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Hypersurfaces of Prescribed k-Order Mean Curvature in Exterior Domains

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Abstract

In this paper, we prove some existence theorems for hypersurfaces of prescribed k- order mean curvature in exterior domains, which lead to the study of the existence of k-order mean curvature equation in exterior domains.

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1 Introduction

In this paper we extend the work of [4] on the Dirichlet problem for the equation of prescribed Gauss curvature in Eucliden space to the case of k-order mean curvature.

Let $\emptyset \neq \mathbb{K} \subset \mathbb{R}^n$, $(n \geq 2)$, be a compact set whose boundary $\partial\mathbb{K}$ is a smooth submanifold of $\mathbb{R}^n$. we are concerned with hypersurfaces M which are graphs of a function u defined over $\mathbb{R}^n \setminus \mathring{\mathbb{K}}$. If $M =$Graph$u$, the induced metric to M in the standard coordinates on $\mathbb{R}^{n+1}$ is given by

$$g_{ij} = \delta_{ij} + D_i u D_j u, \;\; 1 \leq i, j \leq n. \tag{1.1}$$

The second fundamental form of M is given by

$$h_{ij} = \frac{D_{ij}u}{\sqrt{1+|Du|^2}}. \tag{1.2}$$

By definition, the principal curvatures $\lambda_1, \ldots, \lambda_n$ of M are the eigenvalues of $[h_{ij}]$ relative to $[g_{ij}]$, and the k^{th} mean curvature is the k^{th} elementary symmetric function of the principal curvatures, and is denoted by:

$$H_k[u] = \sigma_k(\lambda_1, \ldots, \lambda_n) = \sigma_k. \tag{1.3}$$

We say that M(or u) is k-*admissible* if the principal curvatures $\lambda = (\lambda_1, \cdots, \lambda_n)$ of Graph$u$ at each point are in the positive cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n, :\ \sigma_j(\lambda) > 0\ \ j = 1, \cdots, k\}. \tag{1.4}$$

Thus, We can get the existence of hypersurfaces of prescribed k-order mean curvature by considering the following problem

$$\begin{cases} H_k(u) = \psi(x, u) & in \ \ \mathbb{R}^n \setminus \mathbb{K}, \\ u = u_0 \ \ \text{on } \partial\mathbb{K} \end{cases} \tag{1.5}$$

where $0 < \psi(x, u) \in C^\infty((\mathbb{R}^n \setminus \overset{\circ}{\mathbb{K}}) \times \mathbb{R})$ is the k-order mean curvature, and for strictly convex $u_0 \in C^\infty(\partial\mathbb{K})$.

It is well known that H_k defines a fully nonlinear elliptic operator such that $H_k^{\frac{1}{k}}$ is a concave function of D^2u, if u is k-admissible (for example see [2]).

Remark 1.1
- when $k = 1$, (1.5) is mean curvature equation;
- when $k = 2$, (1.5) is scalar curvature equation;
- when $k = n$, (1.5) is Gauss curvature equation.

Thus, we are concerned with the case of $k \in [2,\ n)$ in this paper.

Hypersurfaces of prescibed curvature have been subject to intensive studies, see [4, 5, 6, 7, 13] and their references, whereas [19] deals with equations of mean curvature type in exterior domains. There are many papers on k-hessian equation in the Euclidean case, see [3, 15], for Minkowski space case, see also [13, 16].

In [4], Finster and Schnürer solved the Dirichlet problem for the prescribed Gauss curvature equation (k = n) in exterior domains when the datas are strictly convex. In [5], Bo Guan solved this problem under the weaker assumption that there is a strictly convex subsolution. He also solved the Plateau problem. Schnürer [13] solved the Dirichlet problem in lorentzian manifolds.

Our approach for the existence of exterior hypersurfaces of prescribed k-order mean curvature relies in a crucial way on works [4, 17] and a maximum principle in [1, 2].

The hypersurfaces should be close to a cone in the sense that

$$\sup |u - r| < \infty, \tag{1.6}$$

with $u = u(x)$ and $r \equiv |x|$. In this paper we will prove:

Theorem 1.1 *Let* $u_0 \in C^\infty(\partial\mathbb{K})$, *and* $0 < \psi(x,u) \in C^\infty((\mathbb{R}^n \setminus \overset{\circ}{\mathbb{K}}) \times \mathbb{R})$ *with*

$$\psi_z \geq 0, \qquad \sup\left(\psi \cdot r^{\frac{k(n+1)}{n}}\right) + \sup\left(\frac{|D\psi| + |D^2\psi|}{\psi}\right) < \infty. \tag{1.7}$$

Then there exists a smooth hypersurface of prescribed k-order mean curvature, which is close to the cone (1.6).

Theorem 1.1 extends the work of [4] on the Dirichlet exterior problem for the equation of prescribed Gauss curvature to the problem of prescribed k-order mean curvature. However, it has the disadvantage that our proof depends on the existence of a strictly convex subsolution which satisfies those decay conditions at infinity (1.5) in [4]. Therefore, we give another existence result which may be better than Theorem 1.1.

Theorem 1.2 *Let* $u_0 \in C^\infty(\partial\mathbb{K})$, *and* $0 < \psi(x,u) \in C^\infty((\mathbb{R}^n \setminus \overset{\circ}{\mathbb{K}}) \times \mathbb{R})$ *with* $\psi_z \geq 0$. *Then there exists a smooth hypersurface of prescribed k-order mean curvature which is close to the cone* (1.6).

2 Construction of Barrier and an exterior solution

We want to construct a subsolution to (1.5) from Theorem 1.1 in [4], and consider the following problem

$$\begin{cases} H_n(u^R) = \psi^{\frac{n}{k}}(x, u^R) \qquad in \quad \mathbb{R}^n \setminus \mathbb{K}, \\ \sup |u - r| < \infty, \\ u = u_0 \qquad \text{on } \partial\mathbb{K}. \end{cases} \tag{2.1}$$

The following result is from Theorem 1.1 in [4].

Lemma 2.1 *Let* $u_0 \in C^\infty(\partial\mathbb{K})$, *and* $0 < \psi(x,u) \in C^\infty((\mathbb{R}^n \setminus \overset{\circ}{\mathbb{K}}) \times \mathbb{R})$ *with*

$$\psi_z \geq 0, \qquad \sup\left(\psi \cdot r^{\frac{k(n+1)}{n}}\right) + \sup\left(\frac{|D\psi| + |D^2\psi|}{\psi}\right) < \infty. \tag{2.2}$$

Suppose that $\underline{u} \in C^\infty(\mathbb{R}^n \setminus \overset{\circ}{\mathbb{K}})$ *is a subsolution which is close to a cone and satisfies the following decay conditions at infinity,*

$$|D(\underline{u} - r)| = \mathcal{O}(\frac{1}{r}), \qquad |D^2(\underline{u} - r)| + |D^3\underline{u}| = \mathcal{O}(\frac{1}{r^2}). \tag{2.3}$$

Then there exists a smooth, strictly convex hypersurface of prescribed Gauss curvature, which is close to the cone (1.6).

We denote hypersurface $\underline{M}$ =Graph$\underline{v}$, thus $\underline{v}$ is a lower barrier for (1.5). Moreover we know $\underline{v}$ satisfy (2.3). i.e. for r sufficiently large,

$$|D(\underline{v} - r)| = \mathcal{O}(\frac{1}{r}), \qquad |D^2(\underline{v} - r)| + |D^3\underline{v}| = \mathcal{O}(\frac{1}{r^2}). \tag{2.4}$$

We choose R_0 such that $\mathbb{K} \subset B_{R_0}$ and set u_R for an k-admissible solution of the following Dirichlet problem for any $R > 4R_0$.

$$\begin{cases} H_k(u^R) = \psi(x, u^R) \qquad in \quad B_R \setminus \mathbb{K}, \\ u = \underline{v} \qquad \text{on } \partial B_R \cup \partial \mathbb{K}. \end{cases} \tag{2.5}$$

We use the solvability of the prescribed mean curvature problem in order to construct an upper barrier. Let $\overline{v}$ be the solution of the Dirchlet problem

$$\begin{cases} H_1(u^R) = \psi^{\frac{1}{k}}(x, u^R) \qquad in \quad \mathbb{R}^n \setminus \mathbb{K}, \\ \sup |u - r| < \infty, \\ u = u_0 \qquad \text{on } \partial \mathbb{K}. \end{cases} \tag{2.6}$$

The existence of solution to (2.6) may come from [19]. Then we set $\overline{v} = |x| + L$, for some L. Moreover, The main idea in [4] is, in a sense, from [19]. One may get the existence of solution to (2.6) by following methods of [4]. Then we can get the following lemma

Lemma 2.2 *As $R \to \infty$, the functions u^R converges locally uniformly to a continuous function u. Moreover, $\underline{v} \le u^R \le \overline{v}$ in $B_R \setminus \mathring{\mathbb{K}}$.*

Proof. From the Mac-Laurin inequalities we have $H_1(u^R) \ge H_1(\overline{v})$ and $H_k(u^R) \ge H_k(\overline{v})$, and using the comparison principle for the mean and k-order mean curvature operator, we have

$$\underline{v} \le u^R \le \overline{v} \quad in \;\; B_R \setminus \mathbb{K}\,.$$

Hence for $R_1 < R_2$,

$$u^{R_1} \le u^{R_2} \quad \text{on } \partial B_{R_1} \cup \partial\, \mathbb{K},$$

and again from the maximum principle,

$$u^{R_1} \le u^{R_2} \quad \text{in } B_{R_1} \setminus \mathbb{K}\,.$$

We conclude that u^R are monotone in R. Their pointwise limit is convex and thus continuous. So they converge locally according to Dini's theorem.

We need also to prove the uniform gradient estimates and second derivatives, with these estimates at hand. Evans-Krylov second derivative Hölder estimate and Schauder interior regularity theory imply a prior bounds for higher-order derivatives, a diagonal process then yields a subsequence u_{R_k}, $R_k \to +\infty$, that locally converges to a solution of (1.5) and (1.6).

From now on we omit the index R and assume that u is a solution of (2.5) with R fixed sufficiently large.

3 The uniform gradient estimate

Lemma 3.1 *$|\nabla u|$ is a prior bounded, uniformly in R.*

Proof. We estimate firstly $|\nabla u|$ at the boundary. Tangential derivatives are uniformly bounded there in view of the Dirichlet boundary conditions. The normal derivatives are estimated as follows. Let $x \in \partial\mathbb{K}$ and ν the outer unit normal to $\partial\mathbb{K}$ at x. We choose $\mu_0 > 0$ independent of R such that the line segment $\{x + \mu\nu, 0 \le \mu \le \mu_0$ is contained in $B_{R_0} \setminus \mathbb{K}$. Using the fact that $\underline{v}$ lies below u and $u(x) = \underline{v}(x)$, we have

$$\nabla_\nu \underline{v}(x) \le \nabla_\nu u(x) \le \frac{u(x+\mu_0\nu) - u(x)}{\mu_0}, \quad (\text{for } x \in \partial\mathbb{K}).$$

For $x \in \partial B_R$ and $\nu = \frac{x}{|x|}$, we consider similarly the line segment $\{\mu\nu, R_0 \le \mu \le R$ and obtain

$$\frac{u(x) - u(R_0\nu)}{R - R_0} \le \nabla_\nu u(x) \le \nabla_\nu \underline{v}(x), \quad (\text{for } x \in \partial B_R).$$

We finally use the uniform C^0 bounded of Lemma 2.2, in particular that $|u(x) - |x||$ is bounded.

In the following content we shall show simply how to estimate the gradient estimate of u in $B_R \setminus \mathbb{K}$ if we know bounds for it on $\partial B_R \bigcup \partial\,\mathbb{K}$.

To estimate $|\nabla u|$ in $B_R \setminus \mathbb{K}$, one may follow the method on page 51 of [2], see also [17, 18]. It will obtain a bound for $z = |\nabla u| \exp(Au)$, where $A = A(\max|\nabla\psi|)$ is a constant to be chosen. If z achieves its maximum on $\partial B_R \bigcup \partial\,\mathbb{K}$, then from above boundary estimate we have a bound and we are through. Assume this is not the case; then it achieves its maximum at a point x in $B_R \setminus \mathbb{K}$. Then $\log|\nabla u| + Au$ take its maximum there. Thus, we can prove Lemma 3.1 by a similar calculation, see to pp. 51-53 in [2].

The next lemma controls the asymptotic behavior of $|\nabla u|$ as R gets large.

Lemma 3.2 *For $\frac{R}{2} \le |x| \le R$ let $\nu = \frac{x}{|x|}$ and τ be unit vectors parallel and orthogonal to x, respectively. Then there are constant C independent of R and $\theta_{R_0} \in (0,1)$ such that*

$$|\nabla_\nu (u - \underline{v})(x)| \le \frac{C}{R}, \tag{3.1}$$

$$|\nabla_\tau u(x)| \le \frac{C}{\sqrt{R}}. \tag{3.2}$$

The proofs of (3.1) and (3.2) follow from Lemma 2.4 in [4].

4 Estimate for some second derivatives

We shall use the arguments in [4], in line with those of [2] and [11] to deriver estimates for second derivatives. We know the fact that one can get the interior second derivative estimates from second derivative estimates at the boundary (see [2, 14]). Thus, we need only to get second derivative estimates at the boundary.

Lemma 4.1 ([4] tangential second derivatives at the outer boundary) *Let $x_0 \in \partial B_R$ and τ_1, τ_2 be tangential directions at x_0. Then we have at x_0,*

$$|u_{\tau_1\tau_2} - |x|_{\tau_1\tau_2}| \le \frac{C}{R^2}. \tag{4.1}$$

Proof. We may assume that $x_0 = R \cdot e_n \equiv (0, ..., 0, 1)$. Then ∂B_R is represented locally as graphω, where

$$\omega : \hat{B}_R \equiv \{\hat{x} \in \mathbb{R}^{n-1} : |\hat{x}| < R\} \to \mathbb{R}, \quad i.e.\ \omega(\hat{x}) = \sqrt{R^2 - |\hat{x}|^2}.$$

According to the Dirichlet boundary conditions,

$$(u - \underline{v})(\hat{x}, \omega(\hat{x})) = 0.$$

We differentiate twice with respect to $\hat{x}^i$, $\hat{x}^j$, $1 \leq i, j \leq n$ and obtain that at x_0,

$$(u - \underline{v})_{ij} + (u - \underline{v})_n \omega_{ij} = 0.$$

According to the decay conditions at infinity (2.4),

$$|\underline{v}_{ij} - |x|_{ij}| = \mathcal{O}(\frac{1}{R^2}),$$

and furthermore

$$\omega_{ij}(x_0) = -\frac{\delta_{ij}}{R}.$$

Thus the result follows in view of Lemma 3.2.

Lemma 4.2 (Mixed second derivatives at the outer boundary). *Let $x_0 \in \partial B_R$ and τ, ν be be unit vectors in tangential and normal directions, respectively. Then*

$$|u_{\tau\nu}(x_0)| \leq \frac{C}{\sqrt{R}}. \tag{4.2}$$

We may assume that $x_0 = R \cdot e_n$ and represent ∂B_R locally as graphω with $\omega(\hat{x}) = \sqrt{R^2 - |\hat{x}|^2}$.

We list some formulas in the following, and use the Einstein summation convention, for $1 \leq i, j, k, l, m \leq n$,

$$\begin{aligned} g_{ij} &= \delta_{ij} - u_i u_j, \quad h_{ij} = \frac{u_{ij}}{\sqrt{1 - |Du|^2}} = \frac{u_{ij}}{w}, \\ g^{ij} &= \delta_{ij} + \frac{u_i u_j}{1 - |Du|^2}, \qquad g_{ij} g^{jk} = \delta_i^k. \end{aligned}$$

For the derivatives of F, we consider

$$F(h_{ij}) = (H_k)^{\frac{1}{k}} = \psi^{\frac{1}{k}} = \phi. \tag{4.3}$$

Thus $F(h_{ij})$ is a concave function of matrix h_{ij} if M is k-admissible. Let

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}.$$

The matrix F^{ij} is diagonal if matrix h_{ij} is diagonal. We differentiate

$$F(h_{ij}) = \phi(x, u),$$

with respect to x_k, for any $k = 1, \cdots, n$ and use

$$h_{ij,k} = \frac{1}{w^3} u^l u_{lk} u_{ij} + \frac{1}{w} u_{ijk},$$

where we denote $\sum_{l=1}^{n} u_l u_{lk}$ by $u^l u_{lk}$, i.e. $u^l = u_j \delta^{jl}$. Thus, using the fact that

$$F^{ij} h_{ij} = F,$$

we obtain

$$\begin{aligned} 0 &= -\frac{\partial \phi}{\partial x_{n+1}} u_k - \frac{\partial \phi}{\partial x_k} + F^{ij} \frac{\partial h_{ij}}{\partial x_k}, \\ &= -\frac{\partial \phi}{\partial x_{n+1}} u_k - \frac{\partial \phi}{\partial x_k} + F^{ij} (\frac{1}{w^3} u^l u_{lk} u_{ij} + \frac{1}{w} u_{ijk}), \\ &= -\frac{\partial \phi}{\partial x_{n+1}} u_k - \frac{\partial \phi}{\partial x_k} + F \frac{1}{w^2} u^l u_{lk} + \frac{1}{w} F^{ij} u_{ijk}. \end{aligned} \tag{4.4}$$

Therefore we introduce the linear differential operator L by

$$L\xi := \frac{1}{w} F^{ij} \xi_{ij} + \frac{1}{w^2} F u^l \xi_l \tag{4.5}$$

and for $t < n$

$$T := \frac{\partial}{\partial x^t} + \omega_{tr}(0) x^r \frac{\partial}{\partial x^n} \equiv \frac{\partial}{\partial x^t} - \frac{x_t}{R} \frac{\partial}{\partial x^n}, \tag{4.6}$$

where we used the convention $\omega_{tn} = 0$(thus we sum over $r = 1, \ldots, n-1$). In what follows we restrict attention to the domain $\Omega_\delta := B_\delta(x_0) \cap B_R$ with $\delta \leq \frac{R}{2}$; notice that $\Omega_\delta \subset B_R \setminus \mathbb{K}$.

Lemma 4.3 *The function $u - \underline{v}$ satisfies the following estimates:*

$$|T(u - \underline{v})| \leq \frac{C}{\sqrt{R}} \qquad in\ \Omega_\delta, \tag{4.7}$$

$$|T(u - \underline{v})| \leq \frac{C}{R^2} |x - x_0|^2 \quad on\ \partial B_R, \tag{4.8}$$

$$|LT(u - \underline{v})| \leq C + \frac{C}{R^2} tr F^{ij} \qquad in\ \Omega_\delta, \tag{4.9}$$

where $tr F^{ij} \equiv F^{ij} \delta_{ij}$.

Proof. Note that $|\omega_i| \leq C$, $|\omega_{ij}| \leq \frac{C}{R}$ and $|\omega_{ijk}| \leq \frac{C}{R^2}$. The first inequality follows directly from the C_1 estimates of Lemma 3.2, whereas for the second inequality we use furthermore that $(u - \underline{v})_t + (u - u)_n \omega_t = 0$ on ∂B_R, the decay properties of ω_{ijk}, and the fact that $u - \underline{v}$ vanishes on ∂B_R.

To prove the last inequality (4.9), from the definition of L and homogeneity, it is easy to see that

$$\begin{aligned} |LT(u-\underline{v})| &= |L((u-\underline{v})_t - \frac{x_t}{R}(u-\underline{v})_n)|, \\ &\leq C\cdot|D\phi| + C\cdot trF^{ij}\cdot\left(|D^3\underline{v}| + \frac{1}{R}|D^2\underline{v}|\right) + C, \\ &\leq C + \frac{C}{R^2}trF^{ij}. \end{aligned} \tag{4.10}$$

Now we use the conditions (1.7) and (2.4).

The function ϑ introduced in the next Lemma will be the main part of a barrier function which we shall construct in what follows.

Lemma 4.4 *There exists a positive constant ε independent of R such that*

$$\vartheta := (u-\underline{v}) + \frac{1}{\sqrt{R}}d - \frac{1}{2R^{\frac{5}{4}}}d^2$$

satisfies the estimates

$$\begin{cases} L\vartheta \leq -CR^{\frac{3}{4n}} - \frac{C}{4R^{\frac{5}{4}}}trF^{ij} & in\ \Omega_\delta, \\ \vartheta \geq 0 & on\ \ \partial\Omega_\delta, \end{cases} \tag{4.11}$$

provided that $\delta = R^{\frac{3}{4}}$ and R is sufficiently large. Here $d = R - |x|$ is the distance from ∂B_R.

Proof.

$$\vartheta = (u-\underline{v}) + \frac{1}{\sqrt{R}}d - \frac{1}{2R^{\frac{5}{4}}}d^2 \geq \frac{1}{\sqrt{R}}d - \frac{1}{2R^{\frac{5}{4}}}d^2 \geq 0 \quad \text{on} \quad \partial\Omega_\delta, \tag{4.12}$$

since $d = R - |x|$ and $\delta = R^{\frac{3}{4}}$. We now fix $x_0 \in \Omega_\delta$ and set $\mu = x_0/|x_0|$. Let τ, τ' belong to an orthonormal basis for the orthogonal complement of μ. Assume that ν and τ, τ' correspond to the indices n and $1, ..., n-1$, respectively. We use the Einstein summation convention for τ, τ'.

As graph $\underline{v}$ is strictly convex, we fix $\varepsilon \in (0, 1]$ such that $\underline{v}_{ij} \geq \varepsilon\delta_{ij}$ in the matrix sense. The matrix F^{ij} is positive, and thus testing with the vectors $\nu \pm \tau$ gives

$$|F^{\nu\tau}| \leq \frac{1}{2}trF^{ij}. \tag{4.13}$$

We introduce the abbreviation

$$trF^{\tau\tau'} = F^{ij}\left(\delta_{ij} - \frac{x_i}{|x|}\frac{x_j}{|x|}\right).$$

Direct computations, using (2.4), (4.13), give

$$\begin{aligned}
Lu &= \frac{1}{w}F^{ij}u_{ij} + \frac{1}{w^2}Fu^l u_l \le C,\\
L\underline{v} &= \frac{1}{w}F^{ij}\underline{v}_{ij} + \frac{1}{w^2}Fu^l\underline{v}_l\\
&\ge -C + \frac{1}{w}F^{\tau\tau'}\underline{v}_{\tau\tau'} + 2\frac{1}{w}F^{\tau\nu}\underline{v}_{\tau\nu} + \nu F^{\nu\nu}\underline{v}_{\nu\nu},\\
&\ge -C + \left(\frac{C}{|x|} - \frac{C}{|x|^2}\right)\frac{1}{w}trF^{\tau\tau'} - \frac{C}{|x|^2}\frac{1}{w}F^{\nu\nu},\\
Ld &= L(R-|x|) = -\frac{1}{w}F^{ij}|x|_{ij} - \frac{1}{w^2}Fu^l|x|_l,\\
&= -\frac{1}{|x|}trF^{\tau\tau'} - \frac{1}{w^2}Fu^l\frac{x_l}{|x|},\\
L\vartheta &\le C\left(1+\frac{1}{\sqrt{R}}\right) - \left(\frac{C}{|x|} - \frac{C}{|x|^2} + \frac{C}{|x|}\left(\frac{1}{\sqrt{R}} - \frac{d}{R^{\frac{5}{4}}}\right)\right)\frac{1}{w}trF^{\tau\tau'}\\
&- \left(\frac{1}{R^{\frac{5}{4}}} - \frac{C}{|x|^2}\right)\frac{1}{w}F^{\nu\nu}. \qquad (4.14)
\end{aligned}$$

Thus for R sufficiently large,

$$L\vartheta \le C - \frac{C}{R}trF^{\tau\tau'} - \frac{1}{2R^{\frac{5}{4}}}F^{\nu\nu}. \qquad (4.15)$$

Hence the Maclaurin inequalities and arithmetic means as well as (1.7) and (2.5) show that for large values of R,

$$\begin{aligned}
L\vartheta &\le C - C\frac{H_{k-1}}{\psi^{k-1}}R^{-\frac{n-1}{n}-\frac{5}{4n}} - \frac{1}{4R^{\frac{4}{5}}}trF^{ij}\\
&\le C - C\psi^{-\frac{(k-1)^2}{k}}R^{-\frac{n-1}{n}-\frac{5}{4n}} - \frac{1}{4R^{\frac{5}{4}}}trF^{ij} \qquad (4.16)\\
&\le -CR^{\frac{3}{4n}} - \frac{1}{4R^{\frac{5}{4}}}trF^{ij}. \qquad (4.17)
\end{aligned}$$

Lemma 4.5 *There exists a positive constant A independent of R such that*

$$\Theta := \vartheta + A\cdot\frac{1}{R^2}|x-x_0|^2 \pm T(u-\underline{v})$$

satisfies the inequalities

$$\begin{cases} L\Theta \le 0 & \text{in } \Omega_\delta,\\ \Theta \ge 0 & \text{on } \partial\Omega_\delta, \end{cases} \qquad (4.18)$$

where $\delta = R^{\frac{3}{4}}$ and ϑ is as in Lemma 4.4.

Proof. According to Lemma 4.4, the condition $\Theta \ge 0$ on $\partial\Omega_\delta$ follows if

$$A\cdot\frac{1}{R^2}\cdot|x-x_0|^2 \pm T(u-\underline{v}) \ge 0 \text{ on } \partial\Omega_\delta.$$

In view of (4.8), this can be arranged by choosing A sufficiently large. The property $L\Theta \le 0$ now follows from the inequality

$$\begin{aligned} L\Theta &= L\vartheta + C\cdot\frac{A}{R} + C + C\cdot\frac{1+A}{R^2}trF^{ij} \\ &\le -CR^{\frac{3}{4n}} - CR^{-\frac{5}{4}}trF^{ij} + C\cdot\frac{A}{R} + C + C\cdot\frac{1+A}{R^2}trF^{ij} \\ &\le 0. \end{aligned} \tag{4.19}$$

Proof of Lemma 4.2. The maximum principle applied to (4.18) yields that $\Theta \ge 0$ in Ω_δ. Since $\Theta(x_0) \ge 0$, it follows that

$$\Theta_\mu(x_0) \ge 0,$$

with $\mu = -\frac{x_0}{|x_0|}$. Thus we obtain

$$\vartheta_\mu(x_0) \ge |(T(u - \underline{v}))_\mu|(x_0),$$

and this finally gives (4.2).

Lemma 4.6 (Double normal C^2-estimates at the outer boundary). *Under the assumptions of Lemma 4.2 and Theorem 1.1,*

$$|u_{\mu\mu}(x_0)| \le C. \tag{4.20}$$

Proof. To prove this estimate, we use the idea of [15], see also [5, 8, 13]. We may assume that the infimum of the invariantly defined function

$$\partial B_R \ni x \mapsto \inf_{0\ne\zeta\in T_x\partial B_R} \frac{h_{ij}\zeta^i\zeta^j(x)}{g_{ij}\zeta^i\zeta^j(x)} \tag{4.21}$$

equals $\frac{h_{11}(x_0)}{g_{11}(x_0)}$.

We intend to establish a positive lower bound for $\frac{h_{11}(x_0)}{g_{11}(x_0)}$, which is independent of R, i.e. we want to prove the strict tangential convexity of our solution. In view of lower order estimates and the strict convexity of the barrier function $\underline{v}$ we know that

$$\underline{v}_{11}(x_0) \ge C > 0. \tag{4.22}$$

Therefore we may assume that

$$u_{11}(x_0) < \frac{1}{2}\underline{v}_{11}(x_0), \tag{4.23}$$

for otherwise the strict tangential convexity of u is proved.

As $\underline{v} - u = 0$ on ∂B_R, we deduce there

$$(\underline{v} - u)_{rs} = 0, \quad (\underline{v} - u)_t = 0, \quad r, s, t < n, \tag{4.24}$$

and furthermore

$$u_{11} = \underline{v}_{11} - B_{11}(u - \underline{v})_n \quad \text{on } \partial B_R, \tag{4.25}$$

where $B_{\alpha\beta} =< \nabla_\alpha e_\beta,\ e_\beta >, 1 \leq \alpha, \beta \leq n-1$. It follows that

$$B_{11}(x_0)(u - \underline{v})_n(x_0) \geq \frac{1}{2}\underline{v}_{11})(x_0) \geq \frac{C}{2}. \tag{4.26}$$

And for $x \in \partial B_R$ near x_0, since $u_{11} \mid_{\partial B_R}$ is minimized at x_0,

$$B_{11}(x)(u - \underline{v})_n(x) \leq \underline{v}_{11}(x) - \underline{v}_{11}(x_0) + B_{11}(x_0)(u - \underline{v})_n(x_0). \tag{4.27}$$

Because B_{11} is smooth near ∂B_R and $0 < (u - \underline{v})_n \leq C$, we must have $B_{11} \geq C > 0$ on $\Omega_\delta := B_\delta(x_0) \cap B_R$, if δ is chosen sufficiently small. Therefore,

$$(u - \underline{v})_n(x) \leq \Psi(x), \text{ for } x \in \Omega_\delta \cap \partial B_R, \quad (u - \underline{v})_n(x_0) \leq \Psi(x_0), \tag{4.28}$$

where $\Psi(x) = B_{11}^{-1}(x)[\underline{v}_{11}(x) - \underline{v}_{11}(x_0) + B_{11}(x_0)(u - \underline{v})_n(x_0)]$.

We define Ξ as Θ in Lemma 4.5

$$\Xi := \vartheta + A \cdot \frac{1}{R^2}|x - x_0|^2 + \Psi(x) - (u - \underline{v})_n.$$

Since (4.28) and $\Psi(x)$ is smooth in Ω_δ, we deduce as the Mixed second derivatives at the outer boundary that

$$\begin{cases} L\Xi \leq 0 & \text{in } \Omega_\delta, \\ \Xi \geq 0 & \text{on } \partial\Omega_\delta. \end{cases} \tag{4.29}$$

As before, the maximum principle then yields

$$\Xi \geq 0 \quad \text{in } \Omega_\delta.$$

Consequently, since $\Xi = 0$ at x_0 , we have

$$u_{nn}(x_0) \leq C.$$

This shows that the eigenvalues of $\{u_{ij}(x_0)\}$ are all bounded (and all positive). Thus each of them must be bounded below from zero by the equation (2.5). In particular, we obtain the estimate

$$u_{11}(x_0) \geq C > 0,$$

which in turn implies

$$u_{nn} \leq C \text{ on } \partial B_R.$$

Lemma 4.7 (Second derivatives at the inner boundary). *Under the assumptions of Theorem 1.1, the second derivatives on $\partial\mathbb{K}$ are bounded uniformly in R.*

The proof of Lemma 4.7 can be found in [1]; we may follow the methods of [9, 5] to consider an arbitrary smooth bounded domain.

5 Proof of Theorem 1.2

We have proved Theorem 1.1 by geting the interior and boundary derivative estimates. On the other hand, we can consider $\mathbb{R}^n$ be approached by an open ball. Thus we can prove Theorem 1.2 using the fact that interior C^2-norms of u^R are uniformly bounded in R, and therefore we can get them without some decay conditions at infinity (2.4), i.e.

$$|D^2(\underline{v} - r)| + |D^3\underline{v}| = \mathcal{O}(\frac{1}{r^2}). \tag{5.1}$$

Based on the interior gradient estimate in section 3 and the construction of exterior solution in in section 2, we need only get the estimates for interior second derivatives. One can use the argument in [14] (see also [1])to deriver estimates for the second derivatives. We have the C^2 estimate :

Proposition 5.1 *There exist C_{R_0} and $R_1 = R_1(R_0)$ such that $\forall R \geq R_1$,*

$$\|u_R\|_{2,B_{R_0}\setminus\mathbb{K}} \leq C_{R_0}. \tag{5.2}$$

Thus, C^2-norms of u^R are uniformly bounded in R, and standard Evans-Krylov and Schauder interior regularity theory imply a prior bounds for higher-order derivatives. A diagonal process then yields a subsequence u_{R_k}, $R_k \to +\infty$, that locally converges to a solution of (1.5) and (1.6).

References

[1] L. Caffarelli, L. Nirenberg, J. Spruck, *Nonlinear second order elliptic equations. IV. Star-shaped compact Weingarten hypersurfaces.* Current topics in partial differential equations, Kinokuniya, Tokyo, (1986), 1-26.

[2] L. Caffarelli, L. Nirenberg, J. Spruck, *Nonlinear second-order elliptic equations, V: The Dirichlet problem for Weingarten hypersurfaces*, Comm. Pure Appl. Math. **41** (1988), 47-70.

[3] K.S. Chou, X.J. Wang, *A variational theorem of the hessian equation*, Comm. Pure and Applied Math. **LIV** (2001), 1029-1064.

[4] F. Finster, O.C. Schnürer, *Hypersurfaces of prescribed Gauss curvature in exterior domains*, Calc. Var. **15** (2002), 67-80.

[5] B. Guan, *The Dirichlet problem for Monge-Ampères in non-convex domains and spacelike hypersurfaces of constant Gauss curvature*, Trans. Amer. Math. Soc. **350** (1998), 4955-4971.

[6] B. Guan, P.F. Guan, *Convex hypersurfaces of prescribed curvatures*, Annals of math. **156** (2002), 655-673.

[7] B. Guan, H.Y. Jian, R. M. Schoen, *Entire spacelike hypersurfaces of prescribed Gauss curvature in Minkowski space*, J. fur die reine und angewandte Math. **595** (2006), 167-188.

[8] B. Guan, Y.Y. Li. *Monge-Ampère equations on Riemannian manifolds*, J. diff. equa. **132** (1996), 126-139.

[9] B. Guan, J. Spruck. *Locally convex hypersurfaces of constant curvature with boundary*, Comm. Pure and Applied Math. **Lvii** (2004), 1311-1331.

[10] D. Gilbarg, N.S. Trudinger. Elliptic Partial Differential Equations of Second Order, second edition, revised third printing, Springer-Verlag, (1998).

[11] N.M. Ivochkina, Mi Lin and N S.Trudinger. *The Dirichlet problem for the prescribed curvature quotient equations with general boundary values.* Geometric analysis and the calculus of variations. Cambridge:Internat. Press, MA, (1996), 125-141.

[12] M. Lin, N.S. Trudinger. *On some inequalities for elementary symmetric functions*, Bull. Austral. Math. Soc. **50** (1994), 317-326.

[13] O.C. Schnürer. *The Dirichlet problem for Weingarten hypersurfaces in Lorentz manifolds*, Math. Z. **242** (2002), 159-181.

[14] W. Sheng, J. Urbas and Xu-Jia Wang. *Interior curvature bounds for a class of curvature equations*, Duke Math. J., **123** (2004), 235-264.

[15] N.S. Trudinger. *On the Dirichlet problem for Hessian equations*, Acta Math. **175** (1995), 151-164.

[16] J. Urbas. *The Dirichlet problem for the equation of prescribed scalar curvature in Minkowski space*, Calc. Var. **18** (2003), 307-316.

[17] Xu-Jia Wang. *Interior gradient estimates for mean curvature equations*, Math. Z. **228** (1998), 73-81.

[18] Xu-Jia Wang. The Monge-Ampère equation and application. Note at NanKai Institute of mathematics, (2004).

[19] I. Walter-Koch. *Equations of mean curvature type on exterior domains*, Manuscripta Math. **95** (1998), 91-105.