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A Multiplicity Result Including Sign-Changing Solutions For a Nonlinear Elliptic Problem in $\mathbf{R}^{\mathbf{N}}$

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Abstract

We show that the problem

$$\begin{cases} -\Delta u + \mu u = Q(x)|u|^{p-1}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

has multiple solutions including sign-changing ones, where $\mu > 0$, $N \geq 3$, $1 < p < (N+2)/(N-2)$ and $Q : \mathbb{R}^N \to \mathbb{R}$ is some continuous function. We obtain sign-changing solutions which are not only local minimum types in $\mathcal{N}_*$ but also mountain pass types in $\mathcal{N}_*$, where $\mathcal{N}_* = \{u = u^+ + u^- \in H^1(\mathbb{R}^N) : u^+ \neq 0, u^- \neq 0, \int_{\mathbb{R}^N} (|\nabla u^\pm|^2 + \mu|u^\pm|^2)\,dx = \int_{\mathbb{R}^N} Q(x)|u^\pm|^{p+1}\,dx\}$.

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1 Introduction

In the present paper, we consider the multiple existence of solutions for the problem

$$\begin{cases} -\Delta u + \mu u = Q(x)|u|^{p-1}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\mu > 0$, $N \geq 3$, $1 < p < (N+2)/(N-2)$ and $Q \in C(\mathbb{R}^N, \mathbb{R})$. Recently, the existence and multiplicity of sign-changing solutions of semilinear elliptic problems has been considered by many authors under various conditions; see [1–4, 7–11, 13–16, 19, 21, 23] and others. In order to show the existence of sign-changing solutions of semilinear elliptic problems, there is a difficulty such that the mapping $u \mapsto \int_{\mathbb{R}^N} |\nabla u^{\pm}|^2 \, dx$ on a first order Sobolev space is not Fréchet differentiable; see [10, 20]. The existence of sign-changing solutions of problem (1.1) was considered in [6]. However, as pointed out by Clapp and Weth [10], the argument contains a gap caused by the difficulty mentioned above.

The purpose of this paper is to show the effect of the shape of the graph of the function Q on the multiple existence of solutions (especially, sign-changing solutions) of problem (1.1). Under some assumptions on Q, we show that there exist not only sign-changing solutions of local minimum types in $\mathcal{N}_*$ but also those of mountain path types in $\mathcal{N}_*$, where $\mathcal{N}_* = \{u = u^+ + u^- \in H^1(\mathbb{R}^N) : u^+ \neq 0, u^- \neq 0, \int_{\mathbb{R}^N} (|\nabla u^{\pm}|^2 + \mu |u^{\pm}|^2) \, dx = \int_{\mathbb{R}^N} Q(x)|u^{\pm}|^{p+1} \, dx\}$. We remark that in [6], the existence of sign-changing solutions were considered only when they are the local minimum types in $\mathcal{N}_*$. We note that some levels of an associated functional corresponding to our sign-changing solutions may not satisfy the Palais-Smale condition. Our first result is the following:

Theorem 1.1. *Let $Q : \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $Q(x) > 1$ for all $x \in \mathbb{R}^N$, $Q(x) \to 1$ as $|x| \to \infty$ and there exist $a^1, a^2 \in \mathbb{R}^N$ such that $a^1 \neq a^2$, $Q(a^j)$ are strict maximuma, $Q(a^j) = M$ for $j = 1, 2$ and $Q(x) < M$ for all $x \in \mathbb{R}^N \setminus \{a^1, a^2\}$, where $M = \max\{Q(x) : x \in \mathbb{R}^N\}$. Then there exists $\mu_0 > 0$ such that for each $\mu \geq \mu_0$, problem (1.1) has at least three positive solutions and at least four pairs of sign-changing solutions.*

Remark 1.1. We can show that at least three of four pairs of sign-changing solutions we obtained have exactly two nodal domains; see Remark 3.2. Here, we say that a subset Ω *of* $\mathbb{R}^N$ *is a nodal domain of* $u : \mathbb{R}^N \to \mathbb{R}$ if Ω is open and connected, u has a fixed sign in Ω and $u(x) = 0$ for all $x \in \partial\Omega$.

Remark 1.2. By our proof in the next section, we can see that if Q has $m(\geq 2)$ distinct points where Q attains its maximum value, then we have at least $m+1$ positive solutions and at least $m + {}_mC_2 + 1$ pairs of sign-changing solutions.

With an additional assumption, we can show that there exists another pair of sign-changing solutions.

Theorem 1.2. *Under the assumptions in the previous theorem, if there exist M_1,*

$$M_1 > 2^{(p-1)/2} M / (M^{2/(p-1)} + 1)^{(p-1)/2},$$

and $R > |a^1 - a^2|/2$ *such that* $Q(x) \geq M_1$ *for* $x \in \mathbb{R}^N$ *with* $\left|x - (a^1 + a^2)/2\right| \leq R$, *and* μ *is sufficiently large, then there is another pair of sign-changing solutions of* (1.1).

2 Preliminaries and existence of positive solutions

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm on $H^1(\mathbb{R}^N)$ defined by $(u,v) = \int_{\mathbb{R}^N}(\nabla u \nabla v + uv)\,dx$ and $\|u\| = (u,u)^{1/2}$ for $u, v \in H^1(\mathbb{R}^N)$, respectively.

We put $\lambda = 1/\sqrt{\mu}$ and $v(x) = \lambda^{2/(p-1)}u(\lambda x)$. Then problem (1.1) becomes

$$\begin{cases} -\Delta v + v = Q(\lambda x)|v|^{p-1}v & \text{in } \mathbb{R}^N, \\ \qquad v \in H^1(\mathbb{R}^N). \end{cases} \tag{2.2}$$

In order to find weak solutions of (2.2), we employ a variational method, and we partially follow the arguments in [6]. We define functionals on $H^1(\mathbb{R}^N)$ corresponding to problem (2.2) by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(\lambda x)|u|^{p+1}\,dx \quad \text{for } u \in H^1(\mathbb{R}^N);$$
$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(\lambda x)(u^+)^{p+1}\,dx \quad \text{for } u \in H^1(\mathbb{R}^N).$$

We know that nontrivial critical points of I_λ and J_λ are nontrivial solutions and positive solutions of (2.2), respectively.

For $\eta > 0$ and $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, we define $C_\eta[x] = \prod_{i=1}^N [x_i - \eta, x_i + \eta]$. We choose $K, l > 0$ such that $C_l[a^1] \cap C_l[a^2] = \emptyset$ and $\bigcup_{j=1}^2 C_l[a^j] \subset \prod_{i=1}^N (-K, K)$. From our assumption, we have $\sup\{Q(x) : x \in \mathbb{R}^N \setminus \bigcup_{j=1}^2 C_l[a^j]\} < M$. For each $\lambda > 0$, we define $\phi_\lambda \in C(\mathbb{R}, \mathbb{R})$ and $\beta_\lambda \in C(H^1(\mathbb{R}^N) \setminus \{0\}, \mathbb{R}^N)$ by

$$\phi_\lambda(t) = \begin{cases} 2K/\lambda & \text{if } t > 2K/\lambda, \\ t & \text{if } -2K/\lambda \leq t \leq 2K/\lambda, \\ -2K/\lambda & \text{if } t < -2K/\lambda, \end{cases}$$
$$\beta_\lambda(u) = \big(\beta_{\lambda,1}(u), \cdots, \beta_{\lambda,N}(u)\big) \quad \text{for } u \in H^1(\mathbb{R}^N) \setminus \{0\},$$

where $\beta_{\lambda,i}(u)$ is defined by

$$\beta_{\lambda,i}(u) = \int_{\mathbb{R}^N} \phi_\lambda(x_i)|u|^{p+1}\,dx \Big/ \int_{\mathbb{R}^N} |u|^{p+1}\,dx \quad \text{for } i = 1, \ldots, N.$$

For each $\lambda > 0$ and $j, k \in \{1, 2\}$, we define subsets of $H^1(\mathbb{R}^N)$ by

$$\mathcal{N}(\lambda) = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : (\nabla I_\lambda(u), u) = 0\};$$
$$\widetilde{\mathcal{N}}(\lambda) = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : (\nabla J_\lambda(u), u) = 0\};$$

$$\begin{aligned}\mathcal{N}(\lambda; j) &= \{u \in \mathcal{N}(\lambda) : \beta_\lambda(u) \in C_{l/\lambda}[a^j/\lambda]\};\\ \widetilde{\mathcal{N}}(\lambda; j) &= \{u \in \widetilde{\mathcal{N}}(\lambda) : \beta_\lambda(u) \in C_{l/\lambda}[a^j/\lambda]\};\\ \mathcal{N}_*(\lambda) &= \{u \in \mathcal{N}(\lambda) : u^+ \in \mathcal{N}(\lambda),\ u^- \in \mathcal{N}(\lambda)\};\\ \mathcal{N}_*(\lambda; j, k) &= \{u \in \mathcal{N}_*(\lambda) : u^+ \in \mathcal{N}(\lambda; j),\ u^- \in \mathcal{N}(\lambda; k)\},\end{aligned}$$

where for $u \in H^1(\mathbb{R}^N)$, $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$, respectively. We define $\tau : H^1(\mathbb{R}^N) \to (0, \infty]$ by

$$\tau(u) = \begin{cases} \text{the unique } t > 0 \text{ with } tu \in \mathcal{N}(\lambda) & \text{if } u \in H^1(\mathbb{R}^N) \setminus \{0\};\\ \infty & \text{if } u = 0.\end{cases}$$

We can easily see that $\tau : H^1(\mathbb{R}^N) \to (0, \infty]$ is continuous and that for each $u \in \mathcal{N}(\lambda)$, we have $u \in \mathcal{N}_*(\lambda)$ if and only if $\tau(u^+) = \tau(u^-)$.

Let $\eta > 0$. Kwong [17] showed that the problem

$$\begin{cases} -\Delta u + u = \eta |u|^{p-1} u & \text{in } \mathbb{R}^N,\\ u \in H^1(\mathbb{R}^N) \end{cases}$$

has a unique radially symmetric, positive solution, which we denote by $\bar{u}_\eta$. We set

$$\bar{c}_\eta = \inf\left\{\frac{1}{2}\|u\|^2 - \frac{\eta}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx : u \in H^1(\mathbb{R}^N), \|u\|^2 = \eta \int_{\mathbb{R}^N} |u|^{p+1}\,dx\right\}.$$

We know that $\bar{u}_\eta$ attains the infimum of the equation above, and we can easily see that

$$\begin{aligned}&\bar{c}_M = \bar{c}_1 \big/ M^{2/(p-1)};\\ &\bar{c}_M < \inf\{I_\lambda(u) : u \in \mathcal{N}(\lambda; j)\} = \inf\{J_\lambda(u) : u \in \widetilde{\mathcal{N}}(\lambda; j)\} \quad \text{for } \lambda > 0 \text{ and } j = 1, 2;\\ &2\bar{c}_M < \inf\{I_\lambda(u) : u \in \mathcal{N}_*(\lambda; j, k)\} \quad \text{for } \lambda > 0 \text{ and } j, k = 1, 2.\end{aligned}$$

We fix $z \in \mathbb{R}^N$ such that $|z| = 1$ and $z \cdot (a^2 - a^1) = 0$, where $\cdot$ is the inner product on $\mathbb{R}^N$. For each $\lambda \in (0, 1/4)$, let $\psi_\lambda \in C^1(\mathbb{R}^N; \mathbb{R})$ be a function such that $|\nabla \psi_\lambda| \le 2$ and

$$\psi_\lambda(x) = \begin{cases} 1 & \text{if } |x| < 1/(2\sqrt{\lambda}) - 1,\\ 0 & \text{if } |x| > 1/(2\sqrt{\lambda}).\end{cases}$$

For each $y \in \mathbb{R}^N$ and $\lambda \in (0, 1/4)$, we define $v_{+,y,\lambda}$, $v_{-,y,\lambda}$, $u_{+,y,\lambda}$ and $u_{-,y,\lambda}$ by

$$\begin{aligned}v_{+,y,\lambda}(x) &= \psi_\lambda\left(x - \left(\frac{y}{\lambda} + \frac{z}{2\sqrt{\lambda}}\right)\right)\bar{u}_{Q(y)}\left(x - \left(\frac{y}{\lambda} + \frac{z}{2\sqrt{\lambda}}\right)\right);\\ v_{-,y,\lambda}(x) &= -\psi_\lambda\left(x - \left(\frac{y}{\lambda} - \frac{z}{2\sqrt{\lambda}}\right)\right)\bar{u}_{Q(y)}\left(x - \left(\frac{y}{\lambda} - \frac{z}{2\sqrt{\lambda}}\right)\right);\\ u_{+,y,\lambda} &= \tau(v_{+,y,\lambda})v_{+,y,\lambda};\\ u_{-,y,\lambda} &= \tau(v_{-,y,\lambda})v_{-,y,\lambda}.\end{aligned}$$

We note that $u_{+,y,\lambda} + u_{-,y,\lambda} \in \mathcal{N}_*(\lambda)$ for each $y \in \mathbb{R}^N$ and $\lambda > 0$.

Lemma 2.1. *For each* $\varepsilon > 0$*, there exists* $\lambda_0 \in (0, 1/4)$ *such that for each* $\lambda \in (0, \lambda_0)$ *and* $y \in \mathbb{R}^N$*, there holds* $I_\lambda(u_{i,y,\lambda}) < \bar{c}_{Q(y)} + \varepsilon$ *for* $i \in \{+, -\}$*.*

Proof. Since

$$\begin{aligned}\tau(v_{+,y,\lambda})^{p-1} &= \frac{\|v_{+,y,\lambda}\|^2}{\int_{\mathbb{R}^N} Q(\lambda x)|v_{+,y,\lambda}|^{p+1}\,dx} \\ &= \frac{Q(y)\|\psi_\lambda \bar{u}_1\|^2}{\int_{\mathbb{R}^N} Q(y + \lambda x + 2\sqrt{\lambda} z)|\psi_\lambda \bar{u}_1|^{p+1}\,dx} \to 1\end{aligned}$$

as $\lambda \to +0$ uniformly for $y \in \mathbb{R}^N$, we have

$$\begin{aligned}I_\lambda(u_{+,y,\lambda}) &= \frac{1}{2}\|\tau(v_{+,y,\lambda})v_{+,y,\lambda}\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} Q(\lambda x)|\tau(v_{+,y,\lambda})v_{+,y,\lambda}|^{p+1}\,dx \\ &= \frac{\tau(v_{+,y,\lambda})^2}{2Q(y)^{\frac{2}{p-1}}}\|\psi_\lambda \bar{u}_1\|^2 - \frac{\tau(v_{+,y,\lambda})^{p+1}}{(p+1)Q(y)^{\frac{p+1}{p-1}}}\int_{\mathbb{R}^N} Q(y + \lambda x + 2\sqrt{\lambda} z)|\psi_\lambda \bar{u}_1|^{p+1}\,dx \\ &\to \bar{c}_{Q(y)}\end{aligned}$$

as $\lambda \to +0$ uniformly for $y \in \mathbb{R}^N$. Hence we obtain the conclusion in the case $i = +$. By a similar way, we can treat the case $i = -$. □

Lemma 2.2. *For each* $\lambda \in (0, \min\{1/4, l^2\})$ *and* $y \in \mathbb{R}^N$ *with* $\max_i |y_i| \le 2K - l$*, there holds* $\beta_\lambda(u_{i,y,\lambda}) \in C_{l/\lambda}[y/\lambda]$ *for* $i \in \{+, -\}$*.*

Proof. Let $\lambda \in (0, \min\{1/4, l^2\})$ and let $y \in \mathbb{R}^N$ with $\max_i |y_i| \le 2K - l$. Fix any $i \in \{1, \dots, N\}$. For each $x \in \mathbb{R}^N$ with $|x_i - y_i/\lambda| > 1/\sqrt{\lambda}$, we have $|x - (y/\lambda + z/(2\sqrt{\lambda}))| > 1/(2\sqrt{\lambda})$ and hence $\psi_\lambda(x - (y/\lambda + z/(2\sqrt{\lambda}))) = 0$. For each $x \in \mathbb{R}^N$ with $|x_i - y_i/\lambda| \le 1/\sqrt{\lambda}$, we have $|x_i| \le 2K/\lambda$ and hence $\phi_\lambda(x_i) = x_i$. Then we have

$$\begin{aligned}&\beta_{\lambda,i}(u_{+,y,\lambda}) = \beta_{\lambda,i}(v_{+,y,\lambda}) \\ &= \frac{\int_{|x_i - y_i/\lambda| \le 1/\sqrt{\lambda}} x_i \left|\psi_\lambda\left(x - \left(\frac{y}{\lambda} + \frac{z}{2\sqrt{\lambda}}\right)\right)\bar{u}_{Q(y)}\left(x - \left(\frac{y}{\lambda} + \frac{z}{2\sqrt{\lambda}}\right)\right)\right|^{p+1} dx}{\int_{|x_i - y_i/\lambda| \le 1/\sqrt{\lambda}} \left|\psi_\lambda\left(x - \left(\frac{y}{\lambda} + \frac{z}{2\sqrt{\lambda}}\right)\right)\bar{u}_{Q(y)}\left(x - \left(\frac{y}{\lambda} + \frac{z}{2\sqrt{\lambda}}\right)\right)\right|^{p+1} dx} \\ &\in [y_i/\lambda - 1/\sqrt{\lambda}, y_i/\lambda + 1/\sqrt{\lambda}] \subset [(y_i - l)/\lambda, (y_i + l)/\lambda],\end{aligned}$$

which implies $\beta_\lambda(u_{+,y,\lambda}) \in C_{l/\lambda}[y/\lambda]$. By a similar way, we can show $\beta_\lambda(u_{-,y,\lambda}) \in C_{l/\lambda}[y/\lambda]$. □

As a direct consequence of the previous two lemmas, we have the following:

Lemma 2.3. *For each* $\varepsilon > 0$*, there exists* $\lambda_0 \in (0, \min\{1/4, l^2\})$ *such that for each* $\lambda \in (0, \lambda_0)$*,*

$$u_{+,a^j,\lambda} \in \widetilde{\mathcal{N}}(\lambda; j), \quad J_\lambda(u_{+,a^j,\lambda}) < \bar{c}_M + \varepsilon \quad \textit{for } j = 1, 2; \tag{2.3}$$

$$\begin{array}{l} u_{+,a^j,\lambda} + u_{-,a^k,\lambda} \in \mathcal{N}_*(\lambda; j, k), \\ I_\lambda(u_{+,a^j,\lambda} + u_{-,a^k,\lambda}) < 2\bar{c}_M + \varepsilon \end{array} \quad \text{for } j = 1, 2 \text{ and } k = 1, 2. \tag{2.4}$$

Lemma 2.4. *If $\{u_n\}$ is a bounded sequence of $H^1(\mathbb{R}^N)$ which satisfies $\sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n|^2\, dx \to 0$, then $\int_{\mathbb{R}^N} |u_n|^{p+1}\, dx \to 0$.*

For the proof of the lemma above, see [22, Lemma 1.21].

Lemma 2.5. *Let $\{\lambda_n\}$ be a positive sequence with $\lambda_n \to +0$ and let $\{u_n\} \subset \mathcal{N}(\lambda_n)$ be a sequence such that $u_n \geq 0$ for each $n \in \mathbb{N}$ and $I_{\lambda_n}(u_n) \to \bar{c}_M$. Then there exist a subsequence $\{u_{n_i}\}$ of $\{u_n\}$, $\{y_i\} \subset \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$ and $j \in \{1, 2\}$ such that $\|u_{n_i} - \bar{u}_M(\cdot + x_0 - y_i)\| \to 0$ and $\lambda_i y_i \to a^j$ as $i \to \infty$.*

Proof. We set $\delta = \underline{\lim}_n \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n|^2\, dx$. By the previous lemma, we have $\delta > 0$. So we can choose $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B(y_n,1)} |u_n|^2\, dx > \delta/2$ for each $n \in \mathbb{N}$. We set $v_n = u_n(\cdot + y_n)$ for each $n \in \mathbb{N}$. We may assume $\{v_n\}$ converges weakly to v in $H^1(\mathbb{R}^N)$. Since $\int_{B(0,1)} |v_n|^2\, dx > \delta/2$ for each $n \in \mathbb{N}$, we have $v \neq 0$. We assume that $\underline{\lim}_n \|v_n - v\| > 0$. We set $w_n = v_n - v$ for each $n \in \mathbb{N}$. Since the Brezis-Lieb lemma says $\int_{\mathbb{R}^N} \big| |v_n|^{p+1} - |w_n|^{p+1} - |v|^{p+1} \big|\, dx \to 0$ (see [22, the proof of Lemma 1.32]), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|v_n|^{p+1}\, dx \\ &\qquad = \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|w_n|^{p+1}\, dx + \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|v|^{p+1}\, dx + o(1). \end{aligned}$$

Using

$$\|v_n\|^2 = \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|v_n|^{p+1}\, dx, \ \|v_n\|^2 = \|w_n\|^2 + \|v\|^2 + o(1)$$

and the equality above, we have

$$\begin{aligned} &\|v\|^2 - \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|v|^{p+1}\, dx \\ &\qquad = -\left(\|w_n\|^2 - \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|w_n|^{p+1}\, dx \right) + o(1). \end{aligned}$$

We may assume

$$\|w_n\|^2 - \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|w_n|^{p+1}\, dx \leq o(1)$$

or

$$\|v\|^2 - \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|v|^{p+1}\, dx \leq o(1).$$

In the former case, we can choose $t_n \in (0, 1 + o(1))$ such that

$$\|t_n w_n\|^2 = \int_{\mathbb{R}^N} Q(\lambda_n x + \lambda_n y_n)|t_n w_n|^{p+1}\, dx,$$

and hence we have

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{p+1}\right)\|v\|^2+\bar{c}_M \\
&\le \left(\frac{1}{2}-\frac{1}{p+1}\right)\|v\|^2+\frac{1}{2}\|t_nw_n\|^2-\frac{1}{p+1}\int_{\mathbb{R}^N} Q(\lambda_n x+\lambda_n y_n)|t_nw_n|^{p+1}\,dx \\
&= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\|v\|^2+\|t_nw_n\|^2\right) \le \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\|v\|^2+\|w_n\|^2\right)+o(1) \\
&= \frac{1}{2}\|v\|^2-\frac{1}{p+1}\int_{\mathbb{R}^N} Q(\lambda_n x+\lambda_n y_n)|v|^{p+1}\,dx \\
&\qquad +\frac{1}{2}\|w_n\|^2-\frac{1}{p+1}\int_{\mathbb{R}^N} Q(\lambda_n x+\lambda_n y_n)|w_n|^{p+1}\,dx+o(1) \\
&= I_{\lambda_n}(u_n)+o(1),
\end{aligned}$$

which contradicts $I_{\lambda_n}(u_n)\to\bar{c}_M$. In the latter case, we can similarly get a contradiction. So we may assume that $\{v_n\}$ converges strongly to v. If $\overline{\lim}_n|\lambda_n y_n|=\infty$, we have $\|v\|^2=\int_{\mathbb{R}^N}|v|^{p+1}\,dx$ and hence

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\left(\frac{1}{2}\|v_n\|^2-\frac{1}{p+1}\int_{\mathbb{R}^N} Q(\lambda_n x+\lambda_n y_n)|v_n|^{p+1}\,dx\right) \\
&\qquad = \frac{1}{2}\|v\|^2-\frac{1}{p+1}\int_{\mathbb{R}^N}|v|^{p+1}\,dx>\bar{c}_M,
\end{aligned}$$

which is a contradiction. So we may assume that $\{\lambda_n y_n\}$ converges to a point $a\in\mathbb{R}^N$. Then we have

$$\|v\|^2=\int_{\mathbb{R}^N} Q(a)|v|^{p+1}\,dx \quad\text{and}\quad \frac{1}{2}\|v\|^2-\frac{1}{p+1}\int_{\mathbb{R}^N} Q(a)|v|^{p+1}\,dx=\bar{c}_M.$$

So there exists $j\in\{1,2\}$ and $x_0\in\mathbb{R}^N$ such that $\lambda_n y_n\to a^j$ and

$$\|v_n-\bar{u}_M(\cdot+x_0)\|\to 0.$$

Hence we obtain

$$\|u_n-\bar{u}_M(\cdot+x_0-y_n)\|\to 0.$$

□

As a direct consequence of the previous lemma, we have the following, which is a slight extension of [6, Lemma 2.4]:

Lemma 2.6. *There exist* $\varepsilon>0$ *and* $\lambda_\varepsilon>0$ *such that for all* $\lambda\in(0,\lambda_\varepsilon)$,

$$\inf\{J_\lambda(u): u\in\widetilde{\mathcal{N}}(\lambda)\setminus\bigcup_{j=1}^{2}\widetilde{\mathcal{N}}(\lambda;j)\}>\bar{c}_M+\varepsilon, \tag{2.5}$$

$$\inf\{I_\lambda(u): u\in\mathcal{N}_*(\lambda)\setminus\bigcup_{j=1}^{2}\bigcup_{k=1}^{2}\mathcal{N}_*(\lambda;j,k)\}>2\bar{c}_M+\varepsilon. \tag{2.6}$$

In the rest of this paper, we fix $\varepsilon > 0$ and $\lambda_0 \in (0, \min\{1/4, l^2\})$ satisfying $\bar{c}_M + \varepsilon < \bar{c}_1$, (2.3), (2.4), (2.5) and (2.6) for all $\lambda \in (0, \lambda_0)$, and we fix $\lambda \in (0, \lambda_0)$. We abbreviate λ when there is no confusion; we write I, J, β, $\mathcal{N}$, $\widetilde{\mathcal{N}}$, $\mathcal{N}(j)$, $\widetilde{\mathcal{N}}(j)$, $\mathcal{N}_*$, $\mathcal{N}_*(j,k)$, $u_{+,y}$ and $u_{-,y}$ instead of I_λ, J_λ, β_λ, $\mathcal{N}(\lambda)$, $\widetilde{\mathcal{N}}(\lambda)$, $\mathcal{N}(\lambda; j)$, $\widetilde{\mathcal{N}}(\lambda; j)$, $\mathcal{N}_*(\lambda)$, $\mathcal{N}_*(\lambda; j, k)$, $u_{+,y,\lambda}$ and $u_{-,y,\lambda}$.

Now, we show the existence of positive solutions for (2.2). The following is a well-known concentration compactness principle. For details, see [22, Theorem 8.4].

Lemma 2.7. *For each $\{u_n\}$ of $H^1(\mathbb{R}^N)$ such that $J(u_n) \to c$ and $\nabla J(u_n) \to 0$, there exist a subsequence $\{u_{n_m}\}$ of $\{u_n\}$, a nonnegative solution v_0 of (2.2), $k \in \mathbb{N} \cup \{0\}$ and sequences $\{y_m^j\}$ of $\mathbb{R}^N$ for $j = 1, \ldots, k$ such that*

$$\Big\| u_{n_m} - v_0 - \sum_{j=1}^{k} \bar{u}_1(\cdot - y_m^j) \Big\| \to 0, \quad c = Jv_0 + k\bar{c}_1,$$
$$|y_m^j| \to \infty, \quad |y_m^j - y_m^{j'}| \to \infty \ \text{ for } j \neq j'.$$

As a direct consequence of the previous lemma and the assumptions for Q, we have the following:

Lemma 2.8. *J satisfies $(\mathrm{PS})_c$ for each $c \in (-\infty, \bar{c}_1)$.*

Since the following is well known, we omit the proof.

Proposition 2.1. *For each $j = 1, 2$, there exists a positive solution u of (2.2) in $\widetilde{\mathcal{N}}(j)$ satisfying $J(u) = \min\{J(v) : v \in \widetilde{\mathcal{N}}(j)\}$.*

Proposition 2.2. *There exists a positive solution u of (2.2) satisfying $J(u) > \min\{J(v) : v \in \widetilde{\mathcal{N}}(j)\}$ for $j = 1, 2$.*

Proof. Let $u_1 \in \widetilde{\mathcal{N}}(1)$, $u_2 \in \widetilde{\mathcal{N}}(2)$ be the positive solutions obtained in the previous proposition. We set

$$\Gamma = \{\gamma \in C([0,1]; \widetilde{\mathcal{N}}) : \gamma(0) = u_1, \gamma(1) = u_2\} \quad \text{and} \quad c = \inf_{\gamma \in \Gamma} \sup_{0 \leq s \leq 1} J(\gamma(s)).$$

By Lemmas 2.1 and 2.2, we have $\max\{Ju_1, Ju_2\} < c < \bar{c}_1$. Since $(\mathrm{PS})_c$ holds for J on $\widetilde{\mathcal{N}}$, by a standard mountain pass theorem, we can show that c is a critical value for J on $\widetilde{\mathcal{N}}$. □

Remark 2.1. Although the Palais-Smale condition is considered for J itself in Lemma 2.8, we use it for the restricted functional J on $\widetilde{\mathcal{N}}$ in the previous proposition. But we can easily see that the restricted functional J on $\widetilde{\mathcal{N}}$ satisfies $(\mathrm{PS})_c$ for each $c \in (-\infty, \bar{c}_1)$. We will find similar situations for I; see Lemma 3.2 and the propositions in the next section. The readers may verify these by the calculation (3.10).

3 Existence of sign-changing solutions

In this section, we show the existence of sign-changing solutions for (2.2). We consider the following problem and its associated functional as follows:

$$\begin{cases} -\Delta u + u = |u|^{p-1}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N); \end{cases} \tag{3.7}$$

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx \quad \text{for } u \in H^1(\mathbb{R}^N).$$

Since $Q(x) \to 1$ as $|x| \to \infty$, problem (3.7) is regarded as the limit problem of (2.2).

The following is also a well-known concentration compactness principle; see also [22, Theorem 8.4].

Lemma 3.1. *For each* $\{u_n\}$ *of* $H^1(\mathbb{R}^N)$ *such that* $I(u_n) \to c$ *and* $\nabla I(u_n) \to 0$, *there exist a subsequence* $\{u_{n_m}\}$ *of* $\{u_n\}$, *a solution* v_0 *of* (2.2), $k \in \mathbb{N} \cup \{0\}$, *nontrivial solutions* $v_1, \dots, v_k$ *of* (3.7) *and sequences* $\{y_m^j\}$ *of* $\mathbb{R}^N$ *for* $j = 1, \dots, k$ *such that*

$$\begin{aligned} &\Big\| u_{n_m} - v_0 - \sum_{j=1}^k v_j(\cdot - y_m^j) \Big\| \to 0, \quad c = Iv_0 + \sum_{j=1}^k I_\infty v_j, \\ &|y_m^j| \to \infty, \quad |y_m^j - y_m^{j'}| \to \infty \ \text{ for } j \neq j'. \end{aligned} \tag{3.8}$$

By the previous lemma, we can see that I does not satisfy Palais-Smale condition at the level $\bar{c}_1$. However, we can show the following Palais-Smale type condition holds.

Lemma 3.2. *For each* $c \in (-\infty, \bar{c}_1 + \bar{c}_M]$, *any sequence* $\{u_n\} \subset \mathcal{N}_*$ *satisfying* $I(u_n) \to c$ *and* $\nabla I(u_n) \to 0$ *has a convergent subsequence.*

Proof. Fix $c \in (-\infty, \bar{c}_1 + \bar{c}_M]$ and $\{u_n\} \subset \mathcal{N}_*$ satisfying $I(u_n) \to c$ and $\nabla I(u_n) \to 0$. By Lemma 3.1, we may assume that there exist a subsequence $\{u_{n_m}\}$ of $\{u_n\}$, a solution v_0 of (2.2), $k \in \mathbb{N} \cup \{0\}$, nontrivial solutions $v_1, \dots, v_k$ of (3.7) and sequences $\{y_m^j\}$ of $\mathbb{R}^N$ for $j = 1, \dots, k$ satisfying (3.8). We know that $I(v_0) > \bar{c}_M$ if $v_0 \neq 0$ and that $I_\infty(v_j) > 2\bar{c}_1$ or $I_\infty(v_j) = \bar{c}_1$ if v_j is sign-changing or not, respectively. So we can infer that one of the following cases occurs:

(i) $k = 1$, v_1 is positive or negative, $v_0 = 0$ and $c = \bar{c}_1$;

(ii) $k = 0$.

By $\{u_n\} \subset \mathcal{N}_*$, we can find $C > 0$ such that $\|u_n^+\| \geq C$ and $\|u_n^-\| \geq C$ for all n. So we can see that the case (i) does not occur. Hence the case (ii) holds, which yields that $\{u_{n_m}\}$ converges to v_0. □

For each $c \in \mathbb{R}$, we set

$$\mathcal{K}_c = \{u \in H^1(\mathbb{R}^N) : \nabla I(u) = 0, I(u) = c\}.$$

As a direct consequence of the previous lemma, we have the following:

Lemma 3.3. *$\mathcal{K}_c \cap \mathcal{N}_*$ is compact or empty for each $c \in (-\infty, \bar{c}_1 + \bar{c}_M]$.*

We recall that $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ for $u \in H^1(\mathbb{R}^N)$. We define $\alpha : [0,1] \times \mathcal{N}_* \to \mathcal{N}$ by

$$\alpha(t,u) = \tau((1-t)u^+ + tu^-)\left((1-t)u^+ + tu^-\right) \quad \text{for } (t,u) \in [0,1] \times \mathcal{N}_*.$$

From the definitions of τ, α and $\mathcal{N}_*$, we have the following:

Lemma 3.4. *For each $(t,u) \in [0,1] \times \mathcal{N}_*$, there hold*

$$\begin{aligned}
\tau((1-t)u^+ + tu^-) &= \left(\frac{(1-t)^2\|u^+\|^2 + t^2\|u^-\|^2}{(1-t)^{p+1}\|u^+\|^2 + t^{p+1}\|u^-\|^2}\right)^{\frac{1}{p-1}};\\
\alpha(t,u) &= \left(\frac{(1-t)^{p+1}\|u^+\|^2 + t^2(1-t)^{p-1}\|u^-\|^2}{(1-t)^{p+1}\|u^+\|^2 + t^{p+1}\|u^-\|^2}\right)^{\frac{1}{p-1}} u^+\\
&\quad + \left(\frac{t^{p-1}(1-t)^2\|u^+\|^2 + t^{p+1}\|u^-\|^2}{(1-t)^{p+1}\|u^+\|^2 + t^{p+1}\|u^-\|^2}\right)^{\frac{1}{p-1}} u^-;\\
I(\alpha(t,u)) &= \|u^+\|^2 f\left(\left(\frac{(1-t)^{p+1}\|u^+\|^2 + t^2(1-t)^{p-1}\|u^-\|^2}{(1-t)^{p+1}\|u^+\|^2 + t^{p+1}\|u^-\|^2}\right)^{\frac{1}{p-1}}\right)\\
&\quad + \|u^-\|^2 f\left(\left(\frac{t^{p-1}(1-t)^2\|u^+\|^2 + t^{p+1}\|u^-\|^2}{(1-t)^{p+1}\|u^+\|^2 + t^{p+1}\|u^-\|^2}\right)^{\frac{1}{p-1}}\right),
\end{aligned}$$

where $f(s) = s^2/2 - s^{p+1}/(p+1)$ for $s \geq 0$.

Lemma 3.5. *For each $c \in \mathbb{R}$ and $\sigma > 0$, there exists $\eta > 0$ such that $I(\alpha(t,u)) \leq I(u) - \eta$ for all $(t,u) \in [0,1] \times \mathcal{N}_*$ with $|t - 1/2| \geq \sigma$ and $I(u) \leq c + \eta$.*

Proof. Fix $c \in \mathbb{R}$. We note that there exists $C > 1$ such that $1/C \leq \|u^\pm\| \leq C$ for all $u \in \mathcal{N}_*$ with $I(u) \leq c + 1$. Since

$$\frac{d}{ds}\left(\frac{\|u^\pm\|^2/\|u^\mp\|^2 + s^2}{\|u^\pm\|^2/\|u^\mp\|^2 + s^{p+1}}\right) = \frac{-(p-1)s^{p+2} + \|u^\pm\|^2/\|u^\mp\|^2\left(-(p+1)s^p + 2s\right)}{\left(\|u^\pm\|^2/\|u^\mp\|^2 + s^{p+1}\right)^2},$$

we can infer that there exist $\nu > 0$ and $\delta \in (0,1)$ such that

$$\frac{d}{ds}\left(\frac{\|u^\pm\|^2/\|u^\mp\|^2 + s^2}{\|u^\pm\|^2/\|u^\mp\|^2 + s^{p+1}}\right) < \begin{cases} -\nu & \text{if } |s-1| \leq \delta;\\ 0 & \text{if } s \geq 1 - \delta \end{cases}$$

for all $u \in \mathcal{N}_*$ with $I(u) \leq c + 1$. Using the previous lemma and the fact that $I(au^+ + bu^-) \leq I(u^+) + I(u^-) = I(u)$ for each $a, b \geq 0$ and $u \in \mathcal{N}_*$, we can easily see that the conclusion holds. $\square$

We define $\Psi : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Psi(u) = (\nabla I(u), u) \quad \text{for each } u \in H^1(\mathbb{R}^N).$$

We remark that $u \in \mathcal{N}$ if and only if $\Psi(u) = 0$ and $u \neq 0$. We also define $\tau_* : \{u \in H^1(\mathbb{R}^N) : u^+ \neq 0 \text{ or } u^- \neq 0\} \to [-\infty, \infty]$ by

$$\tau_*(u) = \tau(u^+) - \tau(u^-) \quad \text{for each } u \in H^1(\mathbb{R}^N) \text{ with } u^+ \neq 0 \text{ or } u^- \neq 0.$$

We remark again that for each $u \in \mathcal{N}$, $u \in \mathcal{N}_*$ if and only if $\tau_*(u) = 0$. For each $A \subset H^1(\mathbb{R}^N)$ and $\delta > 0$, we denote by $B_\delta(A)$ the set $\{u \in H^1(\mathbb{R}^N) : \operatorname{dist}(u, A) \leq \delta\}$.

Now, we show the existence of sign-changing solutions of (2.2) which are local minimum types in $\mathcal{N}_*$. In the following proof, we extend the method employed in [7, 16, 19].

Proposition 3.1. *For each $j, k \in \{1, 2\}$, there exists a pair of sign-changing solutions $\pm u$ of* (2.2) *satisfying $u \in \mathcal{N}_*(j,k)$ and $I(u) = \min\{I(v) : v \in \mathcal{N}_*(j,k)\}$.*

Proof. Fix $j, k \in \{1, 2\}$ and set $c = \inf\{I(u) : u \in \mathcal{N}_*(j,k)\}$. We will show $\mathcal{K}_c \cap \mathcal{N}_*(j,k) \neq \emptyset$. Suppose not. We have $0 < c < 2\bar{c}_M + \varepsilon (< \bar{c}_1 + \bar{c}_M)$. We note that I may not satisfy $(\mathrm{PS})_c$, because c may be equal to $\bar{c}_1$. By $c < \bar{c}_1 + \bar{c}_M$, Lemma 3.3 says $\mathcal{K}_c \cap \mathcal{N}_*$ is compact or empty. So there exists a (possibly empty) neighborhood A of $\mathcal{K}_c \cap \mathcal{N}_*$ in $\mathcal{N}_*$ such that $\operatorname{dist}(A, \mathcal{N}_*(j,k))) > 0$. We choose $\zeta > 0$ such that

$$\operatorname{dist}\left(\mathcal{N}_*(j,k), \bigcup_{j'=1}^{2} \bigcup_{k'=1}^{2} \mathcal{N}_*(j',k') \setminus \mathcal{N}_*(j,k)\right) > 2\zeta. \tag{3.9}$$

By Lemma 3.4, we can choose $\sigma \in (0, 1/2)$ such that

(i) $\|\alpha(t,u) - u\| \leq \zeta$ for each $(t,u) \in [0,1] \times \mathcal{N}_*$ with $|t - 1/2| \leq \sigma$ and $I(u) \leq c+1$.

We set

$$U_\eta = \{\alpha(t,u) : (t,u) \in [0,1] \times (\mathcal{N}_* \setminus A),\ I(u) \leq c + \eta,\ I(\alpha(t,u)) \geq c - \eta\}$$

for each $\eta > 0$. By Lemma 3.2, we can choose $\xi > 0$, $\eta > 0$, $\delta \in (0, \zeta)$ such that

$$\|\nabla I(u)\| \geq \xi \quad \text{and} \quad 0 < \|\nabla \Psi(u)\| \leq 1/\xi \quad \text{for all } u \in B_{2\delta}(U_\eta).$$

By Lemma 3.5, we may assume

(ii) $I(\alpha(t,u)) \leq c - 2\eta$ for each $(t,u) \in [0,1] \times \mathcal{N}_*$ with $|t - 1/2| \geq \sigma$ and $I(u) \leq c + 2\eta$.

We define $X : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ by

$$X(u) = \begin{cases} \dfrac{\operatorname{dist}\left(u, H^1(\mathbb{R}^N) \setminus V\right)}{\operatorname{dist}(u, H^1(\mathbb{R}^N) \setminus V) + \operatorname{dist}(u, W)} \dfrac{\delta}{2} Y(u) & \text{if } u \in V, \\ 0 & \text{if } u \in H^1(\mathbb{R}^N) \setminus V, \end{cases}$$

where $V = \{u \in B_{2\delta}(U_\eta) : |I(u) - c| \le 2\eta\}$, $W = \{u \in B_\delta(U_\eta) : |I(u) - c| \le \eta\}$ and

$$Y(u) = \frac{\nabla I(u)}{\|\nabla I(u)\|} - \left(\frac{\nabla I(u)}{\|\nabla I(u)\|}, \frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|} \right) \frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|} \quad \text{for } u \in V.$$

We define a continuous function $\Phi : [0,1] \times \mathcal{N} \to \mathcal{N}$ by

$$\Phi(0,u) = u, \quad \frac{\partial}{\partial \theta}\Phi(\theta,u) = -X(\Phi(\theta,u)) \quad \text{for } (\theta,u) \in [0,1] \times \mathcal{N}.$$

We note that $\|X(u)\| \le \delta$ for each $u \in H^1(\mathbb{R}^N)$. We can easily see

(iii) $I(\Phi(\theta,u)) \le I(u)$ for each $(\theta,u) \in [0,1] \times \mathcal{N}$;

(iv) $\|\Phi(\theta,u) - u\| \le \delta\theta$ for each $(\theta,u) \in [0,1] \times \mathcal{N}$;

(v) $\Phi(\theta,u) = u$ for each $(\theta,u) \in [0,1] \times \mathcal{N}$ with $I(u) \notin [c - 2\eta, c + 2\eta]$.

Since

$$\begin{aligned}
(\nabla I(u), Y(u)) &= \|\nabla I(u)\| \left[1 - \left(\frac{\nabla I(u)}{\|\nabla I(u)\|}, \frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|} \right)^2 \right] \\
&= \|\nabla I(u)\| \left[1 - \left(\frac{\nabla I(u)}{\|\nabla I(u)\|}, \frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|} - \left(\frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|}, \frac{u}{\|u\|} \right) \frac{u}{\|u\|} \right)^2 \right] \\
&\ge \|\nabla I(u)\| \left[1 - \left\| \frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|} - \left(\frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|}, \frac{u}{\|u\|} \right) \frac{u}{\|u\|} \right\|^2 \right] \\
&= \|\nabla I(u)\| \left(\frac{\nabla \Psi(u)}{\|\nabla \Psi(u)\|}, \frac{u}{\|u\|} \right)^2 = \frac{\|\nabla I(u)\| (-(p-1)\|u\|^2)^2}{\|\nabla \Psi(u)\|^2 \|u\|^2} \\
&\ge (p-1)^2 \xi^3 \|u\|^2
\end{aligned} \tag{3.10}$$

for each $u \in V \cap \mathcal{N}$, we can infer that there exists $\tilde{\eta} \in (0, \eta)$ such that $c + \tilde{\eta} < 2\bar{c}_M + \varepsilon$ and

(vi) $\Phi(1, U_{\tilde{\eta}}) \subset \{u \in \mathcal{N} : I(u) \le c - \tilde{\eta}\}$.

We choose $u_1 \in \mathcal{N}_*(j,k)$ with $I(u_1) \le c + \tilde{\eta}$. Since

$$I(\alpha(t,u_1)) = I(\alpha(t,u_1)^+) + I(\alpha(t,u_1)^-) \le I(u_1^+) + I(u_1^-) = I(u_1),$$

we have $I(\alpha(t,u_1)) \le c + \tilde{\eta}$ for all $t \in [0,1]$. So by (iii) and (vi), we have

$$I(\Phi(1, \alpha(t,u_1))) \le c - \tilde{\eta} \quad \text{for all } t \in [0,1].$$

Since by (ii) and (v), we have

$$\tau_*(\Phi(1, \alpha(0,u_1))) = \tau_*(\alpha(0,u_1)) = \tau(u_1^+) - \tau(0) = -\infty$$

and $\tau_*(\Phi(1,\alpha(1,u_1)))=\infty$, by the Intermediate Value Theorem, we can find $t_1\in[0,1]$ with $|t_1-1/2|\le\sigma$ and $\tau_*(\Phi(1,\alpha(t_1,u_1)))=0$. So we have $\Phi(1,\alpha(t_1,u_1))\in\mathcal{N}_*$ and $I(\Phi(1,\alpha(t_1,u_1)))\le c-\tilde{\eta}$, which and $c-\tilde{\eta}<c<2\bar{c}_M+\varepsilon$ yield $\Phi(1,\alpha(t_1,u_1))\in\bigcup_{j'=1}^2\bigcup_{k'=1}^2\mathcal{N}_*(j',k')$. By (i) and (iv), we have

$$\begin{aligned}\|\Phi(1,\alpha(t,u_1))-u_1\| &\le \|\Phi(1,\alpha(t,u_1))-\alpha(t,u_1)\|+\|\alpha(t,u_1)-u_1\|\\ &\le \delta+\zeta\le 2\zeta\end{aligned}$$

for each $t\in[0,1]$ with $|t-1/2|\le\sigma$. By (3.9) and the inequality above, we consequently have

$$\Phi(1,\alpha(t_1,u_1))\in\mathcal{N}_*(j,k)\quad\text{and}\quad I(\Phi(1,\alpha(t_1,u_1)))\le c-\tilde{\eta},$$

which is a contradiction. This completes the proof. □

Remark 3.1. By a careful argument, we can obtain an additional property such that $\Phi(1,\alpha(t,u_1))$ is sign-changing for each $t\in(0,1)$. However, in the proof above, there may exist $t\in[1/2-\sigma,1/2+\sigma]$ such that $\Phi(1,\alpha(t,u_1))$ is not sign-changing, which means $\tau_*(\Phi(1,\alpha(t,u_1)))$ equals ∞ or $-\infty$. But anyway, by the Intermediate Value Theorem, we can find $t_1\in[1/2-\sigma,1/2+\sigma]$ with $\tau_*(\Phi(1,\alpha(t_1,u_1)))=0$.

Remark 3.2. Since we may assume $\varepsilon<\bar{c}_M$, we can show that each solution obtained in the previous proposition, which belongs to $C(\mathbb{R}^N,\mathbb{R})$ by the elliptic regularity, has exactly two nodal domains. Indeed, let u be one of the solutions and assume that it has at least three nodal domains Ω_i, $i=1,2,3$. Let u_{Ω_i} be the function defined by $u_{\Omega_i}(x)=u(x)$ for $x\in\Omega_i$ and $u_{\Omega_i}(x)=0$ for $x\notin\Omega_i$. Then we have $u_{\Omega_i}\in H^1(\mathbb{R}^N)$ and $\partial u_{\Omega_i}/\partial x_j=\partial u/\partial x_j$ on Ω_i and $\partial u_{\Omega_i}/\partial x_j=0$ on $\mathbb{R}^N\setminus\Omega_i$; see[5, Theorem IX.17, Proposition IX.18]. Hence we have $\int_{\mathbb{R}^N}(|\nabla u_{\Omega_i}|^2+|u_{\Omega_i}|^2)\,dx=\int_{\mathbb{R}^N}Q(x)|u_{\Omega_i}|^{p+1}\,dx$, i.e., $u_{\Omega_i}\in\mathcal{N}$, which yields the contradiction $3\bar{c}_M>2\bar{c}_M+\varepsilon>I(u)\ge 3\bar{c}_M$.

By the previous proposition, we obtained at least three pairs of sign-changing solutions of (2.2). Indeed, the pairs of functions obtained as minimizers in $\mathcal{N}_*(1,2)$ and $\mathcal{N}_*(2,1)$ may be equal.

We define $\mathcal{T}_*:\{u\in H^1(\mathbb{R}^N):u^\pm\ne 0\}\to\mathcal{N}_*$ by

$$\mathcal{T}_*(u)=\tau(u^+)u^++\tau(u^-)u^-\quad\text{for } u\in H^1(\mathbb{R}^N)\text{ with } u^\pm\ne 0.$$

We remark that $\mathcal{T}_*(u)=u$ for each $u\in\mathcal{N}_*$.

Next, we show the existence of a sign-changing solution of (2.2) which is a mountain pass type in $\mathcal{N}_*$. The first author employes our method in another problem; see [14]. Although we already fixed $\varepsilon>0$, $\lambda_0\in(0,\min\{1/4,l^2\})$ and $\lambda\in(0,\lambda_0)$ just after Lemma 2.6, without loss of generality, we may assume that $\sup_{0\le s\le 1}I(\gamma_0(s))<\bar{c}_1+\bar{c}_M$, where $\gamma_0\in C([0,1];\mathcal{N}_*)$ is defined by $\gamma_0(s)=u_{+,a^1}+u_{-,(1-s)a^1+sa^2}$ for $0\le s\le 1$.

Proposition 3.2. *There exists a pair of sign-changing solutions* $\pm u$ *of* (2.2) *satisfying* $I(u)>\min\{I(v):v\in\mathcal{N}_*(j,k)\}$ *for each* $j,k=1,2$.

Proof. We set

$$\Gamma = \{\gamma \in C([0,1];\mathcal{N}_*) : \gamma(0) = u_{+,a^1} + u_{-,a^1},\ \gamma(1) = u_{+,a^1} + u_{-,a^2}\};$$
$$c = \inf_{\gamma\in\Gamma} \sup_{0\le s\le 1} I(\gamma(s)).$$

We will show that $\mathcal{K}_c\cap\mathcal{N}_* \neq \emptyset$. Suppose not. By (2.4), (2.6) and $\sup_{0\le s\le 1} I(\gamma_0(s)) < \bar{c}_1+\bar{c}_M$, we have $\max\{I(u_{+,a^1}+u_{-,a^1}), I(u_{+,a^1}+u_{-,a^2})\} < 2\bar{c}_M+\varepsilon < c < \bar{c}_1+\bar{c}_M$. By (2.6), we also have $c > \min\{I(v) : v \in \mathcal{N}_*(j,k)\}$ for each $j,k = 1,2$. We note again that I may not satisfy $(\mathrm{PS})_c$, because c may equal to $\bar{c}_1$. By $c < \bar{c}_1+\bar{c}_M$ and Lemma 3.3, $\mathcal{K}_c \cap \mathcal{N}_*$ is empty. For each $\eta > 0$, we set

$$U_\eta = \{\alpha(t,\gamma(s)) : \gamma \in \Gamma,\ \max_{0\le r\le 1} I(\gamma(r)) \le c+\eta,$$
$$(t,s) \in [0,1]^2, I(\alpha(t,\gamma(s))) \ge c-\eta\}.$$

By Lemma 3.2, we can choose $\xi > 0$, $\eta > 0$, $\delta > 0$ such that $c-2\eta > 2\bar{c}_M+\varepsilon$ and

$$\|\nabla I(u)\| \ge \xi \quad \text{and} \quad 0 < \|\Psi(u)\| \le 1/\xi \quad \text{for all } u \in B_{2\delta}(U_\eta).$$

By Lemma 3.5, we may assume

(i) $I(\alpha(t,\gamma(s))) \le c-2\eta$ for each $(t,s) \in ([0,1/3]\cup[2/3,1])\times[0,1]$ and $\gamma \in \Gamma$ with $\max_{0\le s\le 1} I(\gamma(s)) \le c+2\eta$.

By a similar argument as in the proof of the previous proposition, we can infer that there exist a continuous function $\Phi : [0,1]\times\mathcal{N} \to \mathcal{N}$ and $\tilde{\eta} \in (0,\eta/2)$ such that

(ii) $I(\Phi(\theta,u)) \le I(u)$ for each $(\theta,u) \in [0,1]\times\mathcal{N}$;

(iii) $\Phi(\theta,u) = u$ for each $(\theta,u) \in [0,1]\times\mathcal{N}$ with $I(u) \notin [c-2\eta, c+2\eta]$;

(iv) $\Phi(1,U_{2\tilde{\eta}}) \subset \{u \in \mathcal{N} : I(u) \le c-2\tilde{\eta}\}$.

We choose $\gamma_1 \in \Gamma$ such that $\sup_{0\le s\le 1} I(\gamma_1(s)) \le c+2\tilde{\eta}$. Since

$$I(\alpha(t,\gamma_1(s))) = I(\alpha(t,\gamma_1(s))^+) + I(\alpha(t,\gamma_1(s))^-)$$
$$\le I(\gamma_1(s)^+) + I(\gamma_1(s)^-) = I(\gamma_1(s)),$$

we have $I(\alpha(t,\gamma_1(s))) \le c+2\tilde{\eta}$ for all $(t,s) \in [0,1]^2$. So by (ii) and (iv), we have

$$I(\Phi(1,\alpha(t,\gamma_1(s)))) \le c-2\tilde{\eta} \quad \text{for all } (t,s) \in [0,1]^2. \tag{3.11}$$

We can find $\zeta > 0$ such that $\max_{0\le s\le 1} I(\gamma(s)) \ge c-\tilde{\eta}$ for all $\gamma \in \Gamma_\zeta$, where Γ_ζ is defined by

$$\Gamma_\zeta = \{\gamma \in C([0,1];\mathcal{N}) : \gamma(s)^+ \neq 0,\ \gamma(s)^- \neq 0 \text{ for all } s \in [0,1],$$
$$\max_{0\le s\le 1} \|\gamma(s) - \mathcal{T}_*(\gamma(s))\| \le \zeta,$$
$$\|\gamma(0) - (u_{+,a^1}+u_{-,a^1})\| \le \zeta,\ \|\gamma(1) - (u_{+,a^1}+u_{-,a^2})\| \le \zeta\}.$$

Indeed, if not, we can find $\{\gamma_n\} \subset C([0,1];\mathcal{N})$ such that

$$\gamma_n \in \Gamma_{1/n} \quad \text{and} \quad \max_{0\le s\le 1} I(\gamma_n(s)) < c - \tilde{\eta} \quad \text{for all } n \in \mathbb{N}.$$

We define $\{\widetilde{\gamma}_n\} \subset \Gamma$ by

$$\widetilde{\gamma}_n(s) = \begin{cases} \mathcal{T}_* \left(3s\gamma_n(0) + (1-3s)(u_{+,a^1} + u_{-,a^1})\right) & \text{if } 0 \le s \le 1/3, \\ \mathcal{T}_* \left(\gamma_n(3s-1)\right) & \text{if } 1/3 \le s \le 2/3, \\ \mathcal{T}_* \left((3-3s)\gamma_n(1) + (3s-2)(u_{+,a^1} + u_{-,a^2})\right) & \text{if } 2/3 \le s \le 1. \end{cases}$$

Since I is uniformly continuous on $\{u \in H^1(\mathbb{R}^N) : I(u) \le c + 2\tilde{\eta}\}$, taking n large enough, we have $\max_{0\le s\le 1} I(\gamma_n(s)) < c$, which is a contradiction. Thus we have shown that we can find such $\zeta > 0$. We can also find $\sigma \in (0, \pi/8)$ such that

(v) for each $v \in \mathcal{N}$ with $|\tan^{-1}(\tau_*(v))| \le 2\sigma$ and $I(v) \le c + 2\tilde{\eta}$, there hold $v^+ \ne 0$, $v^- \ne 0$ and $\|v - \mathcal{T}_*(v)\| \le \zeta$;

(vi) for each $(t,s) \in [0,1] \times \{0,1\}$ with $|\tan^{-1}(\tau_*(\alpha(t,\gamma_1(s))))| \le 2\sigma$, there holds $\|\alpha(t,\gamma_1(s)) - \alpha(1/2,\gamma_1(s))\| \le \zeta$.

We define $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(t,s) = \begin{cases} -\pi/2 & \text{if } t < 0,\ s \in \mathbb{R}, \\ \tan^{-1}(\tau_*(\Phi(1,\alpha(t,\gamma_1(s))))) & \text{if } (t,s) \in [0,1]^2, \\ \pi/2 & \text{if } t > 1,\ s \in \mathbb{R}, \\ \tan^{-1}(\tau_*(\Phi(1,\alpha(t,\gamma_1(1))))) & \text{if } t \in [0,1],\ s > 1, \\ \tan^{-1}(\tau_*(\Phi(1,\alpha(t,\gamma_1(0))))) & \text{if } t \in [0,1],\ s < 0. \end{cases}$$

By (i) and (iii), we can see that h is continuous. Using a suitable standard mollifier $\chi \in C_c^\infty(\mathbb{R}^2;[0,\infty))$, we define $g \in C^\infty(\mathbb{R}^2;\mathbb{R})$ by

$$g(t,s) = \int_{\mathbb{R}^2} \chi(t-r_1, s-r_2) h(r_1,r_2)\, dr_1 dr_2 \quad \text{for each } (t,s) \in \mathbb{R}^2$$

and we obtain $|g(t,s) - h(t,s)| \le \sigma$ for each $(t,s) \in \mathbb{R}^2$. By Sard's theorem, we can find a regular value $b \in \mathbb{R}$ for g with $|b| \le \sigma$. So $g^{-1}(b)$ is a one dimensional submanifold of $\mathbb{R}^2$. Since $h(0,s) = -\pi/2$ and $h(1,s) = \pi/2$ for each $s \in [0,1]$, we have $g(0,s) \le -\pi/2 + \sigma$ and $g(1,s) \ge \pi/2 - \sigma$ for each $s \in [0,1]$. Hence by the Intermediate Value Theorem, we obtain that $([0,1] \times \{s\}) \cap g^{-1}(b) \ne \emptyset$, $(0,s) \notin g^{-1}(b)$ and $(1,s) \notin g^{-1}(b)$ for each $s \in [0,1]$. Since $t \mapsto h(t,0)$ and $t \mapsto h(t,1)$ are strictly increasing on $[0,1]$, we have $\partial g/\partial t(t,0) > 0$ and $\partial g/\partial t(t,1) > 0$ for each $t \in [0,1]$. Hence we obtain that both $g^{-1}(b) \cap ([0,1] \times \{0\})$ and $g^{-1}(b) \cap ([0,1] \times \{1\})$ consist of a single point. Using the classification theorem for one dimensional manifolds (see the appendices of [12, 18]), we can find a connected submanifold A (with boundary) of $g^{-1}(b)$ such that $A \subset [0,1]^2$, $A \cap (\{0\} \times [0,1]) = \emptyset$, $A \cap (\{1\} \times [0,1]) = \emptyset$, both $A \cap ([0,1] \times \{0\})$ and $A \cap ([0,1] \times \{1\})$ consist of a single point and

$\partial A = \big(A \cap ([0,1] \times \{0\})\big) \cup \big(A \cap ([0,1] \times \{1\})\big)$. Hence there is a continuous function $\rho : [0,1] \to (0,1) \times [0,1]$ such that $A = \{\rho(r) : r \in [0,1]\}$, $\rho(0) = g^{-1}(b) \cap ([0,1] \times \{0\})$ and $\rho(1) = g^{-1}(b) \cap ([0,1] \times \{1\})$. Since $|h(\rho(r))| \le |h(\rho(r)) - g(\rho(r))| + |b| \le 2\sigma$ for each $r \in [0,1]$, by (i), (iii), (v) and (vi), we have $\Phi(1, \alpha(\cdot, \gamma_1(\cdot))) \circ \rho \in \Gamma_\zeta$. Hence we have $\max\{I((\Phi(1, \alpha(\cdot, \gamma_1(\cdot))) \circ \rho)(r)) : r \in [0,1]\} \ge c - \tilde{\eta}$, which contradicts (3.11). This completes the proof. □

Using the above propositions, the proof of Theorem 1 follows.

Finally, we give the proof of Theorem 2. Taking $\lambda > 0$ small enough, we may assume

$$\inf\Big\{\sup_{0 \le s \le 1} I(\gamma(s)) : \gamma \in C([0,1]; \mathcal{N}_*),\ \gamma(0) = u_{+,a^1} + u_{-,a^1},\ \gamma(1) = u_{+,a^1} + u_{-,a^2}\Big\}$$
$$< \inf\{I(u) : u \in \mathcal{N}_*,\ u^\pm \notin \mathcal{N}(1) \cup \mathcal{N}(2)\},$$

where the left hand side of the inequality above is the critical value in the previous proposition.

Proof of Theorem 2. We set $h_0 : [0,1]^2 \to H^1(\mathbb{R}^N)$ by

$$h_0(s_1, s_2) = u_{+,(1-s_1)a^1 + s_1 a^2} + u_{-,(1-s_2)a^1 + s_2 a^2} \quad \text{for } (s_1, s_2) \in [0,1]^2.$$

We also set

$$\begin{aligned}\mathcal{F} = \Big\{F : \ & F \text{ is a two dimensional, compact, connected, orientable manifold} \\ & \text{which is embedded into } [0,1]^2 \times \mathbb{R} \text{ such that } \partial F = \partial[0,1]^2 \times \{0\}, \\ & \text{there exists } \xi \in (0, 1/2) \text{ such that} \\ & F \cap (([0,1]^2 \setminus [\xi, 1-\xi]^2) \times \mathbb{R}) = ([0,1]^2 \setminus [\xi, 1-\xi]^2) \times \{0\}, \\ & \text{and the manifold } \widetilde{F} \text{ defined by } \widetilde{F} \cap ([0,1]^2 \times \mathbb{R}) = F \text{ and} \\ & \widetilde{F} \cap ((\mathbb{R}^2 \setminus [0,1]^2) \times R) = (\mathbb{R}^2 \setminus [0,1]^2) \times \{0\} \text{ is of class } C^\infty\Big\}.\end{aligned}$$

For each $F \in \mathcal{F}$, we identify $\partial F = \partial[0,1]^2 \times \{0\}$ with $\partial[0,1]^2$. We define

$$\Gamma = \{(F, \gamma) : F \in \mathcal{F}, \gamma \in C(F; \mathcal{N}_*),\ \gamma = h_0 \text{ on } \partial F\}$$

and

$$c \equiv \inf_{(F,\gamma) \in \Gamma} \sup_{s \in F} I(\gamma(s)).$$

By our assumptions, taking $\lambda > 0$ sufficiently small, we have $\sup_{s \in \partial[0,1]^2} I(h_0(s)) < \sup_{s \in [0,1]^2} I(h_0(s)) = 2\bar{c}_{M_1} + o(1) < \bar{c}_1 + \bar{c}_M$. Fix any $(F, \gamma) \in \mathcal{F}$. We define the mapping $m : F \to [0,1]^2$ by

$$m(s) = \Bigg(\frac{|\beta(\gamma(s)^+) - \beta(u_{+,a^1})|}{|\beta(\gamma(s)^+) - \beta(u_{+,a^1})| + |\beta(\gamma(s)^+) - \beta(u_{+,a^2})|}, \\ \frac{|\beta(\gamma(s)^-) - \beta(u_{-,a^1})|}{|\beta(\gamma(s)^-) - \beta(u_{-,a^1})| + |\beta(\gamma(s)^-) - \beta(u_{-,a^2})|}\Bigg).$$

We recall $\partial F = \partial[0,1]^2$. We note that $m(s_1,s_2) = (s_1,s_2)$ for $s_1, s_2 \in \{0,1\}$, $m(\{s_1\} \times [0,1]) \subset \{s_1\} \times [0,1]$ for $s_1 \in \{0,1\}$ and $m([0,1] \times \{s_2\}) \subset [0,1] \times \{s_2\}$ for $s_2 \in \{0,1\}$. We will show that $m : F \to [0,1]^2$ is surjective. Suppose not. Then there is a mapping $m_1 : m(F) \to \partial[0,1]^2$ which does not move any point on $\partial[0,1]^2$. We denote by $H_1(F)$ and $H_1(\partial[0,1]^2)$ the one-dimensional homology groups of F and $\partial[0,1]^2$ with $\mathbb{Z}_2$-coefficients, respectively. We know that $m_1 \circ m : F \to \partial[0,1]^2$ induces the mapping $(m_1 \circ m)_* : H_1(F) \to H_1(\partial[0,1]^2)$, and we have $(m_1 \circ m)_*([\partial F]) = [\partial[0,1]^2]]$. Since ∂F is the boundary of F, $[\partial F]$ is trivial in $H_1(F)$; however $[\partial[0,1]^2]$ is the generator of $H_1(\partial[0,1]^2) = \mathbb{Z}_2$. Thus we obtain a contradiction. Hence we have shown that $m : F \to [0,1]^2$ is surjective. We can infer that there exists $s \in F$ such that $\gamma(s) \in \{u \in \mathcal{N}_* : u^\pm \notin \mathcal{N}(1) \cup \mathcal{N}(2)\}$. Then we can see that c is greater than the critical value in the previous proposition. Hence, in order to show our theorem, it is enough to show $\mathcal{K}_c \cap \mathcal{N}_* \neq \emptyset$. Suppose not. By $c < \bar{c}_1 + \bar{c}_M$ and Lemma 3.3, $\mathcal{K}_c \cap \mathcal{N}_*$ is empty. For each $\eta > 0$, we set

$$\begin{aligned} U_\eta = \{\alpha(t,\gamma(s)) : (F,\gamma) \in \Gamma,\ &\max_{r\in F} I(\gamma(r)) \le c + \eta, \\ &(t,s) \in [0,1] \times F, I(\alpha(t,\gamma(s))) \ge c - \eta\}. \end{aligned}$$

By Lemma 3.2, we can choose $\xi > 0$, $\eta > 0$, $\delta > 0$ such that

$$\|\nabla I(u)\| \ge \xi \quad \text{and} \quad 0 < \|\Psi(u)\| \le 1/\xi \quad \text{for all } u \in B_{2\delta}(U_\eta).$$

By Lemma 3.5, we may assume

(i) $I(\alpha(t,\gamma(s))) \le c - 2\eta$ for each $(F,\gamma) \in \Gamma$ with $\max_{s\in F} I(\gamma(s)) \le c + 2\eta$ and $(t,s) \in ([0,1/3] \cup [2/3,1]) \times F$.

By similar arguments as in the proof of the previous two propositions, we can infer that there exist a continuous function $\Phi : [0,1] \times \mathcal{N} \to \mathcal{N}$ and $\tilde{\eta} \in (0,\eta/2)$ such that

(ii) $I(\Phi(\theta,u)) \le I(u)$ for each $(\theta,u) \in [0,1] \times \mathcal{N}$;

(iii) $\Phi(\theta,u) = u$ for each $(\theta,u) \in [0,1] \times \mathcal{N}$ with $I(u) \notin [c - 2\eta, c + 2\eta]$;

(iv) $\Phi(1, U_{2\tilde{\eta}}) \subset \{u \in \mathcal{N} : I(u) \le c - 2\tilde{\eta}\}$.

We choose $(F_1,\gamma_1) \in \Gamma$ such that $\sup_{s\in F_1} I(\gamma_1(s)) \le c + 2\tilde{\eta}$. Since

$$\begin{aligned} I(\alpha(t,\gamma_1(s))) &= I(\alpha(t,\gamma_1(s))^+) + I(\alpha(t,\gamma_1(s))^-) \\ &\le I(\gamma_1(s)^+) + I(\gamma_1(s)^-) = I(\gamma_1(s)), \end{aligned}$$

we have $I(\alpha(t,\gamma_1(s))) \le c + 2\tilde{\eta}$ for all $(t,s) \in [0,1] \times F_1$. So by (ii) and (iv), we have

$$I(\Phi(1,\alpha(t,\gamma_1(s)))) \le c - 2\tilde{\eta} \quad \text{for all } (t,s) \in [0,1] \times F_1. \tag{3.12}$$

By a similar argument as in the previous proposition, we can find $\zeta > 0$ such that $\max_{s\in F} I(\gamma(s)) \ge c - \tilde{\eta}$ for all $(F,\gamma) \in \Gamma_\zeta$, where Γ_ζ is defined by

$$\begin{aligned} \Gamma_\zeta = \{(F,\gamma) :\ & F \in \mathcal{F},\ \gamma \in C(F;\mathcal{N}),\ \gamma(s)^+ \neq 0,\ \gamma(s)^- \neq 0 \text{ for all } s \in F, \\ & \max_{s\in F} \|\gamma(s) - \mathcal{T}_*(\gamma(s))\| \le \zeta, \\ & \|\gamma(s) - (u_{+,(1-s_1)a^1+s_1a^2} + u_{-,(1-s_2)a^1+s_2a^2})\| \le \zeta \text{ for } s = (s_1,s_2) \in \partial F\}. \end{aligned}$$

We can also find $\sigma \in (0, \pi/8)$ such that

(v) for each $v \in \mathcal{N}$ with $|\tan^{-1}(\tau_*(v))| \le 2\sigma$ and $I(v) \le c+2\tilde{\eta}$, there hold $v^+ \neq 0$, $v^- \neq 0$ and $\|v - \mathcal{T}_*(v)\| \le \zeta$;

(vi) for each $(t,s) \in [0,1] \times \partial F_1$ with $|\tan^{-1}(\tau_*(\alpha(t,\gamma_1(s)))| \le 2\sigma$, there holds $\|\alpha(t,\gamma_1(s)) - \alpha(1/2,\gamma_1(s))\| \le \zeta$.

Let $\widetilde{F}_1$ be the C^∞-manifold embedded into $\mathbb{R}^3$ which is defined by

$$\widetilde{F}_1 \cap ([0,1]^2 \times \mathbb{R}) = F_1 \quad \text{and} \quad \widetilde{F}_1 \cap ((\mathbb{R}^2 \setminus [0,1]^2) \times R) = (\mathbb{R}^2 \setminus [0,1]^2) \times \{0\};$$

see the definition of $\mathcal{F}$. We define $h : \mathbb{R} \times \widetilde{F}_1 \to \mathbb{R}$ by

$$h(t,s) = \begin{cases} -\pi/2 & \text{if } (t,s) \in (-\infty,0) \times \widetilde{F}_1, \\ \pi/2 & \text{if } (t,s) \in (1,\infty) \times \widetilde{F}_1, \\ \tan^{-1}(\tau_*(\Phi(1,\alpha(t,\gamma_1(s))))) & \text{if } (t,s) \in [0,1] \times F_1, \\ \tan^{-1}(\tau_*(\Phi(1,\alpha(t,\gamma_1(q(s)))))) & \text{if } (t,s) \in [0,1] \times ((\mathbb{R}^2 \setminus [0,1]^2) \times \{0\}), \end{cases}$$

where in the last case, $q(s)$ is the nearest point of s to $[0,1]^2 \times \{0\}$ in $\mathbb{R}^3$. By (i) and (iii), we can see that h is continuous. We define $\chi \in C_c^\infty(R;R)$ by

$$\chi(t) = \begin{cases} \exp(1/(t^2-1)) & \text{if } |t| < 1, \\ 0 & \text{if } |t| \ge 1. \end{cases}$$

We induce the Riemannian structure in $\mathbb{R} \times \widetilde{F}_1$ from the standard one of $\mathbb{R} \times \mathbb{R}^3$, and we denote by d the metric on $\mathbb{R} \times \widetilde{F}_1$ defined by the induced Riemannian structure. Taking $\omega > 0$ sufficiently small (ω should be smaller than the injectivity radius of $\widetilde{F}_1$), we define $g \in C^\infty(\mathbb{R} \times \widetilde{F}_1; \mathbb{R})$ by

$$g(t,s) = \int_{\mathbb{R} \times \widetilde{F}_1} \widetilde{\chi}_{(t,s),\omega}(t',s') h(t',s')\, dt'ds' \quad \text{for each } (t,s) \in \mathbb{R} \times \widetilde{F}_1,$$

where

$$\widetilde{\chi}_{(t,s),\omega}(t',s') = \frac{\chi(d((t,s),(t',s'))/\omega)}{\int_{\mathbb{R} \times \widetilde{F}_1} \chi(d((t,s),(t'',s''))/\omega)\, dt''ds''} \quad \text{for } (t',s') \in \mathbb{R} \times \widetilde{F}_1;$$

then we obtain $|g(t,s) - h(t,s)| \le \sigma$ for each $(t,s) \in \mathbb{R} \times \widetilde{F}_1$. By Sard's theorem, we can find a regular value $b \in \mathbb{R}$ for g with $|b| \le \sigma$. So $g^{-1}(b)$ is a two dimensional, orientable, C^∞-submanifold of $\mathbb{R} \times \widetilde{F}_1$. Since $h(0,s) = -\pi/2$ and $h(1,s) = \pi/2$ for each $s \in F_1$, we have $g(0,s) \le -\pi/2 + \sigma$ and $g(1,s) \ge \pi/2 - \sigma$ for each $s \in F_1$. Hence by the Intermediate Value Theorem, we obtain that $([0,1] \times \{s\}) \cap g^{-1}(b) \neq \emptyset$, $(0,s) \notin g^{-1}(b)$ and $(1,s) \notin g^{-1}(b)$ for each $s \in F_1$. Since $t \mapsto h(t,s)$ is strictly increasing on $[0,1]$ for each $s \in \partial F_1$, we have $\partial g/\partial t(t,s) > 0$ for each $(t,s) \in [0,1] \times \partial F_1$. Hence we obtain that $g^{-1}(b) \cap ([0,1] \times \{s\})$ consists of a single point for each $s \in \partial F_1$. Using the classification theorem for two dimensional, compact, connected, orientable manifolds, we can find a compact, connected submanifold A (with boundary) of $g^{-1}(b)$ such that $g^{-1}(b) \cap ([0,1] \times \partial F_1) \subset A$ and A is homeomorphic to some $F \in \mathcal{F}$ with a homeomorphism $\rho : F \to A$ such that $\rho(s) = g^{-1}(b) \cap ([0,1] \times \{s\})$ for each $s \in \partial F$. Since $|h(\rho(r))| \leq |h(\rho(r)) - g(\rho(r))| + |b| \leq 2\sigma$ for each $r \in F$, by (i), (iii), (v) and (vi), we have $(F, \Phi(1, \alpha(\cdot, \gamma_1(\cdot))) \circ \rho) \in \Gamma_\zeta$. Hence we have $\max\{I((\Phi(1, \alpha(\cdot, \gamma_1(\cdot))) \circ \rho)(r)) : r \in F\} \geq c - \tilde{\eta}$, which contradicts (3.12). This completes the proof. □

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