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Computations of Critical Groups and Applications to Nonlinear Differential Equations With Resonance

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Abstract

In this paper, we develop a method to compute critical groups at degenerate critical points under more general conditions. The abstract results are used to study the existence and multiplicity of nontrivial solutions for nonlinear differential equations with resonance both at infinity and at zero.

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1 Introduction

It is well known that the computations of critical groups plays an important role in studying the existence and multiplicity of the solutions to partial differential equations which

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arise in the calculus of variations. There are some results in the computations of the critical groups: in the nondegenerate case, the critical groups are determined by the Morse index completely; in the degenerate case, we have the Spliting Lemma and Shifting Theorem, which reduce the computation of the critical groups of an isolated critical point of the initial functional to that of a functional defined on the kernel of the Hessian (a finite dimensional space in many cases, see [1,2]). In addition, the computation of critical groups at infinity was introduced, see, for examples [3, 4, 5]. In applications, combining with the critical groups at a real critical point, the critical groups at infinity play a role in dealing with the resonant problems of nonlinear partial differential equations (see, [5, 6, 7]). We would like to mention that there are some results of critical groups concerning the standard minimax method , see, for examples, [8, 9]. Obviously, the applications of Morse theory (or its generalization) essentially depend on the accurate computation of the critical groups. The purpose of this paper is to present a method to compute precisely the critical groups both at the origin and at infinity under more general conditions, which were unknown ever before. Compared with the papers in the literatures on this line, (see [7, 10, 11] and references therein), our conditions are more general. As applications, we shall study the existence and multiplicity of solutions for elliptic boundary value problem and Hamiltonian systems. Certainly, our method is useful for other situations, such as computations of the $\epsilon-$ cohomology critical groups (defined in [12]) and the critical groups for dynamically isolated critical set (defined in [5]), and consequently, we can deal with other problems, such as non-cooperative elliptic systems and wave equations.

The paper is organized as follows: In section 2, we establish some abstract theorems devoted to the computations of the critical groups both at the origin and at infinity. As applications, in Section 3, we study the existence and multiplicity of nontrivial solutions for elliptic boundary value problem and Hamiltonian systems.

2 Computation of critical groups

Let H be a real separable Hilbert space equipped with inner produce $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$. We consider the functional

$$f(x)=\frac{1}{2}\langle Ax,x\rangle+G(x), \tag{2.1}$$

where A is a bounded self-adjoint operator defined on H. We assume that the zero eigenvalue of A is isolated in the spectral set $\sigma(A)$. Then according to the spectral decomposition of A, $H=H_+\oplus H_0\oplus H_-$, where $H_\pm, H_0$ are invariant subspaces corresponding to the positive, negative and zero spectrum, respectively. Recall that (see [1]) the critical groups of an isolated critical point p of f is defined by

$$C_q(f,p)=H_q(f_a\cap U, f_a\cap U\setminus\{p\}),$$

where $a=f(p), f_a=\{x\in H: f(x)\le a\}, U$ is a closed neighborhood of p, and $H_q(\cdot,\cdot)$ is the qth (singular) homology group with coefficients in a field F. We first deal with the critical groups at the origin. Suppose that

(H_1) $A_\pm:=A|_{H_\pm}$ has bounded inverse on $H_\pm$.

(H_2) $r_- := \dim H_-, r_0 = \dim H_0$ are finite.

As to the assumptions for G, we introduce a control function $h_0 : R^+ \to R^+$, which is increasing in t and satisfies

$$2 < \alpha \leq \frac{th_0(t)}{H_0(t)} \leq \beta, \text{ for } t \in R^+,$$

where α, β are constants and $H_0(t) = \int_0^t h_0(s)ds$. Assume:

(H_3) $G(\theta) = 0, G \in C^1(H, R^1)$ has a compact differential, $dG(\theta) = \theta$, and

$$||dG(x)|| \leq c(h_0(||x_0||) + p_0(||x_+|| + ||x_-||)), \text{ as } ||x|| << 1 \text{ and } c > 0,$$

where $p_0 : [0, +\infty] \to [0, +\infty)$, $\lim\limits_{t\to 0} \dfrac{p_0(t)}{t} = 0$.

$(H_4^{\pm})$ $\dfrac{G(x_0)}{h_0^2(||x_0||)} \to \pm\infty$ as $x_0 \in H_0$, $||x_0|| \to 0$.

Here and in the sequel, we always use c to denote indiscriminately various constants where the exact value is irrelevant. Obviously, $h_0(t) = t^\sigma$ with $\sigma > 1$ is a simple example.

Theorem 2.1 *Under the assumptions* $(H_1) - (H_3)$ *and* (H_4^+)*, we have*

$$C_q(f, \theta) = \delta_{q,r_-} F, \ \forall q;$$

under the assumptions $(H_1) - (H_3)$ *and* (H_4^-)*, we have*

$$C_q(f, \theta) = \delta_{q,(r_- + r_0)} F, \ \forall q.$$

Proof. (1) the case of (H_4^-). We take a neighborhood of θ with the following form:

$$N = \{x | ||x_+||^2 - d||x_-||^2 - k\xi(||x_0||) \leq \epsilon r_0^2, ||x_-||^2 + ||x_0||^2 \leq r_0^2\},$$

where $x = x_+ + x_- + x_0 \in H_+ \bigoplus H_- \bigoplus H_0$ and

$$\xi(t) = \begin{cases} \frac{H_0^2(t)}{t^2} & t \neq 0 \\ 0 & t = 0. \end{cases}$$

The positive numbers d, k, ϵ, r_0 are to be determined later . Then the boundary of N consists of two parts:

$$\Gamma_1 = \{x \mid \|x_+\|^2 - d\|x_-\|^2 - k\xi(||x_0||) = \epsilon r_0^2, \ \|x_-\|^2 + \|x_0\|^2 \leq r_0^2\},$$

$$\Gamma_2 = \{x \mid \|x_+\|^2 - d\|x_-\|^2 - k\xi(||x_0||) \leq \epsilon r_0^2, \ \|x_-\|^2 + \|x_0\|^2 = r_0^2\}.$$

And the normal vector on Γ_1 is

$$n = x_+ - dx_- - \frac{k}{2}\xi'(||x_0||)\frac{x_0}{||x_0||}.$$

Now we claim that the negative gradient of f is inward on Γ_1. In fact, letting $m = \inf\{|\langle Ax_\pm, x_\pm\rangle| \mid ||x_\pm|| = 1, x_\pm \in H_\pm\}$, we have

$$\begin{aligned}
&|\langle dG(x), n\rangle| \\
&\le c(h_0(||x_0||) + p_0(||x_+|| + ||x_-||))(||x_+|| + ||x_-|| + k\xi'(||x_0||)) \\
&\le 2c^2h_0^2(||x_0||) + 2c^2p_0^2(||x_+|| + ||x_-||) + k^2(\xi'(||x_0||)^2 \\
&\quad + c_1h_0^2(||x_0||) + \tfrac{1}{4}(m||x_+||^2 + dm||x_-||^2) + cp_0(||x_+|| + ||x_-||)(||x_+|| + ||x_-||) \\
&\le (2c^2 + c_1)h_0^2(||x_0||) + k^2(\xi'(||x_0||))^2 + \tfrac{1}{2}(m||x_+||^2 + dm||x_-||^2)
\end{aligned}$$

and by the definition of $\xi(t)$, it is easy to check that

$$|\xi'(t)| \le c\frac{h_0^2(t)}{t} = o(h_0(t)), \quad h_0^2(t) \le c\frac{H_0^2(t)}{t^2}(\text{as } t \to 0),$$

hence,

$$\begin{aligned}
\langle df(x), n\rangle &= \langle Ax_+, x_+\rangle - d\langle Ax_-, x_-\rangle + \langle dG(x), n\rangle \\
&\ge m||x_+||^2 + dm||x_-||^2 + \langle dG(x), n\rangle \\
&\ge \tfrac{1}{2}(m||x_+||^2 + dm||x_-||^2) - ch_0^2(||x_0||) - k^2(\xi'(||x_0||))^2 \\
&\ge \tfrac{1}{2}(m||x_+||^2 + dm||x_-||^2) - (1+c)h_0^2(||x_0||) \\
&\ge \tfrac{1}{2}(m||x_+||^2 + dm||x_-||^2) - c_2\xi(||x_0||).
\end{aligned}$$

We choose k large enough, say, $k > c_2$ then

$$\begin{aligned}
\langle df(x), n\rangle &\ge \tfrac{1}{2}(m||x_+||^2 + dm||x_-||^2 - k\xi(||x_0||)) \\
&= \tfrac{1}{2}\epsilon r_0^2 > 0.
\end{aligned} \tag{2.2}$$

Next, we study the behavior of f near the boundary Γ_2:

$$\begin{aligned}
f(x) &= \tfrac{1}{2}\langle Ax_+, x_+\rangle + \tfrac{1}{2}\langle Ax_-, x_-\rangle + G(x) \\
&\le \tfrac{1}{2}||A||||x_+||^2 - \tfrac{1}{2}m||x_-||^2 + G(x_0) \\
&\quad + c(h_0(||x_0||) + p_0(||x_+|| + ||x_-||))(||x_-|| + ||x_+||) \\
&\le \tfrac{1}{2}||A||||x_+||^2 - \tfrac{1}{2}m||x_-||^2 + G(x_0) + c(h_0(||x_0||))(||x_-|| + ||x_+||) + L(x) \\
&\le ||A|||x_+|||^2 - \tfrac{1}{4}m||x_-||^2 + G(x_0) + ch_0^2(||x_0||) \\
&\le ||A||\epsilon r_0^2 + (||A||d - \tfrac{1}{4}m)||x_-||^2 + \tfrac{1}{4}G(x_0)
\end{aligned}$$

where $L(x)$ consists of those higher terms w.r.t. $||x_-||^2$ and $||x_+||^2$. Now we take d such that $||A||d - \frac{1}{4}m < 0$. Then for given $r_0 > 0$ and ϵ small enough, we can find two constants $r_1, \delta > 0$ with $r_1 < r_0$ such that

$$\begin{cases}
f(x) \ge -\frac{\delta}{2} & \text{if } ||x_0||^2 + ||x_-||^2 \le r_1^2, \quad x \in N \\
f(x) < 0 & \text{if } ||x_0||^2 + ||x_-||^2 \ge r_1^2, \quad x \in N \\
f(x) \le -\delta & \text{if } ||x_0||^2 + ||x_-||^2 = r_0^2, \quad x \in N.
\end{cases} \tag{2.3}$$

Let

$$N_1 := \{x \in N \mid ||x_0||^2 + ||x_-||^2 \ge r_1^2\} \subset f_0 \cap N \setminus \{0\}.$$

Since the function f satisfies the (PS) condition (see Theorem 2.2 (1)), thus (2.3) enable us to deform N onto $f_0 \cap N$ by using the negative gradient flow σ_1 generated by df.

Again, by the negative gradient flow σ_2 generated by df, $f_0 \cap N \setminus \{0\}$ can be deformed to $f_{-\frac{3}{4}\delta} \cap N \subset N_1$. On the other hand, we can use geometry deformation σ to deform N_1 onto Γ_2, therefore, we have

$$\begin{aligned} C_q(f,\theta) &= H_q(f_0 \cap N, f_0 \cap N \setminus \{\theta\}) \\ &= H_q(N, f_0 \cap N \setminus \{\theta\})\ (\sigma_1) \\ &= H_q(N, \Gamma_2)\ (\sigma_1 \circ \sigma_2) \\ &= H_q(B, \partial B)\ (\sigma) \\ &= \delta_{q, r_- + r_0} F \end{aligned}$$

where $B = \{x_- + x_0 | ||x_-||^2 + ||x_+||^2 \le r_0^2\}$.

(2) The case of (H_4^+). In this case we define

$$N = \{x \mid -d\|x_+\|^2 + \|x_-\|^2 - k\xi(||x_0||) \le \epsilon r_0^2,\ \|x_+\|^2 + \|x_0\|^2 \le r_0^2\}.$$

Let

$$\Gamma_1 = \{x \mid -d\|x_+\|^2 + \|x_-\|^2 - k\xi(||x_0||) = \epsilon r_0^2,\ \|x_+\|^2 + \|x_0\|^2 \le r_0^2\},$$

$$\Gamma_r = \{x \mid -d\|x_+\|^2 + \|x_-\|^2 - k\xi(||x_0||) = \epsilon r_0^2,\ \|x_+\|^2 + \|x_0\|^2 \le r^2\}.$$

Then the normal vector on Γ_1 is $n = -dx_+ + x_- - \frac{k}{2}\xi'(||x_0||)\frac{x_0}{||x_0||}$. Similar to (1), we can prove that the negative gradient of f is outward on Γ_1, that is

$$\begin{aligned} \langle df_n(x), n\rangle &\le \tfrac{m}{2}(d\|x_+\|^2 - \|x_-\|^2 + k\xi(||x_0||)) \\ &= -\tfrac{1}{2}m\epsilon r_0^2 < 0. \end{aligned} \tag{2.4}$$

Moreover, we have

$$f(x) \ge -\|A\|\epsilon r_0^2 - (\|A\|d - \tfrac{1}{4}m)\|x_+\|^2 + \tfrac{1}{4}G(x_0). \tag{2.5}$$

As a result, for r_0 fixed and ϵ small enough, by choosing d satisfying $\|A\|d - \frac{1}{4}m < 0$, we can find $\delta > 0, 0 < r_1 < r_2 < r_0$ such that

$$\begin{cases} f(x) \le \frac{\delta}{2} & \text{if } x \in N, & \|x_0\|^2 + ||x_+||^2 \le r_1^2, \\ f(x) > 0 & \text{if } x \in N, & \|x_0\|^2 + ||x_+||^2 \ge r_1^2, \\ f(x) \ge \delta & \text{if } x \in N, & \|x_0\|^2 + ||x_+||^2 \ge r_2^2. \end{cases} \tag{2.6}$$

Let $N_1 := N \cap \{\|x_0\|^2 + ||x_+||^2 \le r_1^2\} \cup \Gamma_{r_2}$. Then by a geometric deformation σ, we deform N to N_1. We use $\eta(t,x)$ to denote the negative gradient flow. Let $t_1(x)$ be the time of reaching the boundary Γ_1, $t_2(x)$ be the time of reaching the level set of f_0. Obviously, $t_1(x), t_2(x)$ are continuous w.r.t. the variable x and $t_1(x) = 0$ when $x \in f_0$. Let $t(x) = \min\{t_1(x), t_2(x)\}$. We define a flow

$$\sigma_1(s,x) = \begin{cases} \eta(st,x), & x \in N_1, \quad t > 0, \\ x, & x \in N_1, \quad t = 0, \end{cases}$$

then by σ_1, we deform N_1 to $N_2 := N \cap f_0 \cup \Gamma_{r_2}$. At last, the flow

$$\sigma_2(s,u) = \begin{cases} \eta(st_1,x), & x \in N_2 \setminus \{0\}, \quad t_1 > 0, \\ x, & x \in N_2 \setminus \{0\}, \quad t_1 = 0 \end{cases}$$

implies that Γ_{r_2} is a strong deformation retract of $N_2 \setminus \{0\}$. Therefore

$$\begin{aligned} C_q(f,\theta) &= H_q(f_0 \cap N, f_0 \cap N \setminus \{0\}) \\ &\cong H_q(\Gamma_{r_2} \cup (f_0 \cap N), \Gamma_{r_2} \cup (f_0 \cap N) \setminus \{0\}) \\ &\cong H_q(N, \Gamma_{r_2} \cup (f_0 \cap N) \setminus \{0\}) (\text{by } \sigma_1 \circ \sigma) \\ &\cong H_q(N, \Gamma_{r_2}) \\ &= \delta_{q,r_0} F, \end{aligned}$$

and the theorem is proved.

Critical groups at the infinity

Here we denote the control functional and its primitive functional by $h_\infty(t)$ and $H_\infty(t)$ respectively. Assume that

$$0 < \alpha' \le \frac{t h_\infty(t)}{H_\infty(t)} \le \beta' < 2,$$

where α', β' are constants and $H_\infty(t) = \int_0^t h_\infty(s) ds$. In order to deal with the critical groups at the infinity, we assume:

(H_5) $G \in C^1(H, R^1)$ has a compact differential dG satisfying

$$||dG(x)|| \le (1 + h_\infty(||x_0||) + p_\infty(||x_+|| + ||x_-||)),$$

where $p_\infty : [0, +\infty) \to [0, +\infty)$ is a functional satisfying $\lim_{t\to\infty} \frac{p_\infty(t)}{t} \to 0$.

$(H_6^\pm)$ $\dfrac{G(x_0)}{h_\infty^2(||x_0||)} \to \pm\infty$ as $x_0 \in H_0$, $||x_0|| \to \infty$.

Theorem 2.2 *Under the assumptions $(H_1), (H_2)$ and (H_5) we have:*
(1) f satisfies (PS) conditions. Moreover,

(2) $C_q(f,\infty) = \delta_{q,r_- + r_0} F$ if (H_6^-) holds, and

(3) $C_q(f,\infty) = \delta_{q,r_-} F$ if (H_6^+) holds.

Recall that the critical groups at the infinity is defined by

$$C_q(f,\infty) = H_q(H, f_{-a}),$$

for a large enough, as $f_{-a} \cap K = \emptyset$, where $f_a = \{x \in H | f(x) \le a\}$, $K = \{x \in K | f'(x) = 0\}$ is the critical set of f.

Proof. (1) Suppose that $\{x^n\}$ is a (PS) sequence. Then

$$\begin{aligned} ||x_+^n|| &\ge |\langle f'(x^n), x_+^n \rangle| \\ &\ge \langle Ax^n, x_+^n \rangle - |\langle dG(x^n), x_+^n \rangle| \\ &\ge m||x_+^n||^2 - c(1 + h_\infty(||x_0^n||) + p_\infty(||x_+^n|| + ||x_-^n||))||x_+^n|| \\ &\ge \tfrac{1}{2} m||x_+^n||^2 - \tfrac{1}{4} m||x_-^n||^2 - c(h_\infty^2(||x_0^n||) - c \end{aligned}$$

where c denote various positive numbers and $x^n = x_+^n + x_0^n + x_-^n \in H_+ \oplus H_0 \oplus H_-$.

Similarly, we have

$$||x_-^n|| \geq \frac{1}{2}m||x_-^n||^2 - \frac{1}{4}m||x_+^n||^2 - c(h_\infty^2(||x_0^n||) - c.$$

Therefore

$$||x_-^n|| + ||x_+^n|| \geq \frac{1}{4}m(||x_+^n||^2 + ||x_-^n||^2) - ch_\infty^2(||x_0||) - c,$$

hence

$$||x_-^n||^2 + ||x_+^n||^2 \leq c(1 + h_\infty^2(||x_0||)). \tag{2.7}$$

On the other hand,

$$\begin{aligned} |f(x^n)| &= |\tfrac{1}{2}\langle Ax_+^n, x_+^n\rangle + \tfrac{1}{2}\langle Ax_-^n, x_-^n\rangle + G(x^n)| \\ &= |G(x_0^n) + \tfrac{1}{2}\langle Ax_+^n, x_+^n\rangle + \tfrac{1}{2}\langle Ax_-^n, x_-^n\rangle \\ &+ \textstyle\int_0^1 (dG(\tau(x_-^n + x_+^n)), x_-^n + x_+^n)d\tau| \\ &\geq |G(x_0^n)| - c(||x_+^n||^2 + ||x_-^n||^2) \\ &-c(1 + h_\infty(||x_0^n||) + p_\infty(||x_+^n|| + ||x_-^n||))||x_-^n + x_+^n|| \\ &\geq |G(x_0^n)| - c(||x_+^n||^2 + ||x_-^n||^2) - ch_\infty^2(||x_0||) - c \\ &\geq |G(x_0^n)| - ch_\infty^2(||x_0||) - c. \end{aligned}$$

This implies that $\{x_0^n\}$ is bounded. Noting (2.7), we have $\{x^n\}$ is bounded, and a standard arguments yields that f satisfies (PS) condition.

(2) If (H_6^-) holds , we define the following set

$$C_0 = \{x \mid \|x_+\|^2 - d\|x_-\|^2 - k\xi(t) \leq M\},$$

where $\xi(t) = \frac{H_\infty^2(t)}{1+t^2}$, $d, k, M > 0$ will be determined later. Then the normal vector on ∂C_0 is $n = x_+ - dx_- - \frac{k}{2}\xi'(\|x_0\|)\frac{x_0}{\|x_0\|}$. By the definition of h_∞, it is easy to check that

$$\xi'(t) = \frac{2h_\infty(t)H_\infty(t)}{1+t^2} - \frac{2tH_\infty^2(t)}{(1+t^2)^2}, \ |\xi'(t)| \leq \frac{ch_\infty^2(t)}{1+t}. \tag{2.8}$$

In the following, we shall prove that f has no critical points outside C_0 for appropriate k, M, and the negative gradient vector field points inward to C_0 on the boundary ∂C_0. In fact,

$$\begin{aligned} \langle df(x), n\rangle &= \langle Ax_+, x_+\rangle - d\langle Ax_-, x_-\rangle + \langle dG(x), n\rangle \\ &\geq m\|x_+\|^2 + dm\|x_-\|^2 \\ &\quad -c(1 + h_\infty(||x||) + p_\infty(||x_+|| + ||x_-||)) \cdot (\|x_+\| + d\|x_-\| + k\xi'(\|x_0\|) \\ &\geq \tfrac{1}{2}m(\|x_+\|^2 + d\|x_-\|^2) - ch_\infty^2(||x_0||) - k^2|\xi'(||x_0||)|^2 - c \\ &\geq \tfrac{1}{2}m(\|x_+\|^2 + d\|x_-\|^2) - (c+1)h_\infty^2(||x_0||) - c \\ &\geq \tfrac{1}{2}m(\|x_+\|^2 + d\|x_-\|^2) - (c+1)\xi(||x_0||) - c \\ &\geq \tfrac{1}{2}m(||x_+||^2 - d||x_-||^2 - k\xi(||x_0||)) - c \\ &= \tfrac{1}{2}mM - c > 0 (\text{for } M \text{ large}). \end{aligned}$$

Now, we prove that $\forall x \in C_0$,

$$f(x) \to -\infty \Longleftrightarrow \|x_- + x_0\| \to \infty \qquad \text{uniformly in} \quad x_+. \tag{2.9}$$

Indeed, for $x \in C_0$

$$\begin{array}{rl} f(x) & = \frac{1}{2}\langle Ax_+, x_+\rangle + \frac{1}{2}\langle Ax_-, x_-\rangle + G(x) \\ & \leq \frac{1}{2}\|A\|\|x_+\|^2 - \frac{1}{2}m\|x_-\|^2 + G(x_0) \\ & \quad +c(\|x_+ + x_-\|)(1 + h_\infty(||x_0|| + p_\infty(||x_-|| + ||x_+||)) \\ & \leq ||A||||x_+||^2 - \frac{1}{4}m||x_-||^2 + G(x_0) + ch^2_\infty(||x_0||) + c \\ & \leq (||A||d - \frac{1}{4}m)||x_-||^2 + G(x_0) + ch^2_\infty(||x_0||) + c \\ & \leq (||A||d - \frac{1}{4}m)||x_-||^2 + \frac{1}{2}G(x_0) + c. \end{array} \tag{2.10}$$

In (2.10) we have used the assumption(H_6^-) and the fact that the corresponding lower terms w.r.t. $||x_-||^2$ have been absorbed. Hence, if we choose d satisfying $-\frac{1}{4}m + \|A\|d < 0$, then

$$f(x) \to -\infty \quad \text{as } \|x_- + x_0\| \to \infty \qquad \text{uniformly in} \quad x_+.$$

On the other hand, by (H_6^-)

$$\begin{array}{rl} f(x) & \geq \frac{1}{2}m||x_+||^2 - \frac{1}{2}\|A\|\|x_-\|^2 + G(x_0) \\ & \quad -c(1 + h_\infty(||x_0||) + p_\infty(||x_-|| + ||x_+||))(||x_- + x_+||) \\ & \geq \frac{1}{4}m||x_+||^2 - ||A||||x_-||^2 + G(x_0) - ch^2_\infty(||x_0||) - c \\ & \geq -||A|||x_-|||^2 + 2G(x_0) - c. \end{array} \tag{2.11}$$

This implies that $||x_- + x_+|| \to \infty$ as $f(x) \to -\infty$.

Now we choose $a > 0$ large enough such that the critical set K of f satisfies $K \subset \{x \in H : |f(x)| < a\}$. Then the above arguments imply that there exist $b > a$ and $R_1 > R_2 > 0$ such that

$$C_1 := \{x \in C_0|||x_0 + x_-|| \geq R_1\} \subset f_{-b} \cap C_0,$$

$$C_2 := \{x \in C_0|||x_0 + x_-|| \geq R_2\} \subset f_{-a} \cap C_0,$$

and

$$f_{-b} \cap C_0 \subset f_{-a} \cap C_0.$$

By (PS) condition, we know that $K \subset C_0 \setminus f^{-1}[-b, -a]$, then we can use negative gradient flow η deform $f_{-a} \cap C_0$ onto $f_{-b} \cap C_0$. And by a geometric deformation σ, we can deform C_2 onto C_1. Thus $\sigma \circ \eta$ is a strong deformation retract of $f_{-a} \cap C_0$ onto C_1. Therefore

$$\begin{array}{rl} H_q(f, \infty) & = H_q(H, f_{-a}) \cong H_q(C_0, f_{-a} \cap C_0) \\ & \cong H_q(C_0, C_1) \\ & \cong H_q(H_0 \oplus H_-, (H_0 \oplus H_-) \setminus B_{R_1}) \\ & \cong \delta_{q,(r_- + r_0)}F. \end{array}$$

(3) Assume that (H_6^+) holds. In this case we define

$$C_0 = \{||x_-||^2 - d||x_+||^2 - k\xi(||x_0||) \leq M\},$$

where M, d, k are constants to be determined later, ξ is the same as in (2). By a similar way, we have

$$\langle df(x), n\rangle \leq -\frac{m}{2}M + c < 0$$

for M large enough, where $n = x_- - dx_+ - \frac{k}{2}\xi'(||x_0||)\frac{x_0}{||x_0||}$ is the normal vector on ∂C_0. Also, we have $\forall x \in C_0$,

$$f(x) \to +\infty \Longleftrightarrow ||x_+ + x_0|| \to \infty \text{ uniformly in } x_-.$$

Now we choose $a > 0, R > 0$ such that $-a \leq -a_0 = \inf_{x\in C_0} f(x)$, and f has no critical point outside $C_0 \cap C_1 =: \{x \in C_0 | ||x_+ + x_0|| \leq R_1\}$. Then $f_{-a} \subset H \setminus C_0$ and f_{-a} is a strong deformation retract of $H \setminus C_0$. Therefore

$$H_q(H, f_{-a}) \cong H_q(H, H \setminus C_0) \cong \delta_{q,r_-}F$$

3 Applications to resonance problems

As applications, in this section, we first study the existence of nontrivial solution for the following asymptotically linear elliptic boundary value problem with resonance. More precisely, we consider

$$\begin{cases} -\triangle u = p(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary. Let $\lambda_1 < \lambda_2 \leq \lambda_3 ... \leq \lambda_i \leq ...$ be the eigenvalues of $-\triangle$ in Ω with zero boundary condition and $p(x, u)$ be a function in $C(\bar{\Omega} \times R, R)$. We suppose that

$$\lim_{u\to 0} \frac{p(x, u)}{u} = \lambda_j \text{ uniformly in } x \in \bar{\Omega} \tag{3.2}$$

$$\lim_{|u|\to\infty} \frac{p(x, u)}{u} = \lambda_k \text{ uniformly in } x \in \bar{\Omega}. \tag{3.3}$$

Let

$$g_\infty(x, u) = \lambda_k u - p(x, u), \quad G_\infty(u) = \int_0^u g_\infty(x, s)ds,$$

$$g_0(x, u) = \lambda_j u - p(x, u), \quad G_0(u) = \int_0^u g_0(x, s)ds.$$

Take $H = H_0^1(\Omega)$ with norm $||u|| = (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}$, we assume:

(P_1) $g_0(x, 0) = 0$, and there exists constant $M_0 > 0$, such that

$$|g_0(x, u)| < M_0 h_0(|u|), \text{ for } |u| < 1,$$

$(P_2^{\pm})$ $\dfrac{G_0(x, u)}{h_0^2(|u|)} \to \pm\infty$ as $|u| \to 0$ uniformly in $x \in \Omega$,

(P_3) there exist constants $M_\infty > 0$, such that

$$|g_\infty(x,u)| < M_\infty(1 + h_\infty(|u|)), \text{ for } u \in R \text{ and } x \in \Omega,$$

$(P_3^{\pm})$ $\dfrac{G_\infty(x,u)}{h_\infty^2(|u|)} \to \pm\infty$ as $|u| \to \infty$ uniformly in $x \in \Omega$, where h_0, h_∞ are the control functions defined in Section 2.

Theorem 3.1 *Suppose that* $(P_1), (P_3)$ *hold. Then* (3.1) *has at least one nontrivial solutions if one of the following cases occurs:*

(1) $(P_2^+), (P_4^+)$ *hold and* $\lambda_k \neq \lambda_j$;

(2) $(P_2^+), (P_4^-)$ *hold and* $\lambda_{k+1} \neq \lambda_j$;

(3) $(P_2^-), (P_4^+)$ *hold and* $\lambda_k \neq \lambda_{j+1}$;

(4) $(P_2^-), (P_4^-)$ *hold and* $\lambda_k \neq \lambda_j$.

Theorem 3.2 *Assume that* $(P_1), (P_3)$ *hold. If* $x_0 \neq \theta$ *is a nondegerate solution of (3.1), then problem (3.1) has at least another nontrivial solution if one of the following conditions holds:*

(1) $(P_2^+), (P_4^+)$ *hold and* $\lambda_k \neq \lambda_j$;

(2) $(P_2^+), (P_4^-)$ *hold and* $\lambda_{k+1} \neq \lambda_j$;

(3) $(P_2^-), (P_4^+)$ *hold and* $\lambda_k \neq \lambda_{j+1}$;

(4) $(P_2^-), (P_4^-)$ *hold and* $\lambda_k \neq \lambda_j$.

Now we consider the functional

$$\begin{aligned} f(x) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega P(u)dx \\ &= \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_k |u|^2) + \int_\Omega G_\infty(u)dx \\ &= \frac{1}{2}\langle A_\infty u, u\rangle + J_\infty(u) \end{aligned}$$

and

$$\begin{aligned} f(x) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega P(u)dx \\ &= \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_j |u|^2) + \int_\Omega G_0(u)dx \\ &= \frac{1}{2}\langle A_0 u, u\rangle + J_0(u), \end{aligned}$$

where $P(u) = \displaystyle\int_0^u p(x,s)ds$, $A_\infty = Id - \lambda_k(-\triangle)^{-1}$, $A_0 = Id - \lambda_j(-\triangle)^{-1}$. Then it is well known that $f \in C^1(H, R^1)$ and $J_\infty(u), J_0(u)$ have compact differentials $(-\triangle)^{-1}g_\infty(u)$ and $(-\triangle)^{-1}g_0(u)$ respectively. And the solution of (3.1) is equivalent to the critical point of functional f.

Lemma 3.1 *Assume that* $(P_1), (P_2^{\pm})$ *hold. Then*

(1) $||dJ_0(u)|| \leq c(h_0(||u||) + p_0(||u_+|| + ||u_-||))$, *for* $||u|| << 1$.

(2) $\dfrac{J_0(u_0)}{h_0^2(||u_0||)} \to \pm\infty$ *for* $u_0 \in$ *ker*A_0 *and* $||u_0|| \to 0$.

Proof. (1): From (P_1), we get

$$\begin{aligned} ||dJ_0(u)||^2 &= ||(-\triangle)^{-1}g_0(u)||^2 \leq c(\int_\Omega |g_0(u)|^2 dx) \\ &\leq (\int_{|u|\leq\frac{1}{2}} |g_0(u)|^2 dx + \int_{|u|\geq\frac{1}{2}} |g_0(u)|^2 dx). \end{aligned} \tag{3.4}$$

If $||u|| << 1$, we have $||u_0||_{L^\infty} \leq c||u_0|| \leq c||u|| \leq \frac{1}{4}$. If $|u| < \frac{1}{2}$, then $|u_+ + u_-| \leq |u| + |u_0| \leq \frac{3}{4}$, and hence

$$\begin{aligned} \int_{|u|\leq\frac{1}{2}} g_0^2(u)dx &\leq c\int_{|u|\leq\frac{1}{2}} h_0^2(|u|)dx \\ &\leq c\int_{|u|\leq\frac{1}{2}} h_0^2(|u_0|) + h_0^2(|u_- + u_+|)dx \\ &\leq c\int_{|u|\leq\frac{1}{2}} (h_0^2(||u_0||) + |u_- + u_+|^{2(\alpha-1)})dx \\ &\leq c\int_{|u|\leq\frac{1}{2}} (h_0^2(||u_0||) + c|u_- + u_+|^{2q})dx \\ &\leq ch_0^2(||u_0||) + c||u_- + u_+||^{2q}, \end{aligned} \tag{3.5}$$

where $q = \begin{cases} (\alpha-1), & \text{if } (\alpha-1) < \frac{N}{N-2} \\ \frac{N}{N-2}, & \text{if } (\alpha-1) \geq \frac{N}{N-2} \end{cases}$.

If $|u| > \frac{1}{2}$, we have $|u_- + u_+| \geq |u| - |u_0| \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \geq |u_0|$. Hence

$$|u| \leq |u_- + u_+| + |u_0| \leq 2|u_- + u_+|,$$

and it follows that

$$\begin{aligned} \int_{|u|\geq\frac{1}{2}} g_0^2(u)dx &\leq c\int_{|u|\geq\frac{1}{2}} |u|^{2q}dx \\ &\leq \int_{|u|\geq\frac{1}{2}} |u_- + u_+|^{2q}dx \\ &\leq c||u_- + u_+||^{2q}. \end{aligned} \tag{3.6}$$

Combining (3.4),(3.5) and (3.6), we prove (1) with $p_0(t) = ct^{2q}$.

(2) Note that the finite dimensionality and the unique continuous property of the kernel space Ker A_0 imply that $u(x) = 0$, a. e. $x \in \Omega$ if $u \in \text{Ker}A_0, u(x) = 0$ on a set with positive measure. Then similar to the proof of Lemma 3.2 in [13]: for any $\epsilon > 0$, there exists $\delta > 0$ such that meas$\Omega_0 < \epsilon$, where $\Omega_0 = \{x \in \Omega : |u(x)| \leq \delta||u||, u \in \text{Ker}A_0\}$

and $u \neq \theta$. Thus, if (P_2^+) holds and $||u_0|| \to 0$, we take $\epsilon = \frac{1}{2}|\Omega|$, then there exists $\bar{\delta} > 0$ such that $|u(x)| \geq \bar{\delta}||u||_{L^\infty}$ on $\Omega \setminus \Omega_0$, thus, for any $L > 0$,

$$\begin{aligned}
\frac{\int_\Omega G_0(x,u)dx}{h_0^2(||u||)} &= \frac{\int_{\Omega_0} G_0(x,u)dx}{h_0^2(||u||)} + \frac{\int_{\Omega\setminus\Omega_0} G_0(x,u)dx}{h_0^2(||u||)} \\
&\geq \frac{\int_{\Omega\setminus\Omega_0} G_0(x,\bar{\delta}||u||_{L^\infty})dx}{ch_0^2(\bar{\delta}||u||_{L^\infty})} \\
&\geq L \cdot \tfrac{1}{2}|\Omega|,
\end{aligned}$$

provided $||u||_{L^\infty}$ is small enough . This implies that $\dfrac{J_0(u_0)}{h_0^2(||u_0||)} \to +\infty$ as $||u_0|| \to 0$.

Similarly, we have

$$\frac{J_0(u_0)}{h_0^2(||u_0||)} \to -\infty \text{ as } (P_2^-) \text{ holds and } ||u_0|| \to 0.$$

Lemma 3.2 *Assume (P_3) and $(P_4^\pm)$ hold, then*
(1)$||dJ_\infty(u)|| \leq c(1 + h_\infty(||u||) + p_\infty(||u_+|| + ||u_-||))$,
(2) $\dfrac{J_\infty(u_0)}{h_\infty^2(||u_0||)} \to \pm\infty$ as $||u_0|| \to \infty$.

Proof. (1)

$$\begin{aligned}
||dJ_\infty(u)||^2 &= ||(-\Delta)^{-1}g_\infty(u)||^2 \\
&\leq c\int_\Omega |g_\infty(u)|^2 dx \\
&\leq c + c\int_\Omega h_\infty(|u|))^2 dx \\
&\leq c + c\int_\Omega h_\infty^2(u_0) + c\int_\Omega h_\infty^2(u_- + u_+)dx \\
&\leq c + c\int_\Omega h_\infty^2(||u_0||)dx + c\int_\Omega (1 + |u_- + u_+|)^{2(\beta'-1)}dx \\
&\leq c + ch_\infty^2(||u_0||) + c(1 + ||u_-|| + ||u_+||)^{2(\beta'-1)}.
\end{aligned}$$

(2) Note that the finite dimension and the unique continuous property of the kernel space KerA_∞ imply that $|u_0(x)| \to \infty$ for a.e. $x \in \Omega$ if $u_0 \in$ KerA_∞ and $||u_0|| \to \infty$. Then by using the same arguments as in Lemma 3.2, we get the proof of Lemma 3.4.

Proof of Theorem 3.1. We only prove the first case, the other cases are similar. We rewrite the functional f as

$$\begin{aligned}
f(u) &= \tfrac{1}{2}\langle A_0u, u\rangle + J_0(u) \\
&= \tfrac{1}{2}\langle A_\infty u, u\rangle + J_\infty(u).
\end{aligned} \tag{3.7}$$

By Lemma 3.1 and Lemma 3.2, we know that $f(u)$ satisfies all the conditions in Theorem 2.1 and Theorem 2.2. Hence,

$$C_q(f,\theta) \cong \delta_{qr_-^0} F, \forall q,$$

$$C_q(f,\infty) \cong \delta_{qr_-} F, \forall q,$$

where r_-^0 is the dimension of the negative eigenvalue subspace of A_0, and r_- is the dimension of negative eigenvalue subspace of A_∞.

If $\lambda_k \neq \lambda_j$, then

$$C_q(f,\theta) \neq C_q(f,\infty).$$

Now, by the r_-th Morse inequality , f has at least one nontrivial critical point x_1 satisfying $C_{r_-}(f,x_1) \neq 0$. The proof is completed.

Proof of Theorem 3.2 We consider the functional f in (3.7) and suppose that f has no any other critical points except for x_0 and θ. If (1) holds, then

$$C_q(f,\theta) = \delta_{qr_-^0}, \forall q, \quad C_q(f,\infty) = \delta_{qr_-}, \forall q,$$

$$C_q(f,x_0) = \delta_{q\mu}, \forall q,$$

where μ is the Morse index of x_0. By Morse inequality, we have

$$(-1)^{r_-} = (-1)^{r_-^0} + (-1)^{\mu},$$

a contradiction. The end of the proof.

In the following, we consider the asymptotically linear second order Hamiltonian systems and study the periodic solution problem as follows:

$$\begin{cases} \ddot{u} = \nabla V(t,u) \\ u(0) = u(2\pi), \quad \dot{u}(0) = \dot{u}(2\pi) \end{cases} \tag{3.8}$$

where $V \in C^1(R^{N+1}, R)$ is 2π periodic in t. Suppose that there exist $N \times N$ symmetric, 2π periodic matrixes $B_\infty(t)$ and $B_0(t)$ such that

$$|\nabla V(t,x) - B_0(t)x| = o(|x|) \text{ as } |x| \to 0, \tag{3.9}$$

$$|\nabla V(t,x) - B_\infty(t)x| = o(|x|) \text{ as } |x| \to \infty, \tag{3.10}$$

where $|\cdot|$ denotes the norm in R^N. We use $(\cdot,\cdot)$ to denote the inner product in R^N. In addition, we take $H = H^1(S^1)$ with the inner product

$$\langle u,v\rangle = (\int_0^{2\pi} ((u,v) + (\dot{u},\dot{v}))dt)^{\frac{1}{2}} \text{ for } u,v \in H.$$

Furthermore, we make the following assumptions:

(V_1) $\nabla V_0(t,0) = 0$, and there exist constants $A_0 > 0$ such that

$$|\nabla V_0(t,x)| \leq A_0 h_0(|x|) \text{ for } |x| < 1,$$

$(V_2^{\pm})$ $-\dfrac{V_0(t,u_0)}{h_0^2(||u_0||)} \to \pm\infty$ as $u_0 \in H_0^0, ||u_0|| \to 0$,

(V_3) there exist constants $A_\infty > 0$ such that

$$|\nabla V_\infty(t,x)| \leq A_\infty(1 + h_\infty(|x|)) \text{ for } x \in R^N, t \in R,$$

$(V_4^{\pm})$ $-\dfrac{V_\infty(t,u_0)}{h_\infty^2(|u_0|)} \to \pm\infty$ as $u_0 \in H_0, |u_0| \to 0$

where, h_0, h_∞ are the control functionals defined in section 2. And $H_0^0 = \text{Ker}(-\frac{d^2}{dt^2} - B_0(t)), H_0 = \text{Ker}(-\frac{d^2}{dt^2} - B_\infty(t))$. Let $r_0 = \dim H_0, r_0^0 = \dim H_0^0, r_-$ be the dimension of the negative eigenvalue space of $(-\frac{d^2}{dt^2} - B_\infty(t))$, and r_-^0 be the dimension of the negative eigenvalue space of $(-\frac{d^2}{dt^2} - B_0(t))$.

Theorem 3.3 *Suppose that* $(V_1), (V_3)$ *hold. Then* (3.5) *has at least one nontrivial* 2π *periodic solution if one of the following conditions holds:*

(1) $(V_2^+), (V_4^+)$ *hold and* $r_- \neq r_-^0$,

(2) $(V_2^+), (V_4^-)$ *hold and* $r_0 + r_- \neq r_-^0$,

(3) $(V_2^-), (V_4^+)$ *hold and* $r_- \neq r_-^0 + r_0^0$,

(4) $(V_2^-), (V_4^-)$ *hold and* $r_0 + r_- \neq r_-^0 + r_0^0$.

For any $u, v \in H^1(S^1)$, we take

$$\langle A_\infty u, v\rangle := \int_0^{2\pi} ((\dot{u}, \dot{v}) - (B_\infty(t)u, v))dt,$$

$$G_\infty(u) =: -\int_0^{2\pi} V_\infty(t,u)dt,$$

$$\langle A_0 u, v\rangle := \int_0^{2\pi} ((\dot{u}, \dot{v}) - (B_0(t)u, v))dt,$$

$$G_0(u) =: -\int_0^{2\pi} V_0(t,u)dt.$$

It is easy to see that $G_\infty(u), G_0(u)$ has compact differential $(-\frac{d^2}{dt^2} + I)^{-1}\nabla V_\infty(t,u)$ and $(-\frac{d^2}{dt^2} + I)^{-1}\nabla V_0(t,u)$, respectively. We consider the functional

$$f(x) = \frac{1}{2}\langle A_\infty u, u\rangle + G_\infty(u)$$

and

$$f(x) = \frac{1}{2}\langle A_0 u, u\rangle + G_0(u).$$

Then it is well known that $f \in C^1(H, R^1)$ and to find the nontrivial 2π periodic solution of (3.5) is equivalent to find the nontrivial critical points of f.
Proof of Theorem 3.3. We shall use the results of Theorem 2.1 and Theorem 2.2. For this purpose, we first prove that (V_3) implies that

$$||dG_\infty(u)|| \le c(1 + h_\infty(||u_0||) + p_\infty(||u_+|| + ||u_-||)).$$

In fact

$$\begin{aligned} ||dG_\infty(u)||^2 &= ||(-\tfrac{d^2}{dt^2} + Id)^{-1}\nabla V_\infty(t,u)||^2 \\ &\le c(\int_0^{2\pi} |\nabla V_\infty(t,u)|^2 dx) \\ &\le c + c\int_0^{2\pi} h_\infty^2(|u|)dx \\ &\le c + c\int_0^{2\pi} h_\infty^2(|u_0|)dx + c\int_0^{2\pi} h_\infty^2(|u_- + u_+|)dx \\ &\le c + \int_0^{2\pi} h_\infty^2(||u_0||) + C\int_0^{2\pi} (1 + |u_- + u_+|^{2(\beta'-1)})dx \\ &\le ch_\infty(||u_0||) + c(1 + ||u_+|| + ||u_-||)^{2(\beta'-1)}. \end{aligned}$$

By a similar way as in the proof of Lemma 3.2, we can prove that $(V_4^\pm)$ imply that

$$\frac{G_\infty(u_0)}{h_\infty^2(||u_0||)} \to \pm\infty \text{ as } ||u_0|| \to \pm\infty, ||u_0|| \in \text{Ker}A_\infty.$$

And $(V_1), (V_2^\pm)$ imply that(H_3) and $(H_4\pm)$. The rest of the proof is completed by the same arguments of Theorem 3.1.

Theorem 3.4 *Under the conditions of Theorem 3.3, if the nontrivial solution of (3.5) is nondegenerate , then (3.5) has at least two nontrivial solutions.*

Proof. The proof is same as that of Theorem 3.3.

Remark 3.1 Our theorems extend the result of [7], which is the case when we take the control functional $h_0(t) = t^\sigma$ for $\sigma > 1$ and $h_\infty(t) = t^\alpha$ for $\alpha \in (0, 1)$.

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