Advanced Nonlinear Studies **8** (2008), 289–301

Multiplicity Result for Quasilinear Elliptic Problems with General Growth in the Gradient

Boumediene Abdellaoui *

Département de Mathématiques
Université Aboubekr Belkaïd, Tlemcen, Tlemcen 13000, Algeria
e-mail:Boumediene.abdellaoui@uam.es

Received 23 September 2007
Communicated by Ireneo Peral

Abstract

The main result of this work is to get the existence of infinitely many radial positive solutions to the problem

$$-\Delta_p u = |\nabla u|^q + \lambda f(x) \text{ in } \Omega, u|_{\partial\Omega} = 0,$$

where $\Omega = B_1(0)$ and f is a radial positive function. Since, in general when $q \neq p$, a Hopf-Cole type change can not be used, we will consider just the existence and multiplicity of positive radial solutions. The main idea is to get a relation between radial positive solutions of the above equation and a suitable quasilinear family of problems with measures data that we will make precise later.

2000 Mathematics Subject Classification. 35D05; 35D10; 35J20; 35J25; 35J70; 46E30; 46E35.
Key words. Quasilinear problems with gradient term, Existence, Multiplicity of positive solutions, Regularity of solutions.

*The author is partially supported by Projects MTM2004-02223, MEC of Spain and Project A/8174/07, AECI, Spain .

1 Introduction

This work is devoted to the existence and multiplicity of radial positive solutions to the problem

$$\begin{cases} -\Delta_p u \equiv -\text{div}(|\nabla u|^{p-2}\nabla u) &= |\nabla u|^q + \lambda f(x) & \text{in } B_1(0), \\ u &\geq 0 & \text{in } B_1(0), \\ u &= 0 & \text{on } \partial B_1(0), \end{cases} \tag{1.1}$$

under suitable hypotheses on q and f. The case $p = q = 2$ is considered in [1] where a suitable change of variables allows the authors to classify positive solutions of problem (1.1) with respect to a semilinear problem with singular measures data. Following the same argument as in [1], one can consider the case $p = q \not\equiv 2$ in general domains. More precisely by assuming that $1 < p \leq N$, the space of positive Radon singular measures with respect to the classical $\text{Cap}_{1,p}$ capacity associated to the Sobolev space $W_0^{1,p}(\Omega)$, is not empty and then we can find a direct relation between positive solutions to problem (1.1) and a family of quasilinear problems with positive Radon singular measures.

The case $q \neq p$ is more complicated, and hence the problem of multiplicity of positive solutions in general domains is an interesting open problem. The main result of this paper is to give a partial answer to this question if $\Omega = B_1(0)$, for radial data and in the radial setting, by showing the correct space of singular measures to consider.

Existence of positive solution to problem (1.1) in general bounded domain is well know when $q \leq p$, see for instance [6], [10] and the reference therein. In a recent paper by Ferone-Messano, see [10], the authors show that solution to problem (1.1) can be compared, in term of rearrangement, with the solution of a suitable symmetrized problem. The main point to get this comparison is an explicit formula for radial solution to (1.1) where $\Omega \equiv B_r(0)$ and $q \leq p$. In this work we use the same explicit expression to show the existence of infinitely many radial positive solutions to (1.1) when $q \geq \frac{(p-1)N}{N-1}$

The paper is organized as follows. In section 2 we prove a l regularity result for entropy solutions to problem (1.1). In section 3 we prove our main result about the existence of infinitely many positive radial solutions starting from a quasilinear problem with a singular measure datum in a suitable sense given below. In the last section we consider the case $p = q = 2$. Under the presence of a suitable weight, we begin by analyzing the case of a radial weight. We will study the problem

$$\begin{cases} -\Delta u &= a(x)|\nabla u|^2 + \lambda f(x) & \text{in } B_1(0), \\ u &= 0 & \text{on } \partial B_1(0), \end{cases} \tag{1.2}$$

where $a(x)$ is a bounded radial weight. Following the argument of the previous sections we prove the existence of infinitely many radial positive solutions. In subsection 4.1 we consider problem (1.2) with $a = 1$ and under the presence of a first order term $\langle b(x), \nabla u \rangle$. Without using the radial condition, under a suitable hypothesis on b (that is optimal in some sense), we prove the existence of infinitely many positive solutions.

2 Regularity of entropy solutions

The main goal of this section is to obtain a regularity result for general solutions to (1.1). We recall that $T_k(s)$ is defined by

$$T_k(s) = s \text{ if } |s| \leq k \text{ and } T_k(s) = k\frac{s}{|s|} \text{ if } |s| \geq k.$$

Definition 2.1 We say that u is a *solution to problem* (1.1) if $|\nabla u|^q + \lambda f \in L^1(B_1(0))$, $T_k(u) \in W_0^{1,p}(B_1(0))$ and for all $v \in W_0^{1,p}(B_1(0)) \cap L^\infty(B_1(0))$, we have

$$\int_{B_1(0)} |\nabla u|^{p-2}\nabla u \nabla T_k(u-v)dx = \int_{B_1(0)} (|\nabla u|^q + \lambda f)T_k(u-v)dx.$$

The sense of solution given above is the entropy sense defined in [5]. Moreover, using the properties of entropy solutions, it follows that $|\nabla u|^{p-1} \in L^\theta(B_1(0))$ for all $\theta < \frac{N}{N-1}$.

We begin by proving the following regularity result.

Lemma 2.1 *Assume that u is a radial positive solution to problem (1.1), then*

$$\int_0^1 e^{c\int_r^1 |u'(\sigma)|^{q-(p-1)}d\sigma} |u'|^q r^{N-1} dr < \infty \text{ for all } c < 1. \tag{2.3}$$

Proof. Since u is a radial solution to (1.1), then u solves

$$\frac{1}{r^{N-1}}((-u(r))^{p-1}r^{N-1}) = |u'(r)|^q + \lambda f(r). \tag{2.4}$$

Consider $w_\varepsilon(r) = e^{c\int_r^1 (\frac{|u'(\sigma)|}{1+\varepsilon|u'(\sigma)|})^{q-(p-1)}d\sigma} - 1$. Then $w_\varepsilon \in W^{1,\infty}(B_1(0))$. Using w_ε as a test function in (1.1) we obtain that

$$\begin{aligned}
&\int_0^1 ce^{c\int_r^1 (\frac{|u'(\sigma)|}{1+\varepsilon|u'(\sigma)|})^{q-(p-1)}d\sigma} \frac{|u'(r)|^q}{(1+\varepsilon|u'(r)|)^{q-(p-1)}} r^{N-1} dr = \\
&\int_0^1 e^{c\int_r^1 (\frac{|u'(\sigma)|}{1+\varepsilon|u'(\sigma)|})^{q-(p-1)}d\sigma} |u'(r)|^q r^{N-1} dr - \int_0^1 |u'(r)|^q r^{N-1} dr + \\
&\int_0^1 f(r)\Big(e^{c\int_r^1 (\frac{|u'(\sigma)|}{1+\varepsilon|u'(\sigma)|})^{q-(p-1)}d\sigma} - 1\Big) r^{N-1} dr.
\end{aligned}$$

Therefore we get

$$\int_0^1 e^{c\int_r^1 (\frac{|u'(\sigma)|}{1+\varepsilon|u'(\sigma)|})^{q-(p-1)}d\sigma} |u'(r)|^q (1 - \frac{c}{(1+\varepsilon|u'(r)|)^{q-(p-1)}}) r^{N-1} dr \leq \int_0^1 |u'(r)|^q r^{N-1} dr$$

and

$$\int_0^1 f(r)\Big(e^{c\int_r^1 (\frac{|u'(\sigma)|}{1+\varepsilon|u'(\sigma)|})^{q-(p-1)}d\sigma} - 1\Big) r^{N-1} dr \leq \int_0^1 |u'(r)|^q r^{N-1} dr.$$

Hence using the hypotheses on u, by letting $\varepsilon \to 0$, we get the desired result.

Notice that in the case $p = q = 2$ we obtain that

$$\int_0^1 e^{cu(r)}|u'|^2 r^{N-1} dr < \infty \text{ for all } c < 1.$$

Hence $e^{\delta u} - 1 \in W_0^{1,2}(B_1(0))$ for all $\delta < \frac{1}{2}$, that is the regularity result obtained in [1]. ∎

The main result of this section is the following.

Theorem 2.1 *Assume that* $q \geq \frac{(p-1)N}{N-1}$ *and* $\lambda \geq 0$. *Let* f *be a positive radial function such that* $f \in L^\theta(B_1(0))$ *with* $\theta > \frac{(q-(p-1))N}{q}$. *Then there exits a* $\lambda^* > 0$ *such that problem* (1.1) *has no positive solution for* $\lambda > \lambda^*$.

If u *is a radial positive solution to problem* (1.1), *then by setting*

$$v(r) = e^{\frac{1}{\gamma-1}\int_r^1 |u'(\sigma)|^{q-(p-1)} d\sigma} - 1,$$

with $\gamma = \frac{q}{q-(p-1)}$, *we obtain that* $v \in W_0^{1,\alpha}(B_1(0))$ *for all* $\alpha < \frac{(q-1)N}{N-1}$ *and* v *solves the problem*

$$\begin{cases} -\Delta_\gamma v &= \frac{\lambda}{(\gamma-1)^{\gamma-1}} f(v+1)^{\gamma-1} + c_0\delta_0. & \text{in } B_1(0), \\ v &= 0 & \text{on } \partial B_1(0), \end{cases} \tag{2.5}$$

where $-\Delta_\gamma v \equiv -div(|\nabla v|^{p-2}\nabla v)$ *and* c_0 *is a positive constant depending only on* u. *If, in addition,*

$$\int_0^1 e^{\int_r^1 |u'(\sigma)|^{q-(p-1)} d\sigma} |u'|^q r^{N-1} dr < \infty,$$

then $c_0 = 0$.

Proof. The existence of λ^* follows directly, using the same argument as in [12] and [1], where the summability condition on f is also driven.

Consider now the function v defined by

$$v(r) = e^{\frac{1}{\gamma-1}\int_r^1 |u'(\sigma)|^{q-(p-1)} d\sigma} - 1.$$

By the result of the previous lemma and using Hölder inequality, we obtain that $v \in W_0^{1,\alpha}(B_1(0))$ for all $\alpha < \frac{(\gamma-1)N}{N-1}$ where $\gamma = \frac{q}{q-(p-1)}$.

Let $H(s) = \dfrac{s}{1+\varepsilon s}$ and define v_ε by

$$v_\varepsilon(r) = e^{\frac{1}{\gamma-1}\int_r^1 |\frac{d\,H(u(r))}{dr}|^{q-(p-1)} d\sigma} - 1 \equiv e^{\frac{1}{\gamma-1}\int_r^1 (\frac{|u'(\sigma)|}{(1+\varepsilon u(\sigma))^2})^{q-(p-1)} d\sigma} - 1.$$

It is clear that $v_\varepsilon \uparrow v$ as $\varepsilon \to 0$ and $v_\varepsilon \in W_0^{1,\gamma}(B_1(0))$. A direct computation shows that

$$-v_\varepsilon'(r) = \frac{(v_\varepsilon+1)}{\gamma-1}\Big(\frac{-u'(r)}{(1+\varepsilon u(r))^2}\Big)^{q-(p-1)} = \frac{1}{\gamma-1}\Big(-u'(r)H'(u(r))\Big)^{q-(p-1)}(v_\varepsilon+1).$$

Thus,

$$\frac{1}{r^{N-1}}((-v_\varepsilon'(r))^{\gamma-1}r^{N-1}) = \Big(\frac{1}{\gamma-1}\Big)^{\gamma-1}\frac{1}{r^{N-1}}\Big\{\Big(-u'(r)H'(u(r))\Big)^{p-1}(v_\varepsilon+1)^{\gamma-1}r^{N-1}\Big\}.$$

Therefore

$$\tfrac{1}{r^{N-1}}((-v_\varepsilon'(r))^{\gamma-1}r^{N-1}) =$$
$$(\tfrac{1}{\gamma-1})^{\gamma-1}\Big\{(v_\varepsilon+1)^{\gamma-1}(-\Delta_p(H(u)) + (\gamma+1)v'(r)(v+1)^{\gamma-2}\Big(-u'(r)H'(u(r))\Big)^{p-1}\Big\}.$$

Notice that

$$-\Delta_p(H(u)) = (H'(u))^{p-1}(-\Delta_p u) - (p-1)H''(u)(H'(u))^{p-2}|\nabla u|^p; \tag{2.6}$$

hence

$$\begin{aligned}&\frac{1}{r^{N-1}}((-v_\varepsilon'(r))^{\gamma-1}r^{N-1}) = \\ &\Big(\frac{1}{\gamma-1}\Big)^{\gamma-1}\Big\{\frac{\lambda f(r)(v_\varepsilon+1)^{\gamma-1}}{(1+\varepsilon u)^{2(p-1)}} + \frac{|u'|^q(v_\varepsilon+1)^{\gamma-1}}{(1+\varepsilon u)^{2(p-1)}} + \frac{2\varepsilon(p-1)|u'|^p(v_\varepsilon+1)^{\gamma-1}}{(1+\varepsilon u)^{2p-1}}\Big\} - \\ &\Big(\frac{1}{\gamma-1}\Big)^{\gamma-1}(v_\varepsilon+1)^{\gamma-1}\frac{|u'|^q}{(1+\varepsilon u)^{2q}}.\end{aligned} \tag{2.7}$$

Consider

$$I_\varepsilon(r) = \Big(\frac{1}{\gamma-1}\Big)^{\gamma-1}\frac{\lambda f(r)(v_\varepsilon+1)^{\gamma-1}}{(1+\varepsilon u)^{2(p-1)}}$$

and

$$J_\varepsilon(r) = \frac{(v_\varepsilon+1)^{\gamma-1}}{(\gamma-1)^{\gamma-1}}\Big\{\frac{|u'|^q}{(1+\varepsilon u)^{2(p-1)}} + \frac{2\varepsilon(p-1)|u'|^p}{(1+\varepsilon u)^{2p-1}} - \frac{|u'|^q}{(1+\varepsilon u)^{2q}}\Big\}.$$

Then $I_\varepsilon(r), J_\varepsilon(r) \geq 0$. Using the definition of v_ε and the Dominated Convergence Theorem we easily get that

$$I_\varepsilon(r) \to \Big(\frac{1}{\gamma-1}\Big)^{\gamma-1}\lambda f(r)(v_\varepsilon+1)^{\gamma-1}.$$

We claim that J_ε is uniformly bounded in $L^1(B_1(0))$, independently of ε. To see this, we consider the function w_ε defined by

$$w(x) \equiv w_\varepsilon(r) = \frac{(v_\varepsilon+1)^{\gamma-1}}{(1+\varepsilon u)^{2(p-1)}} - 1,$$

then

$$w_\varepsilon'(r) = \frac{2\varepsilon(p-1)(v_\varepsilon+1)^{\gamma-1}|u'|}{(1+\varepsilon u)^{2p+1}} - \frac{(v_\varepsilon+1)^{\gamma-1}|u'|^{q-(p-1)}}{(1+\varepsilon u)^{2q}}.$$

Using w_ε as a test function in (1.1), it follows that

$$\begin{aligned}
&\int_0^1 |u'|^q r^{N-1}dr = \\
&\int_0^1 |u'|^q(w_\varepsilon+1)r^{N-1}dr + \int_0^1 \frac{2\varepsilon(p-1)(v_\varepsilon+1)^{\gamma-1}|u'|^p}{(1+\varepsilon u)^{2p+1}}r^{N-1}dr \\
&-\int_0^1 |u'|^q r^{N-1}dr\frac{(v_\varepsilon+1)^{\gamma-1}|u'|^q}{(1+\varepsilon u)^{2q}} + \int_0^1 f(w_\varepsilon+1)r^{N-1}dr \\
&= \int_0^1 J_\varepsilon(r)r^{N-1}dr + \int_0^1 f(w_\varepsilon+1)r^{N-1}dr.
\end{aligned}$$

Since $\int_0^1 |u'|^q r^{N-1}dr < \infty$, we conclude that $\int_0^1 J_\varepsilon(r)r^{N-1}dr$ is uniformly bounded independently of ε. Thus, there exists a positive radial Radon measure μ such that $J_\varepsilon \rightharpoonup \mu$ in the sense of measures. It is clear that

$$\int_{u\le k} J_\varepsilon(r)r^{N-1}dr \to 0 \text{ as } \varepsilon\to 0 \text{ for all } k\ge 1.$$

Hence we conclude that the measure μ is supported on the set $A\equiv\{u\equiv\infty\}$. Using the fact that u is a radial function such that $u\in W_0^{1,q}(B_1(0))$, it follows that $A=\{0\}$. Thus $\mu=c_0\delta_0$ where $c_0>0$ depends only on u.

Let $\phi\in\mathcal{C}_0^\infty(B_1(0))$. Using ϕ as a test function in (2.7) and passing to the limit as $\varepsilon\to 0$, from the above estimate, we get

$$\int_{B_1(0)}|\nabla v|^{\gamma-2}\nabla v\nabla\phi dx = \frac{\lambda}{(\gamma-1)^{\gamma-1}}\int_{B_1(0)} f(v_\varepsilon+1)^{\gamma-1}\phi dx + c_0\phi(0).$$

Hence the result follows.

In the case where

$$\int_0^1 e^{\int_r^1 |u'(\sigma)|^{q-(p-1)}d\sigma}|u'|^q r^{N-1}dr<\infty,$$

we conclude that $\int_{B_1(0)} J_\varepsilon(r)r^{N-1}dr\to 0$ as $\varepsilon\to 0$. Hence $c_0=0$. ∎

3 Multiplicity result

In this section we will prove that problem (1.1) has infinity many radial positive solutions. We begin by the following definitions.

Definition 3.1 Let $\gamma>1$ and consider $A\subset\subset B_1(0)$, we say that A is *a set of zero capacity with respect to* $W_0^{1,\gamma}(B_1(0))$ if

$$\text{Cap}_{1,\gamma}(A)\equiv\inf\{\int_{B_1(0)}|\nabla\phi|^\gamma dx \mid \phi\in W_0^{1,\gamma}(B_1(0)) \text{ and } \phi\ge 1 \text{ in } A\}=0.$$

As a consequence we have the next definition.

Definition 3.2 Let μ be a bounded positive Radon measure, we say that μ is a *singular measure with respect to* $W_0^{1,\gamma}(B_1(0))$ *capacity* if $A \equiv \text{Supp}(\mu)$ satisfies $\text{Cap}_{1,\gamma}(A) = 0$.

Since we are considering radial solution, then we will deal with radial positive singular measures, notice that by a radial measure we means a bounded Radon measure μ such that $\mu = \lim\limits_{n\to\infty} h_n$ in $\mathcal{M}_0(B_1(0))$, the set of Bounded Radon measure, with $h_n \in L^\infty(B_1(0))$ and h_n is a radial.

The following lemma gives a complete characterization of the set of singular radial positive measures.

Lemma 3.1 *If μ is a singular radial positive Radon measure, then $\mu = c_0\delta_0$, $c_0 \geq 0$.*

Proof. Without loss of generality we can assume that $\mu \neq 0$. Let $\{h_n\}$ a sequence of a bounded radial positive functions such that $h_n \rightharpoonup \mu$ in $\mathcal{M}_0(B_1(0))$, then consider w_n, the unique positive radial solution to problem

$$\begin{cases} -\Delta_\gamma w_n &= h_n & \text{in } B_1(0), \\ w_n &= 0 & \text{on } \partial B_1(0). \end{cases}$$

It is clear that w_n is a radial function with $|w_n(x)| \leq C|x|^{-N}$ for all n. Since μ is a singular measure, it follows that for all $\varepsilon > 0$, there exits an open set U_ε such that $\text{Supp}(\mu) \equiv A \subset\subset U_\varepsilon$ and $\text{cap}_{1,\gamma}(U_\varepsilon) \leq \varepsilon$. Hence we get the existence of $\phi \in \mathcal{C}_0^\infty(B_1(0))$ such that $\phi \geq 0$, $\phi \equiv 1$ in U_ε and $\int_{B_1(0)} |\nabla\phi|^\gamma dx \leq 2\varepsilon$. Using A Picone's type inequality as in [3], it follows that

$$\int_{B_1(0)} |\nabla\phi|^\gamma dx \geq \int_{B_1(0)} \frac{-\Delta_\gamma w_n}{w_n^{\gamma-1}} |\phi|^\gamma dx \geq C \int_{U_\varepsilon} |x|^{(\gamma-1)N} h_n dx.$$

Hence we conclude that

$$\langle \mu, |x|^{(\gamma-1)N} \rangle = 0,$$

since μ is a positive measure, then $\mu = c\delta_0$ and the result follows. ■

Remarks 3.1 If $\gamma > N$, we have $W_0^{1,\gamma}(B_1(0)) \subset \mathcal{C}(\bar{B}_1(0))$, then $\mathcal{M}_0(B_1(0)) \subset W^{-1,\gamma'}$, thus from [7] it follows that

$$\{\mu \in \mathcal{M}_0(B_1(0)) \,|\mu \text{ is a singular measure }\} = \{0\}.$$

The main result in this section is the following.

Theorem 3.1 *Assume that $q \geq \frac{(p-1)N}{N-1}$. Let f be a positive radial function such that $f \in L^\theta(B_1(0))$ with $\theta > \frac{(q-(p-1))N}{q}$. Then there exits $\lambda^* > 0$ such that for all $\lambda < \lambda^*$, problem (1.1) has infinity many positive solutions such that for all $c < 1$, we have*

$$\int_0^1 e^{c\int_r^1 |u'(\sigma)|^{q-(p-1)} d\sigma} |u'|^q r^{N-1} dr < \infty.$$

If $q \geq p$, then

$$e^{cu^{q-(p-1)}} |\nabla u|^q \in L^1(\Omega) \text{ for all } c < 1.$$

Proof. We follow closely the approach used in [1]. Let begin by analyzing the following quasilinear elliptic problem with measure data.

Consider $\gamma = \frac{q}{q-(p-1)} \equiv (\frac{q}{p-1})'$, using the main hypothesis on q, it follows that $1 < \gamma \leq N$. We set $g(x) \equiv (\frac{1}{\gamma-1})^{\gamma-1} f(x)$, then g is a radial function such that $g \in L^{\theta}(B_1(0))$ with $\theta > \frac{(q-(p-1))N}{q} = \frac{N}{\gamma}$. Consider the next quasilinear elliptic problem

$$\begin{cases} -\Delta_{\gamma} v &= \lambda g(x)(v+1)^{\gamma-1} + c_0\delta_0 & \text{in } B_1(0) \\ v &= 0 & \text{on } \partial B_1(0), \end{cases} \tag{3.8}$$

where $c_0 > 0$, notice that since $\gamma \leq N$, then δ_0 is a singular measure in the sense of definition 3.2.

Using the same argument as in [1] and [13], we get the existence of $\lambda^* > 0$ depending only on g, γ and $B_1(0)$ and independent of μ, such that for all $\lambda < \lambda^*$, problem (3.8) has a minimal positive solution v. Notice that $v = \lim\limits_{n\to\infty} v_n$, where v_n is the unique solution to the problem

$$\begin{cases} -\Delta_{\gamma} v_n &= \lambda g(x)(v_n+1)^{\gamma-1} + h_n & \text{in } B_1(0), \\ v_n &= 0 & \text{on } \partial B_1(0), \end{cases} \tag{3.9}$$

and $\{h_n\}$ is a sequence of radial positive bounded functions such that $||h_n||_{L^1} \leq C$ and $h_n \rightharpoonup c_0\delta_0$ in the sense of measures.

Using the classical result about renormalized solutions, we can prove that $T_k(v_n) \to T_k(v)$ strongly in $W_0^{1,\gamma}(B_1(0))$ and $|\nabla v_n|^{\gamma-1} \to |\nabla v|^{\gamma-1}$ strongly in $L^{\alpha}(B_1(0))$, for all $\alpha < \frac{N}{N-1}$.

We set

$$u_n(x) = (\gamma-1)^{\frac{1}{q-(p-1)}} \int_{|x|}^{1} \Big(\frac{-v_n'(\rho)}{v_n(\rho)+1}\Big)^{\frac{1}{q-(p-1)}} d\rho. \tag{3.10}$$

It is clear that $|\nabla u_n| = (\gamma-1)^{\frac{1}{q-(p-1)}} \left(\frac{-v'_n}{v_n+1}\right)^{\frac{1}{q-(p-1)}}$. Thus,

$$-\Delta_p u_n = |\nabla u_n|^q + \lambda f(x) + (\gamma-1)^{\frac{p-1}{q-(p-1)}} \frac{h_n(|x|)}{(v_n+1)^{\gamma-1}}.$$

Since v_n is a radial solution to (3.9), it follows that v_n satisfies the nonlinear differential equation

$$\frac{1}{r^{N-1}} \frac{d}{dr}\Big((-v_n'(r))^{\gamma-1} r^{n-1}\Big) = \lambda g(r)(v_n(r)+1)^{\gamma-1} + h_n(r). \tag{3.11}$$

It is not difficult to prove that $\displaystyle\int_{B_1(0)} |\nabla u_n|^q dx = (\gamma-1)^{\gamma} \int_0^1 \big(\frac{-v_n'(r)}{v_n(r)+1}\Big)^{\gamma} r^{N-1} dr \leq C.$ Hence we get the existence of $u \in W_0^{1,q}(B_1(0))$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(B_1(0))$.

We claim that $\dfrac{h_n}{(v_n+1)^{\gamma-1}} \to 0$ in $\mathcal{D}'(\Omega)$. To prove the claim we argue as in the proof of Lemma 3.1. Using the fact that $\text{cap}_{1,\gamma}(\{0\}) = 0$, it follows that for all $\varepsilon > 0$, there exits an open set U_ε such that $0 \in U_\varepsilon$ and $\text{cap}_{1,\gamma}(U_\varepsilon) \le \varepsilon$. Since $\gamma \le N$, then for all $\varepsilon > 0$, we get the existence of $\phi \in \mathcal{C}_0^\infty(B_1(0))$ such that $\phi \ge 0$, $\phi \equiv 1$ in U_ε and $\int_{B_1(0)} |\nabla\phi|^\gamma dx \le 2\varepsilon$. Using A Picone's type inequality as in [3], it follows that

$$\int_{B_1(0)} |\nabla\phi|^\gamma dx \ge \int\limits_\Omega \frac{-\Delta_\gamma(v_n+1)}{(v_n+1)^{\gamma-1}}|\phi|^\gamma\, dx \ge \int\limits_{U_\varepsilon} \frac{h_n}{(v_n+1)^{\gamma-1}}\, dx.$$

Thus $\displaystyle\int\limits_{U_\varepsilon} \frac{g_n}{(v_n+1)^{\gamma-1}}\, dx \le 2\varepsilon$ for all n. Let $\psi \in \mathcal{C}_0^\infty(\Omega)$ be a test function, we show that $\displaystyle\lim_{n\to\infty} \int\limits_{B_1(0)} |\psi|\frac{g_n}{(v_n+1)^{\gamma-1}}\, dx = 0.$

Notice that

$$\int\limits_{B_1(0)} |\psi|\frac{g_n}{(v_n+1)^{\gamma-1}}\, dx| \le ||\psi||_\infty \int\limits_{U_\varepsilon} \frac{g_n}{(v_n+1)^{\gamma-1}}\, dx + \int\limits_{B_1(0)\backslash U_\varepsilon} |\psi|\, g_n\, dx \le \varepsilon ||\psi||_\infty.$$

Since $g_n \to \mu_s$ in $\mathcal{M}_0(B_1(0))$ and from the fact that μ is concentred on $A \subset U_\varepsilon$, there results

$$\int\limits_{B_1(0)\backslash U_\varepsilon} |\psi|\, g_n\, dx \to 0 \text{ for } n \to \infty.$$

Thus the claim follows.

To finish, we have just to prove that $|\nabla u_n|^q \to |\nabla u|^q$ strongly in $L^1(B_1(0))$. This fact is equivalent to show that

$$\Big(\frac{|\nabla v_n|}{v_n+1}\Big)^\gamma \to \Big(\frac{|\nabla v|}{v+1}\Big)^\gamma \quad \text{strongly in } L^1(B_1(0)).$$

Using the properties of renormalized solutions, see [9], we have $T_k(v_n) \to T_k(v)$ strongly in $W_0^{1,\gamma}(B_1(0))$. Hence $\Big(\frac{|\nabla v_n|}{v_n+1}\Big)^\gamma \to \Big(\frac{|\nabla v|}{v+1}\Big)^\gamma$ a.e in $B_1(0)$, as a consequence, using Vitali's Theorem we have only to prove the equi-integrability of the sequence $\Big\{\Big(\frac{|\nabla v_n|}{v_n+1}\Big)^\gamma\Big\}_n$. Let $E \subset \Omega$ a measurable set, then for all $\delta \in (0, p-1)$ and $k > 0$,

$$\begin{aligned}\int\limits_E \Big(\frac{|\nabla v_n|}{v_n+1}\Big)^\gamma dx &= \int\limits_{E\cap\{v_n\le k\}} \Big(\frac{|\nabla v_n|}{v_n+1}\Big)^\gamma dx + \int\limits_{E\cap\{v_n>k\}} \Big(\frac{|\nabla v_n|}{v_n+1}\Big)^\gamma \\ &\le \int\limits_E |\nabla T_k(v_n)|^\gamma\, dx + \frac{1}{(1+k)^\delta}\int\limits_\Omega \frac{|\nabla v_n|^\gamma}{(v_n+1)^{\gamma-\delta}}\, dx.\end{aligned}$$

Using $1 - \dfrac{1}{(v_n+1)^{\gamma-\delta-1}}$ as a test function in (3.9) where δ is chosen such that $\gamma - \delta > 1$, it follows that the last integral is uniformly bounded independently of n. Thus the corresponding term can be made small if k is large enough. Moreover, since $T_k(v_n) \to T_k(v)$

strongly in $W_0^{1,\gamma}(B_1(0))$ for all $k > 0$, we obtain that the integral $\int_E |\nabla T_k(v_n)|^\gamma\, dx$ is uniformly small if $\mathrm{meas}\,(E)$ is small. Therefore the equi-integrability of $|\nabla u_n|^q$ results directly. Hence $|\nabla u_n|^q \to |\nabla u|^q$ strongly in $L^1(B_1(0))$.

Since $q > p-1$, then it is clear that $-\Delta_p u_n \to -\Delta_p u$ in $\mathcal{D}'(B_1(0))$. Hence u solves problem (1.1) with $|\nabla u|^q = \left(\frac{|\nabla v|}{v+1}\right)^\gamma \equiv |\nabla \log(1+v)|^\gamma$, that insures the multiplicity of the solutions when we change the singular measure in problem (3.9).

We prove now the regularity part. Using the definition of v and u we get

$$u^{q-(p-1)}(x) = (\gamma-1)\Big(\int_{|x|}^1 \Big(\frac{-v'(\rho)}{v(\rho)+1}\Big)^{\frac{1}{q-(p-1)}} d\rho\Big)^{q-(p-1)}.$$

Assume that $a > 0$, using $1 - \dfrac{1}{(1+v)^a}$ as a test function in (3.8) (which follows by approximation), we obtain that

$$\int_{B_1(0)} \frac{|\nabla v|^\gamma}{(1+v)^{1+a}} dx < \infty.$$

From (3.10) and by the fact that u is a radial function, we obtain that

$$u'(r) = -(\gamma-1)^{\frac{1}{q-(p-1)}} \Big(\frac{-v'(r)}{v(r)+1}\Big)^{\frac{1}{q-(p-1)}},$$

and then

$$e^{c\int_r^1 |u'(s)|^{q-(p-1)} ds} = (v+1)^{c(\gamma+1)}.$$

Choosing $c < 1$ and by setting $a = (1-c)\gamma + c - 1 > 0$, it follows that

$$\frac{|\nabla v|^\gamma}{(1+v)^{1+a}} = e^{c\int_r^1 |u'(s)|^{q-(p-1)} ds} |u'|^q.$$

Thus

$$\int_0^1 e^{c\int_r^1 |u'(\sigma)|^{q-(p-1)} d\sigma} |u'|^q r^{N-1} dr \equiv \int_{B_1(0)} \frac{|\nabla v|^\gamma}{(1+v)^{1+a}} dx < \infty.$$

If $q \geq p$, then $q-(p-1) \geq 1$, therefore using Jensen's Inequality we obtain that

$$u^{q-(p-1)}(x) \leq (\gamma-1)(1-r)^{q-(p-1)-1} \int_{|x|}^1 \frac{-v'(\rho)}{v(\rho)+1} d\rho \leq (\gamma-1)\log(v(r)+1).$$

Hence it results that

$$e^{cu^{q-(p-1)}} \leq (v+1)^{c(\gamma-1)} \text{ for all } c > 0.$$

As above, if we consider $c < 1$ and by setting $a = (1-c)\gamma + c - 1 > 0$, we conclude that

$$\int_{B_1(0)} e^{cu^{q-(p-1)}} |\nabla u|^q \leq \int_{B_1(0)} \frac{(\gamma-1)^\gamma |\nabla v|^\gamma}{(v+1)^{1+a}} dx < \infty,$$

and then the result follows. ■

Remark 3.1 In the case $p = 2$, then for $q < \frac{N}{N-1}$ it is proved in [4] that problem (1.1) has a unique positive solution under some condition on f and λ. Hence the multiplicity result of Theorem 3.1 shows that the condition on q is optimal. Notice that if $q < \frac{N}{N-1}$, then $\gamma > N$ and hence the set of singular Radon measure is empty. Therefore it seems to be natural to conjecture that for the general $p - Lapcian$ equation, if we assume that $q < \frac{(p-1)N}{N-1}$, then problem (1.1) has a unique positive solution under some hypothesis on λ and f.

4 The case $p = q = 2$: Some extensions

We deal now with the case where $p = q = 2$ and under the presence of some weight. Namely we assume that a is a positive radial function such that $0 < \alpha_1 \le a(x) \le \alpha_2$ for all $x \in B_1(0)$. We consider the next problem

$$\begin{cases} -\Delta u &= a(x)|\nabla u|^2 + \lambda f(x), & \text{in } B_1(0) \\ u &= 0 & \text{on } \partial B_1(0), \end{cases} \tag{4.12}$$

where f is a radial function such that $f \in L^\theta(B_1(0))$, $\theta > \frac{N}{2}$.

Theorem 4.1 *Assume that a and f satisfy the above hypotheses, then problem (4.12) has infinity many radial positive solutions u, such that $e^{(\delta_1 \inf a)u} - 1 \in W_0^{1,2}(B_1(0))$, for all $\delta_1 < \frac{1}{2}$.*

Proof. Consider the following approximated semilinear problems

$$\begin{cases} -\text{div}(b(x)\nabla v_n) &= \lambda f(v_n + 1) + h_n & \text{in } B_1(0), \\ v &= 0 & \text{on } \partial B_1(0), \end{cases} \tag{4.13}$$

where $b(x) = \frac{1}{a(x)}$ and $\{h_n\}$ is a family of radial positive functions such that $||h_n||_{L^1} \le C$ and $h_n \rightharpoonup c_0\mu_s$ in the sense of measures with $c_0 \ge 0$. It is not difficult to show that $T_k(v_n) \to T_k(v)$ strongly in $W_0^{1,2}(B_1(0))$ and $v_n \to v$ in $W_0^{1,\alpha}(B_1(0))$ for all $\alpha < \frac{N}{N-1}$, where v solves

$$\begin{cases} -\text{div}(b(x)\nabla v) &= \lambda f(v + 1) + c_0\delta, & \text{in } B_1(0) \\ v &= 0 & \text{on } \partial B_1(0). \end{cases} \tag{4.14}$$

We set

$$u_n(x) = \int_{|x|}^{1} b(r)\frac{-v_n'(\rho)}{v_n(\rho) + 1}d\rho.$$

Then

$$-\Delta u_n = a(x)|\nabla u_n|^2 + \lambda f(x) + \frac{h_n(|x|)}{v_n + 1}.$$

Using the argument of the section 2, it follows that $\int_{B_1(0)} |\nabla u_n|^2 dx \le c$ and $\dfrac{h_n}{v_n + 1} \to 0$. Therefore we can prove that $u_n \to u$ strongly in $W_0^{1,2}(B_1(0))$, where u is a radial function that solves problem (4.12) with $|\nabla u|^2 = b^2(x)|\nabla(\log(v + 1))|^2$. Hence the result follows. ∎

4.1 The case of $p = q = 2$ in general domain with the presence of a first order term

Let us now consider the following problem

$$\begin{cases} -\Delta u = |\nabla u|^2 + \langle b(x), \nabla u\rangle + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

where Ω is a bounded domain not necessary a ball and $b(x) \in (L^\alpha(\Omega))^N$ with $\alpha > N$.

Theorem 4.2 *Let f be a non negative function such that $f \in L^p(\Omega)$ for $p > \frac{N}{2}$. Assume that $b(x) \in (L^\alpha(\Omega))^N$ where $\alpha > N$. Then problem (4.15) has infinity many positive solutions.*

Proof. We follow closely the argument used in [1]. Let μ_s be a singular Radon positive measure with respect to the classical capacity Cap_2 associated with the norm of $W_0^{1,2}(\Omega)$. We get the existence of a sequence of bounded positive functions h_n such that $||h_n||_{L^1(\Omega)} \le C$ and $h_n \rightharpoonup \mu_s$ in the sense of measures. Consider the next problem

$$\begin{cases} -\Delta v_n = \lambda f(x)(v_n + 1) + \langle b(x), \nabla v_n\rangle + h_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Then there exits a positive constant λ^* depending only on f such that problem (4.16) has a unique positive solution for all $\lambda < \lambda^*$. We claim that $||v_n||_{W_0^{1,q}(\Omega)} \le C$ for all $q < \frac{N}{N-1}$. To prove the claim we follow closely the argument used in [4], namely we take $z(v_n)$ as a test function in (4.16), where $z(s)$ is defined by

$$z(s) = \begin{cases} 1 & \text{if } s \ge t + s, \\ \frac{(s-t)^+}{s} & \text{in } t \le s \le t + s, \\ 0 & \text{in } 0 \le s \le t. \end{cases}$$

Therefore by Lemma 3.2 in [4], using the same computation as in the proof of Theorem 3.1 in [4], we get the existence of a positive constant C such that $||v_n||_{W_0^{1,q}(\Omega)} \le C$ for all $q < \frac{N}{N-1}$ and the claim follows. Thus we get the existence of $v \in W_0^{1,q}(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $W_0^{1,q}(\Omega)$ for all $q < \frac{N}{N-1}$. It is clear by the main result on $b(x)$ that

$$\langle b(x), \nabla v_n\rangle \to \langle b(x), \nabla v\rangle \text{ strongly in } L^1(\Omega).$$

Hence v solves

$$\begin{cases} -\Delta v = \lambda f(x)(v + 1) + \langle b(x), \nabla v\rangle + \mu_s & \text{in } \Omega, \\ v \in W_0^{1,q}(\Omega) \ \forall q < \frac{N}{N-1}. \end{cases} \tag{4.17}$$

We set $u_n = \log(v_n + 1)$. Then $u_n \in W_0^{1,2}(\Omega)$ and satisfies

$$-\Delta u_n = |\nabla u_n|^2 + \langle b(x), \nabla u_n\rangle + \lambda f(x) + \frac{h_n}{v_n + 1}.$$

Notice that as in [1], using A Picone type identity, we can prove that the last term goes to zero in $\mathcal{D}'(\Omega)$ as $n \to 0$. Moreover, we easily conclude that $u_n \to u$ strongly in $W_0^{1,2}(\Omega)$, where u solves

$$-\Delta u = |\nabla u|^2 + \langle b(x), \nabla u \rangle + \lambda f(x)$$

and Supp$\mu_s \subset \{u = \infty\}$. Changing the measure μ_s we get the desired multiplicity result. ∎

Remark 4.1 Notice that the integrability condition on b is optimal in the sense that without this condition we need to add some additional hypotheses on the measure μ to get the existence of a positive solution to problem (4.17). We refer to [2] where the case $b(x) = \frac{x}{|x|^2}$ is studied. A necessary condition on the behavior of μ near 0 is assumed to ensure the existence of a positive solution.

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral, *Some Remarks on Elliptic Problems with Critical Growth in the Gradient*, J. Differential Equations, **222**, (2006), 21-62

[2] B. Abdellaoui, I. Peral, *A note on a critical problem with natural growth in the gradient*, J. Eur. Math. Soc. **8** (2006), 157-170.

[3] B. Abdellaoui, I. Peral, *Existence and nonexistence results for quasilinear elliptic equations involving the p-laplacian*, Ann. Mat. Pura. Appl. **182**, no. 3 (2003), 247–270.

[4] N.E. Alaa, M. Pierre, *Weak solutions of some quasilinear elliptic equations with data measures*, SIAM J. Math. Anal., **24**, (1993), 23-35.

[5] P. Bénilan, L. Boccardo,T. Gallouët, R. Gariepy, M. Pierre, J. Vázquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22**, (1995), 241-273.

[6] L. Boccardo, T. Gallouet, F. Murat, *A unified presentation of two existence results for problems with natural growth*, Research Notes in Mathematics **296** (1993) 127-137, Longman.

[7] L. Boccardo, T. Gallouët, L. Orsina, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, J. Anal. Math. **73**, (1997), 203-223.

[8] L. Boccardo, F. Murat, J.-P. Puel, *Existence des solutions non bornées pour certains équations quasi-linéaires*, Portugal Math., **41** (1982), 507–534.

[9] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Pisa CI. Sci. **28**, no. 4, (1999), 741-808.

[10] V. Ferone, B. Messano, *Comparison and existence results for classes of nonlinear elliptic rquations with general growth in the gradient*, Preprint.

[11] V. Ferone, F. Murat, *Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small*, Nonlinear Anal. T.M.A. **42**, no. 7 (2000), 1309-1326.

[12] K Hansson, V.G. Maz'ya, I.E. Verbitsky, *Criteria of solvability for multidimensional Riccati equations*, Ark. Mat. **37**, (1999), 87-120.

[13] L. Orsina, *Solvability of linear and semilinear eigenvalue problem with L^1 data,* Rend. Sem. Mat. Univ. Padova **90**, (1993), 207-238.