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Absence of Global Solutions to Systems of Perturbed Parabolic Inequalities with Chipot–Weissler Nonlinearity in the Gradient Term

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Abstract

For a wide class of systems of nonlinear parabolic inequalities with Chipot–Weissler nonlinearity in the gradient term which depend on two nonnegative parameters λ_1 and λ_2, we study the nonexistence of global solutions according to the "test function" method. We establish theorems that, when $\lambda_1 = \lambda_2 = 0$, include and, in some cases, improve, or, in part, yield the results obtained by Escobedo and Herrero; Escobedo and Levine; Galaktionov;Levine, Lieberman and Meier; Mitidieri and Pohozaev; Pohozaev and Tesei.

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1 Introduction

During the last decades many researchers have investigated blow-up theorems for the solutions to nonlinear evolution equations, useful in many fields of Physics. A wide bibliography is present in [6] and, more recently, in [8]. A list of the results is reported in [16].

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Some conditions concerning the nonexistence of global nonnegative solutions to the Cauchy's problem connected with a class of nonlinear parabolic inequalities have been formulated in [9]. In particular, as an answer to an open question in [15], [16], it has been shown that the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + \lambda |\nabla u|^{\frac{2q}{1+q}} = u^q \quad \text{in } \mathbf{R}^N \times]0, +\infty[\\ &\qquad\qquad\qquad\qquad\qquad (0 < \lambda \ll 1,\ q > 1,\ u_0 \in L^1_{\text{loc}}(\mathbf{R}^N),\ u_0 \geq 0) \\ &u \gneqq 0,\ u(x,0) = u_0(x) \quad \text{in } \mathbf{R}^N \end{aligned}$$

is not solvable when $q < 1 + \dfrac{2}{N}$ or when $q = 1 + \dfrac{2}{N}$ if the initial data have a convenient asymptotic behaviour for $|x| \to +\infty$. In other words, it has been observed that, by perturbing the equation

$$\frac{\partial u}{\partial t} - \Delta u = u^q$$

with the nonlinear term in the gradient, the critical exponent of Fujita ([4], [5]) $q_c = 1 + \dfrac{2}{N}$ does not change.

It seemed natural to ask, in the case of the system

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \Delta u_1 + \lambda_1 |\nabla u_1|^{\frac{2q_1}{1+q_1}} = u_2^{q_2} \\ &\qquad\qquad \text{in } \mathbf{R}^N \times]0, +\infty[\qquad (\lambda_i \geq 0,\ q_i > 0,\ u_{i0} \in L^1_{\text{loc}}(\mathbf{R}^N),\ u_{i0} \geq 0) \\ &\frac{\partial u_2}{\partial t} - \Delta u_2 + \lambda_2 |\nabla u_2|^{\frac{2q_2}{1+q_2}} = u_1^{q_1} \\ &u_i \gneqq 0,\ u_i(x,0) = u_{i0}(x) \quad \text{in } \mathbf{R}^N, \end{aligned}$$

what influence the nonlinear terms in the gradient could have on the condition

$$\frac{\max\{q_1, q_2\} + 1}{q_1 q_2 - 1} \geq \frac{N}{2}$$

formulated by Escobedo and Herrero in [1] for the blow-up of solutions to the problem

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \Delta u_1 = u_2^{q_2} \\ &\qquad\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0, +\infty[\\ &\frac{\partial u_2}{\partial t} - \Delta u_2 = u_1^{q_1} \\ &u_i \gneqq 0,\ u_i(x,0) = u_{i0}(x) \quad \text{in } \mathbf{R}^N \end{aligned}$$

when $q_1 q_2 > 1$. We provide an answer to this question in the present paper (the conclusions in section 4), whose aim is the analysis of the nonexistence of solutions to the system

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - \operatorname{div}(A_1(x,t,\nabla u_1)) + \lambda_1 a_1(x,t,\nabla u_1) \geq b_1(x,t,u_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0,+\infty[\\
&\frac{\partial u_2}{\partial t} - \operatorname{div}(A_2(x,t,\nabla u_2)) + \lambda_2 a_2(x,t,\nabla u_2) \geq b_2(x,t,u_1) \\
&u_i \gneqq 0,\ u_i(x,0) = u_{i0}(x) \ \text{ in } \mathbf{R}^N
\end{aligned} \tag{1.1}$$

where

- λ_1 and λ_2 are nonnegative real parameters,
- u_{10} and u_{20} are nonnegative functions locally summable on $\mathbf{R}^N$,
- $A_i(x,t,w_i)$, $a_i(x,t,w_i)$, $b_1(x,t,z_2)$ and $b_2(x,t,z_1)$ are Carathéodory functions defined for almost every $(x,t) \in \mathbf{R}^N \times]0,+\infty[$, for $z_1, z_2 \in [0,+\infty[$ and for $w_i \in \mathbf{R}^N$, with A_i into $\mathbf{R}^N$ and a_i, b_i real functions.

We set

$$1 < \omega < +\infty \qquad \omega' = \frac{\omega}{\omega - 1}$$
$$R > 0 \qquad B_R = \{x \in \mathbf{R}^N \,:\, |x| < R\}.$$

The basic hypotheses are the following:

$i_{11})$ $\left\{\begin{array}{c} \text{there exist } b_0 \in L^1_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[) \text{ with } b_0 > 0 \text{ and } q_1, q_2 > 0 \text{ such that} \\ b_1(x,t,z_2) \geq b_0(x,t) z_2^{q_2},\ b_2(x,t,z_1) \geq b_0(x,t) z_1^{q_1}\ ; \end{array}\right.$

$i_{12})$ $\left\{\begin{array}{c} \text{there exist real Carathéodory functions } A_{i0}, \bar{A}_{i0}, a_{i0} \text{ dependent on} \\ (x,t,w_i) \text{ with } A_{i0}, \bar{A}_{i0} \text{ positive, } a_{i0} \gneqq 0 \text{ and } p_1, p_2 > 1 \text{ such that} \\ A_i \cdot w_i \geq A_{i0}|w_i|^{p_i},\ |A_i|^{p_i'} \leq \bar{A}_{i0}|w_i|^{p_i},\ |a_i| \leq a_{i0}|w_i|^{\frac{p_i q_i}{1+q_i}}\ ; \end{array}\right.$

$i_{13})$ $\left\{\begin{array}{c} \text{there exist } R_0 > 1 \text{ and a nonnegative function } \hat{A}_i \\ \text{measurable on } \mathbf{R}^N \times [0,+\infty[\text{ such that} \\ \hat{A}_i \in L^\infty_{\text{loc}}((\mathbf{R}^N \backslash B_{R_0}) \times [0,+\infty[)\,, \\ \dfrac{\bar{A}_{i0}(x,t,w_i)}{A_{i0}(x,t,w_i)} \leq \hat{A}_i(x,t) \ \text{ for } \ |x| \geq R_0; \end{array}\right.$

$i_{14})$ $\widetilde{A}_i = \sup a_{i0} \left(b_0^{\frac{1}{q_i}} A_{i0} \right)^{-\frac{q_i}{q_i+1}} < +\infty \quad \text{if } \lambda_i > 0.$

The technique we use in this paper is based on the "test function" method ([8], [10], [12]). In order to construct the test functions, systematically, we use a function of the kind

$$\varphi(x,t) = \varphi_0\left(\frac{t^\mu}{R^\beta}\right) \varphi_0\left(\frac{|x|^\beta}{R^\beta}\right) \tag{1.2}$$

where $\varphi_0 = \psi_0^\nu$ with $\psi_0 \in C^1([0,+\infty[)$, $\psi_0(r) = 1$ for $0 \leq r \leq 1$, $0 < \psi_0(r) < 1$ for $1 < r < 2$, $\psi_0(r) = 0$ for $r \geq 2$; β, μ, ν are real parameters with $\beta > 2$ and $\mu, \nu > 1$

such that φ is C^1 in $R^N \times [0,+\infty[$. Additional conditions on ν will be specified when they arise. We will denote with ϑ the ratio $\dfrac{\beta}{\mu}$.

We point out that for the sake of brevity some times the dependence of a_i, A_i, etc. on x, t, ∇u_i, u_1, u_2 will not be made explicitly. Moreover, we will use relations in which only one index is present, for example "i", and others in which "i" and "j" are present. The first ones hold as $i = 1, 2$, the second ones with $i = 1$ and $j = 2$ and with $i = 2$ and $j = 1$.

The absence of weak solutions to problem (1.1) in wide functional classes has been studied without any assumption of regularity on the data.

By using special test functions, we obtain some properties of the weak solutions and inequalities which allow us to estimate suitable integrals and then to establish blow-up results. The nonexistence conditions of weak solutions are expressed in terms of lim inf of sums of integrals dependent on b_0 and $\hat{A}_i$ (Theorems 4.1, 5.1) and, under the assumptions of homogeneity on b_0 and $\hat{A}_i$, by algebraic inequalities (Theorems 4.2, 4.3, 4.4, 5.2, 5.3). The analysis has been possible in the following cases:

- $q_i > \max\{1, p_i - 1\}$ (sections 3 and 4),
- $p_1 = p_2 = 2$, $q_1 q_2 > 1$ and $\lambda_1 = \lambda_2 = 0$ (section 5).

In section 6, by using the same method, we get some results related to problem (1.1) (Theorems 6.1, 6.2, 6.3) when the dependence of b_i with respect to u_i "is not dumb". In fact, assumption i_{11}) is replaced by the following:

$$i'_{11})\quad \begin{cases} \text{there exist } b_0 \in L^1_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[) \\ \text{with } b_0 > 0 \text{ and } q_1, q_2 > 1 \text{ such that} \\ \quad b_1(x,t,z_1,z_2) \geq b_0(x,t) z_1 z_2^{q_2-1}, \\ \quad b_2(x,t,z_1,z_2) \geq b_0(x,t) z_1^{q_1-1} z_2 . \end{cases}$$

The results, completely new,obtained in this paper are compared to those of other Authors ([1], [2], [3], [7], [8], [9], [11], [13], [14]). The comparisons, except the one with [9], deal with the case $\lambda_1 = \lambda_2 = 0$ since in Literature there are no cases in which at least one of the parameters is positive.

2 Weak solutions

Let $\gamma_1, \gamma_2 \in [1,+\infty]$, $\alpha_1, \alpha_2 \in]-1, 0[$ and $u_{10}, u_{20} \in L^1_{\text{loc}}(\mathbf{R}^N)$ with $u_{i0} \geq 0$. Let $S(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ be the set of pairs $(u_1, u_2) \in (W^{1,1}_{\text{loc}}(\mathbf{R}^N \times]0,+\infty[))^2$ with $u_i \gneqq 0$ such that

$$\begin{aligned} &u_i\,, \ \frac{\partial u_i}{\partial t} \in L^1_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[)\,, \ \ |\nabla u_i| \in L^{\gamma_i}_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[)\,, \\ &u_i(x,0) = u_{i0}(x) \ \text{ a.e. on } \mathbf{R}^N\,, \\ &b_i(x,t,u_j) \in L^1_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[)\,, \end{aligned}$$

$$u_i^{\alpha_i-1}A_i(x,t,\nabla u_i)\cdot\nabla u_i \in L^1_{\rm loc}(\mathbf{R}^N\times[0,+\infty[)\,,$$
$$\hat{A}_i^{p_i-1}u_i^{\alpha_i+p_i-1}\in L^1_{\rm loc}((\mathbf{R}^N\backslash B_{R_0})\times[0,+\infty[).$$

For $(u_1,u_2)\in S(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$, let

$$\Omega(u_i)=\{(x,t)\in\mathbf{R}^N\times]0,+\infty[:\,u_i(x,t)>0\}\,,$$
$$\Omega_0(u_i)=\{(x,t)\in\mathbf{R}^N\times]0,+\infty[:\,u_i(x,t)=0\}$$

and let $\tau(u_1,u_2)$ be the set of pairs

$$(v_1,v_2)\in(W^{1,1}(\mathbf{R}^N\times]0,+\infty[)\cap L^\infty(\mathbf{R}^N\times]0,+\infty[))^2\,,$$

with $v_i\geq 0$ and $\operatorname{supp} v_i$ bounded such that

$$|\nabla v_i|\in L^{\gamma_i}(\mathbf{R}^N\times]0,+\infty[)\,,$$
$$A_i(x,t,\nabla u_i)\cdot\nabla v_i\in L^1(\mathbf{R}^N\times]0,+\infty[),$$
$$\lambda_i a_i(x,t,\nabla u_i)v_i\in L^1(\mathbf{R}^N\times]0,+\infty[)\ \ \text{if } \lambda_i>0.$$

We base our considerations on the following

Definition 2.1 A pair of functions $(u_1,u_2)\in S(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$ is called a *weak solution* to problem (1.1) if for any $(v_1,v_2)\in\tau(u_1,u_2)$ it follows that

$$\int_0^{+\infty}\int_{\mathbf{R}^N} b_i(x,t,u_j)v_i\,dxdt\leq$$
$$\leq\lambda_i\int_0^{+\infty}\int_{\mathbf{R}^N} a_i(x,t,\nabla u_i)v_i\,dxdt+$$
$$+\int_0^{+\infty}\int_{\mathbf{R}^N} A_i(x,t,\nabla u_i)\cdot\nabla v_i\,dxdt+\int_0^{+\infty}\int_{\mathbf{R}^N}\frac{\partial u_i}{\partial t}v_i\,dxdt\,.$$

Let us set

$$\begin{aligned}&\alpha_i\in]\max\{-1,1-p_i\},0[&&\text{if } \lambda_i=0,\\ &\alpha_i\in]\max\{-1,1-p_i,-q_i\},0[&&\text{if } \lambda_i>0.\end{aligned}\tag{2.1}$$

Proposition 2.1 *Under conditions* i_{11}) - i_{14})*, if* $(u_1,u_2)\in S(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$ *is a weak solution to (1.1), with* α_i *fixed as in (2.1), then:*

$$\Omega(u_1)=\Omega(u_2)\quad\textit{except for a set with measure equal to zero,}\tag{2.2}$$

$$b_0u_j^{q_j}u_i^{\alpha_i}\in L^1_{loc}(\mathbf{R}^N\times[0,+\infty[)\,,\tag{2.3}$$

$$\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} b_0 u_j^{q_j} u_i^{\alpha_i}\varphi dxdt +$$
$$+\frac{|\alpha_i|}{p_i(1+q_i)}\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R}\varphi u_i^{\alpha_i-1}A_{i0}|\nabla u_i|^{p_i}dxdt \le$$
$$\le \frac{(\lambda_i\widetilde{A}_i)^{1+q_i}}{1+q_i}\left(\frac{p_i}{|\alpha_i|}\right)^{q_i}\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} b_0 u_i^{\alpha_i+q_i}\varphi dxdt +$$
$$+\frac{1}{p_i|\alpha_i|^{p_i-1}}\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}\hat{A}_i^{p_i-1}u_i^{\alpha_i+p_i-1}\varphi^{1-p_i}|\nabla\varphi|^{p_i}dxdt +$$
$$-\frac{1}{\alpha_i+1}\int_{|x|\le 2^{\frac{1}{\beta}}R}u_{i0}^{\alpha_i+1}(x)\varphi(x,0)dx - \frac{1}{\alpha_i+1}\int_{R^\theta}^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R}u_i^{\alpha_i+1}\varphi_t dxdt$$
$$\forall\, R > R_0\,, \tag{2.4}$$

with $\nu > \max\{p_1, p_2\}$ in definition (1.2) of φ.

Proof. Let $u_{i\varepsilon} = u_i + \varepsilon$ $(0 < \varepsilon \le 1)$ and let us choose ν, R as above. Obviously

$$u_{i\varepsilon}^{\alpha_i}\varphi \in W^{1,1}(\mathbf{R}^N\times]0,+\infty[)\cap L^\infty(\mathbf{R}^N\times]0,+\infty[), \tag{2.5}$$

$$\varphi^{1-p_i}|\nabla\varphi|^{p_i} \in L^\infty(\operatorname{supp}|\nabla\varphi|)\,. \tag{2.6}$$

Let us note that

$$\lambda_i|a_i|u_{i\varepsilon}^{\alpha_i}\varphi \le \lambda_i a_{i0}|\nabla u_i|^{\frac{p_i q_i}{1+q_i}}u_{i\varepsilon}^{\alpha_i}\varphi \le$$
$$\le \frac{q_i}{1+q_i}\frac{|\alpha_i|}{p_i}\varphi u_{i\varepsilon}^{\alpha_i-1}A_{i0}|\nabla u_i|^{p_i} + \frac{1}{1+q_i}\left(\frac{p_i}{|\alpha_i|}\right)^{q_i}(\lambda_i\widetilde{A}_i)^{1+q_i}b_0 u_{i\varepsilon}^{q_i+\alpha_i}\varphi, \tag{2.7}$$

$$A_i\cdot\nabla(u_{i\varepsilon}^{\alpha_i}\varphi) = \alpha_i\varphi u_{i\varepsilon}^{\alpha_i-1}A_i\cdot\nabla u_i + u_{i\varepsilon}^{\alpha_i}A_i\cdot\nabla\varphi \tag{2.8}$$

and that for $R \le |x| \le 2^{\frac{1}{\beta}}R$

$$u_{i\varepsilon}^{\alpha_i}|A_i\cdot\nabla\varphi| \le u_{i\varepsilon}^{\alpha_i}|A_i||\nabla\varphi| \le \frac{(p_i-1)|\alpha_i|}{p_i}\varphi u_{i\varepsilon}^{\alpha_i-1}A_{i0}|\nabla u_i|^{p_i} + \frac{1}{p_i|\alpha_i|^{p_i-1}}\hat{A}_i^{p_i-1}u_{i\varepsilon}^{\alpha_i+p_i-1}\varphi^{1-p_i}|\nabla\varphi|^{p_i}. \tag{2.9}$$

We add that

$$u_{i\varepsilon}^{\alpha_i-1}A_{i0}|\nabla u_i|^{p_i} \le u_i^{\alpha_i-1}A_i\cdot\nabla u_i \in L^1_{\text{loc}}(\mathbf{R}^N\times[0,+\infty[) \tag{2.10}$$

and since (2.1) holds

$$b_0 u_{i\varepsilon}^{q_i+\alpha_i} \le b_0(u_i+1)^{q_i+\alpha_i} \in L^1_{\mathrm{loc}}(\mathbf{R}^N \times [0,+\infty[), \tag{2.11}$$

$$\hat{A}_i^{p_i-1} u_{i\varepsilon}^{\alpha_i+p_i-1} \le \hat{A}_i^{p_i-1}(u_i+1)^{\alpha_i+p_i-1} \in L^1_{\mathrm{loc}}((\mathbf{R}^N \backslash B_{R_0}) \times [0,+\infty[). \tag{2.12}$$

Relations (2.7), (2.10), (2.11) imply that

$$\lambda_i a_i u_{i\varepsilon}^{\alpha_i}\varphi \in L^1(\mathbf{R}^N \times]0,+\infty[);$$

from (2.8)–(2.10), (2.12), taking into account (2.6), we get

$$A_i \cdot \nabla(u_{i\varepsilon}^{\alpha_i}\varphi) \in L^1(\mathbf{R}^N \times]0,+\infty[).$$

Then, owing to (2.5), it results in:

$$(u_{1\varepsilon}^{\alpha_1}\varphi, u_{2\varepsilon}^{\alpha_2}\varphi) \in \tau(u_1,u_2)\,.$$

Assuming, in Def. 2.1, $v_i = u_{i\varepsilon}^{\alpha_i}\varphi$, we see that for $R > R_0$

$$\begin{aligned}
&\iint\limits_{\mathrm{supp}\,\varphi\cap\Omega(u_j)\cap\Omega(u_i)} b_0 u_j^{q_j} u_{i\varepsilon}^{\alpha_i}\varphi dxdt + \varepsilon^{\alpha_i} \iint\limits_{\mathrm{supp}\,\varphi\cap\Omega(u_j)\cap\Omega_0(u_i)} b_0 u_j^{q_j}\varphi dxdt + \\
&+\frac{|\alpha_i|}{p_i(1+q_i)} \iint\limits_{\mathrm{supp}\,\varphi} \varphi u_{i\varepsilon}^{\alpha_i-1} A_{i0}|\nabla u_i|^{p_i} dxdt \le \\
&\le \frac{(\lambda_i\widetilde{A}_i)^{1+q_i}}{1+q_i}\left(\frac{p_i}{|\alpha_i|}\right)^{q_i} \iint\limits_{\mathrm{supp}\,\varphi} b_0 u_{i\varepsilon}^{q_i+\alpha_i}\varphi dxdt + \\
&+\frac{1}{p_i|\alpha_i|^{p_i-1}} \iint\limits_{\mathrm{supp}|\nabla\varphi|} \hat{A}_i^{p_i-1} u_{1\varepsilon}^{\alpha_i+p_i-1}\varphi^{1-p_i}|\nabla\varphi|^{p_i} dxdt + \\
&-\frac{1}{\alpha_i+1}\int_{|x|\le 2^{\frac{1}{\beta}}R} (u_{i0}(x)+\varepsilon)^{\alpha_i+1}\varphi(x,0)dx - \frac{1}{\alpha_i+1}\iint\limits_{\mathrm{supp}\,\varphi_t} u_{i\varepsilon}^{\alpha_i+1}\varphi_t dxdt\,.
\end{aligned} \tag{2.13}$$

Since for $\varepsilon \to 0$ the right-hand side of (2.13) is convergent, necessarily

$$\mathrm{meas}(\mathrm{supp}\,\varphi \cap \Omega(u_j) \cap \Omega_0(u_i)) = 0.$$

Thus (2.2) holds. In addition, by (2.13) we get, for $\varepsilon \to 0$, relations (2.3) and (2.4) by virtue of Fatou's Lemma as well. We can broaden the analysis of the weak solutions to (1.1) by pointing out one class of them included in $S(p_1,p_2,\alpha_1,\alpha_2,u_{10},u_{20})$ for any α_i satisfying (2.1).

For $u_{10}, u_{20} \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, with $u_{i0} \ge 0$, let $S(p_1,p_2,u_{10},u_{20})$ be the set of pairs $(u_1,u_2) \in (W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N \times]0,+\infty[))^2$, with $u_i \gneqq 0$, such that

$$u_i,\ \frac{\partial u_i}{\partial t} \in L^1_{\mathrm{loc}}(\mathbf{R}^N \times [0,+\infty[), |\nabla u_i| \in L^{p_i}_{\mathrm{loc}}(\mathbf{R}^N \times [0,+\infty[)\,,$$

$$\begin{aligned}
&u_i(x,0) = u_{i0}(x) \quad \text{a.e. on } \mathbf{R}^N\,,\\
&b_i(x,t,u_j) \in L^1_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[)\,,\\
&\bar{A}_{i0}(x,t,\nabla u_i) \in L^\infty_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[),\\
&\hat{A}_i^{p_i-1} u_i^{p_i-1} \in L^1_{\text{loc}}((\mathbf{R}^N \backslash B_{R_0}) \times [0,+\infty[)
\end{aligned}$$

and moreover

$$\int_0^{+\infty}\int_{\mathbf{R}^N} b_i(x,t,u_j)v_i dxdt \le \lambda_i \int_0^{+\infty}\int_{\mathbf{R}^N} a_i(x,t,\nabla u_i)v_i dxdt + \\ + \int_0^{+\infty}\int_{\mathbf{R}^N} A_i(x,t,\nabla u_i)\cdot \nabla v_i dxdt + \int_0^{+\infty}\int_{\mathbf{R}^N} \frac{\partial u_i}{\partial t} v_i dxdt \tag{2.14}$$

for any $v_i \in C^1(\mathbf{R}^N \times [0,+\infty[)$ with $v_i \ge 0$ and $\operatorname{supp} v_i$ bounded.

Let us note that the integrals in (2.14) are defined, owing to the inequalities

$$\begin{aligned}
&|A_i(x,t,\nabla u_i)| \le \bar{A}_{i0}^{\frac{1}{p_i'}}(x,t,\nabla u_i)|\nabla u_i|^{p_i-1}\,,\\
&|a_i(x,t,\nabla u_i)| \le \frac{q_i}{1+q_i}\bar{A}_{i0}^{\frac{1}{p_i'}}(x,t,\nabla u_i)|\nabla u_i|^{p_i} + \frac{\widetilde{A}_i^{1+q_i}}{1+q_i} b_0(x,t)\,,
\end{aligned}$$

which imply that

$$|A_i(x,t,\nabla u_i)| \in L^{p_i'}_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[)\,,$$

$$|a_i(x,t,\nabla u_i)| \in L^1_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[)\,. \tag{2.15}$$

It is not difficult to verify that (2.14) holds with

$$\begin{aligned}
&v_i \in W^{1,1}(\mathbf{R}^N \times]0,+\infty[) \cap L^\infty(\mathbf{R}^N \times]0,+\infty[)\,, \quad v_i \ge 0\,,\\
&\operatorname{supp} v_i \text{ bounded}, \quad |\nabla v_i| \in L^{p_i}(\mathbf{R}^N \times]0,+\infty[).
\end{aligned} \tag{2.16}$$

Then if in (2.14), $v_i = u_{i\varepsilon}^{\alpha_i}\varphi$ (α_i, $u_{i\varepsilon}$ and φ as in proof of Prop. 2.1), for $R > R_0$:

$$\begin{aligned}
&|\alpha_i| \iint_{\operatorname{supp}\varphi} \varphi u_{i\varepsilon}^{\alpha_i-1} A_i \cdot \nabla u_i dxdt \le\\
&\le \frac{|\alpha_i|}{p_i}\left(\frac{q_i}{1+q_i} + p_i - 1\right) \iint_{\operatorname{supp}\varphi} \varphi u_{i\varepsilon}^{\alpha_i-1} A_{i0}|\nabla u_i|^{p_i} dxdt +\\
&+ \frac{(\lambda_i \widetilde{A}_i)^{1+q_i}}{1+q_i}\left(\frac{p_i}{|\alpha_i|}\right)^{q_i} \iint_{\operatorname{supp}\varphi} b_0 u_{i\varepsilon}^{q_i+\alpha_i}\varphi dxdt +\\
&+ \frac{1}{p_i|\alpha_i|^{p_i-1}} \iint_{\operatorname{supp}|\nabla\varphi|} \hat{A}_i^{p_i-1} u_{i\varepsilon}^{\alpha_i+p_i-1}\varphi^{1-p_i}|\nabla\varphi|^{p_i} dxdt +\\
&- \frac{1}{\alpha_i+1} \iint_{\operatorname{supp}\varphi_t} u_{i\varepsilon}^{\alpha_i+1}\varphi_t dxdt
\end{aligned} \tag{2.17}$$

from which we get

$$\frac{|\alpha_i|}{p_i(1+q_i)} \int\int_{\operatorname{supp}\varphi} \varphi u_{i\varepsilon}^{\alpha_i-1} A_{i0} |\nabla u_i|^{p_i} dxdt \leq$$
$$\frac{(\lambda_i \widetilde{A}_i)^{1+q_i}}{1+q_i} \left(\frac{p_i}{|\alpha_i|}\right)^{q_i} \int\int_{\operatorname{supp}\varphi} b_0 u_{i\varepsilon}^{q_i+\alpha_i} \varphi dxdt +$$
$$+\frac{1}{p_i |\alpha_i|^{p_i-1}} \int\int_{\operatorname{supp}|\nabla\varphi|} \hat{A}_i^{p_i-1} u_{i\varepsilon}^{\alpha_i+p_i-1} \varphi^{1-p_i} |\nabla\varphi|^{p_i} dxdt +$$
$$-\frac{1}{\alpha_i+1} \int\int_{\operatorname{supp}\varphi_t} u_{i\varepsilon}^{\alpha_i+1} \varphi_t dxdt\,. \tag{2.18}$$

Relation (2.18) and Fatou's Lemma imply that

$$\int\int_{\operatorname{supp}\varphi} \varphi u_i^{\alpha_i-1} A_{i0} |\nabla u_i|^{p_i} dxdt < +\infty\,;$$

consequently from (2.17) we get

$$\int\int_{\operatorname{supp}\varphi} \varphi u_i^{\alpha_i-1} A_i \cdot \nabla u_i dxdt < +\infty\,,$$

and thus $u_i^{\alpha_i-1} A_i \cdot \nabla u_i \in L^1_{\rm loc}(\mathbf{R}^N \times [0,+\infty[)$. Thus it follows that $(u_1,u_2) \in S(p_1,p_2,\alpha_1,\alpha_2,u_{10},u_{20})$. Since by (2.15), $\tau(u_1,u_2)$ is the class of pairs (v_1,v_2) with v_i as in (2.16), we can state the following:

Proposition 2.2 *Under conditions $i_{11})$–$i_{14})$, for $(u_1,u_2) \in S(p_1,p_2,u_{10},\, u_{20})$ we get:*

$(u_1,u_2) \in S(p_1,p_2,\alpha_1,\alpha_2,u_{10},u_{20})$ *for any α_i fixed according to (2.1),*
(u_1,u_2) *is a weak solution to (1.1).*

The next example highlights the existence of weak solutions to (1.1) which belong and do not belong to $S(p_1,p_2,u_{10},u_{20})$.

Let us consider the system

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \Delta u_1 + \lambda_1 |\nabla u_1|^{\frac{4}{3}} \geq u_2^{\frac{3}{2}} \\ &\qquad\qquad\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0,+\infty[\\ &\frac{\partial u_2}{\partial t} - \Delta u_2 + \lambda_2 |\nabla u_2|^{\frac{6}{5}} \geq u_1^2 \\ &u_i \gneqq 0 \end{aligned} \tag{2.19}$$

where $N > 5$.

Let $\psi \in C^1([0,+\infty[)$ such that

$$\psi(0) \geq 0\,, \quad \psi'(t) > 0 \ \ \forall\, t > 0\,, \quad \psi(t) \leq 3N - 15 \ \ \forall\, t > 0\,.$$

Let $u_1(x,t) = \psi(t)|x|^{-\frac{5}{2}}$, $u_2(x,t) = \psi(t)|x|^{-3}$ and $u_{i0}(x) = u_i(x,0)$. We have:

- (u_1, u_2) belongs to $S(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$, with $1 \leq \gamma_1 \leq \dfrac{2N}{7}$, $1 \leq \gamma_2 \leq \dfrac{N}{4}$, $\alpha_1, \alpha_2 \in]-1, 0[$, and it is a weak solution to the Cauchy's Problem related to system (2.19) with initial data (u_{10}, u_{20}) ;
- (u_1, u_2) belongs to $S(2, 2, u_{10}, u_{20})$ if and only if $N > 8$.

3 A priori estimates

In this section and in section 4 we study problem (1.1) with the additional hypothesis

$i_{31})$ $\qquad q_i > \max\{1, p_i - 1\}$.

Let

$$\alpha_0 = \max\left\{-1, 1 - p_1, (1-p_2)\frac{q_1}{q_2}, -\frac{q_1}{q_2}\right\}\,,$$

and let us choose

$$\alpha_1 \in]\alpha_0, 0[\,, \qquad \alpha_2 = \frac{q_2}{q_1}\alpha_1\,, \tag{3.1}$$

so that relations (2.1) also hold. Taking into account (2.2), (2.3) and Hölder's inequality, it is possible to estimate the right side of (2.4) through integrals in which $b_0 u_1^{q_1} u_2^{\alpha_2}\varphi$ and $b_0 u_2^{q_2} u_1^{\alpha_1}\varphi$ are present. It is convenient to introduce in advance the following notations:

$$X_{ij} = \int_0^{2^{\frac{1}{\mu}}R^\theta} \int_{|x| \leq 2^{\frac{1}{\beta}}R} b_0 u_i^{q_i} u_j^{\alpha_j} \varphi dx dt\,,$$

$$\widetilde{X}_{ij} = \int_0^{2^{\frac{1}{\mu}}R^\theta} \int_{R \leq |x| \leq 2^{\frac{1}{\beta}}R} b_0 u_i^{q_i} u_j^{\alpha_j} \varphi dx dt\,,$$

$$\widetilde{\widetilde{X}}_{ij} = \int_{R^\theta}^{2^{\frac{1}{\mu}}R^\theta} \int_{|x| \leq 2^{\frac{1}{\beta}}R} b_0 u_i^{q_i} u_j^{\alpha_j} \varphi dx dt$$

and for $R > R_0$, $1 < \omega < \infty$

$$I_0(\omega) = \int_{R^\theta}^{2^{\frac{1}{\mu}}R^\theta} \int_{|x| \leq 2^{\frac{1}{\beta}}R} b_0^{1-\omega} \varphi^{1-\omega} |\varphi_t|^\omega dx dt\,,$$

$$I_i(\omega) = \int_0^{2^{\frac{1}{\mu}}R^\theta} \int_{R \leq |x| \leq 2^{\frac{1}{\beta}}R} b_0^{1-\omega} \hat{A}_i^{\omega(p_i - 1)} \varphi^{1-\omega p_i} |\nabla\varphi|^{\omega p_i} dx dt\,.$$

Since by (3.1)

$$s = \frac{\alpha_1 - q_1}{\alpha_1} = \frac{\alpha_2 - q_2}{\alpha_2} > 1\,,$$

we get

$$\int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{|x|\le 2^{\frac{1}{\beta}} R} b_0 u_i^{q_i+\alpha_i} \varphi dxdt \le X_{ji}^{\frac{1}{s}} X_{ij}^{\frac{1}{s'}}\,.$$

Let

$$\rho_{i1} = \frac{\alpha_1\alpha_2 - q_1q_2}{\alpha_j(\alpha_i + p_i - 1)}\ , \quad \rho_{i2} = \frac{q_1q_2 - \alpha_1\alpha_2}{q_j(\alpha_i + p_i - 1)}\ , \quad \beta_1 = \frac{\alpha_i}{\rho_{i1}}\ , \quad \beta_2 = \frac{q_i}{\rho_{i2}}\,,$$
$$\sigma_{i1} = \frac{\alpha_1\alpha_2 - q_1q_2}{\alpha_j(\alpha_i + 1)}\ , \quad \sigma_{i2} = \frac{q_1q_2 - \alpha_1\alpha_2}{q_j(\alpha_i + 1)}\ , \quad \overline{\beta}_1 = \frac{\alpha_i}{\sigma_{i1}}\ , \quad \overline{\beta}_2 = \frac{q_i}{\sigma_{i2}}\,.$$

Noting (2.2) and the equalities

$$\beta_1 + \beta_2 = \alpha_i + p_i - 1\ , \quad \overline{\beta}_1 + \overline{\beta}_2 = \alpha_i + 1\,,$$

we have

$$u_i^{\alpha_i+p_i-1} = (u_i^{\beta_1}u_j^{\gamma})(u_i^{\beta_2}u_j^{-\gamma})\ , \quad u_i^{\alpha_i+1} = (u_i^{\overline{\beta}_1}u_j^{\overline{\gamma}})(u_i^{\overline{\beta}_2}u_j^{-\overline{\gamma}})\,,$$

where $\gamma = \frac{q_j}{\rho_{i1}} = -\frac{\alpha_j}{\rho_{i2}}$, $\overline{\gamma} = \frac{q_j}{\sigma_{i1}} = -\frac{\alpha_j}{\sigma_{i2}}$.
We add that, owing to (3.1),

$$\rho_{i1} > 1\ , \quad \rho_{i2} > 1\ , \quad \frac{1}{\rho_{i1}} + \frac{1}{\rho_{i2}} < 1\,,$$
$$\sigma_{i1} > 1\ , \quad \sigma_{i2} > 1\ , \quad \frac{1}{\sigma_{i1}} + \frac{1}{\sigma_{i2}} < 1\,;$$

thus with

$$\rho_{i3} = \frac{q_1q_2 - \alpha_1\alpha_2}{q_1q_2 + \alpha_j(p_i - 1) - q_j(\alpha_i + p_i - 1)}\,, \quad \sigma_{i3} = \frac{q_1q_2 - \alpha_1\alpha_2}{q_1q_2 + \alpha_j - q_j(\alpha_i + 1)}$$

it follows that:

$$\rho_{i3} > 1\,, \quad \frac{1}{\rho_{i1}} + \frac{1}{\rho_{i2}} + \frac{1}{\rho_{i3}} = 1\,,$$
$$\sigma_{i3} > 1\,, \quad \frac{1}{\sigma_{i1}} + \frac{1}{\sigma_{i2}} + \frac{1}{\sigma_{i3}} = 1\,.$$

Consequently for $0 \le t < 2^{\frac{1}{\mu}} R^\theta$, $|x| < 2^{\frac{1}{\beta}} R$, we have:

$$\begin{aligned} &\hat{A}_i^{p_i-1} u_i^{\alpha_i+p_i-1} \varphi^{1-p_i} |\nabla\varphi|^{p_i} = \\ &\left(b_0^{\frac{1}{\rho_{i1}}} u_i^{\beta_1} u_j^{\gamma} \varphi^{\frac{1}{\rho_{i1}}}\right) \left(b_0^{\frac{1}{\rho_{i2}}} u_i^{\beta_2} u_j^{-\gamma} \varphi^{\frac{1}{\rho_{i2}}}\right) \left(b_0^{\frac{1}{\rho_{i3}}-1} \hat{A}_i^{p_i-1} \varphi^{\frac{1}{\rho_{i3}}-p_i} |\nabla\varphi|^{p_i}\right)\,, \\ &u_i^{\alpha_i+1}|\varphi_t| = \left(b_0^{\frac{1}{\sigma_{i1}}} u_i^{\overline{\beta}_1} u_j^{\overline{\gamma}} \varphi^{\frac{1}{\sigma_{i1}}}\right) \left(b_0^{\frac{1}{\sigma_{i2}}} u_i^{\overline{\beta}_2} u_j^{-\overline{\gamma}} \varphi^{\frac{1}{\sigma_{i2}}}\right) \left(b_0^{\frac{1}{\sigma_{i3}}-1} \varphi^{\frac{1}{\sigma_{i3}}-1} |\varphi_t|\right)\,. \end{aligned} \tag{3.2}$$

At this point it is natural to make the assumption

$$i_{32})\quad \begin{cases} \text{there exists } \ \alpha_1 \in]\alpha_0, 0[\text{ such that} \\ b_0^{1-\rho_{i3}} \in L^1_{\text{loc}}((\mathbf{R}^N \backslash B_{R_0}) \times [0, +\infty[)\,, \ \ b_0^{1-\sigma_{i3}} \in L^1_{\text{loc}}(\mathbf{R}^N \times [R_0, +\infty[)\,. \end{cases}$$

In fact, if i_{32}) holds, by choosing in Definition (1.2) of φ, $\nu > \max_{i=1,2}\{p_i\rho_{i3}, \sigma_{i3}\}$, for $R > R_0$ we have:

$$I_0(\sigma_{i3}) < +\infty\,, \ \ I_i(\rho_{i3}) < +\infty\,.$$

Then, taking into account (2.3), by (3.2) we get

$$\int_0^{2^{\frac{1}{\mu}}R^\theta} \int_{R \le |x| \le 2^{\frac{1}{\beta}}R} \hat{A}_i^{p_i-1} u_i^{\alpha_i+p_i-1} \varphi^{1-p_i} |\nabla\varphi|^{p_i} dxdt \le \widetilde{X}_{ji}^{\frac{1}{\rho_{i1}}} \widetilde{X}_{ij}^{\frac{1}{\rho_{i2}}} (I_i(\rho_{i3}))^{\frac{1}{\rho_{i3}}}\,,$$

$$\int_{R^\theta}^{2^{\frac{1}{\mu}}R^\theta} \int_{|x| \le 2^{\frac{1}{\beta}}R} u_i^{\alpha_i+1} |\varphi_t| dxdt \le \widetilde{\widetilde{X}}_{ji}^{\frac{1}{\sigma_{i1}}} \widetilde{\widetilde{X}}_{ij}^{\frac{1}{\sigma_{i2}}} (I_0(\sigma_{i3}))^{\frac{1}{\sigma_{i3}}}\,.$$

Proposition 3.1 *Under conditions $i_{11}) - i_{14})$, $i_{31})$, $i_{32})$, if*

$$(u_1, u_2) \in S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20}),$$

with $\alpha_1 \in]\alpha_0, 0[$, is a weak solution to (1.1), then:

$$\begin{aligned} &X_{ji} + \frac{|\alpha_i|}{p_i(1+q_i)} \int_0^{2^{\frac{1}{\mu}}R^\theta} \int_{|x| \le 2^{\frac{1}{\beta}}R} \varphi u_i^{\alpha_i-1} A_{i0} |\nabla u_i|^{p_i} dxdt \le \\ &\le \frac{(\lambda_i \widetilde{A}_i)^{1+q_i}}{1+q_i} \left(\frac{p_i}{|\alpha_i|}\right)^{q_i} X_{ji}^{\frac{1}{s}} X_{ij}^{\frac{1}{s'}} + \frac{1}{p_i|\alpha_i|^{p_i-1}} \widetilde{X}_{ji}^{\frac{1}{\rho_{i1}}} \widetilde{X}_{ij}^{\frac{1}{\rho_{i2}}} (I_i(\rho_{i3}))^{\frac{1}{\rho_{i3}}} + \\ &+ \frac{1}{\alpha_i+1} \widetilde{\widetilde{X}}_{ji}^{\frac{1}{\sigma_{i1}}} \widetilde{\widetilde{X}}_{ij}^{\frac{1}{\sigma_{i2}}} (I_0(\sigma_{i3}))^{\frac{1}{\sigma_{i3}}} - \int_{|x| \le 2^{\frac{1}{\beta}}R} u_{i0}^{\alpha_i+1}(x)\varphi(x,0)dx \qquad \forall\, R > R_0\,, \end{aligned} \tag{3.3}$$

with $\nu > \max_{i=1,2}\{p_i\rho_{i3}, \sigma_{i3}\}$ in Definition (1.2) of φ.

The first consequences of Prop. 3.1 are two results. The first one gives upper limitation formulas for the integrals X_{ji} with terms which do not depend on u_1 and u_2. The second one, with additional conditions on b_0 and $\hat{A}_i$, expresses a property which the weak solutions to (1.1) have to satisfy. In fact, for

$$K(\lambda_1, \lambda_2, \alpha_1) = 1 - \sum_{h=1}^{2} \frac{(\lambda_h \widetilde{A}_h)^{1+q_h}}{1+q_h} \left(\frac{p_h}{|\alpha_h|}\right)^{q_h}\,,$$

the following propositions hold.

Proposition 3.2 *Under the conditions of Prop. 3.1, let*

$$(u_1, u_2) \in S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20}), \quad \textit{with } \alpha_1 \in]\alpha_0, 0[,$$

be a weak solution to (1.1). Then:

- *For* $\lambda_1 > 0$, $\lambda_2 > 0$ *and* $K(\lambda_1, \lambda_2, \alpha_1) > 0$, *we have:*

$$X_{ji} \leq cF(R) - \sum_{h=1}^{2} \int_{|x| \leq 2^{\frac{1}{\beta}} R} u_{h0}^{\alpha_h+1}(x)\varphi(x,0)dx \quad \forall R > R_0, \tag{3.4}$$

where c *is a positive constant independent of* R *and*

$$F(R) = \sum_{h=1}^{2} [I_0(\sigma_{h3}) + I_h(\rho_{h3})].$$

- *For* $\lambda_j = 0$ *we have:*

$$X_{ji} \leq c'_{ji} F_j(R) + c''_{ji} F_{ji}(R) - \int_{|x| \leq 2^{\frac{1}{\beta}} R} u_{i0}^{\alpha_i+1}(x)\varphi(x,0)dx \quad \forall R > R_0, \tag{3.5}$$

where c'_{ji}, c''_{ji} *are constants independent of* R, *with* $c'_{ji} = 0$ *if* $\lambda_i = 0$ *and* $c'_{ji} > 0$, *otherwise* $c''_{ji} > 0$ *and*

$$\begin{aligned}
F_j(R) &\equiv \left[(I_0(\sigma_{j3}))^{\frac{\sigma'_{j1}}{\sigma_{j3}}\left(\frac{\sigma_{j2}}{\sigma'_{j1}}\right)'} + (I_j(\rho_{j3}))^{\frac{\rho'_{j1}}{\rho_{j3}}\left(\frac{\rho_{j2}}{\rho'_{j1}}\right)'} \right], \\
F_{ji}(R) &\equiv \left[(I_j(\rho_{j3}))^{\frac{\rho'_{j1}\rho'_{i1}}{\rho_{j3}\rho_{i2}}} (I_i(\rho_{i3}))^{\frac{\rho'_{i1}}{\rho_{i3}}} \right]^{\left(\frac{\rho_{j2}\rho_{i2}}{\rho'_{j1}\rho'_{i1}}\right)'} + \\
&+ \left[(I_0(\sigma_{j3}))^{\frac{\sigma'_{j1}\rho'_{i1}}{\sigma_{j3}\rho_{i2}}} (I_i(\rho_{i3}))^{\frac{\rho'_{i1}}{\rho_{i3}}} \right]^{\left(\frac{\rho_{i2}\sigma_{j2}}{\rho'_{i1}\sigma'_{j1}}\right)'} + \\
&+ \left[(I_j(\rho_{j3}))^{\frac{\rho'_{j1}\sigma'_{i1}}{\rho_{j3}\sigma_{i2}}} (I_0(\sigma_{i3}))^{\frac{\sigma'_{i1}}{\sigma_{i3}}} \right]^{\left(\frac{\rho_{j2}\sigma_{i2}}{\rho'_{j1}\sigma'_{i1}}\right)'} + \\
&+ \left[(I_0(\sigma_{j3}))^{\frac{\sigma'_{j1}\sigma'_{i1}}{\sigma_{j3}\sigma_{i2}}} (I_0(\sigma_{i3}))^{\frac{\sigma'_{i1}}{\sigma_{i3}}} \right]^{\left(\frac{\sigma_{i2}\sigma_{j2}}{\sigma'_{i1}\sigma'_{j1}}\right)'}.
\end{aligned}$$

Proof. About (3.4), let $X = \max\{X_{12}, X_{21}\}$. By (3.3) we get that for $R > R_0$:

$$X_{21} \leq \frac{(\lambda_1 \widetilde{A}_1)^{1+q_1}}{1+q_1} \left(\frac{p_1}{|\alpha_1|}\right)^{q_1} X +$$

$$
\begin{aligned}
&\quad +c_1\left[X^{1-\frac{1}{\rho_{13}}}(I_1(\rho_{13}))^{\frac{1}{\rho_{13}}}+X^{1-\frac{1}{\sigma_{13}}}(I_0(\sigma_{13}))^{\frac{1}{\sigma_{13}}}\right]+\\
&\quad -\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}(x)\varphi(x,0)dx\,,\\
X_{12} &\le \frac{(\lambda_2\widetilde{A}_2)^{1+q_2}}{1+q_2}\left(\frac{p_2}{|\alpha_2|}\right)^{q_2}X+\\
&\quad +c_2\left[X^{1-\frac{1}{\rho_{23}}}(I_2(\rho_{23}))^{\frac{1}{\rho_{23}}}+X^{1-\frac{1}{\sigma_{23}}}(I_0(\sigma_{23}))^{\frac{1}{\sigma_{23}}}\right]+\\
&\quad -\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{20}^{\alpha_2+1}(x)\varphi(x,0)dx\,,
\end{aligned}
$$

where c_1, c_2 are positive constants independent of R. Then:

$$
\begin{aligned}
K(\lambda_1,\lambda_2,\alpha_1)X &\le \sum_{h=1}^{2}c_h\left[X^{\frac{1}{\rho'_{h3}}}(I_h(\rho_{h3}))^{\frac{1}{\rho_{h3}}}+X^{\frac{1}{\sigma'_{h3}}}(I_0(\sigma_{h3}))^{\frac{1}{\sigma_{h3}}}\right]+\\
&\quad -\sum_{h=1}^{2}\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{h0}^{\alpha_h+1}(x)\varphi(x,0)dx
\end{aligned}
$$

from which, owing to Young's inequality, the thesis follows.

About (3.5), referring, for example, to the case $\lambda_2=0$, relation (3.3) still allows that for $R>R_0$:

$$
\begin{aligned}
X_{21} &\le cX_{21}^{\frac{1}{s}}X_{12}^{\frac{1}{s'}}+c_1\left[\widetilde{X}_{21}^{\frac{1}{\rho_{11}}}\widetilde{X}_{12}^{\frac{1}{\rho_{12}}}(I_1(\rho_{13}))^{\frac{1}{\rho_{13}}}+\widetilde{\widetilde{X}}_{21}^{\frac{1}{\sigma_{11}}}\widetilde{\widetilde{X}}_{12}^{\frac{1}{\sigma_{12}}}(I_0(\sigma_{13}))^{\frac{1}{\sigma_{13}}}\right]+\\
&\quad -\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}(x)\varphi(x,0)dx\,,\\
X_{12} &\le c_2\left[\widetilde{X}_{12}^{\frac{1}{\rho_{21}}}\widetilde{X}_{21}^{\frac{1}{\rho_{22}}}(I_2(\rho_{23}))^{\frac{1}{\rho_{23}}}+\widetilde{\widetilde{X}}_{12}^{\frac{1}{\sigma_{21}}}\widetilde{\widetilde{X}}_{21}^{\frac{1}{\sigma_{22}}}(I_0(\sigma_{23}))^{\frac{1}{\sigma_{23}}}\right], \qquad (3.6)
\end{aligned}
$$

where c, c_1, c_2 are constants independent of R, with $c=0$ if $\lambda_1=0$ and $c>0$ otherwise, $c_1>0$ and $c_2>0$. By using Young's inequality, by (3.6) we get the inequalities:

$$
\begin{aligned}
X_{21} &\le \bar{c}X_{12}+\bar{c}_1\left[\widetilde{X}_{12}^{\frac{\rho'_{11}}{\rho_{12}}}(I_1(\rho_{13}))^{\frac{\rho'_{11}}{\rho_{13}}}+\widetilde{\widetilde{X}}_{12}^{\frac{\sigma'_{11}}{\sigma_{12}}}(I_0(\sigma_{13}))^{\frac{\sigma'_{11}}{\sigma_{13}}}\right]+\\
&\quad -\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}(x)\varphi(x,0)dx\,, \qquad (\bar{c}=0 \text{ if } \lambda_2=0) \qquad (3.7)\\
X_{12} &\le \bar{c}_2\left[\widetilde{X}_{21}^{\frac{\rho'_{21}}{\rho_{22}}}(I_2(\rho_{23}))^{\frac{\rho'_{21}}{\rho_{23}}}+\widetilde{\widetilde{X}}_{21}^{\frac{\sigma'_{21}}{\sigma_{22}}}(I_0(\sigma_{23}))^{\frac{\sigma'_{21}}{\sigma_{23}}}\right]. \qquad (3.8)
\end{aligned}
$$

By substituting the right side of (3.8) in (3.7), we get:

$$
\begin{aligned}
X_{21} \quad &\leq \quad \overline{\overline{c}} \left[\widetilde{X}_{21}^{\frac{\rho'_{21}}{\rho_{22}}} (I_2(\rho_{23}))^{\frac{\rho'_{21}}{\rho_{23}}} + \widetilde{\widetilde{X}}_{21}^{\frac{\sigma'_{21}}{\sigma_{22}}} (I_0(\sigma_{23}))^{\frac{\sigma'_{21}}{\sigma_{23}}} \right] + \\
&+ \overline{\overline{c}}_1 \left[\left(\widetilde{X}_{21}^{\frac{\rho'_{21}}{\rho_{22}}} (I_2(\rho_{23}))^{\frac{\rho'_{21}}{\rho_{23}}} + \widetilde{\widetilde{X}}_{21}^{\frac{\sigma'_{21}}{\sigma_{22}}} (I_0(\sigma_{23}))^{\frac{\sigma'_{21}}{\sigma_{23}}} \right)^{\frac{\rho'_{11}}{\rho_{12}}} (I_1(\rho_{13}))^{\frac{\rho'_{11}}{\rho_{13}}} + \right. \\
&+ \left. \left(\widetilde{X}_{21}^{\frac{\rho'_{21}}{\rho_{22}}} (I_2(\rho_{23}))^{\frac{\rho'_{21}}{\rho_{23}}} + \widetilde{\widetilde{X}}_{21}^{\frac{\sigma'_{21}}{\sigma_{22}}} (I_0(\sigma_{23}))^{\frac{\sigma'_{21}}{\sigma_{23}}} \right)^{\frac{\sigma'_{11}}{\sigma_{12}}} (I_0(\sigma_{13}))^{\frac{\sigma'_{11}}{\sigma_{13}}} \right] + \\
&- \int_{|x| \leq 2^{\frac{1}{\beta}} R} u_{10}^{\alpha_1+1}(x) \varphi(x,0) dx \,. \qquad (\overline{\overline{c}} = 0 \text{ if } \lambda_2 = 0) \qquad (3.9)
\end{aligned}
$$

A simple calculation shows that

$$
\frac{\rho_{22}}{\rho'_{21}} > 1 \,, \quad \frac{\sigma_{22}}{\sigma'_{21}} > 1 \,, \quad \frac{\rho_{12}}{\rho'_{11}} > 1 \,, \quad \frac{\sigma_{12}}{\sigma'_{11}} > 1 \,.
$$

By repeatedly applying Young's inequality in (3.9), the thesis is proved.

Proposition 3.3 *Let assumptions* $i_{11}) - i_{14})$, $i_{31})$ *hold with* $\inf b_0 > 0$ *and* $\hat{A}_i \in L^\infty((\mathbf{R}^N \backslash B_{R_0}) \times]0, +\infty[)$. *If* $(u_1, u_2) \in S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20})$, *with* $\alpha_1 \in]\alpha_0, 0[$, *is a weak solution to (1.1), then:*

$$
\begin{aligned}
&\inf u_j > 0 \Longrightarrow \sup u_i = +\infty \quad \textit{when} \quad K(\lambda_1, \lambda_2, \alpha_1) > 0 \,, \\
&\inf u_2 > 0 \Longrightarrow \sup u_1 = +\infty \quad \textit{when} \quad \lambda_1 = 0 \quad \textit{and} \quad \sup b_0 < +\infty \,, \\
&\inf u_1 > 0 \Longrightarrow \sup u_2 = +\infty \quad \textit{when} \quad \lambda_2 = 0 \quad \textit{and} \quad \sup b_0 < +\infty \,.
\end{aligned}
$$

Proof. The conditions on b_0 and $\hat{A}_i$ imply that

$$
X_{ji} \geq c' (\inf u_j)^{q_j} (\sup u_i)^{\alpha_i} R^{N+\theta} \,, \;\; X_{ij} \leq c'' (\sup u_i)^{q_i} (\inf u_j)^{\alpha_j} R^{N+\theta} \,,
$$

$$
I_0(\sigma_{i3}) \leq c_0 R^{N+\theta-\theta\sigma_{i3}} \,, \;\; I_i(\rho_{i3}) \leq c_i R^{N+\theta-p_i\rho_{i3}} \,,
$$

where c', c'', c_0, c_i are positive constants independent of R. Taking into account (3.3), in the case $K(\lambda_1, \lambda_2, \alpha_1) > 0$ we have:

$$
(\inf u_j)^{q_j} (\sup u_i)^{\alpha_i} \leq c[R^{-\theta\sigma_{13}} + R^{-\theta\sigma_{23}} + R^{-p_1\rho_{13}} + R^{-p_2\rho_{23}}]
$$

$$
\forall\, R > R_0 \,. \qquad (c = \text{const.} > 0 \,\text{indep. of}\, R)
$$

In the case $\lambda_i = 0$, taking into account that by (3.3)

$$
X_{ji} \leq d \left[X_{ij}^{\frac{\rho'_{i1}}{\rho_{i2}}} (I_i(\rho_{i3}))^{\frac{\rho'_{i1}}{\rho_{i3}}} + X_{ij}^{\frac{\sigma'_{i1}}{\sigma_{i2}}} (I_0(\sigma_{i3}))^{\frac{\sigma'_{i1}}{\sigma_{i3}}} \right]
$$

$$
(d = \text{const.} > 0 \text{ indep. of } R)
$$

and that

$$(N+\theta)\frac{\rho'_{i1}}{\rho_{i2}} + (N+\theta-p_i\rho_{i3})\frac{\rho'_{i1}}{\rho_{i3}} = N+\theta-p_i\rho'_{i1}\,,$$
$$(N+\theta)\frac{\sigma'_{i1}}{\sigma_{i2}} + (N+\theta-\theta\sigma_{i3})\frac{\sigma'_{i1}}{\sigma_{i3}} = N+\theta-\theta\sigma'_{i1}\,,$$

we get:

$$\begin{aligned}(\inf u_j)^{q_j}(\sup u_i)^{\alpha_i} \quad &\leq \quad d'[R^{-p_i\rho'_{i1}}((\sup u_i)^{q_i}(\inf u_j)^{\alpha_j})^{\frac{\rho'_{i1}}{\rho_{i3}}} + \\ &+R^{-\theta\sigma'_{i1}}((\sup u_i)^{q_i}(\inf u_j)^{\alpha_j})^{\frac{\sigma'_{i1}}{\sigma_{i3}}}] \\ &\qquad \forall\, R > R_0\,. \qquad (d' = \text{const.} > 0 \ \text{ indep. of}\, R)\end{aligned}$$

Consequently, if $\sup u_i < +\infty$, in both cases it would be $(\inf u_j)^{q_j}(\sup u_i)^{\alpha_i} = 0$.

Remark 3.1 Prop. 3.3 excludes the existence of weak solutions (u_1, u_2) to (1.1) with either $\inf u_1 > 0$ and $u_2 \in L^\infty(\mathbf{R}^N \times]0, +\infty[)$ or $u_1 \in L^\infty(\mathbf{R}^N \times\ ]0, +\infty[)$ and $\inf u_2 > 0$.

4 General results for nonexistence of solutions

Relations (3.3), (3.4), (3.5) allow us to prove the following nonexistence theorem.

Theorem 4.1 *Under the assumptions of Prop. 3.1, problem (1.1) has no weak solutions belonging to* $S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20})$ *with* $\alpha_1 \in]\alpha_0, 0[$ *in the following cases:*

- $\lambda_1 > 0\,,\ \ \lambda_2 > 0$ *and* $K(\lambda_1, \lambda_2, \alpha_1) > 0$ *if* $\lim_{R\to+\infty} \inf F(R) < +\infty$;
- $\lambda_1 > 0$ *and* $\lambda_2 = 0$ *if* $\lim_{R\to+\infty} \inf\,[c'_{21}F_2(R) + c''_{21}F_{21}(R)] < +\infty$;
- $\lambda_1 = 0$ *and* $\lambda_2 > 0$ *if* $\lim_{R\to+\infty} \inf\,[c'_{12}F_1(R) + c''_{12}F_{12}(R)] < +\infty$;
- $\lambda_1 = \lambda_2 = 0$ *if either* $\lim_{R\to+\infty} \inf\, F_{21}(R) < +\infty$ *or* $\lim_{R\to+\infty} \inf\, F_{12}(R) < +\infty$.

Proof. Arguing by contradiction, let $(u_1, u_2) \in S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20})$ be a weak solution to problem (1.1). In the case $\lambda_1 > 0$, $\lambda_2 > 0$ and $K(\lambda_1, \lambda_2, \alpha_1) > 0$, the condition on F assures that there exist $\delta > 0$ and a sequence $\{R_n\}$ positively diverging, with $R_n > R_0$, such that $F(R_n) \leq \delta$. Since

$$\int_0^{R_n^\theta}\int_{|x|\leq R_n} b_0 u_j^{q_j} u_i^{\alpha_i}\,dxdt \leq X_{ji}(R_n)\,,$$

by (3.4), we get

$$b_0 u_j^{q_j} u_i^{\alpha_i} \in L^1(\mathbf{R}^N \times]0, +\infty[)$$

and thus

$$\lim \widetilde{X}_{ji}(R_n) = 0\,, \qquad \lim \widetilde{\widetilde{X}}_{ji}(R_n) = 0\,.$$

Hence by (3.3) it follows that

$$\int_0^{+\infty} \int_{\mathbf{R}^N} b_0 u_j^{q_j} u_i^{\alpha_i} dxdt = 0$$

from which $u_1 \equiv 0$ and $u_2 \equiv 0$ owing to (2.2).

In the case $\lambda_1 > 0$ and $\lambda_2 = 0$, the condition on F_2 and F_{21} implies that there exist $\delta > 0$ and a sequence $\{R_n\}$ positively diverging, with $R_n > R_0$, such that

$$c'_{21} F_2(R_n) + c''_{21} F_{21}(R_n) \leq \delta\,. \tag{4.1}$$

Relations (3.5) and (4.1) imply that

$$b_0 u_2^{q_2} u_1^{\alpha_1} \in L^1(\mathbf{R}^N \times]0, +\infty[).$$

Hence

$$\lim \widetilde{X}_{21}(R_n) = 0\,, \quad \lim \widetilde{\widetilde{X}}_{21}(R_n) = 0\,.$$

Relation (4.1) implies again that the terms depending on R_n:

$$I_0(\sigma_{23})\,, \qquad I_2(\rho_{23})$$

are bounded. We conclude, owing to (3.3), that

$$\int_0^{+\infty} \int_{\mathbf{R}^N} b_0 u_1^{q_1} u_2^{\alpha_2} dxdt = 0\,.$$

The same reasoning can be followed in the remaining two cases.

Corollary 4.1 *Under the assumptions of Prop. 3.1, the conclusions of Theor. 4.1 hold if*

$$b_0^{1-\rho_{i3}} \hat{A}_i^{\rho_{i3}(p_i-1)} \in L^1((\mathbf{R}^N \backslash B_{R_0}) \times]0, +\infty[) \quad \textit{and} \quad b_0^{1-\sigma_{i3}} \in L^1(\mathbf{R}^N \times]R_0, +\infty[)\,.$$

Proof. In fact, for $R > \max\{R_0, R_0^{1/\theta}\}$ we get:

$$I_0(\sigma_{i3}) = I_0(\sigma_{i3}, R) \leq c \int_{R_0}^{+\infty} \int_{\mathbf{R}^N} b_0^{1-\sigma_{i3}} dxdt\,,$$
$$I_i(\rho_{i3}) = I_i(\rho_{i3}, R) \leq c \int_0^{+\infty} \int_{|x| \geq R_0} b_0^{1-\rho_{i3}} \hat{A}_i^{\rho_{i3}(p_i-1)} dxdt\,,$$

where c is a positive constant independent of R.

In order to obtain nonexistence conditions expressed by inequalities, we make the hypotheses:

$$i_{41})\quad \begin{cases} \displaystyle\int_1^2 \int_{|y|\le 2} (b_0(y,\tau))^{1-q_i'}\,dyd\tau < +\infty\,, \quad \hat{A}_i \in L^\infty_{\text{loc}}(\mathbf{R}^N \times [0,+\infty[)\,, \\ \text{there exist } L_0(\theta)\in\mathbf{R} \text{ and } L_i(\theta)\ge 0 \text{ such that for } R>R_0 \\ b_0(Ry,R^\theta\tau) \ge b_0(y,\tau)R^{L_0(\theta)}\,, \quad \hat{A}_i(Ry,R^\theta\tau)\le \hat{A}_i(y,\tau)R^{L_i(\theta)}\,; \end{cases}$$

$$i_{42})\quad \int_0^2 \int_{1\le|y|\le 2} (b_0(y,\tau))^{1-\frac{q_i}{q_i-p_i+1}}\,dyd\tau < +\infty\,.$$

We note in advance that since

$$\rho_{i3} < \frac{q_i}{q_i-p_i+1}\,, \quad \sigma_{i3} < \frac{q_i}{q_i-1}\,,$$

assumption i_{32}) is fulfilled for any $\alpha_1 \in]\alpha_0, 0[$. Hence, by choosing

$$\nu > \max_{i=1,2}\left\{\frac{p_iq_i}{q_i-p_i+1}\,,\ \frac{q_i}{q_i-1}\right\}$$

in Definition (1.2) of φ, the conclusions of Prop. 3.1 and the ones of Prop. 3.2 hold with any $\alpha_1 \in]\alpha_0, 0[$. This fact allows us to create suitable conditions in order to ensure that for $|\alpha_1|$ sufficiently small, the functions $F(R)$, $F_j(R)$ and $F_{ji}(R)$ are infinitesimal for $R \to +\infty$. In fact, by i_{41}), i_{42}), on one hand, we have:

$$I_0(\sigma_{i3}) \le k_{0i}R^{M_0(\sigma_{i3},\theta)}\,,\ I_i(\rho_{i3}) \le k_iR^{M_i(\rho_{i3},\theta)}\,, \tag{4.2}$$

where k_{0i}, k_i are positive constants independent of R and

$$\begin{aligned} M_0(\sigma_{i3},\theta) &= N + (1-\sigma_{i3})\theta + (1-\sigma_{i3})L_0(\theta)\,, \\ M_i(\rho_{i3},\theta) &= N + \theta - p_i\rho_{i3} + (1-\rho_{i3})L_0(\theta) + \rho_{i3}(p_i-1)L_i(\theta)\,. \end{aligned}$$

On the other hand, for

$$\begin{aligned} B_i &= \frac{N+\theta+L_0(\theta)}{q_i}(q_i-p_i+1) - p_i + (p_i-1)L_i(\theta) - L_0(\theta)\,, \\ C_i &= \frac{N(q_i-1)-\theta-L_0(\theta)}{q_i}\,, \end{aligned}$$

we have:

$$\begin{aligned} & M_0(\sigma_{i3},\theta) \underset{\alpha_1\to 0}{\longrightarrow} \frac{q_i}{q_i-1}C_i\,, \\ & M_i(\rho_{i3},\theta) \underset{\alpha_1\to 0}{\longrightarrow} \frac{q_i}{q_i-p_i+1}B_i\,, \\ & M_0(\sigma_{j3},\theta)\frac{\sigma_{j1}'}{\sigma_{j3}}\left(\frac{\sigma_{j2}}{\sigma_{j1}'}\right)' \underset{\alpha_1\to 0}{\longrightarrow} \frac{q_j}{q_j-1}C_j\,, \\ & M_j(\rho_{j3},\theta)\frac{\rho_{j1}'}{\rho_{j3}}\left(\frac{\rho_{j2}}{\rho_{j1}'}\right)' \underset{\alpha_1\to 0}{\longrightarrow} \frac{q_j}{q_j-p_j+1}B_j\,, \end{aligned} \tag{4.3}$$

and moreover

$$\left[M_j(\rho_{j3},\theta)\frac{\rho'_{j1}\rho'_{i1}}{\rho_{j3}\rho_{i2}}+M_i(\rho_{i3},\theta)\frac{\rho'_{i1}}{\rho_{i3}}\right]\left(\frac{\rho_{j2}\rho_{i2}}{\rho'_{j1}\rho'_{i1}}\right)'\underset{\alpha_1\to 0}{\longrightarrow}$$
$$\underset{\alpha_1\to 0}{\longrightarrow}\left[\frac{p_i-1}{q_i}B_j+B_i\right]\frac{q_1q_2}{q_1q_2-(p_1-1)(p_2-1)}\,,$$
$$\left[M_0(\sigma_{j3},\theta)\frac{\sigma'_{j1}\rho'_{i1}}{\sigma_{j3}\rho_{i2}}+M_i(\rho_{i3},\theta)\frac{\rho'_{i1}}{\rho_{i3}}\right]\left(\frac{\sigma_{j2}\rho_{i2}}{\sigma'_{j1}\rho'_{i1}}\right)'\underset{\alpha_1\to 0}{\longrightarrow}$$
$$\underset{\alpha_1\to 0}{\longrightarrow}\left[\frac{p_i-1}{q_i}C_j+B_i\right]\frac{q_1q_2}{q_1q_2-p_i+1}\,,$$
$$\left[M_j(\rho_{j3},\theta)\frac{\sigma'_{i1}\rho'_{j1}}{\sigma_{i2}\rho_{j3}}+M_0(\sigma_{i3},\theta)\frac{\sigma'_{i1}}{\sigma_{i3}}\right]\left(\frac{\sigma_{i2}\rho_{j2}}{\sigma'_{i1}\rho'_{j1}}\right)'\underset{\alpha_1\to 0}{\longrightarrow}$$
$$\underset{\alpha_1\to 0}{\longrightarrow}\left[\frac{1}{q_i}B_j+C_i\right]\frac{q_1q_2}{q_1q_2-p_j+1}\,,$$
$$\left[M_0(\sigma_{j3},\theta)\frac{\sigma'_{j1}\sigma'_{i1}}{\sigma_{j3}\sigma_{i2}}+M_0(\sigma_{i3},\theta)\frac{\sigma'_{i1}}{\sigma_{i3}}\right]\left(\frac{\sigma_{i2}\sigma_{j2}}{\sigma'_{i1}\sigma'_{j1}}\right)'\underset{\alpha_1\to 0}{\longrightarrow}$$
$$\underset{\alpha_1\to 0}{\longrightarrow}\left[\frac{1}{q_i}C_j+C_i\right]\frac{q_1q_2}{q_1q_2-1}\,. \tag{4.4}$$

The natural consequence of inequalities (3.4), (3.5), (4.2)-(4.4) is the following theorem:

Theorem 4.2 *Under assumptions $i_{11})-i_{14})$, $i_{31})$, $i_{41})$, $i_{42})$, problem (1.1) has no weak solutions belonging to $S(\gamma_1,\gamma_2,\alpha_1,q_2\alpha_1/q_1,u_{10},u_{20})$ with $\alpha_1\in]\alpha_0,0[$ and $|\alpha_1|$ sufficiently small in the following cases:*

- $\lambda_1>0$, $\lambda_2>0$ *and* $K(\lambda_1,\lambda_2,\alpha_1)>0$ *if there exists* $\theta>0$ *such that*
$$\max_{i=1,2}\{C_i,B_i\}<0\,;$$
- $\lambda_1>0$ *and* $\lambda_2=0$ *if there exists* $\theta>0$ *such that*
$$\max\{C_2,B_2\}<0\,,$$
$$\max\left\{\frac{p_1-1}{q_1}B_2+B_1\,,\ \frac{p_1-1}{q_1}C_2+B_1\,,\ \frac{1}{q_1}B_2+C_1\,,\ \frac{1}{q_1}C_2+C_1\right\}<0\,;$$
- $\lambda_1=0$ *and* $\lambda_2>0$ *if there exists* $\theta>0$ *such that*
$$\max\{C_1,B_1\}<0\,,$$
$$\max\left\{\frac{p_2-1}{q_2}B_1+B_2\,,\ \frac{p_2-1}{q_2}C_1+B_2\,,\ \frac{1}{q_2}B_1+C_2\,,\ \frac{1}{q_2}C_1+C_2\right\}<0\,;$$

- $\lambda_1 = \lambda_2 = 0$ *if there exists* $\theta > 0$ *such that either*

$$\max\left\{\frac{p_1-1}{q_1}B_2 + B_1\,,\ \frac{p_1-1}{q_1}C_2 + B_1\,,\ \frac{1}{q_1}B_2 + C_1\,,\ \frac{1}{q_1}C_2 + C_1\right\} < 0$$

or

$$\max\left\{\frac{p_2-1}{q_2}B_1 + B_2\,,\ \frac{p_2-1}{q_2}C_1 + B_2\,,\ \frac{1}{q_2}B_1 + C_2\,,\ \frac{1}{q_2}C_1 + C_2\right\} < 0\,.$$

Initial data u_{10} and u_{20} have not been considered up to now, even if they are present in inequalities (3.4) and (3.5). However these data can influence the nonexistence of the solutions when they have a convenient asymptotic behaviour for $|x| \to \infty$. Let us introduce the hypothesis:

$$i_{43})\quad \begin{cases} \text{there exist } i \in \{1,2\},\ c_i > 0 \ \text{ and } \ s_i > -N \ \text{ such that} \\ u_{i0}(x) \geq c_i|x|^{s_i} \ \text{ for } \ |x| \text{ large} \end{cases}$$

and we set

$$\begin{aligned}
N_j(\theta) &= \max\left\{\frac{q_j}{q_j-1}C_j\,,\ \frac{q_j}{q_j-p_j+1}B_j\right\}\,,\\
N_0(\theta) &= \max\{N_1(\theta), N_2(\theta)\}\,,\\
N_{ji}(\theta) &= \max\left\{\left[\frac{p_i-1}{q_i}B_j + B_i\right]\frac{q_1q_2}{q_1q_2-(p_1-1)(p_2-1)}\,,\right.\\
&\qquad \left[\frac{p_i-1}{q_i}C_j + B_i\right]\frac{q_1q_2}{q_1q_2-p_i+1}\,,\\
&\qquad \left.\left[\frac{1}{q_i}B_j + C_i\right]\frac{q_1q_2}{q_1q_2-p_j+1}\,,\ \left[\frac{1}{q_i}C_j + C_i\right]\frac{q_1q_2}{q_1q_2-1}\right\}\,.
\end{aligned}$$

Assumption $i_{43})$ implies that for R large

$$\begin{aligned}
&\int_{|x|\leq 2^{\frac{1}{\beta}}R} u_{i0}^{\alpha_i+1}(x)\varphi(x,0)dx \geq \int_{R\backslash 2\leq |x|\leq R} u_{i0}^{\alpha_i+1}(x)dx \geq \\
&\geq c_i^{\alpha_i+1}\int_{R\backslash 2\leq |x|\leq R} |x|^{s_i(\alpha_i+1)}dx = c_i^{\alpha_i+1}R^{N+s_i(\alpha_i+1)}\int_{1\backslash 2\leq |y|\leq 1} |y|^{s_i(\alpha_i+1)}dy\,.
\end{aligned}$$

Hence, taking into account relations (3.4), (3.5), (4.2)–(4.4), the following theorem holds.

Theorem 4.3 *Under assumptions* $i_{11}) - i_{14})$, $i_{31})$, $i_{41}) - i_{43})$, *problem (1.1) has no weak solutions belonging to* $S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20})$ *with* $\alpha_1 \in]\alpha_0, 0[$ *and* $|\alpha_1|$ *sufficiently small in the following cases:*

- $\lambda_1 > 0$, $\lambda_2 > 0$ *and* $K(\lambda_1, \lambda_2, \alpha_1) > 0$ *if there exists* $\theta > 0$ *such that* $N_0(\theta) < N + s_i$*;*

- $\lambda_1 > 0$ *and* $\lambda_2 = 0$ *if* $i_{43})$ *holds with* $i = 1$ *and there exists* $\theta > 0$ *such that* $\max\{N_2(\theta), N_{21}(\theta)\} < N + s_1$*;*
- $\lambda_1 = 0$ *and* $\lambda_2 > 0$ *if* $i_{43})$ *holds with* $i = 2$ *and there exists* $\theta > 0$ *such that* $\max\{N_1(\theta), N_{12}(\theta)\} < N + s_2$*;*
- $\lambda_1 = \lambda_2 = 0$ *if either* $i_{43})$ *holds with* $i = 1$ *and there exists* $\theta > 0$ *such that* $N_{21}(\theta) < N + s_1$ *or* $i_{43})$ *holds with* $i = 2$ *and there exists* $\theta > 0$ *such that* $N_{12}(\theta) < N + s_2$*.*

As to the case $\lambda_1 = \lambda_2 = 0$, we make an assumption which replaces $i_{42})$, and that allows us to choose $v_i = \varphi$ in Def. 2.1. In this way, we get new *a priori* estimates and, consequently, an improvement of the nonexistence result expressed by Theor. 4.2. Hence let us suppose

$$i_{44})\quad \begin{cases} \text{there exists } \overline{\alpha}_i \in [\max\{-1, 1-p_i, 1-q_i(p_i-1)^{-1}\}, 0[\text{ such that} \\ \displaystyle\int_0^2 \int_{1\le|y|\le 2} (b_0(y,\tau))^{1-\sigma_i'} dy d\tau < +\infty \qquad \forall\, \alpha_i \in]\overline{\alpha}_i, 0[\\ \text{with } \sigma_i = q_i/(1-\alpha_i)(p_i-1)\,. \end{cases}$$

Let us note that $i_{44}) \Rightarrow i_{42})$ since $\sigma_i' > q_i(q_i - p_i + 1)^{-1}$. Let us fix:

$$X_i = \int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{|x|\le 2^{\frac{1}{\beta}} R} b_0 u_i^{q_i} \varphi dx dt\,,$$

$$\overline{X}_i = \max\left\{ \int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{R\le|x|\le 2^{\frac{1}{\beta}} R} b_0 u_i^{q_i} \varphi dx dt\,,\ \int_{R^\theta}^{2^{\frac{1}{\mu}} R^\theta} \int_{|x|\le 2^{\frac{1}{\beta}} R} b_0 u_i^{q_i} \varphi dx dt \right\}$$

and

$$\rho_i = q_i(\alpha_i + 1)^{-1}\,, \quad \omega_i = q_i(\alpha_i + p_i - 1)^{-1} \quad \forall\, \alpha_i \in]\overline{\alpha}_i, 0[\,.$$

Since $\rho_i' < q_i'$ and $\omega_i' < q_i(q_i - p_i + 1)^{-1}$, we get:

$$\int_1^2 \int_{|y|\le 2} (b_0(y,\tau))^{1-\rho_i'} dy d\tau < +\infty\,, \quad \int_0^2 \int_{1\le|y|\le 2} (b_0(y,\tau))^{1-\omega_i'} dy d\tau < +\infty$$

by virtue of $i_{41})$ and $i_{44})$ respectively.

Proposition 4.1 *In the case* $\lambda_1 = \lambda_2 = 0$ *and under conditions* $i_{11}) - i_{13})$*,* $i_{31})$*,* $i_{41})$*,* $i_{44})$*, if* $(u_1, u_2) \in S(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$*, with* $\alpha_i \in]\overline{\alpha_i}, 0[$*, is a weak solution to problem (1.1), then:*

$$\begin{aligned} \int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{R\le|x|\le 2^{\frac{1}{\beta}} R} |A_i \cdot \nabla\varphi| dx dt \quad &\le \quad k \left[(I_i(\omega_i'))^{\frac{1}{p_i'\omega_i'}} (I_i(\sigma_i'))^{\frac{1}{p_i\sigma_i'}} \overline{X}_i^{\frac{p_i-1}{q_i}} + \right. \\ &\left. + (I_0(\rho_i'))^{\frac{1}{p_i'\rho_i'}} (I_i(\sigma_i'))^{\frac{1}{p_i\sigma_i'}} \overline{X}_i^{\frac{2(p_i-1)}{p_i q_i}} \right] \\ &\forall\, R > R_0\,, \end{aligned} \tag{4.5}$$

with k positive constant, independent of R, and $\nu > \max_{i=1,2}\{p_i\sigma'_i, q'_i\}$ in Definition (1.2) of φ.

Proof. Relation (4.5) holds, owing to the inequality

$$\begin{aligned}
&\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}|A_i\cdot\nabla\varphi|dxdt\le\\
&\le\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}A_{i0}^{\frac{1}{p'_i}}|\nabla u_i|^{p_i-1}\hat{A}_i^{\frac{1}{p'_i}}|\nabla\varphi|dxdt\le\\
&\le\left[\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}\varphi u_i^{\alpha_i-1}A_{i0}|\nabla u_i|^{p_i}dxdt\right]^{\frac{1}{p'_i}}\cdot\\
&\quad\cdot\left[\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}\hat{A}_i u_i^{(1-\alpha_i)(p_i-1)}\varphi^{1-p_i}|\nabla\varphi|^{p_i}dxdt\right]^{\frac{1}{p_i}}
\end{aligned}$$

taking into account that by (2.4)

$$\begin{aligned}
&\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}\varphi u_i^{\alpha_i-1}A_{i0}|\nabla u_i|^{p_i}dxdt\le\\
&\le k_1\left[\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}\hat{A}_i^{p_i-1}u_i^{\alpha_i+p_i-1}\varphi^{1-p_i}|\nabla\varphi|^{p_i}dxdt+\right.\\
&\quad\left.+\int_{R^\theta}^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R}u_i^{\alpha_i+1}|\varphi_t|dxdt\right],
\end{aligned}$$

with k_1 positive constant independent of R.

Proposition 4.2 *In the case $\lambda_1=\lambda_2=0$ and under the conditions of Prop. 4.1, if $(u_1,u_2)\in S(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$, with $\alpha_i\in]\overline{\alpha}_i,0[$, is a weak solution to problem (1.1), then:*

$$X_j\le\overline{k}\left[R^{B_i}\overline{X}_i^{\frac{p_i-1}{q_i}}+R^{C_i}\overline{X}_i^{\frac{1}{q_i}}\right]\qquad\forall\, R>R_0\,, \tag{4.6}$$

with $\overline{k}$ positive constant, independent of R.

Proof. Relation (4.5) implies that $(\varphi,\varphi)\in\tau(u_1,u_2)$. By choosing $v_i=\varphi$ in Def. 2.1 and taking into account (4.5), we get:

$$\begin{aligned}
X_j\ \le\ \overline{k}_1&\left[(I_i(\omega'_i))^{\frac{1}{p'_i\omega'_i}}(I_i(\sigma'_i))^{\frac{1}{p_i\sigma'_i}}\overline{X}_i^{\frac{p_i-1}{q_i}}+\right.\\
&\left.+(I_0(\rho'_i))^{\frac{1}{p'_i\rho'_i}}(I_i(\sigma'_i))^{\frac{1}{p_i\sigma'_i}}\overline{X}_i^{\frac{2(p_i-1)}{p_iq_i}}+(I_0(q'_i))^{\frac{1}{q'_i}}\overline{X}_i^{\frac{1}{q_i}}\right],
\end{aligned}$$

with $\overline{k}_1$ positive constant independent of R. A simple check shows that

$$\begin{aligned} (I_i(\omega_i'))^{\frac{1}{p_i'\omega_i'}}(I_i(\sigma_i'))^{\frac{1}{p_i\sigma_i'}} &\leq \overline{k}_2 R^{B_i}\,, \\ (I_0(\rho_i'))^{\frac{1}{p_i'\rho_i'}}(I_i(\sigma_i'))^{\frac{1}{p_i\sigma_i'}} &\leq \overline{k}_3 R^{\frac{1}{p_i}B_i+\frac{1}{p_i'}C_i}\,, \\ (I_0(q_i'))^{\frac{1}{q_i'}} &\leq \overline{k}_4 R^{C_i}\,, \end{aligned}$$

with $\overline{k}_2$, $\overline{k}_3$, $\overline{k}_4$ positive constants independent of R. Then relation (4.6) also holds, owing to Young's inequality.

From (4.6) it is easy to get the inequality:

$$\begin{aligned} X_j \;\leq\; \overline{\overline{k}} \Bigg[& R^{B_i+B_j\frac{p_i-1}{q_i}}\overline{X}_j^{\frac{p_j-1}{q_j}\frac{p_i-1}{q_i}} + R^{B_i+C_j\frac{p_i-1}{q_i}}\overline{X}_j^{\frac{1}{q_j}\frac{p_i-1}{q_i}} + \\ & + R^{C_i+\frac{1}{q_i}B_j}\overline{X}_j^{\frac{1}{q_i}\frac{p_i-1}{q_j}} + R^{C_i+\frac{1}{q_i}C_j}\overline{X}_j^{\frac{1}{q_j}\frac{1}{q_i}}\Bigg] \quad \forall\, R > R_0\,, \end{aligned} \tag{4.7}$$

with $\overline{\overline{k}}$ positive constant, independent of R.

From (4.7) we get the following theorem:

Theorem 4.4 *In the case $\lambda_1 = \lambda_2 = 0$ and under the conditions of Prop. 4.1, problem (1.1) has no weak solutions belonging to $S(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$, with $\alpha_i \in]\overline{\alpha}_i, 0[$, if there exists $\theta > 0$ such that either*

$$\max\left\{\frac{p_1-1}{q_1}B_2+B_1,\ \frac{p_1-1}{q_1}C_2+B_1,\ \frac{1}{q_1}B_2+C_1,\ \frac{1}{q_1}C_2+C_1\right\} \leq 0$$

or

$$\max\left\{\frac{p_2-1}{q_2}B_1+B_2,\ \frac{p_2-1}{q_2}C_1+B_2,\ \frac{1}{q_2}B_1+C_2,\ \frac{1}{q_2}C_1+C_2\right\} \leq 0\,.$$

Now we show some nonexistence conditions of the solutions to problem (1.1) given by algebraic inequalities (dependent only on p_i, q_i and N) which can be easily obtained from theorems 4.2 and 4.4 when b_0 and $\hat{A}_i$ are positive constants.

For the sake of brevity we set

$$F(s) = \frac{N(q_1q_2-1)}{s+1}\,,\ \ G(s,t) = \frac{N(s-2)+st}{t-s+2}\,,$$

$$H(s,t) = \frac{sq_1q_2 - N(q_1q_2-s+1)}{q_1q_2-(s-1)(t+1)}\,,$$

$$\Phi(s,t) = \frac{N(q_1q_2-s+1)-st}{s-1}\,,\ \ \Psi(s,t,r) = \frac{tr(s-1)+sq_1q_2}{q_1q_2-(p_1-1)(p_2-1)}\,,$$

$$I(s,t) = \frac{st}{t-s+1}\,.$$

Proposition 4.3 *Under assumptions* $i_{11}) - i_{14})$ *with* $b_0 =$ *const.* > 0 *and* $\hat{A}_i =$ *const.* > 0, $i_{31})$, *let* $\lambda_1 > 0$ *and* $\lambda_2 > 0$. *If*

$$\max_{i=1,2} N(q_i - 1) < \min_{i=1,2} \left(\frac{p_i q_i}{q_i - p_i + 1} - N \right)$$

then problem (1.1) has no weak solutions in $S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20})$ *when* $\alpha_1 \in]\alpha_0, 0[$, $|\alpha_1|$ *sufficiently small and* $K(\lambda_1, \lambda_2, \alpha_1) > 0$.

Proposition 4.4 *Let the assumptions of Prop. 4.3 be fulfilled and* $\lambda_i > 0$, $\lambda_j = 0$. *Let us suppose that at least one of the following conditions holds:*

- $\begin{cases} q_1q_2 - (p_i - 1)(q_j + 1) > 0\,, \\ \max\{N(q_j - 1),\, F(q_j)\} < \min\{G(p_j, q_j),\, H(p_i, q_j)\}\,; \end{cases}$
- $\begin{cases} q_1q_2 - (p_i - 1)(q_j + 1) = 0\,, \\ N(q_1q_2 - p_i + 1) - p_iq_1q_2 < 0\,, \\ \max\{N(q_j - 1),\, F(q_j)\} < G(p_j, q_j)\,; \end{cases}$
- $\begin{cases} q_1q_2 - (p_i - 1)(q_j + 1) < 0\,, \\ \max\{N(q_j - 1),\, F(q_j),\, H(p_i, q_j)\} < G(p_j, q_j); \end{cases}$
- $\max\{G(p_j, q_j),\, \Phi(p_j, q_j)\} < \min\{\Psi(p_i, p_j, q_j) - N,\, I(p_j, q_j) - N\}$, *where the right-hand side is assumed to be positive.*

Then, problem (1.1) has no weak solutions in $S(\gamma_1, \gamma_2, \alpha_1, q_2\alpha_1/q_1, u_{10}, u_{20})$ *with* $\alpha_1 \in]\alpha_0, 0[$ *and* $|\alpha_1|$ *sufficiently small.*

Proposition 4.5 *Under assumptions* $i_{11}) - i_{13})$ *with* $b_0 =$ *const.* > 0 *and* $\hat{A}_i =$ *const.* > 0, $i_{31})$, *let* $\lambda_1 = \lambda_2 = 0$. *Let us suppose that at least one of the following conditions holds:*

- $\begin{cases} q_1q_2 - (p_i - 1)(q_j + 1) > 0\,, \\ F(q_j) \le \min\{G(p_j, q_j),\, H(p_i, q_j)\}\,; \end{cases}$
- $\begin{cases} q_1q_2 - (p_i - 1)(q_j + 1) = 0\,, \\ N(q_1q_2 - p_i + 1) - p_iq_1q_2 \le 0\,, \\ F(q_j) \le G(p_j, q_j)\,; \end{cases}$
- $\begin{cases} q_1q_2 - (p_i - 1)(q_j + 1) < 0\,, \\ \max\{F(q_j),\, H(p_i, q_j)\} \le G(p_j, q_j)\,; \end{cases}$
- $\max\{G(p_j, q_j),\, \Phi(p_j, q_j)\} \le \Psi(p_i, p_j, q_j) - N$, *where the right-hand side is assumed to be positive.*

Then problem (1.1) has no weak solutions in $S(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ *with* $\alpha_i \in]\max\{-1, 1 - p_i,\, 1 - q_i(p_i - 1)^{-1}\},\, 0[$.

Let us make the following observations:

- The conditions of Prop. 4.5 are the same as those presented by Mitidieri and Pohozaev in ([8], Theor. 40.6, page 193; and Theor. 40.7, page 194) concerning the problem:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - \mathrm{div} A_1(x, u_1, \nabla u_1) \geq u_2^{q_2} \qquad \text{in } \mathbf{R}^N \times]0, +\infty[\\
&\frac{\partial u_2}{\partial t} - \mathrm{div} A_2(x, u_2, \nabla u_2) \geq u_1^{q_1} \qquad \text{in } \mathbf{R}^N \times]0, +\infty[\\
&u_i \gneqq 0\,, \;\; u_i(x,0) = u_{i0}(x) \;\; \text{in } \mathbf{R}^N\,,
\end{aligned}$$

  where $u_{i0} \in L^1_{\mathrm{loc}}(\mathbf{R}^N), u_{i0} \geq 0$, $A_i : \mathbf{R}^N \times [0, +\infty[\times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function such that $A_i(x, z_i, w_i) \cdot w_i \geq c_i |A_i(x, z_i, w_i)|^{p_i'}$ with $c_i =$ const. > 0, $p_i > 1$ and $q_i > \max\{1, p_i - 1\}$.

- Let us suppose $p_1 = p_2 = p$ and $q_1 = q_2 = q$ with $p > 1$ and $q > \max\{1, p-1\}$. Each condition of Prop. 4.4 is then equivalent to the one of Prop. 4.3:

$$q < p - 1 + \frac{p}{N}\,,$$

  while each condition of Prop. 4.5 is equivalent to the inequality:

$$q \leq p - 1 + \frac{p}{N}\,.$$

  It follows, in particular, that the problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \mathrm{div}(|\nabla u|^{p-2} \nabla u) + \lambda |\nabla u|^{\frac{pq}{1+q}} = u^q \qquad \text{in } \mathbf{R}^N \times]0, +\infty[\\
&u \gneqq 0\,, \quad u(x,0) = u_0(x) \;\; \text{in } \mathbf{R}^N\,,
\end{aligned} \tag{4.8}$$

  where $\lambda \geq 0$, $u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $u_0 \geq 0$, has no weak solutions when:

$$\begin{aligned}
&q < p - 1 + \frac{p}{N} \quad \text{if } \; \lambda > 0 \;\text{ is sufficiently small,}\\
&q \leq p - 1 + \frac{p}{N} \quad \text{if } \; \lambda = 0\,.
\end{aligned}$$

  These two statements have been proved in [9] for the scalar case of problem (1.1). The second one of these statements has been proved by Samarskii et al in [14] for problem (4.8), by Pohozaev and Tesei in [11] for a more general problem.

- When $p_1 = p_2 = 2$ and $q_1, q_2 > 1$, the condition of Prop. 4.3 can be written as:

$$q_1 < 1 + \frac{2}{N}\,, \quad q_2 < 1 + \frac{2}{N}\,;$$

  moreover each condition of Prop. 4.4 [resp. Prop. 4.5] holds if and only if:

$$\frac{N}{2} < \min\left\{\frac{1}{q_j - 1}\,, \frac{q_j + 1}{q_1 q_2 - 1}\right\} \qquad \left[\text{resp. } \frac{N}{2} \leq \frac{\max\{q_1, q_2\} + 1}{q_1 q_2 - 1}\right].$$

Let us consider, for example, the problem:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \Delta u_1 + \lambda_1 |\nabla u_1|^{\frac{2q_1}{1+q_1}} = u_2^{q_2} \\ &\qquad\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0,+\infty[\\ &\frac{\partial u_2}{\partial t} - \Delta u_2 + \lambda_2 |\nabla u_2|^{\frac{2q_2}{1+q_2}} = u_1^{q_1} \\ &u_i \gneqq 0\,, \quad u_i(x,0) = u_{i0}(x) \ \text{ in } \mathbf{R}^N\,, \end{aligned} \tag{4.9}$$

where $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 > 0$, $u_{i0} \in L^1_{\text{loc}}(\mathbf{R}^N)$, $u_{i0} \geq 0$. This problem has no weak solutions when:

$$\begin{aligned} &q_1 < 1 + \frac{2}{N}\,, \quad q_2 < 1 + \frac{2}{N} \ \text{ if } \ \lambda_1 > 0 \text{ and } \lambda_2 > 0 \ \text{ are sufficiently small,} \\ &\frac{N}{2} < \min\left\{\frac{1}{q_2 - 1}\,, \frac{q_2 + 1}{q_1 q_2 - 1}\right\} \quad \text{if } \ \lambda_1 > 0 \ \text{ and } \ \lambda_2 = 0\,, \\ &\frac{N}{2} < \min\left\{\frac{1}{q_1 - 1}\,, \frac{q_1 + 1}{q_1 q_2 - 1}\right\} \quad \text{if } \ \lambda_1 = 0 \ \text{ and } \ \lambda_2 > 0\,. \end{aligned}$$

In addition, let us consider the problem

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \operatorname{div}\left(\psi(\sqrt{1 + |\nabla u_1|^2})\nabla u_1\right) = u_2^{q_2} \\ &\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0,+\infty[\\ &\frac{\partial u_2}{\partial t} - \operatorname{div}\left(\psi(\sqrt{1 + |\nabla u_2|^2})\nabla u_2\right) = u_1^{q_1} \\ &u_i \gneqq 0\,, \ \ u_i(x,0) = u_{i0}(x) \ \text{ in } \ \mathbf{R}^N\,, \end{aligned} \tag{4.10}$$

where $\psi : [1\,,+\infty[\to]0,+\infty[$ is a continuous and bounded function, $u_{i0} \in L^1_{\text{loc}}(\mathbf{R}^N)$ and $u_{i0} \geq 0$. Problems (4.9) with $\lambda_1 = \lambda_2 = 0$ and (4.10) have no weak solutions when

$$\frac{N}{2} \leq \frac{\max\{q_1, q_2\} + 1}{q_1 q_2 - 1}\,. \tag{4.11}$$

This result is the same as that obtained by Escobedo and Herrero in [1] for problem (4.9) with $\lambda_1 = \lambda_2 = 0$. Moreover this result, referred to system (4.10), implies that the problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \operatorname{div}\left(\psi(\sqrt{1 + |\nabla u|^2})\nabla u\right) \geq u^q \qquad \text{in } \ \mathbf{R}^N \times]0,+\infty[\\ &u \gneqq 0\,, \quad u(x,0) = u_0(x) \quad \text{ in } \ \mathbf{R}^N\,, \end{aligned} \tag{4.12}$$

where $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$ and $u_0 \geq 0$, has no weak solutions when

$$q \leq 1 + \frac{2}{N}\,.$$

This conclusion broadens the scope of a theorem by Levine, Lieberman and Meier. These Authors, in fact, have proved in [7] that, if $\psi : [1,+\infty[\to]0,+\infty[$ is a $C^{1,\alpha}$

bounded function such that $-1 \le \dfrac{s\psi'(s)}{\psi(s)} \le \omega$ for some $\omega > 0$ and if u_0 is radially symmetric and decreasing, problem (4.12) has no "regular" solutions when $q < 1 + \dfrac{2}{N}$.

Remark 4.1 Condition (4.11) has been formulated by Escobedo and Herrero with $q_1q_2 > 1$ but for "regular" solutions. In section 5 (Example 5.2) we will see that when $q_1q_2 > 1$, the inequality $\dfrac{N}{2} < \dfrac{\max\{q_1, q_2\} + 1}{q_1q_2 - 1}$ allows the nonexistence of weak solutions to problem (4.9), with $\lambda_1 = \lambda_2 = 0$, in a functional class which includes the one of Escobedo and Herrero.

5 Further results

The results of section 4 are based on assumption $i_{31})$, which allowed us to find the estimates of section 3 and, for $\lambda_1 = \lambda_2 = 0$, the ones of section 4. When $p_1 = p_2 = 2$ and $\lambda_1 = \lambda_2 = 0$, it is possible to obtain new results assuming only $q_1q_2 > 1$. In this case Prop. 2.1 still holds, however inequality (2.4) is not useful. Then we set u_{10}, u_{20} in $L^\infty_{\text{loc}}(\mathbf{R}^N)$ and we find a new inequality by which we get useful estimates for the nonexistence of weak solutions in a convenient subset of $S(\gamma_1, \gamma_2, , \alpha_1, \alpha_2, u_{10}, u_{20})$.

We write again problem (1.1) with $\lambda_1 = \lambda_2 = 0$:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \operatorname{div}(A_1(x,t,\nabla u_1)) \ge b_1(x,t,u_2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0,+\infty[\\ &\frac{\partial u_2}{\partial t} - \operatorname{div}(A_2(x,t,\nabla u_2)) \ge b_2(x,t,u_1) \\ &u_i \gneqq 0,\ u_i(x,0) = u_{i0}(x) \ \text{ in } \mathbf{R}^N, \end{aligned} \tag{5.1}$$

where $u_{i0} \in L^\infty_{\text{loc}}(\mathbf{R}^N)$ and $u_{i0} \ge 0$. Let $i_{51})$ be the set of assumptions $i_{11})-i_{13})$ with $p_1 = p_2 = 2$, $q_1q_2 > 1$. Let $S^*(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ [resp. $S^*(2, 2, u_{10}, u_{20})$] be the set of pairs $(u_1, u_2) \in S(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ [resp. $(u_1, u_2) \in S(2, 2, u_{10}, u_{20})$] such that

$$u_i \in L^\infty_{\text{loc}}(\mathbf{R}^N \times [0, +\infty[)\,,$$

$$e_i(u_1, u_2) = \sup \frac{\overline{A}_{i0}(x,t,\nabla u_i)}{A_{10}(x,t,\nabla u_1)A_{20}(x,t,\nabla u_2)} < +\infty\,. \tag{5.2}$$

For each $(u_1, u_2) \in S^*(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ [resp. $(u_1, u_2) \in S^*(2, 2, u_{10}, u_{20})$] let:

$$\delta(u_1, u_2) = \max\{1\,,\ \ (e_1(u_1, u_2))^{1/2}\,,\ \ (e_2(u_1, u_2))^{1/2}\}\,,$$

$$\begin{aligned} D(\delta(u_1,u_2)) \quad &= \quad \{(t_1, t_2) \in]-1, 0[\times]-1, 0[:\ t_1 + t_2 + 1 < \\ &< \quad [(4\delta^2(u_1,u_2))^{-1} - 1]t_1t_2\}\,. \end{aligned}$$

In this section we deal with weak solutions $(u_1,u_2) \in S^*(\gamma_1,\gamma_2,\alpha_1,\ \alpha_2,u_{10},u_{20})$ such that

$$(\alpha_1,\alpha_2) \in D(\delta(u_1,u_2))\,. \tag{5.3}$$

In order to get these results it is necessary that:

$$2\delta(u_1,u_2)(\alpha_1+1)|\alpha_2|^{-1} < [2\delta(u_1,u_2)(\alpha_2+1)]^{-1}|\alpha_1|$$

and this inequality is valid if and only if (5.3) holds.

Let us note that, by virtue of Prop. 2.2, for $(u_1,u_2) \in S^*(2,2,u_{10},u_{20})$, we have:

$$\begin{aligned}&(u_1,u_2) \in S^*(2,2,\alpha_1,\alpha_2,u_{10},u_{20}) \qquad \forall\,(\alpha_1,\alpha_2) \in D(\delta(u_1,u_2))\,,\\ &(u_1,u_2) \ \text{ is a weak solution to problem (5.1).}\end{aligned}$$

Under assumption

i_{52}) $$e_i = \sup \frac{\overline{A}_{i0}(x,t,w_i)}{A_{10}(x,t,w_1)A_{20}(x,t,w_2)} < +\infty,$$

condition (5.2) holds. In addition, let

$$\begin{aligned}&\delta = \max\{1, e_1^{1/2}, e_2^{1/2}\}\,,\\ &D(\delta) = \{(t_1,t_2) \in]-1,0[\times]-1,0[:\ t_1+t_2+1 < [(4\delta^2)^{-1}-1]t_1t_2\}.\end{aligned}$$

For $(\alpha_1,\alpha_2) \in D(\delta)$, each $(u_1,u_2) \in S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$ satisfies (5.3).

Proposition 5.1 *Under condition i_{51}), let $(u_1,u_2) \in S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},\ u_{20})$ with $(\alpha_1,\alpha_2) \in D(\delta(u_1,u_2))$ be a weak solution to problem (5.1). Then for $R > R_0$ and $2\delta(u_1,u_2)(\alpha_1+1)|\alpha_2|^{-1} < \eta < [2\delta(u_1,u_2)(\alpha_2+1)]^{-1}|\alpha_1|$, it follows that:*

$$\begin{aligned}&(\alpha_1+1)\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} b_0 u_2^{q_2+\alpha_2+1}u_1^{\alpha_1}\varphi dxdt +\\ &+(\alpha_2+1)\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} b_0 u_1^{q_1+\alpha_1+1}u_2^{\alpha_2}\varphi dxdt +\\ &+\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}u_{20}^{\alpha_2+1}\varphi(x,0)dx +\\ &+\frac{\alpha_1+1}{2}(|\alpha_1| - 2\delta(u_1,u_2)(\alpha_2+1)\eta)\,\cdot\\ &\cdot\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} \varphi u_1^{\alpha_1-1}u_2^{\alpha_2+1}A_{10}|\nabla u_1|^2 dxdt +\\ &+\frac{\alpha_2+1}{2}(|\alpha_2| - 2\delta(u_1,u_2)(\alpha_1+1)\eta^{-1})\,\cdot\\ &\cdot\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} \varphi u_1^{\alpha_1+1}u_2^{\alpha_2-1}A_{20}|\nabla u_2|^2 dxdt \le\end{aligned}$$

$$\leq \sum_{h=1}^{2} \frac{1}{2|\alpha_h|} \int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{R\leq|x|\leq 2^{\frac{1}{\beta}} R} u_1^{\alpha_1+1} u_2^{\alpha_2+1} \hat{A}_h \varphi^{-1} |\nabla\varphi|^2 dxdt +$$
$$- \int_{R^\theta}^{2^{\frac{1}{\mu}} R^\theta} \int_{|x|\leq 2^{\frac{1}{\beta}} R} u_1^{\alpha_1+1} u_2^{\alpha_2+1} \varphi_t dxdt \,, \tag{5.4}$$

with $\nu > 2$ in Definition (1.2) of φ.

Proof. On one hand, the choice of ν implies that

$$\varphi^{-1}|\nabla\varphi|^2 \in L^\infty(\mathrm{supp}\, |\nabla\varphi|)\,;$$

then if we let $u_{i\varepsilon} = u_i + \varepsilon$ ($0 < \varepsilon \leq 1$), we have:

$$u_{i\varepsilon}^{\alpha_i} u_{j\varepsilon}^{\alpha_j+1} \varphi \in W^{1,1}(\mathbf{R}^N \times]0,+\infty[) \cap L^\infty(\mathbf{R}^N \times]0,+\infty[)\,,$$

$$\begin{aligned} A_i \cdot \nabla(u_{i\varepsilon}^{\alpha_i} u_{j\varepsilon}^{\alpha_j+1}\varphi) \;&=\; \alpha_i \varphi u_{i\varepsilon}^{\alpha_i-1} u_{j\varepsilon}^{\alpha_j+1} A_i \cdot \nabla u_i + \\ &+(\alpha_j+1)\varphi u_{i\varepsilon}^{\alpha_i} u_{j\varepsilon}^{\alpha_j} A_i \cdot \nabla u_j + u_{i\varepsilon}^{\alpha_i} u_{j\varepsilon}^{\alpha_j+1} A_i \cdot \nabla\varphi\,. \end{aligned}$$

On the other hand:

$$\varphi u_{i\varepsilon}^{\alpha_i-1} u_{j\varepsilon}^{\alpha_j+1} A_i \cdot \nabla u_i \leq \varphi u_{j\varepsilon}^{\alpha_j+1} u_i^{\alpha_i-1} A_i \cdot \nabla u_i\,,$$
$$\alpha_i \varphi u_{i\varepsilon}^{\alpha_i-1} u_{j\varepsilon}^{\alpha_j+1} A_i \cdot \nabla u_i \leq \alpha_i \varphi u_{i\varepsilon}^{\alpha_i-1} u_{j\varepsilon}^{\alpha_j+1} A_{i0} |\nabla u_i|^2\,;$$

$$\begin{matrix} |\varphi u_{1\varepsilon}^{\alpha_1} u_{2\varepsilon}^{\alpha_2} A_1 \cdot \nabla u_2| \\ |\varphi u_{1\varepsilon}^{\alpha_1} u_{2\varepsilon}^{\alpha_2} A_2 \cdot \nabla u_1| \end{matrix} \leq \begin{aligned} &\frac{\delta(u_1,u_2)}{2} \eta \varphi u_{1\varepsilon}^{\alpha_1-1} u_{2\varepsilon}^{\alpha_2+1} A_{10} |\nabla u_1|^2 + \\ &+ \frac{\delta(u_1,u_2)}{2\eta} \varphi u_{1\varepsilon}^{\alpha_1+1} u_{2\varepsilon}^{\alpha_2-1} A_{20} |\nabla u_2|^2\,; \end{aligned}$$

$$|u_{i\varepsilon}^{\alpha_i} u_{j\varepsilon}^{\alpha_j+1} A_i \cdot \nabla\varphi| \leq \frac{|\alpha_i|}{2} \varphi u_{i\varepsilon}^{\alpha_i-1} u_{j\varepsilon}^{\alpha_j+1} A_{i0} |\nabla u_i|^2 + + \frac{1}{2|\alpha_i|} u_{i\varepsilon}^{\alpha_i+1} u_{j\varepsilon}^{\alpha_j+1} \hat{A}_i \varphi^{-1} |\nabla\varphi|^2\,;$$

$$\iint_{\mathrm{supp}\,\varphi} u_{1\varepsilon}^{\alpha_1} u_{2\varepsilon}^{\alpha_2+1} \varphi \frac{\partial u_1}{\partial t} dxdt =$$
$$= -\frac{1}{\alpha_1+1} \left[\int_{\mathbf{R}^N} (u_{10}+\varepsilon)^{\alpha_1+1} (u_{20}+\varepsilon)^{\alpha_2+1} \varphi(x,0) dx + \right.$$
$$\left. + \iint_{\mathrm{supp}\,\varphi} \varphi u_{1\varepsilon}^{\alpha_1+1} \frac{\partial u_{2\varepsilon}^{\alpha_2+1}}{\partial t} dxdt + \iint_{\mathrm{supp}\,\varphi_t} u_{1\varepsilon}^{\alpha_1+1} u_{2\varepsilon}^{\alpha_2+1} \varphi_t dxdt \right],$$
$$\iint_{\mathrm{supp}\,\varphi} u_{1\varepsilon}^{\alpha_1+1} u_{2\varepsilon}^{\alpha_2} \varphi \frac{\partial u_2}{\partial t} dxdt = \frac{1}{\alpha_2+1} \iint_{\mathrm{supp}\,\varphi} \varphi u_{1\varepsilon}^{\alpha_1+1} \frac{\partial u_{2\varepsilon}^{\alpha_2+1}}{\partial t} dxdt\,.$$

Hence $(u_{1\varepsilon}^{\alpha_1} u_{2\varepsilon}^{\alpha_2+1}\varphi,\, u_{2\varepsilon}^{\alpha_2} u_{1\varepsilon}^{\alpha_1+1}\varphi) \in \tau(u_1,u_2)$ and relation (5.4) follows by setting in Def. 2.1, $v_i = u_{i\varepsilon}^{\alpha_i} u_{j\varepsilon}^{\alpha_j+1}\varphi$ and passing to the limit for $\varepsilon \to 0$.

Owing to the assumption $q_1 q_2 > 1$, we get an estimate of the right-hand side of relation (5.4). In this estimate the following integrals are present:

$$\widetilde{Y}_{ij} = \int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{R \le |x| \le 2^{\frac{1}{\beta}} R} b_0 u_i^{q_i+\alpha_i+1} u_j^{\alpha_j} \varphi dx dt\,,$$

$$\widetilde{\widetilde{Y}}_{ij} = \int_{R^\theta}^{2^{\frac{1}{\mu}} R^\theta} \int_{|x| \le 2^{\frac{1}{\beta}} R} b_0 u_i^{q_i+\alpha_i+1} u_j^{\alpha_j} \varphi dx dt\,.$$

Let

$$\begin{aligned}
r_1 &= \frac{q_1 q_2 + q_1(\alpha_2+1) + q_2(\alpha_1+1) + \alpha_1 + \alpha_2 + 1}{q_1(\alpha_2+1) + \alpha_1 + 1}\,,\\
r_2 &= \frac{q_1 q_2 + q_1(\alpha_2+1) + q_2(\alpha_1+1) + \alpha_1 + \alpha_2 + 1}{q_2(\alpha_1+1) + \alpha_2 + 1}\,,\\
r_3 &= \frac{q_1 q_2 + q_1(\alpha_2+1) + q_2(\alpha_1+1) + \alpha_1 + \alpha_2 + 1}{q_1 q_2 - 1}\,,\\
\beta_1 &= \frac{\alpha_1}{r_1}\,,\;\; \beta_2 = \frac{q_2+\alpha_2+1}{r_1}\,,\;\; \overline{\beta}_1 = \frac{q_1+\alpha_1+1}{r_2}\,,\;\; \overline{\beta}_2 = \frac{\alpha_2}{r_2}\,,
\end{aligned}$$

it follows that:

$$r_i > 1\,, \qquad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1\,,\;\; \beta_i + \overline{\beta}_i = \alpha_i + 1\,.$$

Then, assuming

$i_{53})$ $\quad b_0^{1-r_3} \in L^1_{\mathrm{loc}}((\mathbf{R}^N \backslash B_{R_0}) \times [0,+\infty[) \cap L^1_{\mathrm{loc}}(\mathbf{R}^N \times [R_0,+\infty[)$

and by choosing $\nu > 2r_3$ in Definition (1.2) of φ, we get:

$$\begin{aligned}
&\int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{R \le |x| \le 2^{\frac{1}{\beta}} R} u_1^{\alpha_1+1} u_2^{\alpha_2+1} \hat{A}_h \varphi^{-1} |\nabla\varphi|^2 dx dt \le \widetilde{Y}_{21}^{\frac{1}{r_1}} \widetilde{Y}_{12}^{\frac{1}{r_2}} (I_h(r_3))^{\frac{1}{r_3}}\,,\\
&\int_{R^\theta}^{2^{\frac{1}{\mu}} R^\theta} \int_{|x| \le 2^{\frac{1}{\beta}} R} u_1^{\alpha_1+1} u_2^{\alpha_2+1} |\varphi_t| dx dt \le \widetilde{\widetilde{Y}}_{21}^{\frac{1}{r_1}} \widetilde{\widetilde{Y}}_{12}^{\frac{1}{r_2}} (I_0(r_3))^{\frac{1}{r_3}}\,.
\end{aligned} \tag{5.5}$$

Prop. 5.1 and inequalities (5.5) imply the proposition:

Proposition 5.2 *Assuming i_{51}), i_{53}), if $(u_1,u_2) \in S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,\, u_{10},u_{20})$ with $(\alpha_1,\alpha_2) \in D(\delta(u_1,u_2))$ is a weak solution to problem (5.1), then for $R > R_0$:*

$$\int_0^{2^{\frac{1}{\mu}} R^\theta} \int_{|x| \le 2^{\frac{1}{\beta}} R} b_0 u_2^{q_2+\alpha_2+1} u_1^{\alpha_1} \varphi dx dt +$$

$$
\begin{aligned}
&+\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} b_0 u_1^{q_1+\alpha_1+1}u_2^{\alpha_2}\varphi dxdt+\\
&+\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}u_{20}^{\alpha_2+1}\varphi(x,0)dx\le\\
&\le d\left[\widetilde{Y}_{21}^{\frac{1}{r_1}}\widetilde{Y}_{12}^{\frac{1}{r_2}}\sum_{h=1}^{2}(I_h(r_3))^{\frac{1}{r_3}}+\widetilde{\widetilde{Y}}_{21}^{\frac{1}{r_1}}\widetilde{\widetilde{Y}}_{12}^{\frac{1}{r_2}}(I_0(r_3))^{\frac{1}{r_3}}\right], \qquad (5.6)
\end{aligned}
$$

where d is a positive constant independent of R and $\nu > 2r_3$ in Definition (1.2) of φ.

A consequence of Prop. 5.2 is the following:

Proposition 5.3 *Let condition i_{51}) hold with* $\inf b_0 > 0$ *and*

$$\hat{A}_i \in L^\infty((\mathbf{R}^N\backslash B_{R_0})\times]0,+\infty[)\,.$$

If $(u_1,u_2)\in S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$, with $(\alpha_1,\alpha_2)\in D(\delta(u_1,u_2))$, is a weak solution to problem (5.1), then:

$$\inf u_i > 0 \quad\Longrightarrow\quad \sup u_j = +\infty\,.$$

In particular, there exist no pairs $(u_1,u_2)\in S^*(2,2,u_{10},u_{20})$ such that either $\inf u_1 > 0$ *and $u_2\in L^\infty(\mathbf{R}^N\times]0,+\infty[)$ or $u_1\in L^\infty(\mathbf{R}^N\times]0,+\infty[)$ and* $\inf u_2>0$.

Another consequence of Prop. 5.2 is the following theorem whose proof is similar to the one of Theor. 4.1.

Theorem 5.1 *Under conditions i_{51}), i_{52}), if for a pair $(\alpha_1,\alpha_2)\in D(\delta)$ condition i_{53}) holds and if*

$$\lim_{R\to+\infty}\inf\left[\sum_{h=1}^{2} I_h(r_3)+I_0(r_3)\right] < +\infty\,,$$

then problem (5.1) has no weak solutions in $S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$.

Corollary 5.1 *Under assumptions i_{51}), i_{52}), the thesis of Theor. 5.1 holds if for a pair $(\alpha_1,\alpha_2)\in D(\delta)$, $b_0^{1-r_3}\hat{A}_i^{r_3}\in L^1((\mathbf{R}^N\backslash B_{R_0})\times]0,+\infty[)$ and $b_0^{1-r_3}\in L^1(\mathbf{R}^N\times]R_0,+\infty[)$.*

We complete the study of problem (5.1) by getting from inequality (5.6), under additional assumptions on b_0 and $\hat{A}_i$, some conditions which ensure that we have $S^*(2,2,u_{10},u_{20})=\emptyset$, ignoring assumption i_{52}).

Let $(u_1,u_2)\in S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$ with $(\alpha_1,\alpha_2)\in D(\delta(u_1,u_2))$ be a weak solution to problem (5.1). Let

$$F(t_1,t_2)=r_3(t_1,t_2)-1=\frac{(q_1+1)(t_2+1)+(q_2+1)(t_1+1)}{q_1q_2-1}\,.$$

We have

$$\max\{F(t_1,t_2)\,:\,(t_1,t_2)\in\overline{D(\delta(u_1,u_2))}\} = \\ = \frac{\max\{q_1,q_2\}+1}{q_1q_2-1} = \begin{cases} F(-1,0) & \text{if } \ q_1\ge q_2 \\ \\ F(0,-1) & \text{if } \ q_1\le q_2\,. \end{cases} \tag{5.7}$$

Hence, if we assume:

i_{54}) $\displaystyle\int_0^2\int_{|y|\le 2}(b_0(y,\tau))^{-\frac{\max\{q_1,q_2\}+1}{q_1q_2-1}}dyd\tau<+\infty\,,$

i_{55}) $\begin{cases} \max\{\hat{A}_1,\hat{A}_2\}\le\hat{A} \ \text{ with } \hat{A}\in L^\infty_{\text{loc}}(\mathbf{R}^N\times[0,+\infty[)\,, \\ \text{there exist } L_0(\theta)\in\mathbf{R} \ \text{ and } \ L(\theta)\ge 0 \ \text{ such that for } \ R>R_0 \\ b_0(Ry,R^\theta\tau)\ge R^{L_0(\theta)}b_0(y,\tau)\,, \quad \hat{A}(Ry,R^\theta\tau)\le R^{L(\theta)}\hat{A}(y,\tau)\,, \end{cases}$

assumption i_{53}) holds and, with $\nu>2\dfrac{q_1q_2+\max\{q_1,q_2\}}{q_1q_2-1}$ in Definition (1.2) of φ, we get:

$$I_0(r_3)\le d_0R^{N+(1-r_3)(\theta+L_0(\theta))}\,,\quad I_i(r_3)\le d_iR^{N+\theta-2r_3+(1-r_3)L_0(\theta)+r_3L(\theta)}\,,$$

where d_0,d_i are positive constants independent of R.
Now we add the assumption

i_{56}) there exists $\overline{\theta}>0$ such that $\overline{\theta}=2-L(\overline{\theta})$

and let us choose β, μ in Definition (1.2) of φ such that $\dfrac{\beta}{\mu}=\overline{\theta}$.

From (5.6) we get that for $R>R_0$:

$$\int_0^{2^{\frac{1}{\mu}}R^{\overline{\theta}}}\int_{|x|\le 2^{\frac{1}{\beta}}R}b_0u_2^{q_2+\alpha_2+1}u_1^{\alpha_1}\varphi dxdt+ \\ +\int_0^{2^{\frac{1}{\mu}}R^{\overline{\theta}}}\int_{|x|\le 2^{\frac{1}{\beta}}R}b_0u_1^{q_1+\alpha_1+1}u_2^{\alpha_2}\varphi dxdt\le \\ \le\overline{d}R^{N+(1-r_3)(\overline{\theta}+L_0(\overline{\theta}))}-\int_{|x|\le 2^{\frac{1}{\beta}}R}u_{10}^{\alpha_1+1}u_{20}^{\alpha_2+1}\varphi(x,0)dx\,, \tag{5.8}$$

where $\overline{d}$ is a positive constant independent of R.

Theorem 5.2 *Under assumptions i_{51}), i_{54}) $-$ i_{56}), we have $S^*(2,2,u_{10},\,u_{20})=\emptyset$ when:*

- $N<\dfrac{\max\{q_1,q_2\}+1}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))\,;$

- $N \geq \dfrac{\max\{q_1,q_2\}+1}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))$ *and* $\lim_{|x|\to+\infty}\inf |x|^{-s_i}u_{i0}(x) > 0$ *with*

$$\begin{aligned}&or\quad s_2 > -\frac{q_1+1}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))\\ &or\quad s_1 > -\frac{q_2+1}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))\\ &or\quad s_1+s_2 > -\frac{q_1+q_2+2}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))\,.\end{aligned}$$

Proof. Arguing by contradiction, let $(u_1,u_2)\in S^*(2,2,u_{10},u_{20})$. If

$$N < \frac{\max\{q_1,q_2\}+1}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))\,,$$

due to (5.7), there exists $(\alpha_1,\alpha_2)\in D(\delta(u_1,u_2))$ such that $N+(1-r_3)(\overline{\theta}+L_0(\overline{\theta}))<0$. Consequently

$$\int_0^{+\infty}\int_{\mathbf{R}^N} b_0u_2^{q_2+\alpha_2+1}u_1^{\alpha_1}dxdt + \int_0^{+\infty}\int_{\mathbf{R}^N} b_0u_1^{q_1+\alpha_1+1}u_2^{\alpha_2}dxdt \leq 0$$

owing to (5.8).

In the second case, for R large

$$\begin{aligned}&\int_{|x|\leq 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}u_{20}^{\alpha_2+1}\varphi(x,0)dx \geq c_0R^{N+(\alpha_1+1)s_1+(\alpha_2+1)s_2}\\ &(c_0=\text{const.}>0 \text{ indep. of} R)\,.\end{aligned}$$

Moreover there exists $(\alpha_1,\alpha_2)\in D(\delta(u_1,u_2))$ such that $(1-r_3)(\overline{\theta}+L_0(\overline{\theta}))<(\alpha_1+1)s_1+(\alpha_2+1)s_2$. Then, from (5.8) we get that

$$\int_0^{+\infty}\int_{\mathbf{R}^N} b_0u_2^{q_2+\alpha_2+1}u_1^{\alpha_1}dxdt + \int_0^{+\infty}\int_{\mathbf{R}^N} b_0u_1^{q_1+\alpha_1+1}u_2^{\alpha_2}dxdt < 0\,.$$

Another simple consequence of relation (5.8) is

Theorem 5.3 *Under assumptions $i_{51})$, $i_{52})$, $i_{54})-i_{56})$, we have:*

- *If* $N < \dfrac{\max\{q_1,q_2\}+1}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))$*, then there exists $(\alpha_1,\alpha_2)\in D(\delta)$ such that problem (5.1) has no weak solutions in $S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$;*
- *If* $N \geq \dfrac{\max\{q_1,q_2\}+1}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))$ *and* $\lim_{|x|\to+\infty}\inf |x|^{-s_i}u_{i0}(x) > 0$ *with*

$$\begin{aligned}&(\alpha_1+1)s_1+(\alpha_2+1)s_2 >\\ &> -\frac{(\alpha_1+1)(q_2+1)+(\alpha_2+1)(q_1+1)}{q_1q_2-1}(\overline{\theta}+L_0(\overline{\theta}))\\ &\text{for some } (\alpha_1,\alpha_2)\in D(\delta)\,,\end{aligned}$$

then problem (5.1) has no weak solutions in $S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$.

Let us suppose assumptions i_{51}), $i_{54}) - i_{56}$) hold. The nonexistence condition obtained from Theor. 5.2:

$$N < \frac{\max\{q_1, q_2\} + 1}{q_1 q_2 - 1} (\overline{\theta} + L_0(\overline{\theta})) \tag{5.9}$$

represents an innovation compared to the results obtained in section 4 when:

one of the exponents q_1, q_2 is ≤ 1;
q_1, $q_2 > 1$ and assumption i_{42}) does not hold.

In fact, from Theor. 4.2 [resp. Theor. 4.4] we deduce that with N as in (5.9) [resp. $N \leq \frac{\max\{q_1, q_2\} + 1}{q_1 q_2 - 1} (\overline{\theta} + L_0(\overline{\theta}))$] it results in $S(2, 2, u_{10}, u_{20}) = \emptyset$.

Example 5.1 Let us consider the problem

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \operatorname{div}((1 + u_1^r)^{-s} \nabla u_1) \geq u_2^{q_2} \\ &\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0, +\infty[\\ &\frac{\partial u_2}{\partial t} - \Delta u_2 \geq u_1^{q_1} \\ &u_i \gneqq 0\,, \quad u_i(x, 0) = u_{i0}(x) \ \text{ in } \mathbf{R}^N\,, \end{aligned}$$

where $r > 0$, $s > 0$, $q_i > 0$, $u_{i0} \in L^1_{\text{loc}}(\mathbf{R}^N)$, $u_{i0} \geq 0$. We have:

- when $q_1, q_2 > 1$, if $\frac{N}{2} \leq \frac{\max\{q_1, q_2\} + 1}{q_1 q_2 - 1}$ then there are no weak solutions in $S(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ with $\alpha_i \in] \max\{-1, 1 - q_i\}, 0[$;
- when one of the exponents q_1, q_2 is ≤ 1 and $u_{i0} \in L^\infty_{\text{loc}}(\mathbf{R}^N)$, if $q_1 q_2 > 1$ and $\frac{N}{2} < \frac{\max\{q_1, q_2\} + 1}{q_1 q_2 - 1}$ then it follows that $S^*(2, 2, u_{10}, u_{20}) = \{(u_1, u_2) \in S(2, 2, u_{10}, u_{20})\colon u_1 \in L^\infty(\mathbf{R}^N \times]0, +\infty[),\ u_2 \in L^\infty_{\text{loc}}(\mathbf{R}^N \times [0, +\infty[)\} = \emptyset$.

Example 5.2 Let us consider the system:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \operatorname{div}(\psi_1((1 + |\nabla u_1|^2 + |\nabla u_2|^2)^\omega) t^\ell |x|^m \nabla u_1) \geq u_2^{q_2} \\ &\qquad\qquad\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0, +\infty[\\ &\frac{\partial u_2}{\partial t} - \operatorname{div}(\psi_2((1 + |\nabla u_1|^2 + |\nabla u_2|^2)^\omega) t^\ell |x|^m \nabla u_2) \geq u_1^{q_1} \\ &u_i \gneqq 0\,, \quad u_i(x, 0) = u_{i0}(x) \quad \text{in } \mathbf{R}^N\,, \end{aligned} \tag{5.10}$$

where $\psi_i : [1, +\infty[\to]0, +\infty[$ is a continuous and bounded function, $\omega > 0$, $\ell \geq 0$, $0 \leq m < 2$, $q_i > 0$, $u_{i0} \in L^1_{\text{loc}}(\mathbf{R}^N)$ and $u_{i0} \geq 0$. We have:

- when $q_1, q_2 > 1$, if $N \leq \frac{\max\{q_1,q_2\}+1}{q_1q_2-1}\frac{2-m}{\ell+1}$, then system (5.10) has no weak solutions in $S(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$ with $\alpha_i \in]\max\{-1,1-q_i\},0[$;

- when one of the exponents q_1, q_2 is ≤ 1 and $u_{i0} \in L^\infty_{\rm loc}(\mathbf{R}^N)$, if

$$e_1 = \sup_{s\in[1,+\infty[} \frac{\psi_1(s)}{\psi_2(s)} < +\infty\,, \quad e_2 = \sup_{s\in[1,+\infty[} \frac{\psi_2(s)}{\psi_1(s)} < +\infty\,,$$

$$q_1q_2 > 1 \quad \text{and} \quad N < \frac{\max\{q_1,q_2\}+1}{q_1q_2-1}\frac{2-m}{\ell+1}\,,$$

  then system (5.10) has no weak solutions in $S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$ for some $(\alpha_1,\alpha_2) \in D(\delta)$.

Remark 5.1 In the following cases

$$\psi_1 = \psi_2 \equiv 1\,, \quad \ell = m = 0\,; \quad \psi_1(s) = \psi_2(s) = \frac{1}{s}\,, \quad \ell = m = 0\,;$$

Pohozaev and Tesei have proved in [13] that system (5.10) has no solutions (in a functional class characterized by different conditions) when $q_1q_2 > 1$ and

$$\frac{N}{2} < \frac{\max\{q_1,q_2\}+1}{q_1q_2-1}.$$

This result is included in the conclusions of Example 5.2.

6 Appendix

Under assumption $i'_{11})$, the technique we have followed for problem (5.1) can also be used when in system (1.1), $\lambda_1+\lambda_2>0$, $p_1=p_2=2$, $u_{i0}\in L^\infty_{\rm loc}(\mathbf{R}^N)$ and $u_{i0}\geq 0$. Therefore, we show only the necessary changes and the consequent results.

Let $i_{61})$ be the set of assumptions $i'_{11})$, $i_{12})-i_{14})$ where $p_1=p_2=2$. We specify that the nonexistence theorems deal with the functional classes:

$$S^{**}(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20}) = \\ = \{(u_1,u_2)\in S^*(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20}) \,:\, \text{meas}\,(\Omega(u_1)\cap\Omega(u_2))>0\}\,,$$

$$S^{**}(2,2,u_{10},u_{20}) = \\ = \{(u_1,u_2)\in S^*(2,2,u_{10},u_{20}) \,:\, \text{meas}\,(\Omega(u_1)\cap\Omega(u_2))>0\}\,.$$

We remark that, reasoning as in section 2 (Prop. 2.2), also under assumption $i_{61})$, for $(u_1,u_2)\in S^{**}(2,2,u_{10},u_{20})$ we have:

$$(u_1,u_2)\in S^{**}(2,2,\alpha_1,\alpha_2,u_{10},u_{20}) \ \text{ for any } \ \alpha_i\in]-1,0[\,,$$
$$(u_1,u_2) \ \text{ is a weak solution to problem (1.1).}$$

We set for $(u_1,u_2)\in S^{**}(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$:

$$Z_{ij}=\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} b_0u_i^{q_i+\alpha_i}u_j^{1+\alpha_j}\varphi dxdt\,,$$

$$\widetilde{Z}_{ij}=\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R} b_0u_i^{q_i+\alpha_i}u_j^{1+\alpha_j}\varphi dxdt\,,$$

$$\widetilde{\widetilde{Z}}_{ij}=\int_{R^\theta}^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R} b_0u_i^{q_i+\alpha_i}u_j^{1+\alpha_j}\varphi dxdt\,,$$

$$\begin{aligned}D'(\delta(u_1,u_2)) &= \{(t_1,t_2)\in]-1,0[\times]-1,0[:\ t_1+t_2+1<\\ &< [(4\delta^2(u_1,u_2)(q_1+1)(q_2+1))^{-1}-1]t_1t_2\}\,,\end{aligned}$$

and

$$\begin{aligned}D'(\delta) &= \{(t_1,t_2)\in]-1,0[\times]-1,0[:\ t_1+t_2+1<\\ &< [(4\delta^2(q_1+1)(q_2+1))^{-1}-1]t_1t_2\}\end{aligned}$$

under assumption i_{52}).

Proposition 6.1 *Under assumption i_{61}), let $(u_1,u_2)\in S^{**}(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$, with $(\alpha_1,\alpha_2)\in D'(\delta(u_1,u_2))$, be a weak solution to problem (1.1). Then for $R>R_0$ and $2\delta(u_1,u_2)(\alpha_1+1)(q_2+1)|\alpha_2|^{-1}<\eta<[2\delta(u_1,u_2)(\alpha_2+1)(q_1+1)]^{-1}|\alpha_1|$ it follows that:*

$$\begin{aligned}&(\alpha_1+1)Z_{21}+(\alpha_2+1)Z_{12}+\frac{\alpha_1+1}{2}\left(\frac{|\alpha_1|}{q_1+1}-2\delta(u_1,u_2)(\alpha_2+1)\eta\right)\cdot\\ &\cdot\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R}\varphi u_1^{\alpha_1-1}u_2^{\alpha_2+1}A_{10}|\nabla u_1|^2dxdt+\\ &+\frac{\alpha_2+1}{2}\left(\frac{|\alpha_2|}{q_2+1}-2\delta(u_1,u_2)(\alpha_1+1)\eta^{-1}\right)\cdot\\ &\cdot\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R}\varphi u_1^{\alpha_1+1}u_2^{\alpha_2-1}A_{20}|\nabla u_2|^2dxdt+\\ &+\int_{|x|\le 2^{\frac{1}{\beta}}R}u_{10}^{\alpha_1+1}u_{20}^{\alpha_2+1}\varphi(x,0)dx\le\\ &\le\frac{\alpha_1+1}{q_1+1}\left(\frac{2}{|\alpha_1|}\right)^{q_1}(\lambda_1\widetilde{A}_1)^{q_1+1}Z_{12}+\frac{\alpha_2+1}{q_2+1}\left(\frac{2}{|\alpha_2|}\right)^{q_2}(\lambda_2\widetilde{A}_2)^{q_2+1}Z_{21}+\\ &-\int_{R^\theta}^{2^{\frac{1}{\mu}}R^\theta}\int_{|x|\le 2^{\frac{1}{\beta}}R}u_1^{\alpha_1+1}u_2^{\alpha_2+1}\varphi_tdxdt+\\ &+\sum_{h=1}^{2}\frac{1}{2|\alpha_h|}\int_0^{2^{\frac{1}{\mu}}R^\theta}\int_{R\le|x|\le 2^{\frac{1}{\beta}}R}u_1^{\alpha_1+1}u_2^{\alpha_2+1}\hat{A}_h\varphi^{-1}|\nabla\varphi|^2dxdt\,,\end{aligned}\tag{6.1}$$

with $\nu > 2$ in Definition (1.2) of φ.

Let us fix $\alpha_1, \alpha_2 \in]-1, 0[$ and

$$\overline{r}_1 = \frac{(\alpha_1+q_1)(\alpha_2+q_2)-(\alpha_1+1)(\alpha_2+1)}{(\alpha_1+q_1)(\alpha_2+1)-(\alpha_1+1)(\alpha_2+1)}\,,$$
$$\overline{r}_2 = \frac{(\alpha_1+q_1)(\alpha_2+q_2)-(\alpha_1+1)(\alpha_2+1)}{(\alpha_2+q_2)(\alpha_1+1)-(\alpha_1+1)(\alpha_2+1)}\,,$$
$$\frac{1}{\overline{r}_3} = 1-\frac{1}{\overline{r}_1}-\frac{1}{\overline{r}_2} = \frac{(q_1-1)(q_2-1)}{(\alpha_1+q_1)(\alpha_2+q_2)-(\alpha_1+1)(\alpha_2+1)}\,,$$

and let us suppose:
$i_{62})$ $\quad b_0^{1-\overline{r}_3} \in L^1_{\text{loc}}((\mathbf{R}^N \backslash B_{R_0}) \times [0,+\infty[) \cap L^1_{\text{loc}}(\mathbf{R}^N \times [R_0,+\infty[)\,.$

Taking into account inequality (6.1), we establish

Proposition 6.2 *Assuming $i_{61})$, $i_{62})$, if $(u_1,u_2) \in S^{**}(\gamma_1,\gamma_2,\alpha_1,\ \alpha_2,u_{10},u_{20})$, with $(\alpha_1,\alpha_2) \in D'(\delta(u_1,u_2))$, is a weak solution to problem (1.1), then for $R > R_0$:*

$$(\alpha_1+1)Z_{21}+(\alpha_2+1)Z_{12}+\int_{|x|\le 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}u_{20}^{\alpha_2+1}\varphi(x,0)dx \le$$
$$\le \frac{\alpha_1+1}{q_1+1}\left(\frac{2}{|\alpha_1|}\right)^{q_1}(\lambda_1\widetilde{A}_1)^{q_1+1}Z_{12}+\frac{\alpha_2+1}{q_2+1}\left(\frac{2}{|\alpha_2|}\right)^{q_2}(\lambda_2\widetilde{A}_2)^{q_2+1}Z_{21}+$$
$$+\widetilde{Z}_{21}^{\frac{1}{\overline{r}_1}}\widetilde{Z}_{12}^{\frac{1}{\overline{r}_2}}\sum_{h=1}^{2}\frac{1}{2|\alpha_h|}(I_h(\overline{r}_3))^{\frac{1}{\overline{r}_3}}+\widetilde{\widetilde{Z}}_{21}^{\frac{1}{\overline{r}_1}}\widetilde{\widetilde{Z}}_{12}^{\frac{1}{\overline{r}_2}}(I_0(\overline{r}_3))^{\frac{1}{\overline{r}_3}}\,, \tag{6.2}$$

with $\nu > 2\overline{r}_3$ in Definition (1.2) of φ.

Let us introduce the notations:

$$K_1(\alpha_1,\alpha_2,\lambda_1) = (\alpha_2+1)-\frac{\alpha_1+1}{q_1+1}\left(\frac{2}{|\alpha_1|}\right)^{q_1}(\lambda_1\widetilde{A}_1)^{q_1+1}\,,$$
$$K_2(\alpha_1,\alpha_2,\lambda_2) = (\alpha_1+1)-\frac{\alpha_2+1}{q_2+1}\left(\frac{2}{|\alpha_2|}\right)^{q_2}(\lambda_2\widetilde{A}_2)^{q_2+1}.$$

Theorem 6.1 *Under conditions $i_{61})$, $i_{52})$, if for a pair $(\alpha_1,\alpha_2) \in D'(\delta)$ condition $i_{62})$ holds and if*

$$\lim_{R\to+\infty}\inf\left[\sum_{h=1}^{2}I_h(\overline{r}_3)+I_0(\overline{r}_3)\right] < +\infty\,,$$

then problem (1.1) has no weak solutions in $S^{**}(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$ whenever $K_i(\alpha_1,\alpha_2,\lambda_i) > 0$.

Corollary 6.1 *Under assumptions $i_{61})$, $i_{52})$, the thesis of Theor. 6.1 holds if for a pair $(\alpha_1,\alpha_2) \in D'(\delta)$*

$$b_0^{1-\overline{r}_3} \in L^1(\mathbf{R}^N \times]R_0,+\infty[)$$

and

$$b_0^{1-\overline{r}_3}\hat{A}_i^{\overline{r}_3} \in L^1((\mathbf{R}^N \backslash B_{R_0})\times]0,+\infty[)\,.$$

Let $(u_1,u_2) \in S^{**}(\gamma_1,\gamma_2,\alpha_1,\alpha_2,u_{10},u_{20})$, with $(\alpha_1,\alpha_2) \in D'(\delta(u_1,u_2))$, be a weak solution to problem (1.1). For the function

$$G(t_1,t_2) = \overline{r}_3(t_1,t_2) - 1 = \frac{(q_2-1)(t_1+1)+(q_1-1)(t_2+1)}{(q_1-1)(q_2-1)}$$

we have:

$$\max\{G(t_1,t_2)\,:\,(t_1,t_2)\in \overline{D'(\delta(u_1,u_2))}\} = \\ = \frac{1}{\min\{q_1,q_2\}-1} = \begin{cases} G(-1,0) & \text{if} \quad q_1 \geq q_2 \\ G(0,-1) & \text{if} \quad q_1 \leq q_2\,. \end{cases}$$

Then, if we suppose that

$i_{63})$ $$\int_0^2 \int_{|y|\leq 2} (b_0(y,\tau))^{-\frac{1}{\min\{q_1,q_2\}-1}}\,dyd\tau < +\infty$$

and assumptions $i_{55})$, $i_{56})$ hold, then from inequality (6.2) we get

$$K_1(\alpha_1,\alpha_2,\lambda_1)Z_{12} + K_2(\alpha_1,\alpha_2,\lambda_2)Z_{21} + \int_{|x|\leq 2^{\frac{1}{\beta}}R} u_{10}^{\alpha_1+1}u_{20}^{\alpha_2+1}\varphi(x,0)dx \leq \\ \leq \widetilde{d}R^{\frac{N+(1-\overline{r}_3)(\overline{\theta}+L_0(\overline{\theta}))}{\overline{r}_3}}\left[\widetilde{Z}_{21}^{\frac{1}{\overline{r}_1}}\widetilde{Z}_{12}^{\frac{1}{\overline{r}_2}} + \widetilde{\widetilde{Z}}_{21}^{\frac{1}{\overline{r}_1}}\widetilde{\widetilde{Z}}_{12}^{\frac{1}{\overline{r}_2}}\right],$$

where $\widetilde{d}$ is a positive constant, independent of R and $\nu > 2\dfrac{\min\{q_1,q_2\}}{\min\{q_1,q_2\}-1}$ in Definition (1.2) of φ.

We can finally formulate the following theorems.

Theorem 6.2 *Under assumptions $i_{61})$, $i_{63})$, $i_{55})$, $i_{56})$, we have $S^{**}(2,2,u_{10},u_{20}) = \emptyset$ when:*

- $q_1 \geq q_2\,,\quad N < \dfrac{\overline{\theta}+L_0(\overline{\theta})}{q_2-1}\,,\quad \lambda_1 \geq 0\,,\ \lambda_2 = 0;$
- $q_1 \leq q_2\,,\quad N < \dfrac{\overline{\theta}+L_0(\overline{\theta})}{q_1-1}\,,\quad \lambda_1 = 0\,,\quad \lambda_2 \geq 0;$
- $N \geq \dfrac{\overline{\theta}+L_0(\overline{\theta})}{\min\{q_1,q_2\}-1}\,,\quad \lim_{|x|\to+\infty}\inf|x|^{-s_i}u_{i0}(x) > 0$
  with $s_2 > -\dfrac{\overline{\theta}+L_0(\overline{\theta})}{q_2-1}$ $\left[\textit{resp. } s_1 > -\dfrac{\overline{\theta}+L_0(\overline{\theta})}{q_1-1}\right]$,
  $\lambda_1 \geq 0\,,\quad \lambda_2 = 0\quad [\textit{resp. } \lambda_1 = 0\,,\ \lambda_2 \geq 0]$*;*

- $N \geq \dfrac{\overline{\theta} + L_0(\overline{\theta})}{\min\{q_1, q_2\} - 1}\,, \quad \lim_{|x|\to+\infty} \inf |x|^{-s_i} u_{i0}(x) > 0$
  with $s_1 + s_2 > -\dfrac{q_1 + q_2 - 2}{(q_1 - 1)(q_2 - 1)}(\overline{\theta} + L_0(\overline{\theta}))\,,$
  $1 - \dfrac{2^{q_i}}{q_i + 1}(\lambda_i \widetilde{A}_i)^{q_i+1} > 0\,.$

Theorem 6.3 *Under assumptions* i_{61}*),* i_{63}*),* i_{52}*),* i_{55}*),* i_{56}*), we have:*

- *if* $q_1 \geq q_2$ *and* $N < \dfrac{\overline{\theta} + L_0(\overline{\theta})}{q_2 - 1}$*, then for any* $\lambda_1 \geq 0$ *there exists* $(\alpha_1, \alpha_2) \in D'(\delta)$ *with* $K_1(\alpha_1, \alpha_2, \lambda_1) > 0$ *such that problem (1.1) has no weak solutions in* $S^{**}(\gamma_1, \gamma_2,\, \alpha_1, \alpha_2, u_{10}, u_{20})$ *when* $K_2(\alpha_1, \alpha_2, \lambda_2) > 0$*;*

- *if* $q_1 \leq q_2$ *and* $N < \dfrac{\overline{\theta} + L_0(\overline{\theta})}{q_1 - 1}$*, then for any* $\lambda_2 \geq 0$ *there exists* $(\alpha_1, \alpha_2) \in D'(\delta)$ *with* $K_2(\alpha_1, \alpha_2, \lambda_2) > 0$ *such that problem (1.1) has no weak solutions in* $S^{**}(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ *when* $K_1(\alpha_1, \alpha_2, \lambda_1) > 0$*;*

- *if* $N \geq \dfrac{\overline{\theta} + L_0(\overline{\theta})}{\min\{q_1, q_2\} - 1}$ *and* $\lim_{|x|\to+\infty} \inf |x|^{-s_i} u_{i0}(x) > 0$ *with* $(\alpha_1 + 1)s_1 +$ $+ (\alpha_2 + 1)s_2 > -\dfrac{(q_2 - 1)(\alpha_1 + 1) + (q_1 - 1)(\alpha_2 + 1)}{(q_1 - 1)(q_2 - 1)}(\overline{\theta} + L_0(\overline{\theta}))$ *for some* $(\alpha_1, \alpha_2) \in D'(\delta)$*, then problem (1.1) has no weak solutions in* $S^{**}(\gamma_1, \gamma_2, \alpha_1, \alpha_2, u_{10}, u_{20})$ *when* $K_i(\alpha_1, \alpha_2, \lambda_i) > 0$*.*

In the case of the system:

$$
\begin{aligned}
&\frac{\partial u_1}{\partial t} - \Delta u_1 = u_1 u_2^{q_2 - 1} \\
&\qquad\qquad\qquad \text{in } \mathbf{R}^N \times]0, +\infty[\quad (q_i > 1,\ u_{i0} \in L^\infty_{\text{loc}}(\mathbf{R}^N),\ u_{i0} \geq 0) \\
&\frac{\partial u_2}{\partial t} - \Delta u_2 = u_1^{q_1 - 1} u_2 \\
&u_i \gneqq 0,\ u_i(x, 0) = u_{i0}(x) \quad \text{in } \mathbf{R}^N\,,
\end{aligned} \tag{6.3}
$$

from Theor. 6.3 we get that $S^{**}(2, 2, u_{10}, u_{20}) = \emptyset$, when

- $\dfrac{N}{2} < \dfrac{1}{\min\{q_1, q_2\} - 1}\,,$

- $\dfrac{N}{2} \geq \dfrac{1}{\min\{q_1, q_2\} - 1}$ and, for example, $\lim_{|x|\to+\infty} \inf u_{i0}(x) > 0\,.$

If u_{i0} is also continuous in $\mathbf{R}^N$ and infinitesimal for $|x| \to +\infty$, a theorem by Escobedo and Levine ([2], [3]) ensures that system (6.3) has no "regular" solutions when $\dfrac{N}{2} \leq \dfrac{1}{\min\{q_1, q_2\} - 1}$.

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