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Classical and Non-Classical Sign-Changing Solutions of a One-Dimensional Autonomous Prescribed Curvature Equation

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Abstract

We discuss existence and multiplicity of solutions of the one-dimensional autonomous prescribed curvature problem

$$-\left(u'/\sqrt{1+u'^2}\right)' = f(u), \quad u(0) = 0, \; u(1) = 0,$$

depending on the behaviour at the origin and at infinity of the function f. We consider solutions that are possibly discontinuous at the points where they attain the value zero.

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1 Introduction

In this paper we discuss existence and multiplicity of solutions of the curvature problem

$$-\left(\varphi(u')\right)' = f(u), \quad u(0) = u(1) = 0, \tag{1.1}$$

where $\varphi(s) = \dfrac{s}{\sqrt{1+s^2}}$, and

(h_0) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(s) \cdot \operatorname{sign}(s) > 0$ for all $s \neq 0$ and $\lim_{s \to \pm\infty} F(s) = +\infty$, with $F(s) = \int_0^s f(\xi)\, d\xi$.

In our study we have in mind the works by M. García-Huidobro, R. Manásevich and F. Zanolin [3, 4, 5], who studied multiplicity and nodal properties of solutions of problem (1.1), with φ an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, based on the analysis of some generalized Fučík spectrum. Our aim is to extend this kind of result to problem (1.1). In particular we get information on existence and multiplicity of solutions with given nodal properties of (1.1) from the knowledge of the limits

$$a_- = \lim_{s \to 0^-} \frac{f(s)}{\varphi(s)}, \quad a_+ = \lim_{s \to 0^+} \frac{f(s)}{\varphi(s)}, \quad A_- = \lim_{s \to -\infty} \frac{f(s)}{\varphi(s)}, \quad A_+ = \lim_{s \to +\infty} \frac{f(s)}{\varphi(s)}.$$

A difficulty in studying the curvature equation derives from the fact that the corresponding homeomorphism φ is not onto $\mathbb{R}$. Non-surjective $\varphi-$Laplacian type equations have recently received some attention in the literature (see [1, 7] and the references therein).

A vast part of the literature on curvature problems follows a variational approach. It is usual to study the relaxed functional naturally associated with (1.1) in the space $BV(0, 1)$ of functions of bounded variation (see e.g. [6, 8]). This implies that the obtained solutions may be highly non-regular. The interest in considering non-regular solutions of (1.1) also stems from the fact that, in this context, functions occur which solve the equation in (1.1) everywhere except at points where the tangent is vertical. This is already evident if we consider the case $f(u) = 1$. Solutions of this kind have been considered for example in [2], where, still in a variational frame, an elliptic regularization approach has been proposed for studying positive solutions of (1.1) with f positive on $]0, +\infty[$. In particular, a positive solution of (1.1) is concave and, hence, vertical tangents may appear only at the endpoints of the domain. Accordingly, the following notion of solution has been introduced in [2].

Definition A *solution* of (1.1) is a function $u : [0, 1] \to \mathbb{R}$, with $u \in C^2(]0, 1[)$ and $u' \in C^0([0, 1], [-\infty, +\infty])$, satisfying $-(\varphi(u'))' = f(u)$ in $]0, 1[$, and $u(0) = u(1) = 0$.

According to this definition, a solution u may be discontinuous only at the endpoints of the interval $[0, 1]$ and, in this case, either $u'(0) = +\infty$ or $u'(1) = -\infty$, while u is C^2 in $]0, 1[$. If we consider only positive solutions, this definition takes into account all possible points with vertical tangent. As we are interested here in solutions which possibly change sign on $[0, 1]$, however, adopting such a definition would prevent a solution from having points with vertical tangent in the interior of the interval. Due to assumption (h_0), a solution u of (1.1) is concave on all intervals where it is positive and it is convex on all intervals where it is negative; hence, we may expect the presence of vertical tangents in the points where the function changes sign. A reasonable way to adapt the definition above might be to say that a function $u : [0, 1] \to \mathbb{R}$ is a solution of (1.1) if there are points $0 = t_0 < t_1 < \ldots < t_N = 1$ such that, for each k, the restriction $u|_{[t_{k-1}, t_k]}$ is a solution in the above sense, with the interval $[0, 1]$ replaced by $[t_{k-1}, t_k]$. Such a solution, however, would lack some properties we would like to keep. For example, if a solution u jumped to 0 at some intermediate point t_k, there would possibly be infinitely many ways to continue u onto the subsequent interval. On the other hand, the conservative nature of the problem suggests to choose, among all possible continuations, the one that preserves the energy.

In this paper we restrict our study to the curvature equation on the unit interval. Our results however might be generalized to problem (1.1) on an interval of any length and for a more general non-surjective and not necessarily odd homeomorphism φ.

We finally mention that the arguments we describe hereby for the Dirichlet problem (1.1) may be used to obtain counterparts of Theorem 2.1 for the periodic problem or for the Neumann problem associated with the equation

$$-\left(\varphi(u')\right)' = f(u).$$

These subjects will be investigated elsewhere.

2 Results

We settle some notation we shall use all along the paper. Given any function $g : \mathbb{R} \to \mathbb{R}$ we denote by g_- and g_+ the restrictions of g on $]-\infty, 0]$ and on $[0, +\infty[$, respectively. The correspondent capital letter will denote the primitive $G(s) = \int_0^s g(\xi)\, d\xi$. In particular $\Phi(s) = \sqrt{1+s^2} - 1$.

For any $s \in \mathbb{R}$ we define

$$\mathcal{L}(s) = \int_0^{\varphi(s)} \varphi^{-1}(\xi)\, d\xi = 1 - \frac{1}{\sqrt{1+s^2}}.$$

For $s \in [0, 1[$ we set

$$\chi(s) = \frac{1-s}{\sqrt{2-s}}.$$

Notice that, for $s \in [0, 1[$, we have

$$\mathcal{L}_+^{-1}(s) = (\mathcal{L}|_{[0,+\infty[})^{-1}(s) = \frac{\sqrt{s}}{\chi(s)} \quad \text{and} \quad \mathcal{L}_-^{-1}(s) = (\mathcal{L}|_{]-\infty,0]})^{-1}(s) = -\frac{\sqrt{s}}{\chi(s)}.$$

We associate with equation (1.1) the energy function

$$E(u, v) = \mathcal{L}(v) + F(u).$$

Definition 2.1 Let $u : [0, 1] \to \mathbb{R}$ be a given function. Assume there exists a decomposition of the interval $[0, 1]$ with nodes $0 = t_0 < t_1 < \ldots < t_N = 1$ such that, for each k, $u(t_{k-1}) = u(t_k) = 0$, the restriction $u_k := u|_{]t_{k-1}, t_k[}$ belongs to $C^2(]t_{k-1}, t_k[)$ and u_k satisfies

$$-\left(\varphi(u_k')\right)' = f(u_k) \quad \text{in }]t_{k-1}, t_k[.$$

Suppose further that $u' \in C^0([0, 1], [-\infty, +\infty])$ and $E(u, u')$ is constant on $]0, 1[\setminus \{t_0, t_1, \ldots, t_N\}$. Then u is said to be a *solution of* (1.1) *with* N *bumps.*

In case $u' \in C^0([0, 1], \mathbb{R})$, and hence $u \in C^2([0, 1])$, the solution u is called *classical*, otherwise it is called *non-classical.*

We compute the time-map $T_f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ for the equation in (1.1) as follows:

$$T_f(r) = \begin{cases} 2\displaystyle\int_r^{F_-^{-1}(F(r)-1)} \frac{1}{\mathcal{L}_+^{-1}(F(r)-F(s))}\,ds & \text{if } r \le F_-^{-1}(1), \\ 2\displaystyle\int_r^{0} \frac{1}{\mathcal{L}_+^{-1}(F(r)-F(s))}\,ds & \text{if } F_-^{-1}(1) < r < 0, \\ 2\displaystyle\int_0^{r} \frac{1}{\mathcal{L}_+^{-1}(F(r)-F(s))}\,ds & \text{if } 0 < r \le F_+^{-1}(1), \\ 2\displaystyle\int_{F_+^{-1}(F(r)-1)}^{r} \frac{1}{\mathcal{L}_+^{-1}(F(r)-F(s))}\,ds & \text{if } r > F_+^{-1}(1). \end{cases} \tag{1.2}$$

For $h > 0$ we also set

$$\sigma_f(h) = T_f(F_+^{-1}(h)); \qquad \tau_f(h) = T_f(F_-^{-1}(h)).$$

Note that T_f is continuous on $\mathbb{R} \setminus \{0\}$ while σ_f and τ_f are continuous on $]0, +\infty[$.

Fix any $r \neq 0$ and let $u :]-\zeta, \zeta[\to]0, +\infty[$ be a positive non-extendible solution, if $r > 0$, or a negative non-extendible solution, if $r < 0$, of the Cauchy problem

$$-(\varphi(u'))' = f(u), \quad u(0) = r, \quad u'(0) = 0. \tag{1.3}$$

Then $T_f(r) = 2\zeta$ computes the life-time of u, $\sigma_f(h)$ is the life-time of a positive solution u with $\max u = F_+^{-1}(h)$ and $\tau_f(h)$ is the life-time of a negative solution u with $\min u = F_-^{-1}(h)$.

Notice that $E(u(t), u'(t)) = F(r)$ for all $t \in]-\zeta, \zeta[$. Further, for each energy level $h \in]0, +\infty[$ there exist a positive solution

$$u_{\text{pos}} : \left]-\frac{1}{2}\sigma_f(h), \frac{1}{2}\sigma_f(h)\right[\to]0, +\infty[$$

of (1.3) and a negative solution

$$u_{\text{neg}} : \left]-\frac{1}{2}\tau_f(h), \frac{1}{2}\tau_f(h)\right[\to]-\infty, 0[$$

of (1.3).

If $h < 1$, then the function u_{pos} will have a C^1 extension onto the closed interval $[-\frac{1}{2}\sigma_f(h), \frac{1}{2}\sigma_f(h)]$ by setting $u_{\text{pos}}(-\frac{1}{2}\sigma_f(h)) = u_{\text{pos}}(\frac{1}{2}\sigma_f(h)) = 0$ and the function u_{neg} has a C^1 extension onto the closed interval $[-\frac{1}{2}\tau_f(h), \frac{1}{2}\tau_f(h)]$ by setting $u_{\text{neg}}(-\frac{1}{2}\tau_f(h)) = u_{\text{neg}}(\frac{1}{2}\tau_f(h)) = 0$.

If $h \ge 1$ then the function u_{pos} has a continuous extension onto the closed interval $[-\frac{1}{2}\sigma_f(h), \frac{1}{2}\sigma_f(h)]$ and the function u_{neg} has a continuous extension onto the closed interval $[-\frac{1}{2}\tau_f(h), \frac{1}{2}\tau_f(h)]$; if $h = 1$ then $u_{\text{pos}}(-\frac{1}{2}\sigma_f(h)) = u_{\text{pos}}(\frac{1}{2}\sigma_f(h)) = 0$ and

$u_{\rm neg}(-\frac{1}{2}\tau_f(h)) = u_{\rm neg}(\frac{1}{2}\tau_f(h)) = 0$, if $h > 1$ then $u_{\rm pos}(-\frac{1}{2}\sigma_f(h)) = u_{\rm pos}(\frac{1}{2}\sigma_f(h)) > 0$ and $u_{\rm neg}(-\frac{1}{2}\tau_f(h)) = u_{\rm neg}(\frac{1}{2}\tau_f(h)) < 0$. In these cases we have $u'_{\rm pos}(-\frac{1}{2}\sigma_f(h)) = +\infty$, $u'_{\rm pos}(\frac{1}{2}\sigma_f(h)) = -\infty$, $u'_{\rm neg}(-\frac{1}{2}\tau_f(h)) = -\infty$ and $u'_{\rm neg}(\frac{1}{2}\tau_f(h)) = +\infty$.

The proof of the following proposition is straightforward.

Proposition 2.1 *Assume f satisfies (h_0) and let $k \in \mathbb{N} \setminus \{0\}$, $h \in]0, +\infty[$.*
(i) There exists a solution u of (1.1) with $2k$ bumps and energy h if and only if

$$k\sigma_f(h) + k\tau_f(h) = 1.$$

(ii) There exists a solution u of (1.1) with $2k-1$ bumps and energy h if and only if either

$$k\sigma_f(h) + (k-1)\tau_f(h) = 1 \quad \text{or} \quad (k-1)\sigma_f(h) + k\tau_f(h) = 1.$$

Remark 2.1 In case (i) there are in fact two solutions. A first solution u_1 is such that $u_1'(0) > 0$ and $u_1'(1) > 0$; the second solution is $u_2(t) = u_1(t + \sigma_f(h))$ on $[0, 1 - \sigma_f(h)[$ and $u_2(t) = u_1(t - \tau_f(h))$ on $[1 - \sigma_f(h), 1]$ and is such that $u_2'(0) < 0$ and $u_2'(1) < 0$.

In case (ii), if $k\sigma_f(h) + (k-1)\tau_f(h) = 1$, the solution u is such that $u'(0) > 0$ and $u'(1) < 0$, if $(k-1)\sigma_f(h) + k\tau_f(h) = 1$, the solution u is such that $u'(0) < 0$ and $u'(1) > 0$.

We now verify the monotonicity of the time-map T_f with respect to f.

Lemma 2.1 (Monotonicity of $T_f(r)$ with respect to f at infinity).
(i) Assume g_1 and g_2 satisfy (h_0) and there exists $M > 0$ such that $g_1(s) \leq g_2(s)$ for all $s \geq M$. Then there exists $M' \geq M$ such that $T_{g_1}(r) \geq T_{g_2}(r)$ for all $r \geq M'$.
(ii) Assume g_1 and g_2 satisfy (h_0) and there exists $M > 0$ such that $g_1(s) \leq g_2(s)$ for all $s \leq -M$. Then there exists $M' \geq M$ such that $T_{g_1}(r) \leq T_{g_2}(r)$ for all $r \leq -M'$.

Proof. We prove (i). Assume $g_1(s) \leq g_2(s)$ for all $s \geq M$. For all $r > M$ and $s \in [M, r[$ we have

$$G_1(r) - G_1(s) = \int_s^r g_1(\xi)\, d\xi \leq \int_s^r g_2(\xi)\, d\xi = G_2(r) - G_2(s). \tag{1.4}$$

Fix $M' \geq (G_2)_+^{-1}(G_2(M) + 1) > M$ and note that $M' > (G_2)_+^{-1}(1)$. Then, for all $r \geq M'$, we have

$$(G_1)_+^{-1}(G_1(r) - 1) \leq (G_2)_+^{-1}(G_2(r) - 1). \tag{1.5}$$

Indeed, as $M \leq (G_2)_+^{-1}(G_2(r) - 1) < r$, by (1.4) we obtain

$$G_1(r) - G_1((G_2)_+^{-1}(G_2(r) - 1)) \leq G_2(r) - G_2((G_2)_+^{-1}(G_2(r) - 1)) = 1.$$

By (1.4) and observing that $(G_2)_+^{-1}(G_2(r)-1)>0$ we get, for all $r \geq M'$,

$$\begin{aligned} T_{g_2}(r) &= 2\int_{(G_2)_+^{-1}(G_2(r)-1)}^{r} \frac{1}{\mathcal{L}_+^{-1}(G_2(r)-G_2(s))}\,ds \\ &\leq 2\int_{(G_2)_+^{-1}(G_2(r)-1)}^{r} \frac{1}{\mathcal{L}_+^{-1}(G_1(r)-G_1(s))}\,ds. \end{aligned}$$

If $G_1(r) < 1$ then

$$T_{g_2}(r) \leq 2\int_0^r \frac{1}{\mathcal{L}_+^{-1}(G_1(r)-G_1(s))}\,ds = T_{g_1}(r);$$

if $G_1(r) \geq 1$ then, by (1.5),

$$T_{g_2}(r) \leq 2\int_{(G_1)_+^{-1}(G_1(r)-1)}^{r} \frac{1}{\mathcal{L}_+^{-1}(G_1(r)-G_1(s))}\,ds = T_{g_1}(r);$$

and in both cases the claim is proved.

The proof of (ii) follows from a symmetric argument.

Lemma 2.2 (Monotonicity of $T_f(r)$ with respect to f at zero).
(i) *Assume g_1 and g_2 satisfy (h_0) and there exists $M>0$ such that $g_1(s) \leq g_2(s)$ for all $s \in [0, M]$. Then there exists $M' \in\,]0, M]$ such that $T_{g_1}(r) \geq T_{g_2}(r)$ for all $r \in [0, M']$.*
(ii) *Assume g_1 and g_2 satisfy (h_0) and there exists $M>0$ such that $g_1(s) \leq g_2(s)$ for all $s \in [-M, 0]$. Then there exists $M' \in\,]0, M]$ such that $T_{g_1}(r) \leq T_{g_2}(r)$ for all $r \in [-M', 0]$.*

Proof. We prove (i). Assume $g_1(s) \leq g_2(s)$ for all $s \in [0, M]$. For all $r \in\,]0, M]$ and $s \in [0, r[$ we have

$$G_1(r) - G_1(s) = \int_s^r g_1(\xi)\,d\xi \leq \int_s^r g_2(\xi)\,d\xi = G_2(r) - G_2(s).$$

Fix $M' \in\,]0, \min\{M, (G_1)_+^{-1}(1), (G_2)_+^{-1}(1)\}[$. Then, for all $r \in [0, M']$, $G_1(r) < 1$, and

$$T_{g_2}(r) = 2\int_0^r \frac{1}{\mathcal{L}_+^{-1}(G_2(r)-G_2(s))}\,ds \leq 2\int_0^r \frac{1}{\mathcal{L}_+^{-1}(G_1(r)-G_1(s))}\,ds = T_{g_1}(r).$$

The proof of (ii) follows from a symmetric argument.

Lemma 2.3 (Limits of $T_{\lambda\varphi}$) *For any $\lambda > 0$*

$$\lim_{r\to\pm\infty} T_{\lambda\varphi}(r) = \frac{2}{\lambda} \qquad \textit{and} \qquad \lim_{r\to 0} T_{\lambda\varphi}(r) = \frac{\pi}{\sqrt{\lambda}}.$$

Proof. We start with the first limit. If $r > \Phi_+^{-1}(\frac{1}{\lambda})$, we compute

$$T_{\lambda\varphi}(r) = 2\int_{\Phi_+^{-1}(\Phi(r)-\frac{1}{\lambda})}^{r} \frac{1}{\mathcal{L}_+^{-1}(\lambda\Phi(r)-\lambda\Phi(s))}\,ds =$$

$$= 2\int_0^1 \frac{\chi(t)}{\sqrt{t}} \frac{1}{\lambda\varphi(\Phi_+^{-1}(\Phi(r)-\frac{t}{\lambda}))}\,dt = \frac{2}{\lambda}\int_0^1 \frac{\chi(t)}{\sqrt{t}} \frac{\sqrt{1+r^2}-\frac{t}{\lambda}}{\sqrt{(\sqrt{1+r^2}-\frac{t}{\lambda})^2-1}}\,dt$$

where we made the change of variable $t = \lambda\Phi(r) - \lambda\Phi(s)$. As

$$\int_0^1 \frac{\chi(t)}{\sqrt{t}}\,dt = 1,$$

we conclude that $\lim_{r\to+\infty} T_{\lambda\varphi}(r) = \frac{2}{\lambda}$. Since $T_{\lambda\varphi}$ is even, $\lim_{r\to-\infty} T_{\lambda\varphi}(r) = \frac{2}{\lambda}$ as well.

We now consider the second limit. If $0 < r < \Phi_+^{-1}(\frac{1}{\lambda})$, we compute

$$\begin{aligned} T_{\lambda\varphi}(r) &= 2\int_0^r \frac{1}{\mathcal{L}_+^{-1}(\lambda\Phi(r)-\lambda\Phi(s))}\,ds \\ &= 2\int_0^1 \frac{r}{\sqrt{\lambda\Phi(r)-\lambda\Phi(rt)}}\chi(\lambda\Phi(r)-\lambda\Phi(rt))\,dt. \end{aligned}$$

Further,

$$\chi(\lambda\Phi(r)-\lambda\Phi(rt)) = \chi\Big(\lambda\frac{r^2-t^2r^2}{\sqrt{1+r^2}+\sqrt{1+t^2r^2}}\Big) \longrightarrow \chi(0) = \frac{1}{\sqrt{2}}$$

as $r \to 0$. Also

$$\frac{r}{\sqrt{\lambda\Phi(r)-\lambda(rt)}} = \frac{1}{\sqrt{\lambda}}\frac{\sqrt{\sqrt{1+r^2}+\sqrt{1+t^2r^2}}}{\sqrt{1-t^2}} \longrightarrow \sqrt{\frac{2}{\lambda}}\frac{1}{\sqrt{1-t^2}}$$

as $r \to 0$. Hence $\lim_{r\to0^+} T_{\lambda\varphi}(r) = \frac{\pi}{\sqrt{\lambda}}$. Since $T_{\lambda\varphi}$ is even, $\lim_{r\to0} T_{\lambda\varphi}(r) = \frac{\pi}{\sqrt{\lambda}}$ as well.

In what follows we shall consider the limits at zero or at infinity of the quotient f/φ. In order to avoid a separate discussion for the cases when these limits are infinite or zero we shall agree on the following convention.

Notation 2.1 For any $\zeta > 0$, whenever we assign to a given parameter $\lambda \in [0,+\infty]$ the value 0, by writing $\frac{\zeta}{\lambda}$ or $\frac{\zeta}{\sqrt{\lambda}}$ we mean $+\infty$, whenever we assign to λ the value $+\infty$, by writing $\frac{\zeta}{\lambda}$ or $\frac{\zeta}{\sqrt{\lambda}}$ we mean 0. We also agree that, for any $\lambda \in [0,+\infty]$, $(+\infty)+\lambda = \lambda+(+\infty) = +\infty$.

Lemma 2.4 (Limits of T_f) *Assume f satisfies (h_0).*

(i) *If* $\lim_{s\to0^+} \frac{f(s)}{\varphi(s)} = a_+$ *exists, then* $\lim_{r\to0^+} T_f(r) = \frac{\pi}{\sqrt{a_+}}$.

(ii) *If* $\lim_{s\to 0^-} \dfrac{f(s)}{\varphi(s)} = a_-$ *exists, then* $\lim_{r\to 0^-} T_f(r) = \dfrac{\pi}{\sqrt{a_-}}$.
(iii) *If* $\lim_{s\to +\infty} \dfrac{f(s)}{\varphi(s)} = A_+$ *exists, then* $\lim_{r\to +\infty} T_f(r) = \dfrac{2}{A_+}$.
(iv) *If* $\lim_{s\to -\infty} \dfrac{f(s)}{\varphi(s)} = A_-$ *exists, then* $\lim_{r\to -\infty} T_f(r) = \dfrac{2}{A_-}$.

Remark 2.2 As $\varphi(s)$ behaves like s at zero and like ± 1 at $\pm\infty$, we can equivalently consider the limits $\lim_{s\to 0^\pm} \frac{f(s)}{s}$ and $\lim_{s\to\pm\infty} f(s)$.

Proof. (i) We consider three cases.
• Assume $a_+ = 0$ and fix $\varepsilon > 0$. Then there exists $M_\varepsilon > 0$ such that

$$f(s) \le \varepsilon\varphi(s) \qquad \text{for all } s \in]0, M_\varepsilon].$$

By Lemma 2.2 there exists $M'_\varepsilon > 0$ such that

$$T_{\varepsilon\varphi}(r) \le T_f(r) \qquad \text{for all } r \in]0, M'_\varepsilon].$$

Since, by Lemma 2.3, $\lim_{r\to 0} T_{\varepsilon\varphi}(r) = \frac{\pi}{\sqrt{\varepsilon}}$, we conclude that $\lim_{r\to 0^+} T_f(r) = +\infty$.
• Assume $0 < a_+ < +\infty$ and fix $0 < \varepsilon < a_+$. Then there exists $M_\varepsilon > 0$ such that

$$(a_+ - \varepsilon)\varphi(s) < f(s) < (a_+ + \varepsilon)\varphi(s) \qquad \text{for all } s \in]0, M_\varepsilon].$$

By Lemma 2.2 there exists $M'_\varepsilon > 0$ such that

$$T_{(a_+ + \varepsilon)\varphi}(r) \le T_f(r) \le T_{(a_+ - \varepsilon)\varphi}(r) \qquad \text{for all } r \in]0, M'_\varepsilon].$$

Therefore we have

$$\lim_{r\to 0^+} T_{(a_+ + \varepsilon)\varphi}(r) \le \liminf_{r\to 0^+} T_f(r) \le \limsup_{r\to 0^+} T_f(r) \le \lim_{r\to 0^+} T_{(a_+ - \varepsilon)\varphi}(r).$$

Since, by Lemma 2.3,

$$\lim_{r\to 0} T_{(a_+ + \varepsilon)\varphi}(r) = \frac{\pi}{\sqrt{a_+ + \varepsilon}} \quad \text{and} \quad \lim_{r\to 0} T_{(a_+ - \varepsilon)\varphi}(r) = \frac{\pi}{\sqrt{a_+ - \varepsilon}},$$

we conclude that $\lim_{r\to 0^+} T_f(r) = \frac{\pi}{\sqrt{a_+}}$.
• Assume $a_+ = +\infty$ and fix $k > 0$. Then there exists $M_k > 0$ such that

$$f(s) \ge k\varphi(s) \qquad \text{for all } s \in]0, M_k].$$

By Lemma 2.2 there exists $M'_k > 0$ such that

$$T_{k\varphi}(r) \ge T_f(r) \qquad \text{for all } r \in]0, M'_k].$$

Since, by Lemma 2.3, $\lim_{r\to 0} T_{k\varphi}(r) = \frac{\pi}{\sqrt{k}}$, we conclude that $\lim_{r\to 0^+} T_f(r) = 0$.

(ii) This is proved as in (i) by a symmetric argument.

(iii) As in (i) we consider three cases.

• Assume $A_+ = 0$ and fix $\varepsilon > 0$. Then there exists $M_\varepsilon > 0$ such that

$$f(s) \le \varepsilon\varphi(s) \qquad \text{for all } s \in [M_\varepsilon, +\infty[.$$

By Lemma 2.1 there exists $M'_\varepsilon > 0$ such that

$$T_{\varepsilon\varphi}(r) \le T_f(r) \qquad \text{for all } r \in [M'_\varepsilon, +\infty[.$$

Since, by Lemma 2.3, $\lim_{r\to+\infty} T_{\varepsilon\varphi}(r) = \frac{2}{\varepsilon}$, we conclude that $\lim_{r\to+\infty} T_f(r) = +\infty$.

• Assume $0 < A_+ < +\infty$ and fix $0 < \varepsilon < A_+$. Then there exists $M_\varepsilon > 0$ such that

$$(A_+ - \varepsilon)\varphi(s) < f(s) < (A_+ + \varepsilon)\varphi(s) \qquad \text{for all } s \in [M_\varepsilon, +\infty[.$$

By Lemma 2.1 there exists $M'_\varepsilon > 0$ such that

$$T_{(A_+ + \varepsilon)\varphi}(r) \le T_f(r) \le T_{(A_+ - \varepsilon)\varphi}(r) \qquad \text{for all } r \in [M'_\varepsilon, +\infty[.$$

Therefore we have

$$\lim_{r\to+\infty} T_{(A_+ + \varepsilon)\varphi}(r) \le \liminf_{r\to+\infty} T_f(r) \le \limsup_{r\to+\infty} T_f(r) \le \lim_{r\to+\infty} T_{(A_+ - \varepsilon)\varphi}(r).$$

Since, by Lemma 2.3,

$$\lim_{r\to\pm\infty} T_{(A_+ + \varepsilon)\varphi}(r) = \frac{2}{A_+ + \varepsilon} \quad \text{and} \quad \lim_{r\to\pm\infty} T_{(A_+ - \varepsilon)\varphi}(r) = \frac{2}{A_+ - \varepsilon},$$

we conclude that $\lim_{r\to+\infty} T_f(r) = \frac{2}{A_+}$.

• Assume $A_+ = +\infty$ and fix $k > 0$. Then there exists $M_k > 0$ such that

$$f(s) \ge k\varphi(s) \qquad \text{for all } s \in [M_k, +\infty[.$$

By Lemma 2.1 there exists $M'_k > 0$ such that

$$T_{k\varphi}(r) \ge T_f(r) \qquad \text{for all } r \in [M'_k, +\infty[.$$

Since, by Lemma 2.3, $\lim_{r\to+\infty} T_{k\varphi}(r) = \frac{2}{k}$, we conclude that $\lim_{r\to+\infty} T_f(r) = 0$.

(iv) This is proved as in (iii) by a symmetric argument.

Lemma 2.5 (Conditions at zero and at infinity) *Assume f satisfies (h_0).*

(i) *Suppose there exist* $\lim_{s\to 0^+} \frac{f(s)}{\varphi(s)} = a_+$, $\lim_{s\to 0^-} \frac{f(s)}{\varphi(s)} = a_-$ *and* $\alpha, \beta \in]0, +\infty[$ *satisfying*

$$\alpha\frac{\pi}{\sqrt{a_+}} + \beta\frac{\pi}{\sqrt{a_-}} < 1.$$

Then there exists $h_* \in]0, +\infty[$ *such that, for all* $h \in]0, h_*]$,

$$\alpha\sigma_f(h) + \beta\tau_f(h) < 1.$$

(ii) *Suppose there exist* $\lim_{s\to 0^+} \frac{f(s)}{\varphi(s)} = a_+$, $\lim_{s\to 0^-} \frac{f(s)}{\varphi(s)} = a_-$ *and* $\alpha, \beta \in]0, +\infty[$ *satisfying*

$$\alpha\frac{\pi}{\sqrt{a_+}} + \beta\frac{\pi}{\sqrt{a_-}} > 1.$$

Then there exists $h_* \in]0, +\infty[$ *such that, for all* $h \in]0, h_*]$,

$$\alpha\sigma_f(h) + \beta\tau_f(h) > 1.$$

(iii) *Suppose there exist* $\lim_{s\to +\infty} \frac{f(s)}{\varphi(s)} = A_+$, $\lim_{s\to -\infty} \frac{f(s)}{\varphi(s)} = A_-$ *and* $\alpha, \beta \in]0, +\infty[$ *satisfying*

$$\alpha\frac{2}{A_+} + \beta\frac{2}{A_-} < 1.$$

Then there exists $h^* \in]0, +\infty[$ *such that, for all* $h \geq h_*$,

$$\alpha\sigma_f(h) + \beta\tau_f(h) < 1.$$

(iv) *Suppose there exist* $\lim_{s\to +\infty} \frac{f(s)}{\varphi(s)} = A_+$, $\lim_{s\to -\infty} \frac{f(s)}{\varphi(s)} = A_-$ *and* $\alpha, \beta \in]0, +\infty[$ *satisfying*

$$\alpha\frac{2}{A_+} + \beta\frac{2}{A_-} > 1.$$

Then there exists $h^* \in]0, +\infty[$ *such that, for all* $h \geq h_*$,

$$\alpha\sigma_f(h) + \beta\tau_f(h) > 1.$$

Proof. We prove (i). Suppose $\alpha\frac{\pi}{\sqrt{a_+}} + \beta\frac{\pi}{\sqrt{a_-}} < 1$. Since, by Lemma 2.4, $\lim_{r\to 0^-} T_f(r) = \frac{\pi}{\sqrt{a_-}}$ and $\lim_{r\to 0^+} T_f(r) = \frac{\pi}{\sqrt{a_+}}$, there are $R_+ > 0$ and $R_- < 0$ such that

$$\alpha T_f(r) + \beta T_f(s) < 1 \qquad \text{for all } R_- \leq s < 0 < r \leq R_+.$$

Pick $h_* \in]0, \min\{F(R_+), F(R_-)\}]$. Then $\alpha\sigma_f(h) + \beta\tau_f(h) < 1$ for all $h \in]0, h_*]$.

The other statements are proved in a similar way.

In our next theorem we consider the following relations, in which $k \in \mathbb{N} \setminus \{0\}$ and $a_-, a_+, A_-, A_+ \in [0, +\infty]$:

(C_1) $$k\frac{2}{A_+} + k\frac{2}{A_-} < 1 < k\frac{\pi}{\sqrt{a_+}} + k\frac{\pi}{\sqrt{a_-}};$$

(C_2) $$(k+1)\frac{2}{A_+} + k\frac{2}{A_-} < 1 < (k+1)\frac{\pi}{\sqrt{a_+}} + k\frac{\pi}{\sqrt{a_-}};$$

$$(C_3) \qquad k\frac{2}{A_+} + (k+1)\frac{2}{A_-} < 1 < k\frac{\pi}{\sqrt{a_+}} + (k+1)\frac{\pi}{\sqrt{a_-}};$$

$$(C_4) \qquad k\frac{\pi}{\sqrt{a_+}} + k\frac{\pi}{\sqrt{a_-}} < 1 < k\frac{2}{A_+} + k\frac{2}{A_-};$$

$$(C_5) \qquad (k+1)\frac{\pi}{\sqrt{a_+}} + k\frac{\pi}{\sqrt{a_-}} < 1 < (k+1)\frac{2}{A_+} + k\frac{2}{A_-};$$

$$(C_6) \qquad k\frac{\pi}{\sqrt{a_+}} + (k+1)\frac{\pi}{\sqrt{a_-}} < 1 < k\frac{2}{A_+} + (k+1)\frac{2}{A_-}.$$

Theorem 2.1 *Assume* (h_0) *and suppose there exist*

$$\lim_{s\to 0^+}\frac{f(s)}{\varphi(s)} = a_+, \quad \lim_{s\to 0^-}\frac{f(s)}{\varphi(s)} = a_-, \quad \lim_{s\to +\infty}\frac{f(s)}{\varphi(s)} = A_+, \quad \lim_{s\to -\infty}\frac{f(s)}{\varphi(s)} = A_-.$$

(i) If either (C_1) *or* (C_4) *holds, then there exist two solutions* u_1 *and* u_2 *of* (1.1) *with* $2k$ *bumps each and such that* $u_1'(0) > 0$ *and* $u_2'(0) < 0$.

(ii) If either (C_2) *or* (C_5) *holds, then there exists a solution* u *of* (1.1) *with* $2k+1$ *bumps and such that* $u'(0) > 0$.

(iii) If either (C_3) *or* (C_6) *holds, then there exists a solution* u *of* (1.1) *with* $2k+1$ *bumps and such that* $u'(0) < 0$.

Proof. Assume (C_1) holds. By Lemma 2.5 there exist h_1 and h_2 in $]0,+\infty[$ such that

$$k\sigma_f(h_1) + k\tau_f(h_1) < 1 < k\sigma_f(h_2) + k\tau_f(h_2).$$

By continuity of the function $k\sigma_f(h) + k\tau_f(h)$, which follows from the continuity of the time-map $T_f(r)$ defined in (1.2), there exists $h > 0$ with $k\sigma_f(h) + k\tau_f(h) = 1$. Hence, the result follows by Proposition 2.1.

The other cases may be studied in a similar way.

Remark 2.3 In case $a_- = a_+ = a$ and $A_- = A_+ = A$, relations $(C_1) - (C_6)$ can be considered as conditions describing the interplay of the limits a, A with the pseudo-spectrum of (1.1) as defined in [3, 4, 5]. The numbers $\{(k\pi)^2,\ k \in \mathbb{N}\setminus\{0\}\}$ and $\{2k,\ k \in \mathbb{N}\setminus\{0\}\}$ are the pseudo-eigenvalues of (1.1) at zero and at infinity respectively. Notice incidentally that $\{(k\pi)^2,\ k \in \mathbb{N}\setminus\{0\}\}$ are the eigenvalues of $-d^2/dt^2$ with Dirichlet boundary conditions on $[0,1]$, in accordance with the fact that $\varphi(v)$ is asymptotic to v at zero. In this context Theorem 2.1 yields the following. Assume there exists $k \in \mathbb{N}\setminus\{0\}$ such that either $a < (k\pi)^2$ and $A > 2k$, or $a > (k\pi)^2$ and $A < 2k$. Then there exists a solution u of (1.1) with k bumps.

Remark 2.4 In Theorem 2.1 existence of solutions with just one bump is not taken into account. The pertinent assumption to get a positive solution is either $A_+ > 2$ and $a_+ < \pi^2$ or $A_+ < 2$ and $a_+ > \pi^2$. Similarly, the pertinent assumption to get a negative solution is either $A_- > 2$ and $a_- < \pi^2$ or $A_- < 2$ and $a_- > \pi^2$.

Remark 2.5 Theorem 2.1 enables to obtain multiplicity results for (1.1). For example, if $A_+ = A_- = +\infty$ and either a_- or a_+ is finite, then (C_1), (C_2) and (C_3) are satisfied for all k sufficiently large; hence (1.1) has infinitely many solutions. Similarly, if $a_+ = a_- = +\infty$ and either A_- or A_+ is finite, then (C_4), (C_5) and (C_6) are satisfied for all k sufficiently large; hence (1.1) has infinitely many solutions.

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