u, v)| \leq C(|u|^{p_2 - 1} + |v|^{q_2 - 1}),$$

for all $(x, u, v) \in \Omega \times \mathbb{R}^2$ and for some constant $C > 0$.

(R_3) There exist $s \in (p, \max\{p_1, p_2\}]$, a nonempty open subset $\Omega_0 \subset \Omega$, and a constant $a > 0$ such that

$$R(x, u, v) \geq a|u|^s \text{ for all } x \in \Omega_0 \text{ and } (u, v) \in \mathbb{R}^2.$$

(R_4) There exists $\mu \in (p, p^*)$ such that

$$\frac{1}{\mu} R_u(x, t, s)t + \frac{1}{\nu} R_v(x, t, s)s - R(x, t, s) \geq 0, \text{ for all } x \in \Omega \text{ and } (t, s) \in \mathbb{R}^2,$$

where ν is given by condition (G_2).

The main result of this section is the following.

Theorem 3.1 *If $(G_1) - (G_2), (R_1) - (R_4)$ are satisfied, then (S_{pq}) possesses a nontrivial solution.*

We observe that $R(u, v) = |u|^s + C|v|^t + \sin|u|^s \sin|v|^t$ satisfies $(R_1) - (R_4)$ with $p_1 = s = p_2$, $q_1 = t = q_2$ for $p_i \in (p, p^*)$, $q_i \in (q, q^*)$, $i = 1, 2$, and p, $q > 1$, $\max\{p, q\} < N$ and $N/2 < p$, where $C > 1$ is a real constant.

Before proving the above theorem, we have to fix some notations. In the sequel V_r stands for the space $W_0^{1,q}(\Omega) \cap L^r(\Omega)$ endowed with the norm

$$\|v\|_r = \|v\|_{W_0^{1,q}(\Omega)} + |v|_r,$$

where $\|v\|_{W_0^{1,q}(\Omega)}$ and $|v|_r$ denote the usual norms in $W_0^{1,q}(\Omega)$ and $L^r(\Omega)$, respectively.

We write X for the space $W_0^{1,p}(\Omega) \times V_r$ endowed with the norm

$$\|(u, v)\|^2 = \|u\|^2_{W_0^{1,p}(\Omega)} + \|v\|^2_r,$$

where $\|u\|_{W_0^{1,p}(\Omega)}$ denotes the usual norm in $W_0^{1,p}(\Omega)$ and $\Phi : X \to \mathbb{R}$ denotes the functional given by

$$\Phi(u, v) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{1}{q} \int_\Omega |\nabla v|^q dx - \int_\Omega H(x, u, v) dx. \tag{3.3}$$

Under the assumptions (G_1) and (R_2), the functional Φ is well defined, belongs to $C^1(X, \mathbb{R})$ and

$$\begin{aligned} \Phi'(u, v)(\phi, \psi) &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi dx - \int_\Omega |\nabla v|^{q-2} \nabla v \nabla \psi dx \\ &- \int_\Omega [H_u(x, u, v)\phi + H_v(x, u, v)\psi] dx \end{aligned} \tag{3.4}$$

for all $(u,v),(\phi,\psi)\in X$. Since $(0,0)$ is a critical point of Φ, we say that (u,v) is a nontrivial solution of (S_{pq}), when it is a critical point of Φ and satisfies $\Phi(u,v)\neq 0$.

In order to apply the linking theorem, we introduce one more piece of notation. Since $(V_r,\|\cdot\|_r)$ is reflexive and separable, from [8] and [21], there exists a sequence $(e_n)\subset V_r$ such that

$$V_r=\overline{\text{span}\,\{e_n : n\in\mathbb{N}\}}. \tag{3.5}$$

Hereafter, for each $n\in\mathbb{N}$ we denote by V_r^n and X_n the following spaces

$$V_r^n=\text{span}\,\{e_j:\ j=1,\cdots,n\}\quad\text{and}\quad X_n=W_0^{1,p}(\Omega)\times V_r^n.$$

The restriction of Φ to X_n will be denoted by Φ_n. Then $\Phi_n : X_n\to\mathbb{R}$ is the functional given by

$$\Phi_n(u,v)=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\frac{1}{q}\int_\Omega|\nabla v|^q dx-\int_\Omega H(x,u,v)dx. \tag{3.6}$$

From the regularity of Φ, it follows that Φ_n belongs to $C^1(X_n,\mathbb{R})$ with

$$\begin{aligned}\Phi_n'(u,v)(\phi,\psi) &= \int_\Omega|\nabla u|^{p-2}\nabla u\nabla\phi dx-\int_\Omega|\nabla v|^{q-2}\nabla v\nabla\psi dx\\ &- \int_\Omega[H_u(x,u,v)\phi+H_v(x,u,v)\psi]dx \end{aligned}\tag{3.7}$$

for all $(u,v),(\phi,\psi)\in X_n$.

In the following, we prove that Φ_n satisfies the hypotheses of Theorem 2.1.

Lemma 3.1 *Under the assumptions $(G_1)-(G_2)$ and $(R_1)-(R_4)$, there exist $\sigma>0$ and $\rho>\sigma$ such that if $u_*\in W_0^{1,p}(\Omega)$ satisfies $\|u_*\|_{W_0^{1,p}(\Omega)}=1$, then*

$$b_n=\inf_{\mathcal{N}_n}\Phi_n>0=\sup_{\partial\mathcal{M}_{u_*}^n}\Phi_n$$

where

$$\mathcal{M}_{u_*}^n=\{(\lambda u_*,v)\in X_n:\ \|(\lambda u_*,v)\|^2\le\rho^2,\ \lambda\ge 0\}$$

and

$$\mathcal{N}_n=\{(u,0)\in X_n:\|u\|_{W_0^{1,p}(\Omega)}=\sigma\}.$$

Proof. By (R_1) and (R_2),

$$\Phi_n(u,0)=\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}-\int_\Omega H(x,u,0)dx\ge\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}-\frac{1}{p^*}|u|_{p^*}^{p^*}-C|u|_{p_1}^{p_1},$$

for some positive constant C. Using the Sobolev embedding theorem,

$$\Phi_n(u,0)\ge\frac{1}{p}\|u\|^p_{W_0^{1,p}(\Omega)}-C_1\|u\|^{p^*}_{W_0^{1,p}(\Omega)}-C_2\|u\|^{p_1}_{W_0^{1,p_1}(\Omega)},$$

for some positive constants C_1 and C_2. Since $p^*, p_1 > p$, there exists $\sigma > 0$ sufficiently small such that

$$\Phi_n(u,0) \geq \frac{1}{2p}\sigma^p, \quad \text{for } \|u\|_{W_0^{1,p}(\Omega)} = \sigma,$$

which implies

$$b_n = \inf_{\mathcal{N}_n} \Phi_n \geq \frac{1}{2p}\sigma^p > 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Now, for $v \in V_n^r$,

$$\Phi_n(0,v) = -\frac{\|v\|^q_{W_0^{1,q}(\Omega)}}{q} - \int_\Omega G(v)dx - \int_\Omega R(x,0,v)dx.$$

From (G_2) and (R_1),

$$\Phi_n(0,v) \leq 0 \ \text{ for all } v \in V_r^n. \tag{3.9}$$

Now, taking $u_* \in W_0^{1,p}(\Omega)$ with $\|u_*\|_{W_0^{1,p}(\Omega)} = 1$, by assumptions (G_2) and (R_1),

$$\Phi_n(\lambda u_*, v) \leq \frac{\lambda^p}{p} - \frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)} - \frac{\lambda^{p^*}}{p^*}|u_*|^{p^*}_{p^*} - \int_\Omega G(v)\,dx, \tag{3.10}$$

for every $\lambda > 0$ and $v \in V_r^n$.

If $r \leq q^*$, then $V_r = W_0^{1,q}(\Omega)$ and the norms $\|\cdot\|_r$ and $\|\cdot\|_{W_0^{1,q}(\Omega)}$ are equivalent. From this, there exists a positive constant C such that

$$\Phi_n(\lambda u_*, v) \leq \frac{\lambda^p}{p} - C\|v\|^q_r - \frac{\lambda^{p^*}}{p^*}|u_*|^{p^*}_{p^*}.$$

Observing that $\|(\lambda u_*, v)\|^2 = \lambda^2 + \|v\|^2_r = \rho^2$ implies that

$$\lambda^2 \geq \frac{\rho^2}{2} \quad \text{or} \quad \|v\|^2_r \geq \frac{\rho^2}{2},$$

it follows that

$$\frac{\lambda^p}{p} - C\|v\|^q_r - \frac{\lambda^{p^*}}{p^*}|u_*|^{p^*}_{p^*} < 0$$

providing ρ is sufficiently large. From (3.10), we conclude that there exists $\rho > \sigma$ such that

$$\Phi_n(\lambda u_*, v) \leq 0 \tag{3.11}$$

for all $(\lambda u_*, v) \in X_n$ such that $\|\lambda u_*\|^2_{W_0^{1,p}(\Omega)} + \|v\|^2_r = \rho^2$ and $\lambda > 0$. By (3.9) and (3.11), $\max\limits_{\partial \mathcal{M}^n_{u_*}} \Phi_n = 0$ and the proof is complete in this case.

Now, if $r > q^*$, by $(G_1)(ii)$ there is a positive constant C such that

$$\Phi_n(\lambda u_*, v) \leq \frac{\lambda^p}{p} - \frac{1}{q}\|v\|^q_{W_0^{1,q}(\Omega)} - \frac{\lambda^{p^*}}{p^*}|u_*|^{p^*}_{p^*} - C|v|^r_r.$$

Observing that $\|(\lambda u_*, v)\|^2 = \lambda^2 + \|v\|_r^2 = \rho^2$ implies that

$$\lambda^2 \geq \frac{\rho^2}{2}, \quad \|v\|^2_{W_0^{1,q}(\Omega)} \geq \frac{\rho^2}{4} \quad \text{or} \quad |v|_r^2 \geq \frac{\rho^2}{4},$$

the same argument used in the former case implies that for $\rho > 0$ large enough

$$\Phi_n(\lambda u_*, v) \leq 0$$

for all $(\lambda u_*, v) \in X_n$ such that $\|\lambda u_*\|^2_{W_0^{1,p}(\Omega)} + \|v\|^2 = \rho^2$ and $\lambda > 0$, which completes the proof of Lemma 3.1.

Lemma 3.2 *Suppose that G satisfies $(G_1)-(G_2)$ and R satisfies $(R_1)-(R_4)$. Then there exists $u_* \in W_0^{1,p}(\Omega)$ with $\|u_*\|_{W_0^{1,p}(\Omega)} = 1$ and $A \in (0, \frac{1}{N}S^{\frac{N}{p}})$ such that*

$$d_n = \sup_{\mathcal{M}^n_{u_*}} \Phi_n \leq A \text{ for all } n \in \mathbb{N}$$

where S denotes the best Sobolev constant of the embedding $W_0^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$.

Proof. Considering Ω_0 given by (R_3), we take $x_0 \in \Omega_0$ and $r_0 > 0$ such that $B_{2r_0}(x_0) \subset \Omega_0$. Choose $\phi \in C^\infty(\mathbb{R}^N), 0 \leq \phi \leq 1, \phi \equiv 1$ on $B_{r_0}(x_0)$ and $\phi \equiv 0$ on $\mathbb{R}^N \setminus B_{2r_0}(x_0)$. Given $\epsilon > 0$, consider

$$u_\epsilon(x) = \frac{\phi(x) w_\epsilon(x)}{|\phi w_\epsilon|_{p^*}}, \text{ for all } x \in \Omega$$

where

$$w_\epsilon(x) = \frac{\{\epsilon N(\frac{N-p}{p-1})^{p-1}\}^{\frac{N-p}{p^2}}}{(\epsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}, \text{ for all } x \in \mathbb{R}^N.$$

As proved in [9],

$$\int_\Omega |\nabla u_\epsilon|^p \leq S + O(\epsilon^{\frac{N-p}{p}}), \text{ as } \epsilon \to 0.$$

From (G_2) and (R_3),

$$\Phi_n(\lambda u_\epsilon, v) \leq \frac{\lambda^p}{p} \int_\Omega |\nabla u_\epsilon|^p - \frac{\lambda^{p^*}}{p^*} - a\lambda^s \int_\Omega |u_\epsilon|^s.$$

Arguing as in [17, Proposition 5.1], there exists $\epsilon > 0$ such that

$$A \doteq \max_{\lambda \geq 0} \left\{ \frac{\lambda^p}{p} \int_\Omega |\nabla u_\epsilon|^p - \frac{\lambda^{p^*}}{p^*} - a\lambda^s \int_\Omega |u_\epsilon|^s \right\} < \frac{1}{N} S^{\frac{N}{p}}.$$

Therefore,

$$d_n \leq A \quad \text{for all } n \in \mathbb{N},$$

and the proof of the lemma is completed by taking $u_* = \frac{u_\epsilon}{\|u_\epsilon\|_{W_0^{1,p}(\Omega)}}$.

From Lemmas 3.1 and 3.2, we can apply the linking theorem to functional Φ_n using the point $z_n = (u_*, 0)$ and the sets

$$Y_n = \{0\} \times V_r^n, \quad Z = W_0^{1,p}(\Omega) \times \{0\} \quad \text{and} \quad \mathcal{N}_n = \{(u,0) \in Z : \|u\|_{W_0^{1,p}(\Omega)} = \sigma\}.$$

Then, there exists a sequence $(u_k, v_k) \subset X_n$ with

$$\Phi_n(u_k, v_k) \to c_n \in [b_n, d_n] \quad \text{and} \quad \Phi_n'(u_k, v_k) \to 0, \text{ as } k \to +\infty. \tag{3.12}$$

Lemma 3.3 *The sequence (u_k, v_k) is bounded in X_n.*

Proof. From (3.12),

$$\Phi_n(u_k, v_k) - \Phi_n'(u_k, v_k)(\frac{1}{\mu}u_k, \frac{1}{\nu}v_k) = c_n + o_k(1)\|(u_k, v_k)\|. \tag{3.13}$$

By (G_2) and (R_4),

$$\Phi_n(u_k, v_k) - \Phi_n'(u_k, v_k)(\frac{1}{\mu}u_k, \frac{1}{\nu}v_k) \geq (\frac{1}{p} - \frac{1}{\mu})\|u_k\|^p_{W_0^{1,p}(\Omega)} + (\frac{1}{\nu} - \frac{1}{q})\|v_k\|^q_{W_0^{1,q}(\Omega)}.$$

From this,

$$(\frac{1}{p} - \frac{1}{\mu})\|u_k\|^p_{W_0^{1,p}(\Omega)} + (\frac{1}{\nu} - \frac{1}{q})\|v_k\|^q_{W_0^{1,q}(\Omega)} \leq c_n + o_k(1)\|(u_k, v_k)\|.$$

Since V_r^n is a finite dimensional space, the norms $\|\cdot\|_{W_0^{1,q}(\Omega)}$ and $\|\cdot\|_r$ are equivalent. Thus, there is a positive constant $C = C(n)$ such that

$$(\frac{1}{p} - \frac{1}{\mu})\|u_k\|^p_{W_0^{1,p}(\Omega)} + C\|v_k\|_r^q \leq c_n + o_k(1)\|(u_k, v_k)\|,$$

which implies that there exists a constant $K = K(n)$ such that

$$\|(u_k, v_k)\| \leq K, \text{ for all } k \in \mathbb{N}.$$

This concludes the proof of Lemma 3.3.

From Lemma 3.3, we may assume that there exists a subsequence of (u_k, v_k), still denoted by itself, and $(w_n, y_n) \in X_n$ such that

$$(u_k, v_k) \rightharpoonup (w_n, y_n) \text{ weakly in } X_n, \text{ as } k \to +\infty.$$

Hence,

$$\begin{cases} u_k \rightharpoonup w_n \text{ weakly in } W_0^{1,p}(\Omega), \\ v_k \rightharpoonup y_n \text{ weakly in } V_r^n. \end{cases}$$

Lemma 3.4 *The sequence (w_n, y_n) is bounded in X. Moreover,*

$$\Phi_n(w_n, y_n) = c_n \quad \textit{and} \quad \Phi_n'(w_n, y_n) = 0 \textit{ in } X_n^*.$$

Proof. Since V_r^n is a finite dimensional space, (v_k) converges strongly to y_n in V_r^n. Therefore,

$$\int_\Omega |\nabla v_k|^q\,dx \to \int_\Omega |\nabla y_n|^q\,dx, \tag{3.14}$$

$$\int_\Omega |\nabla v_k|^{q-2}\nabla v_k \nabla\psi\,dx \to \int_\Omega |\nabla y_n|^{q-2}\nabla y_n \nabla\psi\,dx, \tag{3.15}$$

$$\int_\Omega g(v_k)\psi\,dx \to \int_\Omega g(y_n)\psi\,dx, \tag{3.16}$$

as $k \to \infty$, for all $\psi \in V_r^n$. From the Sobolev embedding and by concentration-compactness principle due to Lions [13], we can assume

$$u_k \text{ converges to } w_n \text{ in } L^t(\Omega), \text{for all } t \in [1, p^*) \text{ and a.e.,} \tag{3.17}$$

$$|u_k|^{p^*} \rightharpoonup \zeta = |w_n|^{p^*} + \sum_{i\in I} \zeta_i \delta_{x_i}, \quad \zeta_i > 0,$$

$$|\nabla u_k|^p \rightharpoonup \chi \geq |\nabla w_n|^p + \sum_{i\in I} \chi_i \delta_{x_i}, \quad \chi_i > 0,$$

$$\zeta_i^{p/p^*} \leq \frac{\chi_i}{S}, \quad \text{for all } i \in I, \tag{3.18}$$

where ζ and χ are nonnegative measures on Ω.

In the sequel we prove that I is an empty set. In fact, suppose that there exists $\zeta_i > 0$ for some $i \in I$. Let ϕ be a cut off function satisfying $\phi = 1$ on the ball $B_1(0)$, $\phi = 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $0 \leq \phi \leq 1$. Given $\epsilon > 0$ and x_i a singular point, consider $\phi_\epsilon(x) = \phi(\frac{x-x_i}{\epsilon})$. Since

$$\Phi_n'(u_k, v_k)(u_k\phi_\epsilon, 0) = o_k(1),$$

a well known argument used in the scalar case shows that $\zeta_i = \chi_i$ (see for instance [9, Lemma 2.3]). Combining this with (3.18) we reach that $\zeta_i \geq S^{N/p}$. Now, (R_1), (R_4), (G_2) together with

$$\Phi_n(u_k, v_k) - \Phi_n'(u_k, v_k)(\frac{1}{p}u_k, \frac{1}{\nu}v_k) = c_n + o_k(1)\|(u_k, v_k)\|$$

imply that

$$c_n \geq \frac{1}{N} \lim_{k\to\infty} \int_\Omega |u_k|^{p^*}\,dx \geq \frac{1}{N}\zeta_i \geq \frac{1}{N}S^{N/p}.$$

On the other hand, from Lemma 3.2, $c_n < \frac{1}{N}S^{N/p}$, which is a contradiction.

Now, using that $I = \emptyset$, it follows that u_k converges strongly to w_n in $L^{p^*}(\Omega)$ and thus

$$\lim_{k\to\infty} \int_\Omega |u_k|^{p^*} = \lim_{k\to\infty} \int_\Omega |u_k|^{p^*-2}u_k w_n = \int_\Omega |w_n|^{p^*} \tag{3.19}$$

and

$$\lim_{k\to\infty} \int_\Omega R_u(x, u_k, v_k)w_n = \lim_{k\to\infty} \int_\Omega R_u(x, u_k, v_k)u_k = \int_\Omega R_u(x, w_n, y_n)w_n. \tag{3.20}$$

Since

$$\int_\Omega \langle |\nabla u_k|^{p-2}\nabla u_k - |\nabla w_n|^{p-2}\nabla w_n, \nabla u_k - \nabla w_n\rangle\, dx = \int_\Omega |u_k|^{p^*} -$$
$$- \int_\Omega |u_k|^{p^*-2}u_k w_n - \int_\Omega R_u(x,u_k,v_k)w_n + \int_\Omega R_u(x,u_k,v_k)u_k + o_k(1),$$

it follows from (3.19) and (3.20) that

$$\lim_{k\to\infty} \int_\Omega \langle |\nabla u_k|^{p-2}\nabla u_k - |\nabla w_n|^{p-2}\nabla w_n, \nabla u_k - \nabla w_n\rangle = 0.$$

According to

$$\langle |a|^{s-2}a - |b|^{s-2}b, a-b\rangle \geq \begin{cases} C_s|a-b|^s & \text{if } s\geq 2,\\ C_s\dfrac{|a-b|^2}{(|a|+|b|)^{2-s}} & \text{if } 1<s<2, \end{cases} \tag{3.21}$$

for every $a,b\in\mathbb{R}^N$ (see [18]), if $p\geq 2$, we have that u_k converges strongly to w_n in $W_0^{1,p}(\Omega)$. Now, if $1<p<2$, we conclude that

$$\lim_{k\to\infty} C_p \int_\Omega \frac{|\nabla u_k - \nabla w_n|^2}{(|\nabla u_k| + |\nabla w_n|)^{2-p}} = 0. \tag{3.22}$$

By the Hölder inequality,

$$\begin{aligned}\int_\Omega |\nabla u_k - \nabla w_n|^p &= \int_\Omega \frac{|\nabla u_k - \nabla w_n|^p}{(|\nabla u_k| + |\nabla w_n|)^{p(2-p)/2}} (|\nabla u_k| + |\nabla w_n|)^{\frac{p(2-p)}{2}}\\ &\leq \left(\int_\Omega \frac{|\nabla u_k - \nabla w_n|^2}{(|\nabla u_k| + |\nabla w_n|)^{2-p}}\right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_k| + |\nabla w_n|)^p\right)^{\frac{2-p}{2}}.\end{aligned}$$

Combining this inequality with (3.22) and the boundedness of (u_k), we obtain again that u_k converges strongly to w_n in $W_0^{1,p}(\Omega)$. Hence,

$$(u_k,v_k)\to(w_n,y_n) \quad \text{in } X_n.$$

The last limit implies that

$$\Phi_n(w_n,y_n) = c_n \in [b_n,d_n] \quad \text{and} \quad \Phi_n'(w_n,y_n) = 0 \quad \text{in } X_n^*.$$

From this,

$$\Phi_n(w_n,y_n) - \Phi_n'(w_n,y_n)(\frac{1}{\mu}w_n, \frac{1}{\nu}y_n) = c_n,$$

which combined with (R_4), (G_2) and (ii) of (G_1), if $r>q^*$, implies that (w_n,y_n) is bounded in X and proof is complete.

3.1 Proof of Theorem 3.1

The proof of Theorem 3.1 will be carried out in two lemmas. We start observing that since X is reflexive, there is no loss of generality in assuming that

$$(w_n, y_n) \rightharpoonup (u, v) \quad \text{in } X.$$

The same arguments used in the proof of Lemma 3.4 can be repeated to show that

$$w_n \to u \quad \text{in } W_0^{1,p}(\Omega). \tag{3.23}$$

However, similar limit involving the sequence (y_n) and v in V_r requires a careful analysis, which is the content of the next lemma.

Lemma 3.5 *The sequence (y_n) verifies the following limit*

$$y_n \to v \quad in \quad V_r.$$

Proof. From (3.5), there is $(\xi_k) \subset V_r$ such that

$$\xi_k \to v \quad \text{in} \quad V_r$$

and

$$\xi_k = \sum_{i=1}^{j(k)} \alpha_i e_i \in V_r^{j(k)}$$

where $j(k) \in \mathbb{N}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, it follows that

$$V_r^{j(k)} \subseteq V_r^n \quad \text{for all} \quad n \geq n_o$$

for some $n_o \geq j(k)$.

If $q \geq 2$ and $r \geq q^*$, from (3.21) and (G_1), we have that there is $C > 0$ such that

$$C \int_\Omega [|\nabla y_n - \nabla \xi_k|^q + |y_n - \xi_k|^r] \leq \int_\Omega \langle |\nabla y_n|^{q-2} \nabla y_n - |\nabla \xi_k|^{q-2} \nabla \xi_k, \nabla y_n - \nabla \xi_k \rangle$$
$$+ \int_\Omega (g(y_n) - g(\xi_k))(y_n - \xi_k).$$

Since $\Phi_n'(w_n, y_n) = 0$ in X_n^*, we derive that

$$\int_\Omega \langle |\nabla y_n|^{q-2} \nabla y_n - |\nabla \xi_k|^{q-2} \nabla \xi_k, \nabla y_n - \nabla \xi_k \rangle + \int_\Omega (g(y_n) - g(\xi_k))(y_n - \xi_k) =$$
$$- \int_\Omega |\nabla \xi_k|^{q-2} \nabla \xi_k (\nabla y_n - \nabla \xi_k) - \int_\Omega R_v(x, w_n, y_n) y_n + \int_\Omega R_v(x, w_n, y_n) \xi_k$$
$$- \int_\Omega g(\xi_k)(y_n - \xi_k).$$

This equality leads to

$$C\int_\Omega[|\nabla y_n-\nabla\xi_k|^q+|y_n-\xi_k|^r]\le-\int_\Omega|\nabla\xi_k|^{q-2}\nabla\xi_k(\nabla y_n-\nabla\xi_k)$$

$$-\int_\Omega R_v(x,w_n,y_n)y_n+\int_\Omega R_v(x,w_n,y_n)\xi_k-\int_\Omega g(\xi_k)(y_n-\xi_k).$$

Taking the limit as $n\to+\infty$, we reach

$$\limsup_{n\to+\infty}\int_\Omega[|\nabla y_n-\nabla\xi_k|^q+|y_n-\xi_k|^r]\le\frac{1}{C}\big[-\int_\Omega|\nabla\xi_k|^{q-2}\nabla\xi_k(\nabla v-\nabla\xi_k)$$

$$-\int_\Omega R_v(x,u,v)v+\int_\Omega R_v(x,u,v)\xi_k-\int_\Omega g(\xi_k)(v-\xi_k)\big].$$

Now, given $\delta>0$ there is $k_0\in\mathbb{N}$ such that

$$\tfrac{1}{C}\big[-\int_\Omega|\nabla\xi_k|^{q-2}\nabla\xi_k(\nabla v-\nabla\xi_k)-\int_\Omega R_v(x,u,v)v$$

$$+\int_\Omega R_v(x,u,v)\xi_k-\int_\Omega g(\xi_k)(v-\xi_k)\big]<\frac{\delta}{2}$$

for all $k\ge k_0$. Hence,

$$\limsup_{n\to+\infty}\int_\Omega[|\nabla y_n-\nabla\xi_k|^q+|y_n-\xi_k|^r]\le\frac{\delta}{2},\quad\text{for all }k\ge k_0.\tag{3.24}$$

Now, if $1<q<2$, then by (3.21) we obtain

$$\limsup_{n\to+\infty}\int_\Omega[\frac{|\nabla y_n-\nabla\xi_k|^2}{(|\nabla y_n|+|\nabla\xi_k|)^{2-q}}+|y_n-\xi_k|^r]\le\frac{\delta}{2},\quad\text{for all }k\ge k_0.\tag{3.25}$$

Employing the Hölder inequality to the first term of the left-hand side of (3.25) as in the proof of Lemma 3.4 and using the boundedness of (y_n) and (ξ_k), we can find a positive constant C such that

$$\limsup_{n\to+\infty}\int_\Omega[|\nabla y_n-\nabla\xi_k|^q+|y_n-\xi_k|^r]\le\frac{C\delta}{2},\quad\text{for all }k\ge k_0.\tag{3.26}$$

As a consequence of (3.24) and (3.26), there is a constant $C>0$ such that

$$\limsup_{n\to+\infty}\|y_n-\xi_k\|_r\le C\left[\delta^{1/q}+\delta^{1/r}\right]=o(1),\quad\text{for all }k\ge k_0.$$

Given $\epsilon>0$, for δ sufficiently small, it follows that

$$\limsup_{n\to+\infty}\|y_n-\xi_k\|_r\le\frac{\epsilon}{4},\quad\text{for all }k\ge k_0.$$

Fixing $k \geq k_0$ sufficiently large such that

$$\|\xi_k - v\|_r < \frac{\epsilon}{4},$$

we obtain

$$\|y_n - v\|_r \leq \|y_n - \xi_k\|_r + \frac{\epsilon}{4},$$

which implies that

$$\limsup_{n\to+\infty} \|y_n - v\|_r \leq \limsup_{n\to+\infty} \|y_n - \xi_k\|_r + \frac{\epsilon}{4} < \frac{\epsilon}{2}.$$

From this, $y_n \to v$ in V_r as $n \to \infty$. The case $r < q^*$ can be handled in much the same way with minor modifications.

Lemma 3.6 *The pair (u, v) is a nontrivial critical point of Φ.*

Proof. Fixing $k, n \in \mathbb{N}$ with $n \geq k$, we have $X_k \subset X_n$. Thus, for $(\phi, \psi) \in X_k$, it follows that

$$\Phi_n'(w_n, y_n)(\phi, \psi) = 0, \quad \text{for all } n \geq k,$$

because, by Lemma 3.4, $\Phi_n'(w_n, y_n) = 0$. Combining Lemma 3.5 with (3.23) we get

$$\Phi'(u, v)(\phi, \psi) = 0, \quad \text{for all } (\phi, \psi) \in X_k. \tag{3.27}$$

We claim that

$$\Phi'(u, v)(\phi, \psi) = 0, \quad \text{for all } (\phi, \psi) \in X. \tag{3.28}$$

In fact, we start observing that for all $\phi \in W_0^{1,p}(\Omega)$, the pair $(\phi, 0) \in X_k$ for all k. Hence, $\Phi'(u, v)(\phi, 0) = 0$. On the other hand, for $\psi \in V_r$, there exists $\psi_n \in V_r^{k(n)}$ such that

$$\lim_{n\to\infty} \psi_n = \psi, \quad \text{in } V_r.$$

From (3.27),

$$\Phi'(u, v)(0, \psi_n) = 0, \quad \text{for all } n \in \mathbb{N},$$

which implies after passage to the limit as $n \to \infty$ that

$$\Phi'(u, v)(0, \psi) = 0, \quad \text{for all } \psi \in V_r.$$

Thus, (3.28) is proved. Using the fact that $(w_n, y_n) \to (u, v)$ in X and that $\Phi(w_n, y_n) \geq b_n \geq \frac{1}{2p}\sigma^p > 0$ for all $n \in \mathbb{N}$, we have that $\Phi(u, v) \geq \frac{1}{2p}\sigma^p > 0$, from where it follows that (u, v) is a nontrivial solution for (S_{pq}), and the proof is complete.

4 Systems with critical growth for $N = p = q$

Our next application of Theorem 2.1 deals with systems

$$(S_{NN}) \qquad \begin{cases} -\Delta_N u = H_u(x,u,v), \ x \text{ in } \Omega \\ -\Delta_N v = -H_v(x,u,v), \ x \text{ in } \Omega \\ u = v = 0, \ \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $H : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function of the form

$$H(x,u,v) = F(x,u) + G(x,v) + R(x,u,v)$$

where F, G and R satisfy some subsequent conditions. In order to treat variationally (S_{NN}) in $W^{1,N}(\Omega) \times W^{1,N}(\Omega)$, we use the inequalities of Trudinger and Moser (see [14], [20]), which provide

$$\exp(\alpha|u|^{N/(N-1)}) \in L^1(\Omega), \quad \text{for all } u \in W_0^{1,N}(\Omega) \text{ and } \alpha > 0 \tag{4.29}$$

and there exists a constant $C(\Omega) > 0$ such that

$$\sup_{\|u\| \leq 1} \int_\Omega \exp(\alpha|u|^{N/(N-1)})\, dx \leq C(\Omega), \text{ for all } \alpha \leq \alpha_N \text{ and } u \in W_0^{1,N}(\Omega) \tag{4.30}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the $N-1$-dimensional surface of the unit sphere.

Hereafter, $F'(x,s) = f(x,s)$, $G'(x,s) = g(x,s)$ with f and g verifying the following conditions

(F_1) There exists a continuous function b verifying

$$f(x,s) = b(x,s)\exp(\alpha_N|s|^{N/(N-1)}),$$

with

$$c_p|s|^{p-2}s \leq b(x,s) \leq d_p|s|^{p-2}s, \ \text{for all } x \in \overline{\Omega}, \ s \in \mathbb{R},$$

for some $p > N$ and constants $c_p, d_p > 0$.

(G_1) There exists a constant $C > 0$ such that

$$|g(x,s)| \leq C\exp(\alpha_N|s|^{N/(N-1)}), \text{ for all } x \in \overline{\Omega}, s \in \mathbb{R}.$$

(G_2) There exist $\nu \in (0,N)$ and $\mu > N$ such that

$$0 \leq \nu G(x,s) \leq g(x,s)s, \text{ for all } s \in \mathbb{R} \text{ and } x \in \Omega.$$

and

$$0 \leq \mu B(x,s) \leq b(x,s)s, \text{ for all } s \in \mathbb{R} \text{ and } x \in \Omega.$$

where $B(x,s) = \int_0^s b(x,t)\,dt$.

(G_3) The constants c_p, ν, μ given by conditions (F_1) and (G_2) satisfy

$$\max\left\{\frac{\nu N}{N-\nu}, \frac{\mu N}{\mu - N}\right\}\left(\frac{p-N}{pN}\right)\left(\frac{S_p}{c_p}\right)^{\frac{N}{p-N}} < 1,$$

where

$$S_p = \inf_{u \in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\|u\|^N}{|u|_p^N}.$$

Related to function R, we assume that $R \in C^1(\overline{\Omega}\times\mathbb{R}^2, \mathbb{R})$ and that the following conditions hold:

(R_1) $R_u(x,0,0) = R_v(x,0,0) = 0$ and $R(x,u,v) \geq 0$ for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R}^2$.

(R_2) For any $\alpha, \beta > 0$

$$\lim_{|(u,v)|\to+\infty} \frac{R_u(x,u,v)}{\exp(\alpha|u|^{N/(N-1)}) + \exp(\beta|v|^{N/(N-1)})} = 0$$

and

$$\lim_{|(u,v)|\to+\infty} \frac{R_v(x,u,v)}{\exp(\alpha|u|^{N/(N-1)}) + \exp(\beta|v|^{N/(N-1)})} = 0.$$

(R_3) For ν and μ given by condition (G_3), we assume that

$$\frac{1}{\mu}R_u(x,t,s)t + \frac{1}{\nu}R_v(x,t,s)s - R(x,t,s) \geq 0, \text{ for all } x \in \Omega \text{ and } (t,s) \in \mathbb{R}^2,$$

where ν is given by condition (G_3).

We observe that condition (F_1) implies that

$$\lim_{|t|\to\infty} \sup_{x\in\overline{\Omega}} |b(x,t)| \exp(-\epsilon|t|^{N/(N-1)}) = 0$$

and

$$\lim_{|t|\to\infty} \sup_{x\in\overline{\Omega}} |b(x,t)| \exp(\epsilon|t|^{N/(N-1)}) = \infty,$$

for every $\epsilon > 0$, which characterizes the growth on f as critical exponential. On the other hand, the assumption (R_2) guarantees that the function R has subcritical exponential growth. For more details we refer the reader to [1] and [6].

The main result of this section is:

Theorem 4.1 *If the assumptions $(F_1), (G_1) - (G_2), (R_1) - (R_3)$ are satisfied, the system (S_{NN}) possesses a nontrivial solution.*

We note that the hypotheses $(R_1)-(R_3)$ are satisfied by the function given by $R(u,v)=|u|^s e^{|u|^\alpha}|v|^t e^{|v|^\beta}$, where $1<\alpha,\ \beta<2$, s and t are positive real numbers such that $s/\mu+t/\nu\geq 1$, where μ and ν are given by conditions (G_2) and (G_3).

We follow the same notation used in Section 3. By X we denote the space $W_0^{1,N}(\Omega)\times W_0^{1,N}(\Omega)$ endowed with the norm

$$\|(u,v)\|^2=\|u\|^2+\|v\|^2,$$

where $\|\cdot\|$ denotes the usual norm in $W_0^{1,N}(\Omega)$ and we write $\Phi : X\to\mathbb{R}$ the functional given by

$$\Phi(u,v)=\frac{1}{N}\int_\Omega |\nabla u|^N dx-\frac{1}{N}\int_\Omega |\nabla v|^N dx-\int_\Omega H(x,u,v)dx. \tag{4.31}$$

Under the assumptions (G_1) and (R_2), the functional Φ is well defined, belongs to $C^1(X,\mathbb{R})$ and

$$\begin{aligned}\Phi'(u,v)(\phi,\psi) &= \int_\Omega |\nabla u|^{N-2}\nabla u\nabla\phi dx-\int_\Omega |\nabla v|^{N-2}\nabla v\nabla\psi dx\\ &- \int_\Omega [H_u(x,u,v)\phi+H_v(x,u,v)\psi]dx\end{aligned} \tag{4.32}$$

for all $(u,v),(\phi,\psi)\in X$. In order to apply the linking theorem, in the next we fix some notations. Since $(W_0^{1,N}(\Omega),\|\cdot\|)$ is reflexive and separable, by using again [8] and [21], there exists a sequence $(e_n)\subset W_0^{1,N}(\Omega)$ such that

$$W_0^{1,N}(\Omega)=\overline{\operatorname{span}\{e_n : n\in\mathbb{N}\}}. \tag{4.33}$$

Hereafter, for each $n\in\mathbb{N}$ we denote by V^n and X_n the following spaces

$$V^n=\operatorname{span}\{e_j:\ j=1,\cdots,n\}\quad\text{and}\quad X_n=W_0^{1,p}(\Omega)\times V^n.$$

Let $\Phi_n : X_n\to\mathbb{R}$ be the functional given by

$$\Phi_n(u,v)=\frac{1}{N}\int_\Omega |\nabla u|^N dx-\frac{1}{N}\int_\Omega |\nabla v|^N dx-\int_\Omega H(x,u,v)dx. \tag{4.34}$$

Under the assumptions (G_1) and (R_2), the functional Φ_n is well defined and belongs to $C^1(X_n,\mathbb{R})$. Furthermore,

$$\begin{aligned}\Phi_n'(u,v)(\phi,\psi) &= \int_\Omega |\nabla u|^{N-2}\nabla u\nabla\phi dx-\int_\Omega |\nabla v|^{N-2}\nabla v\nabla\psi dx\\ &- \int_\Omega [H_u(x,u,v)\phi+H_v(x,u,v)\psi]dx\end{aligned} \tag{4.35}$$

for all $(u,v),(\phi,\psi)\in X_n$.

The following results establish some limits that are crucial in the proof of results later on.

Proposition 4.1 *Let (φ_j) be a sequence of functions in $W_0^{1,N}(\Omega)$ converging to φ weakly in $W_0^{1,N}(\Omega)$. Assume that $\|\varphi_j\|^{N/(N-1)} \leq \delta < 1$ and $l \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies*

$$|l(x,s)| \leq C \exp\left(\alpha_N |s|^{N/(N-1)}\right), \text{ for all } (x,s) \in \overline{\Omega} \times \mathbb{R}$$

and for some $C > 0$. Then,

$$\lim_{j \to +\infty} \int_\Omega l(x,\varphi_j) w dx = \int_\Omega l(x,\varphi) w dx, \tag{4.36}$$

for every $w \in W_0^{1,N}(\Omega)$, and

$$\lim_{j \to +\infty} \int_\Omega l(x,\varphi_j)\varphi_j dx = \int_\Omega l(x,\varphi)\varphi dx. \tag{4.37}$$

Proof. Consider $q > 1$ so that $q\delta < 1$. From the hypothesis on l,

$$\int_\Omega |l(x,\varphi_j)|^q dx \leq C \int_\Omega e^{q\alpha_N |\varphi_j|^{N/(N-1)}} dx = C \int_\Omega e^{q\alpha_N \|\varphi_j\|^{N/(N-1)} (\frac{|\varphi_j|}{\|\varphi_j\|})^{N/(N-1)}} dx$$

$$\leq C \int_\Omega e^{q\alpha_N \delta (\frac{|\varphi_n|}{\|\varphi_j\|})^{N/(N-1)}} dx.$$

By Trudinger and Moser inequality, there exists $M_1 > 0$ such that

$$\int_\Omega |l(x,\varphi_j)|^q dx \leq M_1, \ \ \forall n \in \mathbb{N}. \tag{4.38}$$

Combining Sobolev embeddings with Egoroff theorem, given $\epsilon > 0$ there exists $E \subset \Omega$ such that $|E| < \epsilon$ and $\varphi_j(x) \to \varphi(x)$ uniformly on $\Omega \setminus E$. By Hölder inequality and using (4.38), we get

$$\Big| \int_\Omega (l(x,\varphi_j) - l(x,\varphi)) w dx \Big| \leq \int_{\Omega \setminus E} |l(x,\varphi_j) - l(x,\varphi)||w| dx + o_\epsilon(1)$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$. As $\epsilon > 0$ is arbitrary and $l(x,\varphi_j) \to l(x,\varphi)$ uniformly on $\Omega \setminus E$, we conclude the proof of (4.36). Similar argument shows that the limit (4.37) holds.

Lemma 4.1 *Suppose that R satisfies the condition (R_2) and let (φ_j, ξ_j) be a sequence weakly convergent to (φ, ξ) in $W_0^{1,N}(\Omega) \times W_0^{1,N}(\Omega)$. Then,*

$$\int_\Omega R(x,\varphi_j,\xi_j)\, dx \to \int_\Omega R(x,\varphi,\xi)\, dx,$$

$$\int_\Omega R_u(x,\varphi_j,\xi_j)\varphi_j\, dx \to \int_\Omega R_u(x,\varphi,\xi)\varphi\, dx,$$

$$\int_\Omega R_u(x,\varphi_j,\xi_j)\phi\, dx \to \int_\Omega R_u(x,\varphi,\xi)\phi\, dx,$$

$$\int_\Omega R_v(x,\varphi_j,\xi_j)\xi_j\, dx \to \int_\Omega R_v(x,\varphi,\xi)\xi\, dx$$

and

$$\int_\Omega R_v(x,\varphi_j,\xi_j)\psi\,dx \to \int_\Omega R_v(x,\varphi,\xi)\psi\,dx,$$

for all $\phi,\psi \in W_0^{1,N}(\Omega)$.

Proof. Since (φ_j,ξ_j) is weakly convergent, there is $M>0$ such that

$$\|\varphi_j\|,\ \|\xi_j\| \le M \quad \text{for all } j\in\mathbb{N}.$$

Now from (R_2), given $0<\alpha,\beta<M^{-N/(N-1)}\alpha_N$, there exists a constant $C>0$ such that

$$|R_u(x,\varphi_j,\xi_j)| \le C\left(e^{\alpha|\varphi_j|^{N/(N-1)}}+e^{\beta|\xi_j|^{N/(N-1)}}\right), \tag{4.39}$$

and

$$|R_v(x,\varphi_j,\xi_j)| \le C\left(e^{\alpha|\varphi_j|^{N/(N-1)}}+e^{\beta|\xi_j|^{N/(N-1)}}\right). \tag{4.40}$$

As a consequence,

$$|R(x,\varphi_j,\xi_j)| \le C\left(e^{\alpha|\varphi_j|^{N/(N-1)}}+e^{\beta|\xi_j|^{N/(N-1)}}\right)(|\varphi_j|+|\xi_j|). \tag{4.41}$$

Taking $q>1$ such that $q\alpha M^{N/(N-1)}, q\beta M^{N/(N-1)}<\alpha_N$, from Trudinger and Moser inequality there exists $K>0$ such that

$$\int_\Omega e^{\alpha q|\varphi_j|^{N/(N-1)}},\int_\Omega e^{\beta q|\xi_j|^{N/(N-1)}} \le K \ \ \forall n\in\mathbb{N}.$$

This combining with (4.39)-(4.41) and Sobolev embeddings imply that the above limits hold. This concludes the proof.

In the following, we prove that Φ_n satisfies the hypotheses of Theorem 2.1.

Lemma 4.2 *Suppose that the assumptions* $(F_1),(G_1)-(G_2),(R_1)-(R_3)$ *hold. Then there exist* $\sigma>0$ *and* $\rho>\sigma$ *such that if* $u_*\in W_0^{1,N}(\Omega)$ *is a nonnegative function and satisfies* $\|u_*\|_{W_0^{1,N}(\Omega)}=1$, *then*

$$b_n=\inf_{\mathcal{N}_n}\Phi_n>0=\sup_{\partial\mathcal{M}^n_{u_*}}\Phi_n$$

where

$$\mathcal{M}^n_{u_*}=\{(\lambda u_*,v)\in X_n:\ \|(\lambda u_*,v)\|^2\le\rho^2,\ \lambda\ge 0\}$$

and

$$\mathcal{N}_n=\{(u,0)\in X_n:\|u\|=\sigma\}.$$

Proof. We start observing that, from (F_1),

$$|F(x,t)|\le d_p|t|^p e^{\alpha_N|t|^{N/(N-1)}},\ \text{for all } x\in\overline{\Omega},\ t\in\mathbb{R}.$$

Thus, from (4.29),

$$\Phi_n(u,0) = \tfrac{1}{N}\|u\|^N - \int_\Omega F(x,u)dx \geq \frac{1}{N}\|u\|^N - d_p \int_\Omega |u|^p e^{\alpha_N |u|^{N/(N-1)}} dx$$

$$\geq \tfrac{1}{N}\|u\|^N - d_p |u|_{2p}^p \Big\{ \int_\Omega e^{2\alpha_N |u|^{N/(N-1)}} dx \Big\}^{\frac{1}{2}}$$

$$= \tfrac{1}{N}\|u\|^N - d_p |u|_{2p}^p \Big\{ \int_\Omega e^{2\alpha_N \|u\|^{N/(N-1)} (\frac{|u|}{\|u\|})^{N/(N-1)}} dx \Big\}^{\frac{1}{2}}.$$

By Trudinger and Moser inequality (4.30), if $\|u\|^{N/(N-1)} < \frac{1}{2}$, then

$$\Phi_n(u,0) \geq \frac{1}{N}\|u\|^N - C|u|_{2p}^p \geq \frac{1}{N}\|u\|^N - C\|u\|^p$$

Hence, by choosing σ sufficiently small, we derive that

$$\Phi_n(u,0) \geq \frac{1}{4}\sigma^N > 0 \ \text{ if } \ \|u\| = \sigma$$

and, therefore, $b_n \geq \frac{1}{4}\sigma^N > 0$ for all $n \in \mathbb{N}$.

Now fixing a nonnegative function $u_* \in W_0^{1,N}(\Omega)$ with $\|u_*\| = 1$, from (F_1), there is a constant $C > 0$ such that

$$\Phi_n(\lambda u_*, v) \leq \frac{\|\lambda u_*\|^N}{N} - \frac{\|v\|^N}{N} - C|\lambda u_*|_p^p.$$

On $\partial\mathcal{M}_{z_*}^n$ we have that $\|\lambda u_*\|^2 + \|v\|^2 = \rho^2$, either $\|\lambda u_*\| \geq \frac{\rho^2}{2}$ or $\|v\|^2 \geq \frac{\rho^2}{2}$. If $\|v\|^2 \geq \frac{\rho^2}{2}$ holds, we obtain

$$\Phi_n(\lambda u_*, v) \leq \frac{\|\lambda u_*\|^N}{N} - C|\lambda u_*|_p^p - \frac{\rho^N}{N 2^{N/2}}.$$

Considering

$$Q(t) = \frac{t^N}{N} - \widetilde{C}t^p, \ \text{ for } t \geq 0,$$

where $\widetilde{C} = C|u_*|_p^p$. It is straightforward to show directly that there is $M > 0$ such that

$$Q(t) \leq M, \quad \text{for all } t \geq 0.$$

From this,

$$\Phi_n(\lambda u_*, v) \leq M - \frac{\rho^N}{N 2^{N/2}} < 0$$

provided that ρ is sufficiently large. Now assuming that $\|\lambda u_*\|^2 \geq \frac{\rho^2}{2}$, it follows that

$$\Phi_n(\lambda u_*, v) \leq \frac{\|\lambda u_*\|^N}{N} - C|\lambda u_*|_p^p = \frac{\lambda^N}{N} - \widetilde{C}\lambda^p.$$

Since $\lambda^2 \geq \frac{\rho^2}{2}$ and $p > N$, it follows that

$$\Phi_n(\lambda u_*, v) < 0$$

provided that ρ is sufficiently large.

Lemma 4.3 *Suppose that the assumptions $(F_1), (G_1) - (G_2), (R_1) - (R_3)$ hold. Then there exists a nonnegative function $u_* \in W_0^{1,N}(\Omega)$ with $\|u_*\| = 1$ such that*

$$d_n = \sup_{\mathcal{M}_{u_*}^n} \Phi_n \leq (\frac{1}{N} - \frac{1}{p}) \frac{S_p^{\frac{p}{p-N}}}{c_p^{\frac{N}{p-N}}}. \tag{4.42}$$

Proof. In the sequel, $\widetilde{u}$ is a nonnegative function verifying

$$S_p = \frac{\|\widetilde{u}\|^N}{|\widetilde{u}|_p^N}.$$

From (F_1),

$$F(x,s) \geq \frac{c_p}{p}|s|^p, \text{ for all } (x,s) \in \overline{\Omega} \times \mathbb{R}.$$

Thus,

$$\Phi_n(\lambda \widetilde{u}, v) \leq \frac{\lambda^N}{N}\|\widetilde{u}\|^N - \frac{c_p}{p}\lambda^p |\widetilde{u}|_p^p \leq \max_{\lambda \geq 0}\left\{\frac{\lambda^N}{N}\|\widetilde{u}\|^N - \frac{c_p}{p}\lambda^p|\widetilde{u}|_p^p\right\}$$

$$= (\frac{1}{N} - \frac{1}{p}) \frac{1}{c_p^{\frac{N}{p-N}}} \Big(\frac{\|\widetilde{u}\|^N}{|\widetilde{u}|_p^N}\Big)^{\frac{p}{p-N}}$$

for all $\lambda > 0$ and $v \in W_0^{1,N}(\Omega)$. This gives

$$\sup_{\lambda \geq 0,\, v \in W_0^{1,N}(\Omega)} \Phi_n(\lambda \widetilde{u}, v) \leq (\frac{1}{N} - \frac{1}{p}) \frac{1}{c_p^{\frac{N}{p-N}}} \Big(\frac{\|\widetilde{u}\|^N}{|\widetilde{u}|_p^N}\Big)^{\frac{p}{p-N}}.$$

Therefore, the proof of the lemma is completed by taking $u_* = \frac{\widetilde{u}}{\|\widetilde{u}\|}$.

From Lemmas 4.2 and 4.3, we can apply the linking theorem for the functional Φ_n, the point $z_n = (u_*, 0)$ and the sets

$$Y_n = \{0\} \times V^n, \quad Z = W_0^{1,N}(\Omega) \times \{0\} \;\text{ and }\; \mathcal{N}_n = \{(u,0) \in Z : \|u\|_{W_0^{1,N}(\Omega)} = \sigma\}.$$

Then, there exists a sequence $(u_k, v_k) \subset X_n$ with

$$\Phi_n(u_k, v_k) \to c_n \in [b_n, d_n], \quad \Phi_n'(u_k, v_k) \to 0, \text{ as } k \to +\infty. \tag{4.43}$$

The principal significance of the following lemma is that it allows us to apply the Trudinger and Moser inequalities to the sequences (u_k) and (v_k).

Lemma 4.4 *The sequence* (u_k, v_k) *is bounded in* X_n. *Moreover, there is* $k_0 \in \mathbb{N}$ *and* $m \in (0,1)$ *such that*

$$\|u_k\|^{N/(N-1)},\ \|v_k\|^{N/(N-1)} \leq m, \quad \textit{for all } k \geq k_0. \tag{4.44}$$

Proof. We start observing that the condition (G_2) implies that

$$0 \leq \mu F(x,s) \leq f(x,s)s, \text{ for all } s \in \mathbb{R} \text{ and } x \in \Omega.$$

As in the proof of Lemma 3.3,

$$(\frac{1}{N} - \frac{1}{\mu})\|u_k\|^N + (\frac{1}{\nu} - \frac{1}{N})\|v_k\|^N \leq d_n + o_k(1)\|(u_k, v_k)\|,$$

from where it follows that (u_k, v_k) is bounded in X_n. Consequently,

$$\limsup_{k\to\infty} \|u_k\|^N \leq \frac{\mu N d_n}{\mu - N}$$

and

$$\limsup_{k\to\infty} \|v_k\|^N \leq \frac{\nu N d_n}{N - \nu}.$$

From (G_3) and (4.42), we get

$$\limsup_{k\to\infty} \|u_k\|^N,\ \limsup_{k\to\infty} \|v_k\|^N < 1.$$

Therefore, there are $k_0 \in \mathbb{N}$ and $m \in (0,1)$ such that

$$\|u_k\|^{N/(N-1)},\ \|v_k\|^{N/(N-1)} \leq m, \quad \text{for all } k \geq k_0,$$

which proves the lemma.

From Lemma 4.4, we may assume that there exists a subsequence of (u_k, v_k), still denoted by itself, and $(w_n, y_n) \in X_n$ such that

$$(u_k, v_k) \rightharpoonup (w_n, y_n) \text{ weakly in } X_n, \text{ as } k \to +\infty.$$

Hence,

$$\begin{cases} u_k \rightharpoonup w_n \text{ weakly in } W_0^{1,N}(\Omega) \\ v_k \rightharpoonup y_n \text{ weakly in } V^n. \end{cases}$$

Lemma 4.5 *The sequence* (w_n, y_n) *is bounded in* X. *Moreover,*

$$\Phi_n(w_n, y_n) = c_n \quad \textit{and} \quad \Phi_n'(w_n, y_n) = 0 \textit{ in } X_n^*.$$

Proof. Let m be the constant given by Lemma 4.4. Since m is independent of n, the weak convergence implies that

$$\|w_n\|^{N/(N-1)},\ \|y_n\|^{N/(N-1)} \le m, \quad \text{for all } n \in \mathbb{N}. \tag{4.45}$$

Since V^n is a finite dimensional space, for some subsequence, still denoted by itself, (v_k) converges strongly to y_n in V^n. We now verify that (u_k) converges strongly to w_n. To this end, we start recalling that

$$\int_\Omega \langle |\nabla u_k|^{N-2}\nabla u_k - |\nabla w_n|^{N-2}\nabla w_n, \nabla u_k - \nabla w_n\rangle\, dx = \int_\Omega f(x,u_k)u_k -$$

$$\int_\Omega f(x,u_k)w_n + \int_\Omega R_u(x,u_k,v_k)u_k - \int_\Omega R_u(x,u_k,v_k)w_n + o_k(1).$$

On the other hand,

$$\|u_k\|^{N/(N-1)} \le m < 1, \quad \text{for all } k \ge k_0,$$

thus, by Proposition 4.1 and Lemma 4.1 it follows that

$$\int_\Omega f(x,u_k)w_n dx \to \int_\Omega f(x,w_n)w_n dx,$$

$$\int_\Omega f(x,u_k)u_k\, dx \to \int_\Omega f(x,w_n)w_n dx,$$

$$\int_\Omega R_u(x,u_k,v_k)u_k\, dx \to \int_\Omega R_u(x,w_n,y_n)w_n\, dx$$

and

$$\int_\Omega R_u(x,u_k,v_k)w_n\, dx \to \int_\Omega R_v(x,w_n,y_n)w_n\, dx,$$

as $k \to \infty$. Consequently,

$$\lim_{k\to\infty} \int_\Omega \langle |\nabla u_k|^{N-2}\nabla u_k - |\nabla w_n|^{N-2}\nabla w_n, \nabla u_k - \nabla w_n\rangle = 0.$$

Combining this with (3.21) yields

$$\int_\Omega |\nabla u_k - \nabla w_n|^N \to 0, \quad \text{as } k \to \infty,$$

that is, (u_k) converges strongly to w_n in $W_0^{1,N}(\Omega)$. The strong convergence of (u_k,v_k) to (w_n,y_n) together with (4.43) lead to $\Phi_n(w_n,y_n) = c_n$ and $\Phi_n'(w_n,y_n) = 0$, and the proof is complete.

4.1 Proof of Theorem 4.1

As in Section 3.1, the proof of Theorem 4.1 is divided into two lemmas. We start observing that since X is reflexive, without loss of generality, we may assume that

$$(w_n, y_n) \rightharpoonup (u, v) \quad \text{in } X.$$

The same arguments used in the proof of Lemma 4.5 yield

$$w_n \to u \quad \text{in } W_0^{1,N}(\Omega). \tag{4.46}$$

On the other hand, as observed in Section 2, the convergence of the sequence (y_n) to v in V_r is not immediate. The following lemma establishes this convergence.

Lemma 4.6 *The sequence (y_n) verifies the following limit*

$$y_n \to v \quad in \quad W_0^{1,N}(\Omega).$$

Proof. From (3.5), there is $(\xi_k) \subset W_0^{1,N}(\Omega)$ such that

$$\xi_k \to v \quad \text{in} \quad W_0^{1,N}(\Omega)$$

and

$$\xi_k = \sum_{i=1}^{j(k)} \alpha_i e_i \in V^{j(k)}$$

where $j(k) \in \mathbb{N}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, it follows that

$$V^{j(k)} \subseteq V^n \quad \text{for all} \quad n \geq n_o$$

for some $n_o \geq j(k)$. From 3.21, there is $C > 0$ such that

$$C \int_\Omega [|\nabla y_n - \nabla \xi_k|^N] \leq \int_\Omega \langle |\nabla y_n|^{N-2} \nabla y_n - |\nabla \xi_k|^{N-2} \nabla \xi_k, \nabla y_n - \nabla \xi_k \rangle.$$

Since $\Phi_n'(w_n, y_n) = 0$ in X_n^*, we derive that

$$\int_\Omega \langle |\nabla y_n|^{N-2} \nabla y_n - |\nabla \xi_k|^{N-2} \nabla \xi_k, \nabla y_n - \nabla \xi_k \rangle = - \int_\Omega |\nabla \xi_k|^{N-2} \nabla \xi_k (\nabla y_n - \nabla \xi_k) +$$

$$\int_\Omega R_v(x, w_n, y_n) y_n - \int_\Omega R_v(x, w_n, y_n) \xi_k + \int_\Omega g(x, y_n)(\xi_k - y_n).$$

This equality leads to

$$C \int_\Omega [|\nabla y_n - \nabla \xi_k|^N] \leq - \int_\Omega |\nabla \xi_k|^{N-2} \nabla \xi_k (\nabla y_n - \nabla \xi_k) + \int_\Omega R_v(x, w_n, y_n) y_n -$$

$$- \int_\Omega R_v(x, w_n, y_n) \xi_k + \int_\Omega g(x, y_n)(\xi_k - y_n).$$

From (4.45) together with Proposition 4.1 and Lemma 4.1, we reach

$$\limsup_{n\to+\infty}\int_\Omega [|\nabla y_n - \nabla\xi_k|^N] \le \frac{1}{C}\big[-\int_\Omega |\nabla\xi_k|^{N-2}\nabla\xi_k(\nabla v - \nabla\xi_k) + \int_\Omega R_v(x,u,v)v - \int_\Omega R_v(x,u,v)\xi_k + \int_\Omega g(x,v)(\xi_k - v)\big].$$

Now, given $\delta > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\tfrac{1}{C}\big[-\int_\Omega |\nabla\xi_k|^{N-2}\nabla\xi_k(\nabla v - \nabla\xi_k) + \int_\Omega R_v(x,u,v)v - \int_\Omega R_v(x,u,v)\xi_k + \int_\Omega g(x,v)(\xi_k - v)\big] < \frac{\delta}{2}$$

for all $k \ge k_0$, from where it follows that

$$\limsup_{n\to+\infty}\int_\Omega [|\nabla y_n - \nabla\xi_k|^N] \le \frac{\delta}{2}, \quad \text{for all } k \ge k_0.$$

As a resut,

$$\limsup_{n\to+\infty}\|y_n - \xi_k\| \le \left(\frac{\delta}{2}\right)^{1/N} = o(1).$$

Hence, given $\epsilon > 0$, for δ sufficiently small, it follows

$$\limsup_{n\to+\infty}\|y_n - \xi_k\| \le \frac{\epsilon}{4}, \quad \text{for all } k \ge k_0.$$

Fixing $k \ge k_0$ such that

$$\|\xi_k - v\| < \frac{\epsilon}{4},$$

we get

$$\|y_n - v\| \le \|y_n - \xi_k\| + \frac{\epsilon}{4}.$$

Thus

$$\limsup_{n\to+\infty}\|y_n - v\| \le \limsup_{n\to+\infty}\|y_n - \xi_k\| + \frac{\epsilon}{4} < \frac{\epsilon}{2}$$

which implies that $y_n \to v$ in $W_0^{1,N}(\Omega)$ as $n \to \infty$. The proof of the lemma is complete.

Lemma 4.7 *The pair (u,v) is a nontrivial critical point of Φ.*

Proof. The proof follows by applying the same arguments used in the proof of Lemma 3.6.

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