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Sobolev Inequalities and Ellipticity of Planar Linear Hamiltonian Systems

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Abstract

In this paper we will establish two different classes of ellipticity criteria, called the L^p criteria and the L^p-L^q criteria respectively, for planar linear Hamiltonian systems with periodic coefficients. The criteria are explicitly expressed using the L^p and L^q norms of coefficients and some known Sobolev constants. These results can be considered as the extensions of the famous Lyapunov stability criterion for Hill's equations.

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1 Introduction and the L^1_+ criterion

Let $a(t)$ be a T-periodic, integrable potential, i.e., $a(t) \in L^1(\mathbb{S}_T)$, $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$. Consider the Hill's equation

$$x'' + a(t)x = 0, \qquad x \in \mathbb{R}. \tag{1.1}$$

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The famous stability criterion of Lyapunov asserts that when $a(t)$ satisfies the following two conditions

$$\text{either} \quad \overline{a} > 0, \qquad \text{or,} \quad \overline{a} = 0 \text{ and } a(t) \not\equiv 0, \tag{1.2}$$

$$T \cdot \|a_+\|_{1,T} \le 4, \tag{1.3}$$

equation (1.1) is elliptic and therefore is stable in the sense of Lyapunov [8]. Here and henceforth $a_+(t) = \max(a(t), 0)$ is the non-negative part of $a(t)$, $\|u\|_{p,I} = \|u\|_{L^p(I)}$ is the L^p norm for $p \in [1, \infty]$, and $\overline{a} = (1/T)\int_0^T a(t)\,\mathrm{d}t$ is the mean value. For a T-periodic function $a(t) \in L^p(\mathbb{S}_T)$, $\|u\|_{p,T}$ is $\|u\|_{L^p(\mathbb{S}_T)}$.

We call (1.2)-(1.3) the L^1 criterion for Hill's equations. This criterion can be explained using eigenvalues of Hill's operators. Recall that the T-periodic and T-anti-periodic eigenvalues of

$$x'' + (\lambda + a(t))x = 0 \tag{1.4}$$

can be written as a whole

$$-\infty < \overline{\lambda}_0(a) < \underline{\lambda}_1(a) \le \overline{\lambda}_1(a) < \cdots < \underline{\lambda}_m(a) \le \overline{\lambda}_m(a) < \cdots$$

where $\underline{\lambda}_m(a)$, $\overline{\lambda}_m(a)$ are T-periodic eigenvalues (T-anti-periodic eigenvalues, respectively) of (1.4) when m is even (when m is odd, respectively). See [4, 9]. Now condition (1.2) implies that the smallest T-periodic eigenvalue $\overline{\lambda}_0(a) < 0$, while condition (1.3) implies that the smallest T-anti-periodic eigenvalue $\underline{\lambda}_1(a) > 0$. Thus $0 \in (\overline{\lambda}_0(a), \underline{\lambda}_1(a))$ and equation (1.1) is actually in the first stability zone. For the details, see [25]. Under assumption (1.2) on $a(t)$, the bound 4 in (1.3) is optimal to guarantee that (1.1) is in the first stability zone. The constant 4 in (1.3) is actually the best Sobolev constant of the inequality

$$C\|u\|^2_{\infty,[0,1]} \le \|u'\|^2_{2,[0,1]} \qquad \text{for all } u \in H^1_0(0,1).$$

In [16, Lemma 5.2], the authors have used some different Sobolev inequalities to establish the following so-called L^1_+ criterion for the canonical form of planar linear Hamiltonian systems [7, Section 7.1]

$$x' = b(t)y, \qquad y' = -a(t)x. \tag{1.5}$$

That is, if $a(t),\ b(t) \in L^\infty(\mathbb{S}_T)$ satisfy $\operatorname{ess\,inf}_t a(t) > 0$, $\operatorname{ess\,inf}_t b(t) > 0$ and

$$\|a\|_{1,T} \cdot \|b\|_{1,T} \le 4, \tag{1.6}$$

then (1.5) is elliptic and is therefore Lyapunov stable. Moreover, the constant 4 in condition (1.6) is also optimal. Note that when $b(t) \equiv 1$, system (1.5) is equivalent to equation (1.1), while condition (1.6) is almost the same as (1.3).

Instead of the L^1 norms used in (1.3), the optimal ellipticity conditions using the L^p, $p \in (1, \infty]$, norms of a_+ have been established [25]. Some further generalizations to the so-called p-Laplacian and higher order stability zones can be found in [23].

The purpose of this paper is to give the corresponding ellipticity criteria for planar linear Hamiltonian systems (1.5) using some L^p and L^q norms of coefficients. The criteria obtained are called respectively the L^p criteria (Theorem 3.1) and the L^p-L^q criteria (Theorem 4.1). The bounds in the L^p criteria of Theorem 3.1 we have found are also optimal. In order to describe the L^p-L^q criteria, write

$$\mathcal{N}_{r,S}(\psi) := S^{1-2/r}\|\psi\|_{r,S}^2/|\overline{\psi}| \tag{1.7}$$

for $\psi(t) \in L^r(\mathbb{R}/S\mathbb{Z})$ with $\overline{\psi} \neq 0$, $r \in [1,\infty]$ and $S > 0$. Let $a(t) \in L^p(\mathbb{S}_T)$ and $b(t) \in L^q(\mathbb{S}_T)$ for some $p,\ q \in [1,\infty]$. The L^p-L^q criteria are simply

$$\overline{a}\cdot\overline{b} > 0 \text{ and } \mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b) < \mathbf{L}(p^*,q^*) \Longrightarrow (1.5) \text{ is elliptic.} \tag{1.8}$$

Here p^* and q^* are conjugate exponents of p and q, and $\mathbf{L}(u,v)$ can be expressed explicitly using some Sobolev constants in Section 2. When the sign of the first condition of (1.8) is reversed, the second condition yields the hyperbolicity of (1.5), see also Theorem 4.1. Different from the L^p criteria, the bounds $\mathbf{L}(p^*,q^*)$ of (1.8) are not optimal. In fact, as $p,\ q$ vary over $[1,\infty]$, $\mathbf{L}(p^*,q^*)$ varies on the interval $[1,4]$. Although the optimal bounds, denoted by $\mathbf{R}(p,q)$, for the L^p-L^q criteria are unknown to the author, $\mathbf{R}(p,q)$ must satisfy $\mathbf{R}(p,q) < \pi^2$. Hence criteria (1.8) are fairly satisfactory.

In Section 5, we will give some methods to improve the L^p-L^q criteria. This will lead to some non-standard eigenvalue problems which are linear for the case $p = q = 2$. See Theorem 5.1 and Corollary 5.1. Some problems concerning the optimal L^p-L^q criteria remain open.

Theoretically, ellipticity of (1.5) can be explained using the location of eigenvalues of the corresponding Dirac operator of (1.5). For details, see Section 4.2. Of course eigenvalues are involved both coefficients $a(t)$ and $b(t)$. It is remarkable that in the construction of the L^p-L^q criteria, we have successfully decomposed the problem into two eigenvalue problems which are modifications of the Hill's operators, each depending on only one of the coefficients $a(t)$ and $b(t)$. See Section 5.

In recent years, the basic Sobolev inequalities and Sobolev constants used in this paper have been shown to have many interesting applications to different problems involving equations (not necessarily linear) such as the non-degeneracy of linear ODEs, the so-called p-Laplacian and linear delay equations and corresponding PDEs [10, 16, 23], the existence and multiplicity of boundary value problems of semilinear and some superlinear equations [6, 16], lower and upper bounds of rotation numbers [2, 3], and the Lyapunov stability of linear and nonlinear Lagrangian and Hamiltonian systems of degree of freedom 3/2 [2, 11, 12, 13, 14, 15, 20, 24, 25]. The ellipticity results for planar linear Hamiltonian systems in this paper will be helpful for the understanding of these problems, especially the Lyapunov stability of periodic solutions of planar nonlinear Hamiltonian systems. Some extensions of these Sobolev inequalities have also many interesting applications for PDEs [6, 17].

2 Sobolev inequalities and canonical forms of Hamiltonian systems

2.1 Sobolev inequalities and Sobolev constants

Given an exponent $q \in [1, \infty]$. The best constant in the Sobolev inequality

$$C\|u\|_{q,[0,1]} \le \|u'\|_{2,[0,1]} \qquad \text{for all } u \in H_0^1(0,1) = W_0^{1,2}(0,1) \tag{2.1}$$

is denoted by $\mathbf{M}(q)$. That is,

$$\mathbf{M}(q) = \min_{u \in H_0^1(0,1),\ u \neq 0} \frac{\|u'\|_{2,[0,1]}}{\|u\|_{q,[0,1]}}. \tag{2.2}$$

The explicit formula for $\mathbf{M}(q)$ is known. See, for example, [19] and [22]. That is,

$$\mathbf{M}(q) = \begin{cases} \left(\frac{2\pi}{q}\right)^{1/2} \left(\frac{2}{q+2}\right)^{1/2-1/q} \frac{\mathbf{\Gamma}(1/q)}{\mathbf{\Gamma}(1/2+1/q)} & \text{for } 1 \le q < \infty, \\ 2 & \text{for } q = \infty, \end{cases}$$

where $\mathbf{\Gamma}(\cdot)$ is the Gamma function of Euler. In particular, $\mathbf{M}(1) = \sqrt{12}$, $\mathbf{M}(2) = \pi$ and $\mathbf{M}(\infty) = 2$. Moreover, as a function of the exponent, $\mathbf{M}(q)$ is strictly decreasing in $q \in [1, \infty]$.

For the case $q \in [1, \infty)$, the minimizers of (2.2) are smooth. However, in case $q = \infty$, the corresponding minimizers are $c\,u_\infty(t)$, $c \neq 0$, where $u_\infty(t) = t$ for $t \in [0, 1/2|$, and $u_\infty(t) = 1 - t$ for $t \in [1/2, 1]$. Hence, if $u \in H_0^1(0,1)\backslash\{0\}$ is C^1 on $[0, 1]$, one has actually the strict inequality $2\|u\|_{\infty,[0,1]} < \|u'\|_{2,[0,1]}$. This is why inequality (1.3) is allowed not to be strict for $p = 1$, and inequality (3.1) in the next section should be strict for $p \in (1, \infty]$.

We need some Sobolev inequalities more general than (2.1) and their constants. Let $q,\ p \in [1, \infty]$. Introduce the best Sobolev constant

$$\mathbf{N}(q,p) = \min_{u \in W_0^{1,p}(0,1),\ u \neq 0} \frac{\|u'\|_{p,[0,1]}}{\|u\|_{q,[0,1]}}. \tag{2.3}$$

Evidently, $\mathbf{M}(q) = \mathbf{N}(q, 2)$.

- Case $p \in (1, \infty)$. The explicit formula of $\mathbf{N}(q,p)$ is known [19, 22]. That is,

$$\mathbf{N}(q,p) = \begin{cases} \frac{2(q+p^*)^{1/q+1/p^*-1}}{(p^*)^{1/q} q^{1/p^*}} \mathbf{B}(1/q, 1/p^*) & \text{for } 1 \le q < \infty, \\ 2 & \text{for } q = \infty. \end{cases} \tag{2.4}$$

where $\mathbf{B}(\cdot,\cdot)$ is the Beta function of Euler.

- Case $p = \infty$. Let $u \in W_0^{1,\infty}(0,1)$. As $u(0) = u(1) = 0$, one has

$$\begin{aligned} |u(t)| &= \left|\int_0^t u'(s)\,\mathrm{d}s\right| \le t\|u'\|_{\infty,[0,1]} && \text{for } t \in [0, 1/2], \\ |u(t)| &= \left|\int_t^1 u'(s)\,\mathrm{d}s\right| \le (1-t)\|u'\|_{\infty,[0,1]} && \text{for } t \in [1/2, 1]. \end{aligned}$$

Thus

$$\begin{aligned}\|u\|_{q,[0,1]} &\le \left(\int_0^{1/2} t^q\,\mathrm{d}t + \int_{1/2}^1 (1-t)^q\,\mathrm{d}t\right)^{1/q} \|u'\|_{\infty,[0,1]}\\ &= 2^{-1}(1+q)^{-1/q}\|u'\|_{\infty,[0,1]} \quad \text{for } q\in[1,\infty),\\ \|u\|_{\infty,[0,1]} &\le 2^{-1}\|u'\|_{\infty,[0,1]}.\end{aligned}$$

On the other hand, for $u = u_\infty(t) \in W_0^{1,\infty}(0,1)$, both of these inequalities become equalities. Hence

$$\mathbf{N}(q,\infty) = 2(1+q)^{1/q} \ \ \big(= \lim\nolimits_{p\uparrow\infty}\mathbf{N}(q,p)\big), \qquad q\in[1,\infty], \tag{2.5}$$

and the corresponding minimizers of (2.3) with $p=\infty$ are still $c\,u_\infty(t)$, $c\neq 0$. Here we understand $\mathbf{N}(\infty,\infty) = \lim_{q\uparrow\infty}\mathbf{N}(q,\infty) = 2$.

- Case $p=1$. Let $u\in W_0^{1,1}(0,1)$. One has, for $t\in[0,1]$,

$$|u(t)| \le \int_0^t |u'(s)|\,\mathrm{d}s =: U, \qquad |u(t)| \le \int_t^1 |u'(s)|\,\mathrm{d}s =: V.$$

Then $U+V \equiv \|u'\|_{1,[0,1]}$ and $|u(t)| \le (U+V)/2 \equiv \|u'\|_{1,[0,1]}/2$. Hence $\|u\|_{\infty,[0,1]} \le \|u'\|_{1,[0,1]}/2$ and $\mathbf{N}(\infty,1)\ge 2$. On the other hand, let $u=u_\infty(t)\in W_0^{1,1}(0,1)$. One has $\|u'_\infty\|_{1,[0,1]} = 1$ and $\|u_\infty\|_{\infty,[0,1]} = 1/2$. Hence $\mathbf{N}(\infty,1) = 2$ and the corresponding minimizers of (2.3), with $(q,p)=(\infty,1)$, are again $c\,u_\infty(t)$, $c\neq 0$. As $\mathbf{N}(q,1)$ is non-increasing in $q\in[1,\infty]$, one has $\mathbf{N}(q,1)\ge\mathbf{N}(\infty,1) = 2$ for all $q\in[1,\infty)$. Let $q\in[1,\infty)$ be given. Take a smooth function $u(t)\in W_0^{1,1}(0,1)$ such that (i) $u(1-t)\equiv u(t)$, (ii) $u(t)$ is increasing in $t\in[0,1/2]$, and (iii) $u(0)=0$ and $u(t)=1/2$ for $\varepsilon < t \le 1/2$ with $\varepsilon>0$ small. Then we have

$$\begin{aligned}\|u\|_{q,[0,1]} &= \left(2\int_0^{1/2} u^q(t)\,\mathrm{d}t\right)^{1/q} = \left(2\int_0^{\varepsilon} u^q(t)\,\mathrm{d}t + 2\int_\varepsilon^{1/2} u^q(t)\,\mathrm{d}t\right)^{1/q}\\ &= \big(O(\varepsilon) + 2(1/2-\varepsilon)/2^q\big)^{1/q} = 1/2 + O(\varepsilon),\\ \|u'\|_{1,[0,1]} &= 2\int_0^{1/2} u'(t)\,\mathrm{d}t = 2(u(1/2)-u(0)) = 1.\end{aligned}$$

By (2.3) again, one has $\mathbf{N}(q,1)\le 2$. Hence $\mathbf{N}(q,1) = 2$ for all $q\in[1,\infty)$. In conclusion,

$$\mathbf{N}(q,1) = 2 \ \ \big(= \lim\nolimits_{p\downarrow 1}\mathbf{N}(q,p)\big) \qquad \text{for } q\in[1,\infty]. \tag{2.6}$$

Lemma 2.1 *The Sobolev constants* $\mathbf{N}(q,p)$ *are given by* (2.4)*,* (2.5) *and* (2.6)*.*

By the Hölder inequality and the defining equality (2.3), $\mathbf{N}(q,p)$ is non-increasing in $q\in[1,\infty]$ and is non-decreasing in $p\in[1,\infty]$. Hence we have from (2.5) and (2.6)

$$2 = \mathbf{N}(\infty,1) \le \mathbf{N}(q,p) \le \mathbf{N}(1,\infty) = 4 \qquad \text{for all } q,\ p\in[1,\infty].$$

Let $q,\ p \in [1,\infty]$. Suppose that $I=[a,b]$ is a finite interval of length $|I|=b-a$. By a scaling technique, the corresponding Sobolev constant

$$\mathbf{N}(q,p,I)=\mathbf{N}(q,p,|I|):=\min_{u\in W_0^{1,p}(I),\ u\neq 0}\frac{\|u'\|_{p,I}}{\|u\|_{q,I}}$$

is given by

$$\mathbf{N}(q,p,|I|)=\mathbf{N}(q,p)/|I|^{1/q+1/p^*}. \tag{2.7}$$

Thus we have the following optimal Sobolev inequality

$$\frac{\mathbf{N}(q,p)}{|I|^{1/q+1/p^*}}\|u\|_{q,I}\le \|u'\|_{p,I} \qquad \text{for all } u\in W_0^{1,p}(I),\ p\in[1,\infty]. \tag{2.8}$$

2.2 Canonical forms of Hamiltonian systems

Consider the planar linear Hamiltonian system

$$x'=JA(t)x, \tag{2.9}$$

where

$$J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \qquad A(t)=\begin{pmatrix} \alpha(t) & \gamma(t)\\ \gamma(t) & \beta(t)\end{pmatrix}.$$

Here $\alpha(t),\ \beta(t),\ \gamma(t)$ are smooth T-periodic functions. The (period T-) Poincaré matrix is

$$M_T=\begin{pmatrix} \phi_1(T) & \phi_2(T)\\ \psi_1(T) & \psi_2(T)\end{pmatrix},$$

where $(\phi_1(t),\psi_1(t))$ and $(\phi_2(t),\psi_2(t))$ are solutions of (2.9) satisfying $\phi_1(0)=1$, $\phi_2(0)=0$ and $\psi_1(0)=0$, $\psi_2(0)=1$, respectively. The eigenvalues $\nu_{1,2}$ of M_T are called the Floquet multipliers of (2.9). Obviously $\nu_1\cdot\nu_2=1$. We say that (2.9) is parabolic, hyperbolic, or, elliptic, if $\nu_1=\nu_2=\pm1$, $|\nu_i|\neq 1$, or, $|\nu_i|=1$ and $\nu_i\neq\pm1$, respectively.

It is well-known that (2.9) is Lyapunov stable iff (2.9) is either elliptic or is parabolic with the additional property that all solutions of (2.9) are T-periodic in case $\nu_1=\nu_2=1$, or T-anti-periodic in case $\nu_1=\nu_2=-1$. See, for example, [4, Theorem 7.2]. Here the T-anti-periodicity of a solution $x(t)$ is $x(t+T)\equiv -x(t)$. We say that system (2.9) is non-degenerate if $\nu_{1,2}\neq 1$. The non-degeneracy is the same as that equation (2.9) has only the trivial T-periodic solution.

Lemma 2.2 [7] *There exists a T-periodic, smooth function $t\to\varphi(t)$ such that the change of variables $x=R_{\varphi(t)}z$ will transform* (2.9) *into the canonical form*

$$z'=JB(t)z$$

with $B(t)=\operatorname{diag}(a(t),b(t))$ being diagonal. Explicitly, in coordinates $z=(x,y)^T$, this reads as (1.5).

Here, for $\theta \in \mathbb{R}$, R_θ denotes the rigid rotation

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The function $\varphi(t)$ is constructed explicitly in [7, Section 7.1]. In case $A(t)$ is T-periodic, by checking the construction there, one can see that $\varphi(t)$ and $a(t)$, $b(t)$ are also T-periodic.

Example 2.1 Let $a_0,\ b_0 \in \mathbb{R}$ be constants and consider the following system

$$x' = b_0 y, \qquad y' = -a_0 x. \tag{2.10}$$

Suppose first that $a_0 \cdot b_0 \neq 0$. Denote $\gamma = |a_0 \cdot b_0|^{1/2} > 0$ and $\delta = |b_0/a_0|^{1/2} > 0$. The solution of (2.10) with $(x(0), y(0)) = (x_0, y_0)$ is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\gamma t) & \delta\sin(\gamma t) \\ -\delta^{-1}\sin(\gamma t) & \cos(\gamma t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

in case $a_0 \cdot b_0 > 0$, and by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cosh(\gamma t) & \delta\sinh(\gamma t) \\ \delta^{-1}\sinh(\gamma t) & \cosh(\gamma t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

in case $a_0 \cdot b_0 < 0$.

When $a_0 = 0$ or $b_0 = 0$, the solutions can be expressed using the corresponding limits.

Let us now consider (2.10) as a T-periodic system. Denote $\theta = T|a_0 \cdot b_0|^{1/2}$. From the expression of general solutions, one has

• System (2.10) is hyperbolic iff $a_0 \cdot b_0 < 0$. In this case, the Floquet multipliers are $\nu_{1,2}(0) = \cosh\theta \pm \sinh\theta$.

• System (2.10) is parabolic iff $Ta_0 \cdot Tb_0 = ((n-1)\pi)^2$ for some $n \in \mathbb{N}$.

• System (2.10) is elliptic iff $((n-1)\pi)^2 < Ta_0 \cdot Tb_0 < (n\pi)^2$ for some $n \in \mathbb{N}$. In this case, the Floquet multipliers are $\nu_{1,2}(0) = \cos\theta \pm i\sin\theta$. In particular, if

$$0 < Ta_0 \cdot Tb_0 < \pi^2, \tag{2.11}$$

the T-periodic system (2.10) is elliptic. Here the upper bound π^2 of condition (2.11) is optimal.

• System (2.10) is degenerate iff $Ta_0 \cdot Tb_0 = (2(n-1)\pi)^2$ for some $n \in \mathbb{N}$.

3 The L^p criteria for Hamiltonian systems

By Lemma 2.2, in the following we need only to consider linear systems of the form (1.5) where $a(t)$ and $b(t)$ are T-periodic, integrable functions. A crucial assumption in this section is that at least one of the coefficients $a(t)$ and $b(t)$ of system (1.5) does not change sign. By the transformation $t \to -t$ and/or the transformation $(x, y) \to$

$(-y, x)$, without loss of generality, we can assume that $b(t)$ satisfies $\mathrm{ess\,inf}_t b(t) \geq 0$. More restrict, we always assume $b(t)$ in (1.5) is from the following set

$$L^1_+(\mathbb{S}_T) := \left\{ b(t) \in L^1(\mathbb{S}_T) : \mathrm{ess\,inf}_t b(t) > 0 \right\}.$$

Note that $b \in L^1_+(\mathbb{S}_T)$ implies that $1/b \in L^\infty(\mathbb{S}_T)$.

Recall that the extension of criterion (1.2)-(1.3) to L^p potentials is as follows.

Lemma 3.1 [25, Theorem 1] **(L^p criteria for Hill's equations)** *Let $a(t) \in L^p(\mathbb{S}_T)$, $p \in (1, \infty]$. If $a(t)$ satisfies* (1.2) *and*

$$T^{1+1/p^*} \cdot \|a_+\|_{p,T} < \mathbf{M}^2(2p^*), \tag{3.1}$$

equation (1.1) *is in the first stability zone. Moreover, the bounds of* (3.1) *are also optimal.*

Note that when $p = 1$, $p^* = \infty$ and $\mathbf{M}^2(\infty) = 4$. That is, (3.1) with $p = 1$ is essentially same as (1.3).

Let us now go back to system (1.5) where $b(t) \in L^1_+(\mathbb{S}_T)$. Define

$$B(t) = \int_0^t b(s)\, \mathrm{d}s, \qquad \hat{T} = B(T) = \|b\|_{1,T} > 0.$$

Then $B(t)$ is increasing in t. Moreover, by the T-periodicity of $b(t)$, one has $B(t + T) \equiv B(t) + \hat{T}$. Using the change of times $s = B(t)$, system (1.5) is transformed into

$$\frac{\mathrm{d}\hat{x}(s)}{\mathrm{d}s} = \hat{y}(s), \qquad \frac{\mathrm{d}\hat{y}(s)}{\mathrm{d}s} = -\hat{q}(s)\hat{x}(s), \tag{3.2}$$

where $\hat{x}(s) = x(B^{-1}(s))$, $\hat{y}(s) = y(B^{-1}(s))$, and the coefficient

$$\hat{q}(s) = a(B^{-1}(s))/b(B^{-1}(s))$$

is $\hat{T}$-periodic. It is obvious that system (3.2) shares the same ellipticity with the original system (1.5). Note that system (3.2) is equivalent to the following Hill's equation

$$\frac{\mathrm{d}^2\hat{x}(s)}{\mathrm{d}s^2} + \hat{q}(s)\hat{x}(s) = 0. \tag{3.3}$$

One has the following trivial relations.

Lemma 3.2 *Let $b(t) \in L^1_+(\mathbb{S}_T)$ and $a(t) \in L^1(\mathbb{S}_T)$. Then*

- *Hamiltonian system* (1.5) *is stable iff Hill's equation* (3.3) *is stable.*
- *Hamiltonian system* (1.5) *is elliptic iff Hill's equation* (3.3) *is elliptic.*

Based on Lemma 3.2, we can apply Lemma 3.1 to obtain the following ellipticity criteria for systems (1.5).

Theorem 3.1 *Let* $b(t) \in L^1_+(\mathbb{S}_T)$ *and* $a(t) \in L^p(\mathbb{S}_T)$*,* $p \in [1,\infty]$*.*

- L^1 **criterion:** *For* $p = 1$*, system* (1.5) *is elliptic if* $a(t)$ *satisfies* (1.2) *and* $a(t)$*,* $b(t)$ *satisfy*

$$\|b\|_{1,T} \cdot \|a_+\|_{1,T} \le \mathbf{M}^2(\infty) = 4. \tag{3.4}$$

- L^p **criteria:** *For* $p \in (1,\infty]$*, system* (1.5) *is elliptic if* $a(t)$ *satisfies* (1.2) *and* $a(t)$*,* $b(t)$ *satisfy*

$$\|b\|_{1,T}^{1+1/p^*} \cdot \|a_+/b^{1/p^*}\|_{p,T} < \mathbf{M}^2(2p^*). \tag{3.5}$$

Furthermore, the bounds $\mathbf{M}^2(2p^*)$ *are optimal.*

Proof. Note that

$$\int_0^{\hat{T}} \hat{q}(s)\,\mathrm{d}s = \int_0^{\hat{T}} \frac{a(B^{-1}(s))}{b(B^{-1}(s))}\,\mathrm{d}s = \int_0^T a(t)\,\mathrm{d}t, \tag{3.6}$$

following the change of variables $s = B(t)$. If $a(t)$ satisfies (1.2), then $\hat{q}(s)$ satisfies (1.2) as well, with the period T replaced by $\hat{T}$. Similarly, one has

$$\|\hat{q}_+\|_{1,\hat{T}} = \|a_+\|_{1,T}, \qquad \|\hat{q}_+\|_{\infty,\hat{T}} = \|a_+\|_{\infty,T}, \tag{3.7}$$

$$\begin{aligned} \|\hat{q}_+\|_{p,\hat{T}} &= \left(\int_0^{\hat{T}} \left(\frac{a_+(B^{-1}(s))}{b(B^{-1}(s))} \right)^p \mathrm{d}s \right)^{1/p} \qquad \text{(by setting } s = B(t)\text{)} \\ &= \left(\int_0^T \frac{a_+^p(t)}{b^{p-1}(t)}\,\mathrm{d}t \right)^{1/p} = \|a_+/b^{1/p^*}\|_{p,T}, \qquad p \in (1,\infty). \end{aligned} \tag{3.8}$$

Note that $\hat{T} = \|b\|_{1,T}$. Now criteria (3.4) and (3.5) follow immediately Lemmas 3.1 and 3.2 and equalities (3.6)–(3.8). □

Criterion (3.4) corresponds essentially to (3.5) with $p = 1$. It is evident that the L^1 criterion (3.4), where $a(t)$ is allowed to be sign-changing, is a generalization of the L^1_+ criterion (1.6).

4 The L^p-L^q criteria for Hamiltonian systems

4.1 The L^p-L^q criteria

In Section 3, the ellipticity criteria have a severe restriction on $a(t)$ and $b(t)$. That is, at least one of $a(t)$ and $b(t)$ does not change sign. In this case, system (1.5) can be reduced to a Hill's equation. From this section, we will give some ellipticity criteria for systems (1.5) without such a restriction on coefficients.

Consider system (1.5) where $a(t)$ and $b(t)$ satisfy, for some $p,\ q \in [1,\infty]$,

$$a(t) \in L^p(\mathbb{S}_T), \qquad b(t) \in L^q(\mathbb{S}_T), \qquad \overline{a} \cdot \overline{b} \ne 0. \tag{4.1}$$

Suppose that $(x(t), y(t))$ is a T-periodic solution of (1.5). Observe that, under (4.1),

$$x(t) \in W^{1,q}(\mathbb{S}_T), \quad y(t) \in W^{1,p}(\mathbb{S}_T). \tag{4.2}$$

Let us write $(x(t), y(t)) = (\overline{x} + \tilde{x}(t), \overline{y} + \tilde{y}(t))$. By (4.2), $\tilde{x} \in \tilde{W}^{1,q}(\mathbb{S}_T)$ and $\tilde{y} \in \tilde{W}^{1,p}(\mathbb{S}_T)$, where

$$\tilde{W}^{1,r}(\mathbb{S}_T) := \left\{ u \in W^{1,r}(\mathbb{S}_T) : \overline{u} = 0 \right\}, \qquad r \in [1, \infty].$$

Using $(\tilde{x}, \tilde{y})$, system (1.5) reads as

$$\tilde{x}' = b(t)\tilde{y} + \overline{y}\, b(t), \qquad \tilde{y}' = -a(t)\tilde{x} - \overline{x}\, a(t). \tag{4.3}$$

By integrating two equations in (4.3) over one period, we get

$$\overline{x} = -\overline{a\tilde{x}}/\overline{a} = -\left(\int_0^T a\tilde{x} \right) / (T\overline{a}), \qquad \overline{y} = -\overline{b\tilde{y}}/\overline{b} = -\left(\int_0^T b\tilde{y} \right) / (T\overline{b}). \tag{4.4}$$

Now system (4.3) can be rewritten as

$$\tilde{x}'(t) = \frac{-b(t)}{T\overline{b}} \int_0^T b(s)(\tilde{y}(s) - \tilde{y}(t))\, \mathrm{d}s, \quad \tilde{y}'(t) = \frac{a(t)}{T\overline{a}} \int_0^T a(s)(\tilde{x}(s) - \tilde{x}(t))\, \mathrm{d}s. \tag{4.5}$$

The L^p-L^q criteria below are based on the following estimates. From (4.2) and the first equation of (4.5), we have

$$\begin{aligned} T|\overline{b}| \cdot \|\tilde{x}'\|_{q,T} &= \left\| b(\cdot) \cdot \int_0^T b(s)(\tilde{y}(s) - \tilde{y}(\cdot))\, \mathrm{d}s \right\|_{q,T} \\ &\le \|b\|_{q,T} \cdot \max_t \left| \int_0^T b(s)(\tilde{y}(s) - \tilde{y}(t))\, \mathrm{d}s \right| \qquad (4.6) \\ &\le \|b\|_{q,T} \cdot \max_t \left(\|b\|_{q,T} \|\tilde{y}(\cdot) - \tilde{y}(t)\|_{q^*,T} \right) \qquad (4.7) \\ &\le \|b\|_{q,T} \cdot \max_t \left(\|b\|_{q,T} \frac{T^{1/p^*+1/q^*}}{\mathbf{N}(q^*, p)} \|\tilde{y}'\|_{p,T} \right) \qquad (4.8) \\ &= \frac{T^{1/p^*+1/q^*}}{\mathbf{N}(q^*, p)} \|b\|_{q,T}^2 \|\tilde{y}'\|_{p,T}. \qquad (4.9) \end{aligned}$$

Here both (4.6) and (4.7) are obtained from the Hölder inequality, and (4.8) is resulted from the Sobolev inequality (2.8) because $\tilde{y}(\cdot) - \tilde{y}(t) \in W_0^{1,p}(t, t+T)$ and

$$\|\tilde{y}(\cdot) - \tilde{y}(t)\|_{q^*,T} = \|\tilde{y}(\cdot) - \tilde{y}(t)\|_{q^*,[t,t+T]} \le \frac{T^{1/p^*+1/q^*}}{\mathbf{N}(q^*, p)} \|\tilde{y}'\|_{p,T}.$$

From the second equation of (4.5), we can obtain an estimate similar to (4.9)

$$T|\overline{a}| \cdot \|\tilde{y}'\|_{p,T} \le \frac{T^{1/q^*+1/p^*}}{\mathbf{N}(p^*, q)} \|a\|_{p,T}^2 \|\tilde{x}'\|_{q,T}. \tag{4.10}$$

Using the notation $\mathcal{N}_{r,S}(\psi)$ in (1.7), we get from (4.9) and (4.10)

$$\begin{aligned}\|\tilde{x}'\|_{q,T} &\le \frac{T^{1/p^*+1/q^*}}{\mathbf{N}(q^*,p)}\frac{\|a\|_{p,T}^2}{T|\overline{a}|}\frac{T^{1/q^*+1/p^*}}{\mathbf{N}(p^*,q)}\frac{\|b\|_{q,T}^2}{T|\overline{b}|}\|\tilde{x}'\|_{q,T} \\ &= \frac{1}{\mathbf{N}(q^*,p)\cdot\mathbf{N}(p^*,q)}\frac{T^{1-2/p}\|a\|_{p,T}^2}{|\overline{a}|}\frac{T^{1-2/p}\|b\|_{q,T}^2}{|\overline{b}|}\|\tilde{x}'\|_{q,T} \\ &= \frac{\mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b)}{\mathbf{N}(p^*,q)\cdot\mathbf{N}(q^*,p)}\|\tilde{x}'\|_{q,T}. \end{aligned} \tag{4.11}$$

Lemma 4.1 *Suppose that $a(t)$ and $b(t)$ satisfy* (4.1)*. System* (1.5) *is non-degenerate if*

$$\mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b) < \mathbf{N}(p^*,q)\cdot\mathbf{N}(q^*,p). \tag{4.12}$$

Proof. Under condition (4.12), we get from (4.11) that $\|\tilde{x}'\|_{q,T} = \|\tilde{y}'\|_{p,T} = 0$. Thus $\tilde{x}(t) = \tilde{y}(t) \equiv 0$, because $\tilde{x}(t)$ and $\tilde{y}(t)$ have the mean values 0. By (4.4), we have $\overline{x} = \overline{y} = 0$. Hence $x(t) \equiv y(t) \equiv 0$. That is, system (1.5) has only the trivial T-periodic solution. □

Lemma 4.1 asserts that when (4.12) is satisfied, the Floquet multipliers of (1.5) (of period T) satisfy $\nu_{1,2} \neq 1$. In order to deduce the non-parabolicity of (1.5), we must exclude the case $\nu_{1,2} = -1$. In case $\nu_{1,2} = -1$, we can consider (1.5) as a Hamiltonian system of period $2T$ which is degenerate. Using the invariance of mean values and the equalities $\|a\|_{p,2T}^2 = 2^{2/p}\|a\|_{p,T}^2$ and $\|b\|_{q,2T}^2 = 2^{2/q}\|b\|_{q,T}^2$, one has from (1.7)

$$\begin{aligned}&\mathcal{N}_{q,2T}(b) = 2\mathcal{N}_{q,T}(b), \quad \mathcal{N}_{p,2T}(a) = 2\mathcal{N}_{p,T}(a), \\ &\mathcal{N}_{p,2T}(a)\cdot\mathcal{N}_{q,2T}(b) = 4\mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b).\end{aligned} \tag{4.13}$$

Now the non-degeneracy condition $\mathcal{N}_{p,2T}(a)\cdot\mathcal{N}_{q,2T}(b) < \mathbf{N}(p^*,q)\cdot\mathbf{N}(q^*,p)$ for $2T$-periodic Hamiltonian system (1.5) is the same as

$$\mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b) < \mathbf{N}(p^*,q)\cdot\mathbf{N}(q^*,p)/4. \tag{4.14}$$

As (4.14) is stronger than (4.12), we know that condition (4.14) has excluded both $\nu_{1,2} = +1$ and $\nu_{1,2} = -1$. This yields the following result.

Lemma 4.2 *Under assumptions* (4.1) *on $a(t)$ and $b(t)$, system* (1.5) *is not parabolic if $a(t)$ and $b(t)$ satisfy* (4.14)*.*

Remark 4.1 From formulas (2.4)-(2.5)-(2.6), let us define

$$\mathbf{L}(u,v) = \mathbf{L}(v,u) := \begin{cases} \left(\frac{(u+v)^{1/u+1/v-1}}{u^{1/v}v^{1/u}}\mathbf{B}(1/u,1/v)\right)^2 & \text{for } u,\ v \in [1,\infty), \\ 1 & \text{for } u = \infty \text{ or } v = \infty. \end{cases} \tag{4.15}$$

One has then the equality

$$\mathbf{N}(p^*,q)\cdot\mathbf{N}(q^*,p)/4 \equiv \mathbf{L}(p^*,q^*). \tag{4.16}$$

From the monotonicity of $\mathbf{N}(q,p)$ of Section 2, the function $\mathbf{L}(u,v)$ is decreasing in both u and v. Hence one has

$$1 \le \mathbf{L}(\infty,\infty) \le \mathbf{L}(u,v) \le \mathbf{L}(1,1) = 4, \qquad (u,v) \in [1,\infty]^2. \tag{4.17}$$

By (4.14) and equality (4.16), we can give the second class of ellipticity criteria.

Theorem 4.1 *Let* $a(t) \in L^p(\mathbb{S}_T)$ *and* $b(t) \in L^q(\mathbb{S}_T)$ *for some* $p,\ q \in [1,\infty]$.

- L^p-L^q **ellipticity criteria:** *System* (1.5) *is elliptic if*

$$\overline{a}\cdot\overline{b} > 0 \qquad \text{and} \qquad \mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b) < \mathbf{L}(p^*,q^*). \tag{4.18}$$

- L^p-L^q **hyperbolicity criteria:** *System* (1.5) *is hyperbolic if*

$$\overline{a}\cdot\overline{b} < 0 \qquad \text{and} \qquad \mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b) < \mathbf{L}(p^*,q^*). \tag{4.19}$$

Proof. By the Hölder inequality, we have $|\overline{a}| \le T^{-1/p}\|a\|_{p,T}$ and $\|\overline{a}\|_{p,T} \le \|a\|_{p,T}$. Let use choose a homotopy $a_\tau(t) = \tau a(t) + (1-\tau)\overline{a}$, $\tau \in [0,1]$. Then

$$\begin{aligned}\mathcal{N}_{p,T}(a_\tau) &= T^{1-2/p}\|a_\tau\|_{p,T}^2/|\overline{a_\tau}| \equiv T^{1-2/p}\|a_\tau\|_{p,T}^2/|\overline{a}| \\ &\le T^{1-2/p}\left(\tau\|a\|_{p,T} + (1-\tau)\|\overline{a}\|_{p,T}\right)^2/|\overline{a}| \\ &\le T^{1-2/p}\|a\|_{p,T}^2/|\overline{a}| = \mathcal{N}_{p,T}(a).\end{aligned} \tag{4.20}$$

Similarly, $b_\tau(t) = \tau b(t) + (1-\tau)\overline{b}$ satisfies $\mathcal{N}_{q,T}(b_\tau) \le \mathcal{N}_{q,T}(b)$ for all $\tau \in [0,1]$. Under assumption (4.14) on $a(t)$ and $b(t)$, we know that $a_\tau(t)$, $b_\tau(t)$ also satisfy (4.14) for all $\tau \in [0,1]$. By Lemma 4.2, the following Hamiltonian system

$$x' = b_\tau(t)y, \qquad y' = -a_\tau(t)x, \qquad \tau \in [0,1], \tag{4.21}$$

is not parabolic for each $\tau \in [0,1]$. That is, the Floquet multipliers $\nu_{1,2}(\tau)$ of (4.21) satisfy $\nu_{1,2}(\tau) \ne \pm 1$ for all $\tau \in [0,1]$. As $\nu_{1,2}(\tau)$ are continuous in $\tau \in [0,1]$, we know that either

$$\nu_{1,2}(\tau) \in \mathbb{S}^1\backslash\{\pm 1\} \qquad \text{for all } \tau \in [0,1], \tag{4.22}$$

or

$$|\nu_{1,2}(\tau)| \ne 1 \qquad \text{for all } \tau \in [0,1]. \tag{4.23}$$

For $\tau = 0$, system (4.21) is the constant system (2.10), where a_0 and b_0 are $\overline{a}$ and $\overline{b}$ respectively. Let us notice that $\mathcal{N}_{p,T}(\overline{a}) = T|\overline{a}|$ and $\mathcal{N}_{q,T}(\overline{b}) = T|\overline{b}|$. By (4.14), (4.17) and (4.20), we have

$$0 < |T\overline{a}\cdot T\overline{b}| \le \mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b) < \mathbf{L}(p^*,q^*) \le 4. \tag{4.24}$$

Let $\theta = |T\overline{a}\cdot T\overline{b}|^{1/2}$. Then $\theta \in (0,2) \subset (0,\pi)$. From Example 2.1, the corresponding Floquet multipliers are

$$\nu_{1,2}(0) = \cos\theta \pm i\sin\theta \qquad \text{for } \overline{a}\cdot\overline{b} > 0,$$

$$\nu_{1,2}(0) = \cosh\theta \pm \sinh\theta \qquad \text{for } \overline{a}\cdot\overline{b} < 0.$$

Thus (2.10) is elliptic if $\overline{a}\cdot\overline{b} > 0$ (see condition (2.11)), and is hyperbolic if $\overline{a}\cdot\overline{b} < 0$, provided that (4.14) is satisfied. Now conclusions (4.18) and (4.19) can be deduced from (4.22) and (4.23) respectively. $\square$

Remark 4.2 (i) Note from (4.15) that $\mathbf{L}(u,v)$ is symmetric in (u,v). Hence both conditions $\overline{a}\cdot\overline{b}>0$ and $\mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b)<\mathbf{L}(p^*,q^*)$ of (4.18) are symmetric with respect to (a,p) and (b,q). These are consistent with the invariance of ellipticity for systems (1.5) under the change of variables $(x,y)\mapsto(y,-x)$.

(ii) In case $p=q=1$, one has $\mathbf{L}(1^*,1^*)=\mathbf{L}(\infty,\infty)=1$ and the second condition of (4.18) is $\mathcal{N}_{1,T}(a)\cdot\mathcal{N}_{1,T}(b)<1$. In this case, as in the L^1 criterion for Hill's equations, the strict inequality can be weakened as $\mathcal{N}_{1,T}(a)\cdot\mathcal{N}_{1,T}(b)\le 1$. When both $a(t)$ and $b(t)$ do not change sign, one has $\mathcal{N}_{1,T}(a)=\|a\|_{1,T}$ and $\mathcal{N}_{1,T}(b)=\|b\|_{1,T}$. Hence, if $\overline{a}\cdot\overline{b}>0$ and $\|a\|_{1,T}\cdot\|b\|_{1,T}\le 1$, then (1.5) is elliptic. The constant 1 is smaller than the optimal constant 4 in the L^1_+ criterion (1.6). Hence our L^p-L^q criteria (4.18) are a partial generalization of (1.6).

4.2 Eigenvalues of Dirac operators and ellipticity zones

Theoretically, ellipticity of (1.5) is related with eigenvalues. Precisely, system (1.5) defines a Dirac operator with the corresponding eigenvalue problem [7, Part two]

$$x'=(b(t)-\lambda)y, \qquad y'=-(a(t)-\lambda)x. \tag{4.25}$$

It is well-known that there exist

$$\cdots<\underline{\lambda}_{-1}(a,b)\le\overline{\lambda}_{-1}(a,b)<\underline{\lambda}_0(a,b)\le\overline{\lambda}_0(a,b)<\underline{\lambda}_1(a,b)\le\overline{\lambda}_1(a,b)<\cdots$$

such that $\underline{\lambda}_m(a,b)$ and $\overline{\lambda}_m(a,b)$ are T-periodic eigenvalues (T-anti-periodic eigenvalues, respectively) of (4.25) for m to be even (m to be odd, respectively). Now system (1.5) is elliptic iff there exists some $m\in\mathbb{Z}$ such that

$$\overline{\lambda}_{-m}(a,b)<0<\underline{\lambda}_{-m+1}(a,b). \tag{4.26}$$

When (a,b) satisfies (4.26), we say that system (1.5) is in the mth ellipticity zone.

Theorem 4.2 *Suppose that $a(t)$ and $b(t)$ satisfy $\mathcal{N}_{p,T}(a)\cdot\mathcal{N}_{q,T}(b)<\mathbf{L}(p^*,q^*)$. Then system* (1.5) *is in the first ellipticity zone if $\overline{a}>0$ and $\overline{b}>0$, and system* (1.5) *is in the zeroth ellipticity zone if $\overline{a}<0$ and $\overline{b}<0$.*

Proof. In the proof of Theorem 4.1, we have obtained a homotopy equation (4.21) without changing the ellipticity zone. For the constant system (2.10), one has

$$\begin{aligned}
&\underline{\lambda}_0(\overline{a},\overline{b})=\min(\overline{a},\overline{b}), \qquad \overline{\lambda}_0(\overline{a},\overline{b})=\max(\overline{a},\overline{b}),\\
&\underline{\lambda}_m(\overline{a},\overline{b})=\overline{\lambda}_m(\overline{a},\overline{b})=\big(\overline{a}+\overline{b}+((\overline{a}-\overline{b})^2+(m\pi/T)^2)^{1/2}\big)/2 \qquad \text{for } m>0,\\
&\underline{\lambda}_m(\overline{a},\overline{b})=\overline{\lambda}_m(\overline{a},\overline{b})=\big(\overline{a}+\overline{b}-((\overline{a}-\overline{b})^2+(m\pi/T)^2)^{1/2}\big)/2 \qquad \text{for } m<0.
\end{aligned}$$

Moreover, the L^p-L^q criteria (4.18) imply that $0<T\overline{a}\cdot T\overline{b}<2^2<\pi^2$. See (4.24).

From these formulas for eigenvalues, it is easy to use the fact $0<T\overline{a}\cdot T\overline{b}<\pi^2$ to verify the following conclusions

$$\overline{a}>0,\ \overline{b}>0\Longrightarrow\overline{\lambda}_{-1}(\overline{a},\overline{b})<0<\underline{\lambda}_0(\overline{a},\overline{b}), \tag{4.27}$$

$$\overline{a} < 0,\ \overline{b} < 0 \Longrightarrow \overline{\lambda}_0(\overline{a},\overline{b}) < 0 < \underline{\lambda}_1(\overline{a},\overline{b}). \tag{4.28}$$

By the homotopy system (4.21), conclusions (4.27) and (4.28) hold as well when the constant coefficients $\overline{a}$, $\overline{b}$ are replaced by $a(t)$ and $b(t)$ respectively. □

Similarly, ellipticity criteria (3.4) and (3.5) imply that system (1.5) is in the first ellipticity zone. This explanation also applies to the operator norm criteria of Theorem 5.1 below.

5 The operator norms and non-standard eigenvalue problems

5.1 The operator norm criteria

Let us pay more attention to the L^p-L^q criteria. Given $b(t) \in L^q(\mathbb{S}_T)$, let us introduce a linear operator $K_b : \tilde{W}^{1,p}(\mathbb{S}_T) \to L^q(\mathbb{S}_T)$ by

$$K_b u(t) = \left(\int_0^T b\right) b(t)u(t) - b(t)\left(\int_0^T bu\right). \tag{5.1}$$

Integrating (5.1) over one period, one knows that $K_b u$ has mean value 0. Hence K_b is actually an operator from $\tilde{W}^{1,p}(\mathbb{S}_T)$ to $\tilde{L}^q(\mathbb{S}_T)$. The operator norm is

$$\hat{\mathcal{K}}_{q,p,T}(b) = \|K_b\|_{\mathcal{L}\left(\tilde{W}^{1,p}(\mathbb{S}_T),\tilde{L}^q(\mathbb{S}_T)\right)} := \max_{u\in\tilde{W}^{1,p}(\mathbb{S}_T),\ \|u'\|_{p,T}=1} \|K_b u\|_{q,T}. \tag{5.2}$$

Some preliminary properties of $\hat{\mathcal{K}}_{q,p,T}(b)$ are as follows.

• $\hat{\mathcal{K}}_{q,p,T}(b)$ is homogeneous in b: $\hat{\mathcal{K}}_{q,p,T}(kb) = k^2\hat{\mathcal{K}}_{q,p,T}(b)$ for all $k \in \mathbb{R}$.

• Let $b(t) = 1$. By (5.1), $K_b u(t) \equiv Tu(t)$ for $u \in \tilde{W}^{1,p}(\mathbb{S}_T)$ because $\int_0^T u = 0$. Thus

$$\hat{\mathcal{K}}_{q,p,T}(1) = \max_{u\in\tilde{W}^{1,p}(\mathbb{S}_T),\ u\neq 0} T\|u\|_{q,T}/\|u'\|_{p,T}. \tag{5.3}$$

The computation of $\hat{\mathcal{K}}_{q,p,T}(1)$ is then reduced to another kind of Sobolev constants which are analogous to (2.7) with T-periodic functions of mean value 0. It is well-known that $\hat{\mathcal{K}}_{2,2,T}(1) = T^2/(2\pi)$ and $\hat{\mathcal{K}}_{\infty,2,T}(1) = T^{3/2}/\sqrt{12}$. See, e.g., [10, §3]. The inequalities

$$\|u\|_{q,T} \le T^{-1}\hat{\mathcal{K}}_{q,p,T}(1)\|u'\|_{p,T}, \qquad u \in \tilde{W}^{1,p}(\mathbb{S}_T),$$

resulted from (5.3), are also called the Wirtinger inequalities.

• The main estimate (4.9) used in the L^p-L^q criteria can be stated as the following upper bound for $\hat{\mathcal{K}}_{q,p,T}(b)$

$$\hat{\mathcal{K}}_{q,p,T}(b) \le \left(T^{1/p^*+1/q^*}/\mathbf{N}(q^*,p)\right)\|b\|_{q,T}^2, \qquad b(t) \in L^q(\mathbb{S}_T). \tag{5.4}$$

As in (1.7), let us introduce the following notation

$$\mathcal{K}_{q,p,T}(b) := \hat{\mathcal{K}}_{q,p,T}(b)/(T|\overline{b}|) \qquad \text{for } b(t) \in L^q(\mathbb{S}_T) \text{ with } \overline{b} \neq 0. \tag{5.5}$$

One has the following observation on non-degeneracy of systems (1.5)

Lemma 5.1 *Let $a(t) \in L^p(\mathbb{S}_T)$ and $b(t) \in L^q(\mathbb{S}_T)$. Then the T-periodic system* (1.5) *is non-degenerate if*

$$\overline{a} \cdot \overline{b} \neq 0 \qquad \text{and} \qquad \mathcal{K}_{p,q,T}(a) \cdot \mathcal{K}_{q,p,T}(b) < 1. \tag{5.6}$$

Moreover, under condition (5.6)*, system* (1.5) *has only the trivial T/k-periodic solution for all $k \in \mathbb{N}$.*

Proof. Let $(x(t), y(t)) = (\overline{x} + \tilde{x}(t), \overline{y} + \tilde{y}(t))$ be a T-periodic solution of (1.5). Using the operator K_b, the first equation of (4.5) is $\tilde{x}'(t) = -(T\overline{b})^{-1} K_b \tilde{y}(t)$. By the defining equalities (5.2) and (5.5) for $\hat{\mathcal{K}}_{q,p,T}(b)$ and $\mathcal{K}_{q,p,T}(b)$, one has

$$\|\tilde{x}'\|_{q,T} \leq \mathcal{K}_{q,p,T}(b) \|\tilde{y}'\|_{p,T}.$$

Similarly, the second equation of (4.5) is $\tilde{y}'(t) = (T\overline{a})^{-1} K_a \tilde{x}(t)$. Thus one has another inequality

$$\|\tilde{y}'\|_{p,T} \leq \mathcal{K}_{p,q,T}(a) \|\tilde{x}'\|_{q,T}.$$

Under condition (5.6), we have necessarily $\|\tilde{x}'\|_{q,T} = \|\tilde{y}'\|_{p,T} = 0$. Hence, $\tilde{x} = \tilde{y} = 0$. By (4.4) again, $\overline{x} = \overline{y} = 0$. Thus $(x(t), y(t)) \equiv (0, 0)$, which proves the non-degeneracy of (1.5). □

In order to use the operator norms $\mathcal{K}_{q,p,T}(b)$ to deduce ellipticity criteria for (1.5), we need to exclude non-zero periodic solutions of periods T and $2T$, as we did in the proof of Theorem 4.1. However, for the scaling of periods, the quantities $\mathcal{K}_{q,p,T}(b)$ have no simple scaling result like (4.13) for $\mathcal{N}_{q,T}(b)$. Thus we have to consider $a(t) \in L^p(\mathbb{S}_T)$ and $b(t) \in L^q(\mathbb{S}_T)$ as $2T$-periodic functions. Based on the L^p-L^q criteria of Theorem 4.1, we can use $\mathcal{K}_{q,p,2T}(b)$ to obtain the following ellipticity and hyperbolicity criteria for systems (1.5).

Theorem 5.1 *Suppose that $a(t) \in L^p(\mathbb{S}_T)$ and $b(t) \in L^q(\mathbb{S}_T)$.*

- **Operator norm ellipticity criteria:** *The T-periodic system* (1.5) *is elliptic if*

$$\overline{a} \cdot \overline{b} > 0 \qquad \text{and} \qquad \mathcal{K}_{p,q,2T}(a) \cdot \mathcal{K}_{q,p,2T}(b) < 1. \tag{5.7}$$

- **Operator norm hyperbolicity criteria:** *The T-periodic system* (1.5) *is hyperbolic if*

$$\overline{a} \cdot \overline{b} < 0 \qquad \text{and} \qquad \mathcal{K}_{p,q,2T}(a) \cdot \mathcal{K}_{q,p,2T}(b) < 1. \tag{5.8}$$

Proof. As in the proof of Theorem 4.1, we will use the homotopy technique. However, the homotopy $a_\tau(t) = \tau a(t) + (1 - \tau)\overline{a}$ does not work, because, for quantities $\mathcal{K}_{p,q,2T}(a)$, it seems that there is no inequality like (4.20).

This can be overcome by choosing a simple homotopy $a_\tau(t) := \tau a(t)$ and $b_\tau(t) := \tau b(t)$, $\tau \in (0, 1]$. That is, we consider the family of systems

$$x' = \tau b(t) y, \qquad y' = -\tau a(t) x, \qquad \tau \in (0, 1]. \tag{5.9}$$

Under assumption $\mathcal{K}_{p,q,2T}(a) \cdot \mathcal{K}_{q,p,2T}(b) < 1$, we can use the homogeneity of $\hat{\mathcal{K}}_{q,p,2T}(b)$ to obtain

$$\mathcal{K}_{p,q,2T}(\tau a) \cdot \mathcal{K}_{q,p,2T}(\tau b) \equiv \tau^2 \mathcal{K}_{p,q,2T}(a) \cdot \mathcal{K}_{q,p,2T}(b) \leq \mathcal{K}_{p,q,2T}(a) \cdot \mathcal{K}_{q,p,2T}(b) < 1$$

for all $\tau \in (0,1]$. Now, by Lemma 5.1, the Floquet multipliers $\nu_k(\tau)$ of (5.9) still satisfy either (4.22) or (4.23), where $\tau \in [0,1]$ is replaced by $\tau \in (0,1]$.

Note that for any $\tau \in (0,1]$, one has $\mathrm{sign}(\overline{\tau a} \cdot \overline{\tau b}) = \mathrm{sign}(\tau^2 \overline{a} \cdot \overline{b}) = \mathrm{sign}(\overline{a} \cdot \overline{b})$. On the other hand, for $\tau > 0$ small enough, one has necessarily

$$\mathcal{N}_{p,T}(\tau a) \cdot \mathcal{N}_{q,T}(\tau b) \equiv \tau^2 \mathcal{N}_{p,T}(a) \cdot \mathcal{N}_{q,T}(b) < \mathbf{L}(p^*, q^*).$$

Now we can apply Theorem 4.1 to know that, for $\tau > 0$ small enough, system (5.9) is elliptic if $\overline{a} \cdot \overline{b} > 0$ and is hyperbolic if $\overline{a} \cdot \overline{b} < 0$. Since we have either (4.22) or (4.23), now the continuity of $\nu_{1,2}(\tau)$ shows that system (1.5) shares the same classification with systems (5.9) with $\tau > 0$ small. Hence conclusions (5.7) and (5.8) can be obtained from the homotopy argument. □

It is obvious that Theorem 5.1 is an improvement of Theorem 4.1. Theorem 5.1 shows that the quantities $\hat{\mathcal{K}}_{q,p,2T}(b)$ and $\mathcal{K}_{q,p,2T}(b)$ and their estimates are important in the study of ellipticity of Hamiltonian systems. In Lemma 5.3 below, we will give some upper bounds for $\mathcal{K}_{2,2,2T}(b)$.

Remark 5.1 Instead of using the $2T$-periodicity of $a(t)$ and $b(t)$, one can do as follows. Let $\hat{x}(t) = x(2t)$ and $\hat{y}(t) = y(2t)$. System (1.5) is now transformed into $\hat{x}' = \hat{b}(t)\hat{y}$, $\hat{y}' = -\hat{a}(t)\hat{x}$, where $\hat{a}(t) = 2a(2t)$ and $\hat{b}(t) = 2b(2t)$. Note that the T-periodic and $2T$-periodic solutions of (1.5) are transformed into $T/2$-periodic and T-periodic solutions respectively. The second condition in (5.7) and (5.8) is now the same as $\mathcal{K}_{p,q,T}(\hat{a}) \cdot \mathcal{K}_{q,p,T}(\hat{b}) < 1$.

5.2 Non-standard eigenvalue problems

In the following, we always assume that $q = p = 2$ and will give a characterization and some upper bound for $\hat{\mathcal{K}}_{2,2,T}(b)$.

Let $b(t) \in L^2(\mathbb{S}_T)$ be given. From (5.1), one has

$$\|K_b u\|_{2,T}^2 = c_1^2 \int_0^T b^2 u^2 + c_2 \left(\int_0^T bu \right)^2 - 2c_1 \left(\int_0^T bu \right) \left(\int_0^T b^2 u \right) =: Q_b(u),$$

where the constants $c_k := \int_0^T b^k(t)\,\mathrm{d}t$, $k = 1,\ 2$, are determined by $b(t)$. Note that $Q_b(u)$ can be extended to the Hilbert space $H^1(\mathbb{S}_T)$. Under the constraints

$$\|u'\|_{2,T}^2 = \int_0^T u'^2(s)\,\mathrm{d}s = 1, \qquad \int_0^T u = 0,$$

a standard argument shows that $Q_b(u)$ attains its maximum. Moreover, the maximizers $u(t)$ of $Q_b(u)$ satisfy, by the Lagrangian multiplier method, the following linear equation (in u):

$$\mu u''(t) + I_b u(t) = \nu \tag{5.10}$$

for some multipliers μ and ν. Here

$$I_b u(t) := c_1^2 b^2(t) u(t) + c_2 b(t) \int_0^T bu - c_1 b^2(t) \int_0^T bu - c_1 b(t) \int_0^T b^2 u$$

is the differential of $Q_b(u)/2$ at the point $u \in \tilde{H}^1(\mathbb{S}_T)$. It is easy to see that $I_b u$ has mean value 0 for any u. In order that (5.10) has T-periodic solutions $u(t)$, it is necessary that $\nu = 0$. Now equation (5.10) becomes

$$\mu u''(t) + I_b u(t) = 0. \tag{5.11}$$

This is a modified Hill's equation. As $I_b(u+c) = I_b u$ for any constant c, one knows that if $u(t)$ is a solution of (5.11), then so is $u(t)+c$. In particular, all constant functions are solutions of (5.11).

Considering the constraint $\bar{u} = 0$, we say that μ is an eigenvalue of (5.11) if equation (5.11) has non-zero T-periodic solutions $u(t)$ such that $\bar{u} = 0$. As I_b is symmetric in the following sense

$$(I_b u,\, v)_{L^2(\mathbb{S}_T)} = (u,\, I_b v)_{L^2(\mathbb{S}_T)} \qquad u,\ v \in \tilde{H}^1(\mathbb{S}_T),$$

we know that all eigenvalues of (5.11) are real. Multiplying (5.11) by $u(t)$ and integrating over one period, we can get $\mu \|u'\|_{2,T}^2 = \|K_b u\|_{2,T}^2$. Hence all eigenvalues of (5.11) are non-negative. Let us use $\mu_1(b)$ to denote the maximal eigenvalue of (5.11). For example, if $b(t) \equiv 1$, eigenvalue problem (5.11) is reduced to

$$\mu u'' + T^2 u = 0,$$

because $c_1 = T$ and other terms of $I_b u$ are 0. Now all eigenvalues are $T^4/(2m\pi)^2$, $m \in \mathbb{N}$. Hence $\mu_1(1) = T^4/(2\pi)^2$.

By (5.2), we have the following characterization on the norm $\hat{\mathcal{K}}_{2,2,T}(b)$.

Lemma 5.2 *Let $b(t) \in L^2(\mathbb{S}_T)$. Then*

$$\hat{\mathcal{K}}_{2,2,T}(b) = \max\nolimits_{u \in \tilde{H}^1(\mathbb{S}_T),\ \|u'\|_{2,T}=1} \|K_b u\|_{2,T} = \sqrt{\mu_1(b)}.$$

This shows that the computation of $\hat{\mathcal{K}}_{2,2,T}(b)$ can be reduced to a non-standard eigenvalue problem (5.11). When $b(t) \equiv 1$, we have

$$\hat{\mathcal{K}}_{2,2,T}(1) = (\mu_1(1))^{1/2} = T^2/(2\pi). \tag{5.12}$$

In the following, we will give some upper bounds for $\hat{\mathcal{K}}_{2,2,T}(b)$. First, from general result (5.4), we have the following upper bound

$$\hat{\mathcal{K}}_{2,2,T}(b) \le (T/\pi)\|b\|_{2,T}^2, \qquad b \in L^2(\mathbb{S}_T). \tag{5.13}$$

Next, we will derive another simple upper bound which is useful in studying oscillatory coefficients $b(t)$. To this end, let us define

$$B_k(t) = \int_0^t b^k(s)\,\mathrm{d}s, \qquad k = 1,\ 2. \tag{5.14}$$

For $u \in \tilde{H}^1(\mathbb{S}_T)$, expression (5.1) can be written as $K_b u(t) = b(t) \cdot w(t)$, where

$$\begin{aligned} w(t) &= \int_0^T b(s)(u(s) - u(t))\,\mathrm{d}s \\ &= \int_0^T (u(s) - u(t))\,\mathrm{d}B_1(s) \\ &= B_1(T)(u(T) - u(t)) - \int_0^T B_1(s)u'(s)\,\mathrm{d}s \\ &= \int_t^T B_1(T)u'(s)\,\mathrm{d}s - \int_0^T B_1(s)u'(s)\,\mathrm{d}s \\ &=: \int_0^T B_3(t,s)u'(s)\,\mathrm{d}s, \qquad t \in [0,T]. \end{aligned}$$

Here

$$B_3(t,s) = \begin{cases} -B_1(s) & \text{for } 0 \le s \le t \le T, \\ B_1(T) - B_1(s) & \text{for } 0 \le t < s \le T. \end{cases} \tag{5.15}$$

Denote

$$B_4(t,s) \equiv b(t)B_3(t,s) \in L^2([0,T]^2). \tag{5.16}$$

Thus

$$K_b u(t) = \int_0^T B_4(t,s)u'(s)\,\mathrm{d}s, \qquad u \in \tilde{H}^1(\mathbb{S}_T).$$

By the Cauchy-Schwartz inequality, one has

$$|K_b u(t)|^2 = \left(\int_0^T B_4(t,s)u'(s)\,\mathrm{d}s\right)^2 \le \|u'\|_{2,T}^2 \int_0^T B_4^2(t,s)\,\mathrm{d}s.$$

Hence

$$\|K_b u\|_{2,T}^2 \le \|u'\|_{2,T}^2 \int_0^T \int_0^T B_4^2(t,s)\,\mathrm{d}t\,\mathrm{d}s = \|B_4\|_{L^2([0,T]^2)}^2 \|u'\|_{2,T}^2.$$

Let us introduce

$$\begin{aligned} \hat{\mathcal{M}}_T(b) &:= \|B_4\|_{L^2([0,T]^2)} && \text{for } b \in L^2(\mathbb{S}_T), \\ \mathcal{M}_T(b) &:= \hat{\mathcal{M}}_T(b)/|T\bar{b}| && \text{for } b \in L^2(\mathbb{S}_T) \text{ with } \bar{b} \ne 0, \end{aligned}$$

where $B_4(t,s)$ is determined by $b(t)$ from (5.14), (5.15) and (5.16). Now we have the following result.

Lemma 5.3 *For any $b \in L^2(\mathbb{S}_T)$, we have the following inequalities*

$$\hat{\mathcal{K}}_{2,2,T}(b) \le \hat{\mathcal{M}}_T(b) \qquad \text{and} \qquad \mathcal{K}_{2,2,T}(b) \le \mathcal{M}_T(b). \tag{5.17}$$

Let us give a comparison between (5.13) and (5.17).

Example 5.1 • Constant coefficients: Let $b(t) \equiv 1$. The norm $\hat{\mathcal{K}}_{2,2,T}(1)$ is $T^2/(2\pi)$. See (5.12). The upper bounds (5.13) and (5.17) are respectively

$$\hat{\mathcal{K}}_{2,2,T}(1) \le T^2/\mathbf{N}(2,2) = T^2/\pi,$$
$$\hat{\mathcal{K}}_{2,2,T}(1) \le \hat{\mathcal{M}}_T(1) = T^2/\sqrt{6}\,.$$

Hence (5.13) is better than (5.17) for this example.

• Oscillatory coefficients: We take the period $T = 1$ and consider $b(t) = \varphi_{\alpha,\beta}(t) := \alpha + \beta \sin(2\pi t)$, α, $\beta \in \mathbb{R}$. The upper bounds (5.13) and (5.17) are respectively

$$\hat{\mathcal{K}}_{2,2,1}(\varphi_{\alpha,\beta}) \le \|\varphi_{\alpha,\beta}\|_{2,1}^2/\pi = (2\alpha^2 + \beta^2)/(2\pi),$$
$$\hat{\mathcal{K}}_{2,2,1}(\varphi_{\alpha,\beta}) \le \hat{\mathcal{M}}_1(\varphi_{\alpha,\beta}) = \frac{\sqrt{8\pi^2\alpha^4 - (57 - 4\pi^2)\alpha^2\beta^2 + 9\beta^4}}{\pi\sqrt{48}}.$$

When $|\beta|/|\alpha| > R_0 \approx 0.6802$, one has $\hat{\mathcal{M}}_1(\varphi_{\alpha,\beta}) < \|\varphi_{\alpha,\beta}\|_{2,1}^2/\pi$, i.e., the upper bound (5.17) is better than (5.13).

This example shows that the following results, obtained simply from Theorem 5.1 and Lemma 5.3, are also useful.

Corollary 5.1 *Let $a(t)$, $b(t) \in L^2(\mathbb{S}_T)$. Then system* (1.5) *is elliptic if $\overline{a} \cdot \overline{b} > 0$ and $\mathcal{M}_{2T}(a) \cdot \mathcal{M}_{2T}(b) < 1$, and system* (1.5) *is hyperbolic if $\overline{a} \cdot \overline{b} < 0$ and $\mathcal{M}_{2T}(a) \cdot \mathcal{M}_{2T}(b) < 1$.*

In order to apply Corollary 5.1, we give an explicit formula for $\mathcal{M}_{2T}(b)$. In addition to the functions $B_k(t)$ given in (5.14), we introduce another function associated with $b(t)$. For $b(t) \in L^2(\mathbb{S}_T)$ with $\overline{b} \neq 0$, let

$$\check{B}_1(t) = \check{B}_{1,T}(t) := \frac{\int_0^t b}{\int_0^T b} = \frac{B_1(t)}{B_1(T)}. \tag{5.18}$$

It is a normalization of $B_1(t)$ such that $\check{B}_1(T) = 1$.

Lemma 5.4 *Let $b(t) \in L^2(\mathbb{S}_T)$ with $\overline{b} \neq 0$. Then*

$$\mathcal{M}_T(b) = \left(\int_0^T \left(B_2(T)(\check{B}_1(s))^2 - 2\check{B}_1(s)B_2(s) + B_2(s) \right) \mathrm{d}s \right)^{1/2}, \tag{5.19}$$

$$\mathcal{M}_{2T}(b) = \left((\mathcal{M}_T(b))^2 + TB_2(T)/2 \right)^{1/2}, \tag{5.20}$$

where $B_2(s)$ and $\check{B}_1(s)$ are given by (5.14) *and* (5.18) *respectively.*

Proof. By (5.14), (5.15) and (5.16), we have

$$\begin{aligned}(\hat{\mathcal{M}}_T(b))^2 &= \|B_4\|^2_{L^2([0,T]^2)} = \int_0^T \mathrm{d}s \int_0^T (B_4(t,s))^2\,\mathrm{d}t \\ &= \int_0^T \left(\int_0^s (B_1(T)-B_1(s))^2 b^2(t)\,\mathrm{d}t + \int_s^T (B_1(s))^2 b^2(t)\,\mathrm{d}t\right)\mathrm{d}s \\ &= \int_0^T \big((B_1(T)-B_1(s))^2 B_2(s) + (B_1(s))^2(B_2(T)-B_2(s))\big)\,\mathrm{d}s \\ &= \int_0^T \big(B_2(T)(B_1(s))^2 - 2B_1(T)B_1(s)B_2(s) + (B_1(T))^2 B_2(s)\big)\,\mathrm{d}s.\end{aligned}$$

Note that $T|\bar{b}| = |B_1(T)|$. Now formula (5.19) for $\mathcal{M}_T(b) = \hat{\mathcal{M}}_T(b)/|B_1(T)|$ can be obtained from this equality.

Since $b(t)$ is T-periodic, $B_k(s+T) \equiv B_k(T) + B_k(s)$ and $B_k(2T) = 2B_k(T)$. When $b(t)$ is considered as a $2T$-periodic function, $\mathcal{M}_{2T}(b)$ is given by (5.19) where T is changed to $2T$, $\check{B}_1(s)$ is changed to $\check{B}_{1,2T}(s) = \check{B}_1(s)/2$ (see (5.18)), and $B_2(s)$ is unchanged. Thus

$$\begin{aligned}(\mathcal{M}_{2T}(b))^2 &= \int_0^{2T} \left(\frac{1}{2}B_2(T)(\check{B}_1(s))^2 - \check{B}_1(s)B_2(s) + B_2(s)\right)\mathrm{d}s \\ &= \left\{\int_0^T + \int_T^{2T}\right\}\left(\frac{1}{2}B_2(T)(\check{B}_1(s))^2 - \check{B}_1(s)B_2(s) + B_2(s)\right)\mathrm{d}s, \qquad (5.21)\end{aligned}$$

where the equality $B_2(2T) = 2B_2(T)$ is used. Note that

$$\begin{aligned}&\frac{1}{2}B_2(T)(\check{B}_1(T+s))^2 - \check{B}_1(T+s)B_2(T+s) + B_2(T+s) \\ &= \frac{1}{2}B_2(T)(1+\check{B}_1(s))^2 - (1+\check{B}_1(s))(B_2(T)+B_2(s)) + B_2(T) + B_2(s) \\ &= \frac{1}{2}B_2(T)(\check{B}_1(s))^2 - \check{B}_1(s)B_2(s) + \frac{1}{2}B_2(T).\end{aligned}$$

Hence

$$\begin{aligned}&\int_T^{2T} \left(\frac{1}{2}B_2(T)(\check{B}_1(s))^2 - \check{B}_1(s)B_2(s) + B_2(s)\right)\mathrm{d}s \\ &= \int_0^T \left(\frac{1}{2}B_2(T)(\check{B}_1(s))^2 - \check{B}_1(s)B_2(s)\right)\mathrm{d}s + \frac{1}{2}TB_2(T).\end{aligned}$$

Substituting into (5.21), we can obtain (5.20). □

We remark that for general $q,\ p \in (1,\infty)$, the norm $\hat{\mathcal{K}}_{q,p,T}(b)$ can be also reduced to some eigenvalue problems. However, in case $(q,p) \neq (2,2)$, these eigenvalue problems are nonlinear. We will not develop this and refer readers to [22].

5.3 General remarks and problems

From the explanation in the preceding section, ellipticity criteria can be established by finding reasonable estimates for eigenvalues of the Dirac operators. For the Hill's operators, there are many works on the estimates of eigenvalues and ellipticity. Some typical works are [1, 5, 18]. In literature, general eigenvalue theory for planar linear Hamiltonian systems has been established. However, to the knowledge of the author, practical criteria for ellipticity of linear systems are seldom found. In this sense, the L^p criteria and L^p-L^q criteria of this paper constitute an attempt toward this important problem.

Of course, eigenvalues of Dirac operators are dependent on both coefficients $a(t)$ and $b(t)$. One of the novelties of this paper is that the problem can be reduced into two eigenvalue problems like (5.11), each involving of only one of $a(t)$ and $b(t)$. Moreover, sign-changing coefficients $a(t)$ and $b(t)$ are allowed, for which the systems cannot be reduced into Hill's equations.

In the deduction of the L^p-L^q criteria (4.18), we have used both the Hölder inequality and the Sobolev inequality. Note that the bounds $\mathbf{L}(p^*, q^*)$ for $\mathcal{N}_{p,T}(a) \cdot \mathcal{N}_{q,T}(b)$ in the L^p-L^q criteria (4.18) are not optimal. Let us introduce

$$\mathbf{R}(p,q) = \sup\big\{C > 0 : \overline{a} \cdot \overline{b} > 0 \ \& \ \mathcal{N}_{p,T}(a) \cdot \mathcal{N}_{q,T}(b) < C \ \Rightarrow \ \text{ellipticity of (1.5)}\big\}.$$

The meaning of $\mathbf{R}(p,q)$ is as follows.

Lemma 5.5 (Optimal L^p-L^q criteria) *Let $a(t) \in L^p(\mathbb{S}_T)$ and $b(t) \in L^q(\mathbb{S}_T)$. Then system* (1.5) *is elliptic if $\overline{a} \cdot \overline{b} > 0$ and $\mathcal{N}_{p,T}(a) \cdot \mathcal{N}_{q,T}(b) < \mathbf{R}(p,q)$.*

The L^p-L^q criteria in Theorem 4.1 assert that $\mathbf{R}(p,q) \geq \mathbf{L}(p^*, q^*)$. By taking constant coefficients, we know from (2.11) that $\mathbf{R}(p,q) \leq \pi^2$ for all $p,\ q \in [1, \infty]$. Hence there is a gap between our L^p-L^q criteria and the optimal ones. It is then an interesting problem to find the corresponding Sobolev inequalities and the exact values of $\mathbf{R}(p,q)$.

References

[1] F. V. Atkinson, *Estimation of an eigen-value occuring in a stability problem*, Math. Z. **68** (1958), 82-99.

[2] J. Chu and M. Zhang, *Rotation numbers and Lyapunov stability of elliptic periodic solutions*, Discrete Contin. Dyn. Syst. **21** (2008), 1071-1094.

[3] H. Feng and M. Zhang, *Optimal estimates on rotation number of almost periodic systems*, Z. Angew. Math. Phys. **57** (2006), 183-204.

[4] J. K. Hale, Ordinary Differential Equations, 2nd Edition, Robert E. Krieger Publishing Co., New York, 1980.

[5] M. G. Krein, *On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability*, Amer. Math. Soc. Transl. Ser. 2, Vol. 1, pp. 163-187, Amer. Math. Soc., Providence, RI, 1955.

[6] M. Kunze and R. Ortega, *On the number of solutions to semilinear boundary value problems*, Adv. Nonlinear Stud. **4** (2004), 237-249.

[7] B. M. Levitan and L. S. Sargsjan, Sturm-Liouville and Dirac Operators, Math. and Appl. (Soviet Ser.), Vol. 59, Kluwer Academic Publishers, Dordrecht, 1991.

[8] A. M. Lyapunov, *Probléme Géneral de la Stabilité du Mouvement*, Comm. Soc. Math., Kharkov, 1892; Ann. Math. Studies, No. 17, Princeton Univ. Press, Princeton, NJ, 1947.

[9] W. Magnus and S. Winkler, Hill's Equation, Dover, New York, 1979.

[10] G. Meng, P. Yan, X. Lin and M. Zhang, *Non-degeneracy and periodic solutions of semilinear differential equations with deviation*, Adv. Nonlinear Stud. **6** (2006), 563-590.

[11] D. Núñez and R. Ortega, *Parabolic fixed points and stability criteria for nonlinear Hill's equation*, Z. Angew. Math. Phys. **51** (2000), 890-911.

[12] D. Núñez and P. J. Torres, *KAM dynamics and stabilization of a particle sliding over a periodically driven curve*, Appl. Math. Lett. **20** (2007), 610-615.

[13] R. Ortega, *The twist coefficient of periodic solutions of a time-dependent Newton's equation*, J. Dynam. Differential Equations **4** (1992), 651-665.

[14] R. Ortega, *The stability of equilibrium of a nonlinear Hill's equation*, SIAM J. Math. Anal. **25** (1994), 1393-1401.

[15] R. Ortega, *Periodic solution of a Newtonian equation: stability by the third approximation*, J. Differential Equations **128** (1996), 491-518.

[16] R. Ortega and M. Zhang, *Optimal bounds for bifurcation values of a superlinear periodic problem*, Proc. Royal Soc. Edinburgh Sect. A **135** (2005), 119-132.

[17] L. Saloff-Coste, *Aspects of Sobolev Inequalities*, Cambridge Univ. Press, Cambridge, 2002.

[18] V. M. Starzinskii, *Survey of works on conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients*, Amer. Math. Soc. Transl. Ser. 2, Vol. 1, pp. 189-237, Amer. Math. Soc., Providence, RI, 1955.

[19] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.

[20] P. J. Torres, *Existence and stability of periodic solutions for second order semilinear differential equations with a singular nonlinearity*, Proc. Royal Soc. Edinburgh Sect. A **137** (2007), 195-201.

[21] K. Yosida, Functional Analysis, Reprint of the 6th (1980) edition, Springer-Verlag, Berlin, 1995.

[22] M. Zhang, *Nonuniform nonresonance of semilinear differential equations*, J. Differential Equations **166** (2000), 33-50.

[23] M. Zhang, *Certain classes of potentials for p-Laplacian to be non-degenerate*, Math. Nachr. **278** (2005), 1823-1836.

[24] M. Zhang, J. Chu and X. Li, *Lyapunov stability of periodic solutions of the quadratic Newtonian equation*, Math. Nachr., in press. `http://faculty.math.tsinghua.edu.cn/~mzhang/publs/publ.htm`

[25] M. Zhang and W. Li, *A Lyapunov-type stability criterion using L^α norms*, Proc. Amer. Math. Soc. **130** (2002), 3325-3333.