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Nonexistence of Positive Solutions for Polyharmonic Systems in $\mathbb{R}^N$

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Abstract

This work is devoted to the nonexistence of positive solutions for polyharmonic systems

$$(-\Delta)^m u = f(u,v), \qquad (-\Delta)^m v = g(u,v).$$

By using the method of moving plane combined with integral inequalities and Hardy's inequality, we prove some new Liouville type theorems for the above semilinear polyharmonic systems in $\mathbb{R}^N$.

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1 Introduction

We consider the nonexistence of positive solutions for the problem:

$$\begin{cases} (-\Delta)^m u &= f(u,v) \\ (-\Delta)^m v &= g(u,v) \quad \text{in } \mathbb{R}^N. \\ u>0 & v>0 \end{cases} \tag{1.1}$$

Throughout this paper, we assume $N > 2m$. In the paper of Wei and Xu [9], the authors considered the classification of the solutions for polyharmonic equations $(-\Delta)^m u = f(u)$. They proved, among other things, the non-existence of positive solutions under the assumptions that $f(t)$ is locally Lipschtz continuous with the essential hypotheses on monotonicity and sub-criticality. In a recent paper of Guo, Liu [5], the authors proved the non-existence of positive solutions under the assumptions that f is continuous with the essential hypotheses on monotonicity.

In this paper, we investigate the nonexistence of positive solutions for a more complicated system (1.1) in $\mathbb{R}^N$. Note that, if the nonlinearities f, g are only assumed to be continuous, we can not conclude that the weak solutions of the system are of C^{2m} class. Hence, the method of moving plane for classical solutions is not able to be applied directly, some other new techniques are needed. To overcome this difficulty, we use the method of moving plane combined with integral inequalities, an idea originally due to S. Terracini [8], where singular elliptic problem in $\mathbb{R}^N$ or in the half space $\mathbb{R}^N_+$ with nonlinear Neumann data on the boundary are treated. This method was later used by Damascelli and Gladiali [4]. On the other hand, since we do not have Maximum principle for polyharmonic systems in general, we are forced to prove the superharmonic properties: $(-\Delta)^i u \geq 0, (-\Delta)^i v \geq 0, i = 1, 2, ..., m-1$ in $\mathbb{R}^N$. To do this, we use the method of moving plane combined with Sobolev's inequality in narrow domain. Another difficulty stems from the fact that the nonlinearities are coupled each other, we are not clear if or where we can start the method of moving plane. In developing the method of moving plane, we also use Hardy's inequality which helps us to start the method.

We would like to mention that the Liouville Theorems for (1.1) play a very important role in the study of the existence of the solutions for non-variational polyharmonical systems, which leads naturally to the questions answered by theorems of Liouville type. For more details, we refer the readers to [3]. On the other hand, polyharmonic equations (systems) have received great attention recently since, besides the analytic importance, it is conformally invariant and has a geometric root; we refer to [2] and references therein for more details on this aspect.

Let f, g be continuous functions on $\mathbb{R}^2$. We say that $(u, v) \in E := H^m_{loc}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N) \times H^m_{loc}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ is a *weak solution* of (1.1) if

$$\begin{aligned} \int_{\mathbb{R}^N} \Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} \varphi dx &= \int_{\mathbb{R}^N} f(u,v)\varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N) \\ \int_{\mathbb{R}^N} \Delta^{\frac{m}{2}} v \Delta^{\frac{m}{2}} \varphi dx &= \int_{\mathbb{R}^N} g(u,v)\varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N) \end{aligned} \quad \text{for } m \text{ even,} \tag{1.2}$$

$$\left.\begin{aligned}\int_{\mathbb{R}^N} \nabla(\Delta^{\frac{m-1}{2}} u)\nabla(\Delta^{\frac{m-1}{2}}\varphi)dx &= \int_{\mathbb{R}^N} f(u,v)\varphi dx, \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N)\\ \int_{\mathbb{R}^N} \nabla(\Delta^{\frac{m-1}{2}} v)\nabla(\Delta^{\frac{m-1}{2}}\varphi)dx &= \int_{\mathbb{R}^N} g(u,v)\varphi dx, \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N)\end{aligned}\right. \text{for } m \text{ odd.} \tag{1.3}$$

Theorem 1.1 *Let $(u,v)\in E$ be a weak solution of the problem:*

$$\begin{cases} (-\Delta)^m u &= f(v) \\ (-\Delta)^m v &= g(u) \quad in\ \mathbb{R}^N. \\ u>0 & \quad v>0 \end{cases} \tag{1.4}$$

Suppose that

(1)$f,g:[0,\infty)\to[0,\infty)$ are continuous and nondecreasing;

(2) $h(t)=\frac{f(t)}{t^{\frac{N+2m}{N-2m}}}$, $k(t)=\frac{g(t)}{t^{\frac{N+2m}{N-2m}}}$ are nonincreasing in $(0,\infty)$;

(3) there are constants $p,q>1$ and $T_1>0$ such that $f(t)\geq t^p$ and $g(t)\geq t^q$ for $t\geq T_1$.

Then (1.4) has no (weak) solutions unless $h(t)=m, k(t)=l$, where $m,l>0$ are constants in the range of v,u respectively. Moreover, in the later case, if $m=l=1$, then (u,v) has the form

$$u(x)=u(|x|)=C_{N,m}(1+|x|^2)^{-\frac{N-2m}{2}},\ v(x)=v(|x|)=\tilde{C}_{N,m}(1+|x|^2)^{-\frac{N-2m}{2}},$$

after the translation and dilation for some constants $C_{N,m},\tilde{C}_{N,m}$.

Remark 1.1 Note that under the continuity assumptions on the functions f,g, every weak solution of (1.4) is of $W^{2m,p}_{loc}(\mathbb{R}^N)$ class for $p<+\infty$ and hence of $C^{2m-1,\alpha}(\mathbb{R}^N)$ class for $0<\alpha<1$. In fact, firstly by the difference quotient method we can show the $H^{2m}_{loc}(\mathbb{R}^N)$ regularity, then by the L^p theory for elliptic equations we know that u,v belong to $W^{2m,p}_{loc}(\mathbb{R}^N)$ for $p<+\infty$, and hence to $C^{2m-1,\alpha}(\mathbb{R}^N)$ for $0<\alpha<1$.

Theorem 1.1 can be extended to more general cases. We mention the following corollary as an example.

Corollary 1.1 *Let $(u,v)\in E$ be a weak solution of the problem:*

$$\begin{cases} (-\Delta)^m u &= \sum_{i=1}^n u^{p_{1i}}v^{q_{1i}}, \\ (-\Delta)^m v &= \sum_{i=1}^n u^{p_{2i}}v^{q_{2i}}, \\ u>0 &, \quad v>0,\ in\ \mathbb{R}^N. \end{cases} \tag{1.5}$$

Suppose that $p_{11}=q_{21}=0, p_{11},q_{21}>1, p_{ji},q_{ji}\geq 0$, $p_{ji}+q_{ji}\leq\frac{N+2m}{N-2m}, j=1,2,\ i=1,2,...,n$. Then (1.5) has no solutions unless $p_{ji}+q_{ji}=\frac{N+2m}{N-2m}, j=1,2,\ i=1,2,...,n$.

Remark 1.2 Indeed, by using the strong maximum principle, in many cases we can get stronger conclusion by considering non-negative solution instead of positive solution. Since we are not intent to stress this, in this paper we just consider the positive solution.

The paper is organized as follows. In section 2, we begin with the simpler system (1.4) and give a detailed proof of Theorem 1.1. In section 3, we will consider more general systems (3.1)(Theorem 3.1) and (1.5) (Corollary 1.1). Since the proof of Theorem 1.1 can be repeated with necessary modifications in these more general systems, we pay special attention on the outline and the key steps of their proofs. At the end of the paper, we also give a nonexistence result for system (3.1) (Theorem 3.2) under stronger smoothness assumptions on f, g but with simpler other conditions.

2 Preliminaries and the proof of Theorem 1.1

We begin with the superharmonicity property of the solutions for (1.4) so that the Maximum principle can be applied.

Proposition 2.1 *Let (u, v) be a weak solution of (1.4). Then we have*

$$(-\Delta)^i u(x) \geq 0,\ (-\Delta)^i v(x) \geq 0,\ i = 1, 2, ..., m-1. \tag{2.1}$$

Proof. Let $u_i = (-\Delta)^{i-1}u,\ v_i = (-\Delta)^{i-1}v, i = 1, 2, ..., m$ with $u_1 = u, v_1 = v$. We first prove $u_m \geq 0, v_m \geq 0$ by an indirect argument. Suppose that there exists $x_0 \in \mathbb{R}^N$ such that $u_m(x_0) < 0$. Without loss of generality, we assume that $x_0 = 0$. We introduce the sphere average of a function u by

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{\partial B_r(0)} u d\sigma,$$

where ω_N denotes the area of the unit sphere in $\mathbb{R}^N$. Since $u, v \in W^{2m,p}_{loc}(\mathbb{R}^N)$ and satisfy system (1.4) in the weak sense, then the average $(\bar{u}, \bar{v})$ of (u, v) is of C^{2m} class; moreover, it holds that

$$\begin{cases} (-\Delta)^{m-1}\bar{u} &= \bar{u}_m \\ -\Delta \bar{u}_m &= \overline{f(v)} \\ (-\Delta)^{m-1}\bar{v} &= \bar{v}_m \\ -\Delta \bar{v}_m &= \overline{g(u)}. \end{cases} \tag{2.2}$$

Since $\bar{u}_m(0) = u_m(0) < 0$ and $-\Delta \bar{u}_m = \overline{f(v)} \geq 0$, we have

$$\bar{u}_m(r) \leq \bar{u}_m(0) < 0,$$
$$(-\Delta)^{m-1}\bar{u}(r) = \bar{u}_m(r) \leq \bar{u}_m(0). \tag{2.3}$$

By repeating integrations, we have

$$\begin{aligned} \bar{u}(r) \quad &\leq \bar{u}(0) + \tfrac{\Delta \bar{u}(0)}{2N} r^2 + ... + \frac{\Delta^{m-2}\bar{u}(0)}{\Pi_{k=1}^{m-2} 2k(N+2k-2)} r^{2(m-2)} \\ &+ \frac{\bar{u}_m(0)}{\Pi_{k=1}^{m-2} 2k(N+2k-2)} r^{2(m-1)},\ \text{for } m \text{ odd}, \end{aligned} \tag{2.4}$$

$$\begin{aligned}\bar{u}(r) \quad & \geq \bar{u}(0) + \frac{\Delta \bar{u}(0)}{2N} r^2 + ... + \frac{\Delta^{m-2}\bar{u}(0)}{\Pi_{k=1}^{m-2} 2k(N+2k-2)} r^{2(m-2)} \\ & - \frac{\bar{u}_m(0)}{\Pi_{k=1}^{m-2} 2k(N+2k-2)} r^{2(m-1)}, \text{ for } m \text{ even.}\end{aligned} \tag{2.5}$$

If m is odd, we deduce from (2.4) that $\bar{u}(r) \to -\infty$ as $r \to +\infty$ and arrive at a contradiction. If m is even, we deduce from (2.5) that $\bar{u}(r) \to +\infty$ as $r \to +\infty$. In this case from (2.2) we obtain that there exist $A_0 > 0$ and $r_0 > 0$ such that

$$\bar{u}(r) \geq A_0, \text{ for all } r \geq r_0,$$

$$\frac{d}{dr} \Delta^{m-i} \bar{u}|_{r=r_0} \geq 0, \ \Delta^{m-i} \bar{u}(r_0) \geq 0, \ i = 1, 2, ..., m,$$

again, it follows from (2.2)

$$-\Delta \bar{v}_m = \overline{g(u)} \geq \overline{u^q - c} \geq \bar{u}^q - c \geq \frac{1}{2} \bar{u}^q, \ r \geq r_0$$

and

$$\Delta^m \bar{v}(r) = -\Delta \bar{v}_m \geq \frac{1}{2} \bar{u}^p \geq \frac{1}{2} A_0^p, \text{ for } r \geq r_0. \tag{2.6}$$

By repeating the above process, we get

$$\bar{v}(r) \geq B_0, \text{ for all } r \geq r_0,$$

$$\frac{d}{dr} \Delta^{m-i} \bar{v}|_{r=r_0} \geq 0, \ \Delta^{m-i} \bar{v}(r_0) \geq 0, \ i = 1, 2, ..., m,$$

$$\Delta^m \bar{u}(r) = -\Delta \bar{u}_m \geq \frac{1}{2} \bar{v}^p \geq \frac{1}{2} B_0^p, \text{ for } r \geq r_0. \tag{2.7}$$

Again we will obtain a contradiction, see [9], [6] for the details. Hence $u_m \geq 0, v_m \geq 0$. By induction on i, we can show that $u_{m-i+1} \geq 0, v_{m-i+1} \geq 0, i = 2, ..., m-1$. Finally, since $u > 0, v > 0$ we have $u_i = (-\Delta)^{i-1} u > 0, v_i = (-\Delta)^{i-1} v > 0, i = 1, 2, ..., m$.

Lemma 2.1 *For* $0 < t_0 < 1$, *we define two functions* α, β *by*

$$\alpha(t) = \begin{cases} f(1) t^{\frac{N+2m}{N-2m}}, & 0 \leq t \leq t_0, \\ f(1) t_0^{\frac{N+2m}{N-2m}} + f(1) \frac{N+2m}{N-2m} t_0^{\frac{4m}{N-2m}} (t - t_0), & t > t_0, \end{cases} \tag{2.8}$$

$$\beta(t) = \begin{cases} g(1) t^{\frac{N+2m}{N-2m}}, & 0 \leq t \leq t_0, \\ g(1) t_0^{\frac{N+2m}{N-2m}} + g(1) \frac{N+2m}{N-2m} t_0^{\frac{4m}{N-2m}} (t - t_0), & t > t_0. \end{cases} \tag{2.9}$$

Then α, β *are continuous, increasing and convex, moreover*

$$\alpha(t) \leq f(t), \ \beta(t) \leq g(t), \ for \, t \geq 0 \tag{2.10}$$

provided t_0 *is sufficiently small.*

Proof. We only prove $\alpha(t) \le f(t)$. The proof of $\beta(t) \le g(t)$ can be obtained by a similar way. Let $T_0 = \max\{T, f^{1/p}(1), 1\}$, choose $t_0 < 1$ such that $\frac{N+2m}{N-2m} t_0^{\frac{4m}{N-2m}} = \frac{1}{T_0}$. Then for $0 \le t \le t_0,\ \alpha(t) = f(1) t^{\frac{N+2m}{N-2m}} \le f(t)$ (by (2));
for $t_0 \le t \le 1$,

$$\begin{aligned} \alpha(t) &= f(1)\tfrac{N+2m}{N-2m} t_0^{\frac{4m}{N-2m}}(t-t_0) + f(1) t_0^{\frac{N+2m}{N-2m}} \\ &\le f(1) t^{\frac{N+2m}{N-2m}} \le f(t) \quad \text{(by (2))}; \end{aligned}$$

for $1 \le t \le T_0$,

$$\begin{aligned} \alpha(t) &= f(1)\tfrac{N+2m}{N-2m} t_0^{\frac{4m}{N-2m}}(t-t_0) + f(1) t_0^{\frac{N+2m}{N-2m}} \\ &\le f(1)\tfrac{N+2m}{N-2m} t_0^{\frac{4m}{N-2m}}(T_0-t_0) + f(1) t_0^{\frac{N+2m}{N-2m}} \\ &= f(1) - f(1)\tfrac{4m}{N-2m} t_0^{\frac{N+2m}{N-2m}} < f(1) \le f(t) \quad \text{(by (1))}; \end{aligned}$$

for $t \ge T_0$,

$$\begin{aligned} \alpha(t) &= f(1)\tfrac{N+2m}{N-2m} t_0^{\frac{4m}{N-2m}}(t-t_0) + f(1) t_0^{\frac{N+2m}{N-2m}} \\ &= f(1)\tfrac{1}{T_0}(t-T_0) + \alpha(T_0) \\ &\le p T_0^{p-1}(t-T_0) + T_0^p \\ &\le t^p \le f(t). \end{aligned}$$

Lemma 2.2 $\displaystyle\int_1^\infty r^{2m-1}\bar{u}(r)^{\frac{N+2m}{N-2m}} dr < +\infty,\ \int_1^\infty r^{2m-1}\bar{v}(r)^{\frac{N+2m}{N-2m}} dr < +\infty.$

Proof. Since $-\Delta\bar{v}(r) = \bar{v}_2(r) \ge 0, \bar{v}'(r) \le 0$ and $\bar{v}(r) \le \bar{v}(0)$. There is a constant C_1 such that

$$\alpha(\bar{v}(r)) \ge C_1 \bar{v}(r)^{\frac{N+2m}{N-2m}}, r \ge 0. \tag{2.11}$$

Similarly, there is a constant C_2 such that

$$\beta(\bar{u}(r)) \ge C_2 \bar{u}(r)^{\frac{N+2m}{N-2m}}, r \ge 0. \tag{2.12}$$

Thus

$$-\Delta\bar{u}_m = \overline{f(v)} \ge \overline{\alpha(v)} \ge \alpha(\bar{v}) \ge C_1 \bar{v}^{\frac{N+2m}{N-2m}}, \tag{2.13}$$

since $\bar{u}_i \ge 0 (i = 1, 2, ..., m-1)$, by Lemma 3.1 in [10],

$$\bar{u}_m(r) \ge C_1 r^2 \bar{v}(r)^{\frac{N+2m}{N-2m}}. \tag{2.14}$$

It follows from $(-\Delta)^{m-1}\bar{u} = \bar{u}_m$ that $\bar{u}(r) \ge C r^{2m-2}\bar{u}_m$, and hence

$$\bar{u}(r) \ge C r^{2m} \bar{v}(r)^{\frac{N+2m}{N-2m}},$$

$$\bar{v}(r) \ge C r^{2m} \bar{u}(r)^{\frac{N+2m}{N-2m}}.$$

In the above and in what follows, we always use the same C to denote different constants. Hence $\bar{u}^{\frac{N+2m}{N-2m}}(r) \le Cr^{-2m-\frac{N-2m}{2}}, \bar{v}^{\frac{N+2m}{N-2m}}(r) \le Cr^{-2m-\frac{N-2m}{2}}$, and therefore

$$\int_1^\infty r^{2m-1}\bar{u}^{\frac{N+2m}{N-2m}}(r)dr < +\infty, \int_1^\infty r^{2m-1}\bar{v}^{\frac{N+2m}{N-2m}}(r)dr < +\infty.$$

Let $w(x) = \frac{1}{|x|^{N-2m}}u(\frac{x}{|x|^2}), z(x) = \frac{1}{|x|^{N-2m}}v(\frac{x}{|x|^2})$ be the Kelvin's transform, then we have $\bar{w}(r) = \frac{1}{r^{N-2m}}\bar{u}(\frac{1}{r})$, and $\bar{z}(r) = \frac{1}{r^{N-2m}}\bar{v}(\frac{1}{r})$. A direct computation shows that

$$\int_0^1 r^{N-1}\bar{w}^{\frac{N+2m}{N-2m}}dr = \int_1^\infty r^{2m-1}\bar{u}^{\frac{N+2m}{N-2m}}dr < +\infty, \tag{2.15}$$

$$\int_0^1 r^{N-1}\bar{z}^{\frac{N+2m}{N-2m}}dr = \int_1^\infty r^{2m-1}\bar{v}^{\frac{N+2m}{N-2m}}dr < +\infty. \tag{2.16}$$

Moreover

$$(-\Delta)^m w(x) = \frac{1}{|x|^{N+2m}}f(|x|^{N-2m}z), \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{2.17}$$

$$(-\Delta)^m z(x) = \frac{1}{|x|^{N+2m}}g(|x|^{N-2m}w), \text{ in } \mathbb{R}^N \setminus \{0\}. \tag{2.18}$$

Now let $w_i = (-\Delta)^{i-1}w, i = 1, 2, ..., m$. Then for some $c_i > 0$,

$$w_i(x) \sim c_i|x|^{2m-2i+2-N}, i = 1, 2, ..., m, \text{ as } |x| \to \infty,$$

$$|x|^{-m+2i-2}w_i \sim c_i|x|^{m-N} \in L^2(\mathbb{R}^N \setminus B_r), \tag{2.19}$$

$$|x|^{-m+2i-1}w_i \sim c_i|x|^{m+1-N} \in L^{\frac{2N}{N-2}}(\mathbb{R}^N \setminus B_r), \text{ for all } r > 0. \tag{2.20}$$

And the same results hold for $z_i = (-\Delta)^{i-1}z, i = 1, 2, ..., m$.

Proposition 2.2 $w_i = (-\Delta)^{i-1}w \ge 0, z_i = (-\Delta)^{i-1}z \ge 0, i = 1, 2, ..., m-1$.

Proof. By (2.19) and (2.20), it is sufficient to prove Proposition 2.2 in a ball, say $B_1(0) \setminus \{0\}$. Let $\chi \in C_0^\infty(B_1)$ be a nonnegative function. We prove that

$$\int_{B_1} \chi(-\Delta)^{m-1}w\, dx \ge 0. \tag{2.21}$$

Define a positive φ by

$$\begin{cases} -\Delta\varphi = \chi & \text{in } B_1(0) \\ \varphi = 0 & \text{in } \partial B_1(0). \end{cases}$$

For any $\epsilon > 0$, we take $\eta_\epsilon \in C_0^\infty$ such that $\eta_\epsilon(x) \equiv 1$ for $|x| \geq 2\epsilon$ and $\eta_\epsilon(x) \equiv 0$ for $|x| \leq \epsilon$, and $|D^j \eta_\epsilon| \leq c\epsilon^{-j}$. Thus

$$\begin{aligned} 0 &\leq \int_{B_1} \varphi\eta_\epsilon |x|^{-N-2m} f(|x|^{N-2m} z)dx \\ &= \int_{B_1} \varphi\eta_\epsilon (-\Delta)^m w dx \\ &= \int_{B_1} (-\Delta)(\varphi\eta_\epsilon)(-\Delta)^{m-1} w dx \\ &= \int_{B_1} (-\Delta)^{m-1} w(-\Delta\varphi)\eta_\epsilon dx + \int_{B_1} (-\Delta)^{m-1} w(-2\nabla\varphi\nabla\eta_\epsilon - \varphi\Delta\eta_\epsilon)dx. \end{aligned} \tag{2.22}$$

Letting $\psi = 2\nabla\varphi\nabla\eta_\epsilon + \varphi\Delta\eta_\epsilon$, we have

$$\begin{aligned} |\int_{B_1} (-\Delta)^{m-1} w\psi dx| &\leq \int_{B_1} w|\Delta^{m-1}\psi| dx \\ &\leq \epsilon^{-2m} \int_{\epsilon\leq|x|\leq 2\epsilon} w dx \\ &= C\epsilon^{-2m} \int_\epsilon^{2\epsilon} \bar{w}(r) r^{N-1} dr \\ &\leq C\epsilon^{-2m} (\int_\epsilon^{2\epsilon} \bar{w}(r)^{\frac{N+2m}{N-2m}} r^{N-1} dr)^{\frac{N-2m}{N+2m}} (\int_\epsilon^{2\epsilon} r^{N-1} dr)^{\frac{4m}{N+2m}} \\ &\leq C\epsilon^{\frac{2m(N-2m)}{N+2m}} \to 0, \text{as } \epsilon \to 0. \end{aligned}$$

Therefore

$$\int_{B_1} \chi(-\Delta)^{m-1} w dx = \int_{B_1} (-\Delta)^{m-1} w(-\Delta)\varphi dx \geq 0, \forall \chi \in C_0^\infty(B_1), \chi \geq 0.$$

This implies that $(-\Delta)^{m-1} w \geq 0$. Similarly, we can prove that $(-\Delta)^{m-1} z \geq 0$. By induction on i, we finish the proof of Proposition 2.2.

In what follows, we shall use the method of moving plane. We start by considering planes parallel to $x^1 = 0$, coming from $+\infty$. For each $\lambda > 0$, we write $x = (x^1, x')$ with $x' = (x^2, ..., x^N) \in \mathbb{R}^{N-1}$ and define $\Sigma_\lambda := \{x = (x^1, x')|x^1 > \lambda\}$, $T_\lambda := \partial\Sigma_\lambda = \{x = (x^1, x')|x^1 = \lambda\}$. For $x = (x^1, x') \in \Sigma_\lambda$, let $x_\lambda = (2\lambda - x^1, x')$ be the reflected point with respect to the hyperplane T_λ, let $\tilde{\Sigma}_\lambda = \Sigma_\lambda \setminus \{e_\lambda\}$ with $e_\lambda = (2\lambda, 0, ..., 0)$. We define the reflected functions by

$$w^\lambda(x) = w(x_\lambda),\ z^\lambda(x) = z(x_\lambda), x \in \tilde{\Sigma},$$

$$z_i^\lambda = (-\Delta)^{i-1} z^\lambda, w_i^\lambda = (-\Delta)^{i-1} w^\lambda, i = 1, 2, ..., m.$$

Then we have

$$\begin{cases} -\Delta w_1 = & w_2 \\ \vdots & \vdots \\ -\Delta w_m = & \frac{1}{|x|^{N+2m}} f(|x|^{N-2m} z_1) \\ -\Delta z_1 = & z_2 \\ \vdots & \vdots \\ -\Delta z_m = & \frac{1}{|x|^{N+2m}} f(|x|^{N-2m} w_1), \end{cases} \qquad \begin{cases} -\Delta w_1^\lambda = & w_2^\lambda \\ \vdots & \vdots \\ -\Delta w_m^\lambda = & \frac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m} z_1^\lambda) \\ -\Delta z_1^\lambda = & z_2^\lambda \\ \vdots & \vdots \\ -\Delta z_m^\lambda = & \frac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m} w_1^\lambda). \end{cases} \tag{2.23}$$

Proposition 2.3 *For λ large enough, $w_i \le w_i^\lambda, z_i \le z_i^\lambda$ in $\tilde{\Sigma}_\lambda, i = 1, 2, ..., m$.*

Proof. Let η_ϵ be smooth function such that $\eta_\epsilon = 1$ if $2\epsilon \le |x - e_\lambda| \le \frac{1}{\epsilon}; \eta_\epsilon = 0$, for $|x - e_\lambda| \le \epsilon$ or $|x - e_\lambda| \ge \frac{2}{\epsilon}$. Test the equation $-\Delta(w_1 - w_1^\lambda) = w_2 - w_2^\lambda$ by $\varphi = (w_1 - w_1^\lambda)_+(|x|^{-m+1}\eta_\epsilon)^2$, where $(w_1 - w_1^\lambda)_+ = \max\{w_1 - w_1^\lambda, 0\}$, we have

$$\begin{aligned} \int_{\Sigma_\lambda} |\nabla((w_1 - w_1^\lambda)_+|x|^{-m+1}\eta_\epsilon)|^2 dx &= \int_{\Sigma_\lambda} (w_2 - w_2^\lambda)(w_1 - w_1^\lambda)_+(|x|^{-m+1}\eta_\epsilon)^2 dx \\ &\quad + \int_{\Sigma_\lambda} (w_1 - w_1^\lambda)_+^2 |\nabla(|x|^{-m+1}\eta_\epsilon)|^2 dx \\ &=: I + II. \end{aligned} \tag{2.24}$$

$$\begin{aligned} I &\le (\int_{\Sigma_\lambda} ((w_2 - w_2^\lambda)_+|x|^{-m+2}\eta_\epsilon)^2 dx)^{1/2} (\int_{\Sigma_\lambda} ((w_1 - w_1^\lambda)_+|x|^{-m}\eta_\epsilon)^2 dx)^{1/2} \\ &\le C(\int_{\Sigma_\lambda} ((w_2 - w_2^\lambda)_+|x|^{-m+2}\eta_\epsilon)^2 dx)^{1/2} (\int_{\Sigma_\lambda} |\nabla((w_1 - w_1^\lambda)_+|x|^{-m+1}\eta_\epsilon)|^2 dx)^{1/2} \end{aligned} \tag{2.25}$$

where we have used Hardy's inequality:

$$(\frac{N-2}{2})^2 \int_{R^N} \frac{u^2}{|x|^2} dx \le \int_{R^N} |\nabla u|^2 dx, u \in H^1(\mathbb{R}^N).$$

And for $\delta > 0$,

$$\begin{aligned} II &= \int_{\Sigma_\lambda} (w - w_1^\lambda)_+^2 |(-m+1)|x|^{-m-1} x\eta_\epsilon + |x|^{-m+1}\nabla\eta_\epsilon|^2 dx \\ &\le (1+\delta)(m-1)^2 \int_{\Sigma_\lambda} ((w_1 - w_1^\lambda)_+|x|^{-m}\eta_\epsilon)^2 dx \\ &\quad + c_\delta \int_{\Sigma_\lambda} ((w_1 - w_1^\lambda)_+|x|^{-m+1})^2 |\nabla\eta_\epsilon|^2 dx \\ &=: II_1 + II_2. \end{aligned} \tag{2.26}$$

By Hardy's inequality

$$II_1 \le \mu \int_{\Sigma_\lambda} |\nabla((w_1 - w_1^\lambda)_+|x|^{-m+1}\eta_\epsilon)|^2 dx, \tag{2.27}$$

where $\mu = (1+\delta)(m-1)^2(\frac{N-2}{2})^{-2} < 1$ if δ small.

$$
\begin{aligned}
II_2 &\le C(\int_{\Sigma_\lambda \cap \{\nabla\eta_\epsilon \neq 0\}} (((w_1 - w_1^\lambda)_+ |x|^{-m+1})^{\frac{2N}{N-2}}) dx)^{\frac{N-2}{N}} (\int_{\Sigma_\lambda} |\nabla \eta_\epsilon|^N)^{\frac{2}{N}} \\
&\le C(\int_{\Sigma_\lambda \cap \{\nabla\eta_\epsilon \neq 0\}} (((w_1 - w_1^\lambda)_+ |x|^{-m+1})^{\frac{2N}{N-2}}) dx)^{\frac{N-2}{N}} \to 0, \text{ as } \epsilon \to 0,
\end{aligned}
$$

since $(w_1 - w_1^\lambda)_+ |x|^{-m+1} \le w_1 |x|^{-m+1} \in L^{\frac{2N}{N-2}}$.

Letting $\epsilon \to 0$ in (2.24), by using (2.25), (2.26), (2.27), we have

$$
\int_{\Sigma_\lambda} |\nabla (w_1 - w_1^\lambda)_+ |x|^{-m+1}|^2 dx \le C \int_{\Sigma_\lambda} |(w_2 - w_2^\lambda)_+ |x|^{-m+2}|^2 dx. \tag{2.28}
$$

Similarly, for $i = 2, ..., m-1$

$$
\begin{aligned}
\int_{\Sigma_\lambda} |\nabla (w_i - w_i^\lambda)_+ |x|^{-m+2i-1}|^2 dx &\le C \int_{\Sigma_\lambda} |(w_{i+1} - w_{i+1}^\lambda)_+ |x|^{-m+2i}|^2 dx \\
&\le C \int_{\Sigma_\lambda} |\nabla (w_{i+1} - w_{i+1}^\lambda)_+ |x|^{-m+2i+1}|^2 dx.
\end{aligned} \tag{2.29}
$$

Finally, if $z_1 \le z_1^\lambda$,

$$
\begin{aligned}
-\Delta (w_m - w_m^\lambda) &= |x|^{-N-2m} f(|x|^{N-2m} z_1) - |x_\lambda|^{-N-2m} f(|x_\lambda|^{N-2m} z_1^\lambda) \\
&\le |x|^{-N-2m} f(|x|^{N-2m} z_1) - |x_\lambda|^{-N-2m} f(|x_\lambda|^{N-2m} z_1) \text{ (by (1))} \\
&\le 0 \text{ (by (2))};
\end{aligned}
$$

if $z_1 > z_1^\lambda$,

$$
\begin{aligned}
-\Delta (w_m - w_m^\lambda) &\le |x|^{-N-2m} f(|x|^{N-2m} z_1)(1 - (\frac{z_1^\lambda}{z_1})^{\frac{N+2m}{N-2m}}) \\
&\le \frac{N+2m}{N-2m} \frac{f(|x|^{N-2m} z_1)}{|x|^{N-2m} z_1} \frac{1}{|x|^{4m}} (z_1 - z_1^\lambda) \\
&\le C_\lambda \frac{1}{|x|^{4m}} (z_1 - z_1^\lambda),
\end{aligned}
$$

where C_λ is nonincreasing in λ.

Testing the equation

$$
-\Delta (w_m - w_m^\lambda) = |x|^{-N-2m} f(|x|^{N-2m} z_1) - |x_\lambda|^{-N-2m} f(|x_\lambda|^{N-2m} z_1^\lambda)
$$

by $\varphi = (w_m - w_m^\lambda)_+ (|x|^{m-1} \eta_\epsilon)^2$, we have

$$
\begin{aligned}
&\int_{\Sigma_\lambda} |\nabla (w_m - w_m^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2 dx \\
&\le C_\lambda \int_{\Sigma_\lambda} \frac{1}{|x|^{4m}} (z_1 - z_1^\lambda)_+ (w_m - w_m^\lambda)_+ (|x|^{m-1} \eta_\epsilon)^2 dx \\
&\quad + \int_{\Sigma_\lambda} (w_m - w_m^\lambda)_+^2 |\nabla |x|^{m-1} \eta_\epsilon|^2 dx \\
&=: I + II
\end{aligned}
$$

$$
\begin{aligned}
I \quad &\le C_\lambda\Big(\int_{A_\lambda}\frac{dx}{|x|^{(m+1)N}}\Big)^{\frac{2}{N}}\times\Big(\int_{\Sigma_\lambda}((z_1-z_1^\lambda)_+|x|^{-m+1}\eta_\epsilon)^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{2N}}\\
&\quad\times\Big(\int_{\Sigma_\lambda}((w_m-w_m^\lambda)_+|x|^{m-1}\eta_\epsilon)^{\frac{2N}{N-2}}dx\Big)^{\frac{N-2}{2N}}\\
&\le C_\lambda\Big(\int_{A_\lambda}\frac{1}{|x|^{(m+1)N}}dx\Big)^{\frac{2}{N}}\times\Big(\int_{\Sigma_\lambda}|\nabla(z_1-z_1^\lambda)_+|x|^{-m+1}\eta_\epsilon|^2dx\Big)^{\frac{1}{2}}\\
&\quad\times\Big(\int_{\Sigma_\lambda}|\nabla(w_m-w_m^\lambda)_+(|x|^{m-1}\eta_\epsilon|)^2dx\Big)^{\frac{1}{2}},
\end{aligned}
\tag{2.30}
$$

where $A_\lambda=\{x\in\Sigma_\lambda|z_1(x)>z_1^\lambda(x)\}$.

$$
\begin{aligned}
II \quad &\le (1+\delta)(m-1)^2\int_{\Sigma_\lambda}((w_m-w_m^\lambda)_+|x|^{m-2}\eta_\epsilon)^2dx\\
&\quad + C_\delta\int_{\Sigma_\lambda}((w_m-w_m^\lambda)_+|x|^{m-1})^2|\nabla\eta_\epsilon|^2dx\\
&\le \mu\int_{\Sigma_\lambda}|\nabla(w_m-w_m^\lambda)_+|x|^{m-1}\eta_\epsilon|^2dx+o(1).
\end{aligned}
$$

Let $\epsilon\to 0$,

$$
\begin{aligned}
&\int_{\Sigma_\lambda}|\nabla(w_m-w_m^\lambda)_+|x|^{m-1}|^2dx\\
&\le C_\lambda\Big(\int_{A_\lambda}\frac{1}{|x|^{(m+1)N}}dx\Big)^{\frac{2}{N}}\int_{\Sigma_\lambda}|\nabla(z_1-z_1^\lambda)_+|x|^{-m+1}|^2dx.
\end{aligned}
$$

Similarly, we have for $i=1,2,...,m-1$

$$
\begin{aligned}
\int_{\Sigma_\lambda}|\nabla(z_i-z_i^\lambda)_+|x|^{-m+2i-1}|^2dx \quad &\le C\int_{\Sigma_\lambda}|(z_{i+1}-z_{i+1}^\lambda)_+|x|^{-m+2i}|^2dx\\
&\le C\int_{\Sigma_\lambda}|\nabla(z_{i+1}-z_{i+1}^\lambda)_+|x|^{-m+2i+1}|^2dx.
\end{aligned}
\tag{2.31}
$$

and

$$
\begin{aligned}
&\int_{\Sigma_\lambda}|\nabla(z_m-z_m^\lambda)_+|x|^{m-1}|^2dx\\
&\le C_\lambda\Big(\int_{\tilde{A}_\lambda}\frac{1}{|x|^{(m+1)N}}dx\Big)^{\frac{2}{N}}\int_{\Sigma_\lambda}|\nabla(w_1-w_1^\lambda)_+|x|^{-m+1}|^2dx.
\end{aligned}
$$

Where $\tilde{A}_\lambda=\{x\in\Sigma_\lambda|w_1(x)>w_1^\lambda(x)\}$. Note that, for λ large, $C_\lambda\int_{A_\lambda}\frac{dx}{|x|^{(m+1)N}}\le C_\lambda\int_{\Sigma_\lambda}\frac{dx}{|x|^{(m+1)N}}<1$, $C_\lambda\int_{\tilde{A}_\lambda}\frac{dx}{|x|^{(m+1)N}}\le C_\lambda\int_{\Sigma_\lambda}\frac{dx}{|x|^{(m+1)N}}<1$; hence $(w_i-w_i^\lambda)_+=0,(z_i-z_i^\lambda)_+=0$ in $\tilde{\Sigma}_\lambda$, $i=0,1,...,m$.

Let $\Lambda=\inf\{\lambda|w_i\le w_i^\mu,\ z_i\le z_i^\mu,\ \text{in }\tilde{\Sigma}_\mu,i=1,2,...,m,\mu>\lambda\}$.

Proposition 2.4 *If* $\Lambda>0$, *then* $w_1=w_1^\Lambda,z_1=z_1^\Lambda$ *in* Σ_Λ.

Proof. By continuity, $w_i\le w_i^\Lambda,z_i\le z_i^\Lambda,i=1,2,...,m$ in $\tilde{\Sigma}_\Lambda$, and

$$
\begin{cases}
-\Delta(w_1-w_1^\Lambda) & =w_2-w_2^\Lambda\le 0\\
\quad w_1-w_1^\Lambda & \le 0
\end{cases}
\quad \text{in }\tilde{\Sigma}_\Lambda;
$$

by the Maximum principle, either $w_1 \equiv w_1^\Lambda$ or $w_1 < w_1^\Lambda$ in $\tilde{\Sigma}_\Lambda$. Let χ_S be the characteristic function of S. Then $\chi_{A_\lambda} \to 0$ a.e. as $\lambda \to \Lambda$, and

$$\int_{A_\lambda} \frac{dx}{|x|^{(m+1)N}} = \int_{\mathbb{R}^N} \chi_{A_\lambda} \frac{dx}{|x|^{(m+1)N}} \to 0, \text{ as } \lambda \to \Lambda.$$

Similarly,

$$\int_{\tilde{A}_\lambda} \frac{dx}{|x|^{(m+1)N}} = \int_{\mathbb{R}^N} \chi_{\tilde{A}_\lambda} \frac{dx}{|x|^{(m+1)N}} \to 0, \text{ as } \lambda \to \Lambda.$$

Hence, there is a constant $\delta > 0$ such that for $\lambda \in [\Lambda - \delta, \Lambda]$, $C_\lambda \int_{A_\lambda} \frac{1}{|x|^{(m+1)N}} < 1$, $C_\lambda \int_{\tilde{A}_\lambda} \frac{1}{|x|^{(m+1)N}} < 1$. By (2.30) and (2.31), we know that $w_i \leq w_i^\lambda, z_i \leq z_i^\lambda$ in $\tilde{\Sigma}_\lambda$ for $\lambda \in [\Lambda - \delta, \Lambda]$, which contradicts with the definition of Λ.

Proof of Theorem 1.1. If for all direction $\Lambda = 0$, we conclude by continuity that $w(x) \leq w_0(x)$ and $z(x) \leq z_0(x)$ for all $x \in \Sigma_0$. We perform the moving plane procedure from the left and get $w_0(x) \leq w(x)$ and $z_0(x) \leq z(x)$ for all $x \in \Sigma_0$. So w, z and hence u, v are radially symmetric around the pole of the transform. Since the pole is arbitrary, u, v are constants.

If $\Lambda \neq 0$, then w, z are regular at the origin, and u, v are regular at infinity. Since $w_1 \equiv w_1^\Lambda, z_1 \equiv z_1^\Lambda$, by repeating the proof of Proposition 2.4, we have $w_m \equiv w_m^\Lambda, z_m \equiv z_m^\Lambda$, which implies that

$$|x|^{-N-2m} f(|x|^{N-2m} w_1) = |x_\Lambda|^{-N-2m} f(|x_\Lambda|^{N-2m} w_1^\Lambda). \tag{2.32}$$

Note that for any $x \in \Sigma_\Lambda$, $|x| > |x_\Lambda|$, and $f(t)/t^{\frac{N+2m}{N-2m}}$ is nonincreasing. It follows from (2.32) that $f(t)/t^{\frac{N+2m}{N-2m}}$ is constant in a left neighborhood of t with the form $t = |x|^{N-2m} w(x) = u(\frac{x}{|x|^2}), x^1 > \Lambda$. Similarly $f(t)/t^{\frac{N+2m}{N-2m}}$ is constant in any right neighborhood of $t = u(\frac{x}{|x|^2}), x^1 < \Lambda$, in particular it is true for t close to 0 since $t = u(\frac{x}{|x|^2})$ convergence to 0 at infinity. Therefore we conclude that if $\Lambda > 0$, $f(t)/t^{\frac{N=2m}{N-2m}}$ is constant in $u(\mathbb{R}^N)$. By the same argument, if $\Lambda > 0$, $g(t)/t^{\frac{N+2m}{N-2m}}$ is constant in $v(\mathbb{R}^N)$.

Assume $f(t) = t^{\frac{N+2m}{N-2m}}, g(t) = t^{\frac{N+2m}{N-2m}}$. Then

$$(-\Delta)^m u = v^{\frac{N+2m}{N-2m}}, \ (-\Delta)^m u = v^{\frac{N+2m}{N-2m}},$$

since (u, v) is regular at infinity, we have

$$\int_{\mathbb{R}^N} |\Delta^{\frac{m}{2}}(u - v)|^2 + \int_{\mathbb{R}^N} (u^{\frac{N+2m}{N-2m}} - v^{\frac{N+2m}{N-2m}})(u - v) = 0, \text{ for } m \text{ is even}$$

$$\int_{\mathbb{R}^N} |\nabla(\Delta^{\frac{m-1}{2}}(u - v))|^2 + \int_{\mathbb{R}^N} (u^{\frac{N+2m}{N-2m}} - v^{\frac{N+2m}{N-2m}})(u - v) = 0, \text{ for } m \text{ is odd.}$$

Hence $u = v$, By Lemma 4.3 in [9], u, v have the desired forms.

3 Extensions

In this section, we will show that how the techniques used in Section 2 can be applied to a more general system:

$$\begin{cases} (-\Delta)^m u &= \sum_{i=1}^n f_i(u,v) \\ (-\Delta)^m v &= \sum_{i=1}^n g_i(u,v) \quad \text{in } \mathbb{R}^N. \\ u>0 \quad , & v>0 \end{cases} \tag{3.1}$$

Theorem 3.1 *Let $(u,v) \in E$ be a weak solution of problem (3.1). Suppose that $f = \sum_i^n f_i$, $g = \sum_i^n g_i$, $f_i, g_i : [0,\infty) \times [0,\infty) \to \mathbb{R}, i = 1,2,...,n$ are continuous functions satisfying:*

(i) $f(s,t)$ is positive and nondecreasing in t,

(ii) there exist $p > 1$ and $T > 0$ such that $f(s,t) \geq t^p$, for $t > T$ and all s,

(iii) for $i \geq 1, f_i(s,t) \geq 0$ and there exist $p_{1i} \geq 0, q_{1i} \geq 0, p_{1i} + q_{1i} = \frac{N+2m}{N-2m}$ such that $\dfrac{f_i(s,t)}{s^{p_{1i}} t^{q_{1i}}}$ is nonincreasing in (s,t),

(iv) $g(s,t)$ is positive and nondecreasing in s,

(v) there exist $q \geq 1$ and $S > 0$ such that $g(s,t) \geq s^q$, for $s > S$ and all t,

(vi) for $i \geq 1, g_i(s,t) \geq 0$ and there exist $p_{2i} \geq 0, q_{2i} \geq 0, p_{2i} + q_{2i} = \frac{N+2m}{N-2m}$ such that $\dfrac{g_i(s,t)}{s^{p_{2i}} t^{q_{2i}}}$ is nonincreasing in (s,t).

Then (3.1) has no (weak) solutions unless there exist $m_i, l_i > 0$ such that $f_i(s,t) = m_i s^{p_{1i}} t^{q_{1i}}, g_i(s,t) = l_i s^{p_{2i}} t^{q_{2i}}, i = 1,2,...,n,$ and in the later case u, v are radially symmetric and regular at infinity. Moreover $u = aU, v = bU$, where $a = f(a,b), b = g(a,b)$ and $U > 0$ satisfies $(-\Delta)^m U + U^{\frac{N+2m}{N-2m}} = 0$ in $\mathbb{R}^N$.

We remark that the proof of Theorem 3.1 consists in the usual steps in the method of moving plane and will be completed similarly to that of Theorem 1.1. The key steps are proving analogues of Proposition 2.3 and Proposition 2.4. We use the same notations as in Section 2. Let (w,z) be the Kelvin's transform of (u,v). Then we have:

$$\begin{cases} -\Delta w_1(x) &= w_2(x) \\ \quad\vdots & \\ -\Delta w_m(x) &= \frac{1}{|x|^{N+2m}} f(|x|^{N-2m} w(x), |x|^{N-2m} z(x)) \\ -\Delta z_1(x) &= z_2(x) \\ \quad\vdots & \\ -\Delta z_m(x) &= \frac{1}{|x|^{N+2m}} g(|x|^{N-2m} w(x), |x|^{N-2m} z(x)) \end{cases} \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{3.2}$$

and

$$\begin{cases} -\Delta w_1^\lambda(x) &= w_2^\lambda(x) \\ \vdots & \\ -\Delta w_m^\lambda(x) &= \frac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m} w_1^\lambda(x), |x_\lambda|^{N-2m} z_1^\lambda(x)) \\ -\Delta z_1^\lambda(x) &= z_2^\lambda(x) \\ \vdots & \\ -\Delta z_m^\lambda(x) &= \frac{1}{|x_\lambda|^{N+2m}} g(|x_\lambda|^{N-2m} w_1^\lambda(x), |x_\lambda|^{N-2m} z_1^\lambda(x)). \end{cases} \quad x \in \mathbb{R}^N \setminus \{e_\lambda\} \tag{3.3}$$

We first mention that the superharmonic properties of u, v are guaranteed by condition (ii) and (v). Meanwhile, these two conditions allow us to have an analogue of Lemma 2.1. That is

Lemma 3.1 *$\alpha(s,t) \le f(s,t), \beta(s,t) \le g(s,t)$ for $s, t > 0$, where*

$$\alpha(s,t) = \begin{cases} a t^{\frac{N+2m}{N-2m}}, & 0 \le t \le T, \forall s, \\ a T^{\frac{N+2m}{N-2m}} + a \frac{N+2m}{N-2m} T^{\frac{4m}{N-2m}} (t - T), & t > T, \forall s \end{cases} \tag{3.4}$$

$$\beta(s,t) = \begin{cases} a s^{\frac{N+2m}{N-2m}}, & 0 \le s \le S, \forall t \\ a S^{\frac{N+2m}{N-2m}} + a \frac{N+2m}{N-2m} S^{\frac{4m}{N-2m}} (s - S), & s > S, \forall t \end{cases} \tag{3.5}$$

In the above, a is a constant to be chosen later.

Proof. Indeed, for $t < T$,

$$\begin{aligned} f(s,t) &\ge \sum f_i(s,T)(\tfrac{t}{T})^{q_i} \\ &\ge \sum f_i(s,T)(\tfrac{t}{T})^{\frac{N+2m}{N-2m}} \\ &= f(s,T)(\tfrac{t}{T})^{\frac{N+2m}{N-2m}} \\ &> T^{p-\frac{N+2m}{N-2m}} t^{\frac{N+2m}{N-2m}} > \alpha(s,t) \end{aligned}$$

for $t > T$, $f(s,t) > t^p > \alpha(s,t)$ provided a is small enough, since $\alpha(s,t)$ is a linear function in t for $t > T$.

In a similar way, we have

$$\beta(s,t) \le g(s,t), \forall s, t \ge 0.$$

In order to make the proof of Theorem 3.1 transparent and clear, having proved the superharmonity, in the following we assume that the summing on the right hand of (3.1) has only one term. Then with necessary modification, one can easily obtain the corresponding results for (3.1). More precisely, we consider a system with supperharmonity

$$\begin{cases} (-\Delta)^m u = & f(u,v) \\ (-\Delta)^m v = & g(u,v) \\ (-\Delta)^i u \ge & 0, \ (-\Delta)^i v \ge 0, i = 0, 1, 2, ...m-1. \end{cases} \tag{3.6}$$

Suppose

(i) $f(s,t)$ is positive and non-decreasing in t and there exist $p_1 \geq 0, q_1 > 0, p_1 + q_1 \leq \frac{N+2m}{N-2m}$ such that $\frac{f(s,t)}{s^{p_1} t^{q_1}}$ is non-increasing in (s,t).

(ii) $g(s,t)$ is positive and non-decreasing in s and there exist $p_2 \geq 0, q_2 > 0, p_2 + q_2 \leq \frac{N+2m}{N-2m}$ such that $\frac{g(s,t)}{s^{p_2} t^{q_2}}$ is non-increasing in (s,t).

We use the same notation as in Section 2.

Lemma 3.2 *There exists* $\lambda_0 > 0$ *such that* $z_i \leq z_i^\lambda, w_i \leq w_i^\lambda, i = 1, 2, ..., m,\ \lambda > \lambda_0$.

Proof. It is sufficient to prove, for $i = 1, ..., m-1$,

$$\int_{\Sigma_\lambda} |\nabla(w_i - w_i^\lambda)_+ |x|^{-m+2i-1}|^2 dx \leq C_\lambda \int_{\Sigma_\lambda} |\nabla(w_{i+1} - w_{i+1}^\lambda)_+ |x|^{-m+2i+1}|^2 dx, \quad (3.7)$$

$$\begin{aligned} &\int_{\Sigma_\lambda} |\nabla(w_m - w_m^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2 \\ &\leq C_\lambda (\int_{A_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} \int_{\Sigma_\lambda} |\nabla(w_1 - w_1^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2 \\ &+ C_\lambda (\int_{A_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} (\int_{\Sigma_\lambda} |\nabla(w_1 - w_1^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2)^{\frac{1}{2}} \\ &\times (\int_{\Sigma_\lambda} |\nabla(z_1 - z_1^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2)^{\frac{1}{2}} \end{aligned} \quad (3.8)$$

and

$$\int_{\Sigma_\lambda} |\nabla(z_i - z_i^\lambda)_+ |x|^{-m+2i-1}|^2 dx \leq C_\lambda \int_{\Sigma_\lambda} |\nabla(z_{i+1} - z_{i+1}^\lambda)_+ |x|^{-m+2i+1}|^2 dx, \quad (3.9)$$

$$\begin{aligned} &\int_{\Sigma_\lambda} |\nabla(z_m - z_m^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2 \\ &\leq C_\lambda (\int_{\tilde{A}_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} \int_{\Sigma_\lambda} |\nabla(z_1 - z_1^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2 \\ &+ C_\lambda (\int_{\tilde{A}_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} (\int_{\Sigma_\lambda} |\nabla(w_1 - w_1^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2)^{\frac{1}{2}} \\ &\times (\int_{\Sigma_\lambda} |\nabla(z_1 - z_1^\lambda)_+ |x|^{m-1} \eta_\epsilon|^2)^{\frac{1}{2}}. \end{aligned} \quad (3.10)$$

The proof of (3.7) is the same as that of Proposition 2.3. We only prove (3.8).
If $z_1 \leq z_1^\lambda$, we have

$$\begin{aligned} &f(|x_\lambda|^{N-2m} w_1^\lambda(x), |x_\lambda|^{N-2m} z_1^\lambda(x)) \\ &\geq f(|x_\lambda|^{N-2m} w_1^\lambda(x), |x_\lambda|^{N-2m} z_1(x) \tfrac{w_1^\lambda(x)}{w_1(x)}) \\ &\geq f(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x)) \tfrac{|x_\lambda|^{N+2m}}{|x|^{N+2m}} (\tfrac{w_1^\lambda(x)}{w_1(x)})^{p_1}, \end{aligned}$$

$$\begin{aligned}
-\Delta(w_m - w_m^\lambda) &\leq \frac{1}{|x|^{N+2m}} f(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x))(1 - (\frac{w_1^\lambda}{w_1})^{p_1}) \\
&\leq \frac{1}{|x|^{N+2m}} f^+(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x))(1 - (\frac{w_1^\lambda}{w_1})^{\frac{N+2m}{N-2m}}) \\
&\leq \frac{1}{|x|^{N+2m}} f^+(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x)) \frac{N+2m}{N-2m}(1 - \frac{w_1^\lambda}{w_1}) \\
&= \frac{N+2m}{N-2m} \frac{f^+(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x))}{|x|^{N-2m} w_1(x)} \frac{1}{|x|^{4m}} (w_1 - w_1^\lambda) \\
&\leq \frac{C_\lambda}{|x|^{4m}} (w_1 - w_1^\lambda),
\end{aligned} \tag{3.11}$$

for some constant C_λ.

If $z_1 \geq z_1^\lambda$, we have

$$\begin{aligned}
&f(|x_\lambda|^{N-2m} w_1^\lambda(x), |x_\lambda|^{N-2m} z_1^\lambda(x)) \\
&\geq f(|x_\lambda|^{N-2m} w_1^\lambda(x) \frac{z_1^\lambda(x)}{z_1(x)}, |x_\lambda|^{N-2m} z_1^\lambda(x) \frac{w_1^\lambda(x)}{w_1(x)}) \\
&\geq f(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x)) (\frac{|x_\lambda|^{N-2m} w_1^\lambda(x)}{|x|^{N-2m} w_1(x)})^{p_1} (\frac{|x_\lambda|^{N-2m} z_1^\lambda(x)}{|x|^{N-2m} z_1(x)})^{q_1} \\
&= f(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x)) \frac{|x_\lambda|^{N+2m}}{|x|^{N+2m}} (\frac{w_1^\lambda(x)}{w_1(x)})^{p_1} (\frac{z_1^\lambda(x)}{z_1(x)})^{q_1},
\end{aligned}$$

$$\begin{aligned}
&-\Delta(w_m - w_m^\lambda) \\
&\leq \frac{1}{|x|^{N+2m}} f(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x))(1 - (\frac{w_1^\lambda}{w_1})^{p_1} (\frac{z_1^\lambda}{z_1})^{q_1}) \\
&\leq \frac{1}{|x|^{N+2m}} f^+(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x))(1 - (\frac{w_1^\lambda}{w_1})^{\frac{N+2m}{N-2m}} (\frac{z_\lambda}{z_1})^{\frac{N+2m}{N-2m}}) \\
&\leq \frac{1}{|x|^{N+2m}} f^+(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x)) \frac{N+2m}{N-2m}[(1 - \frac{w_1^\lambda}{w_1}) + (1 - \frac{z_1^\lambda}{z_1})] \\
&= \frac{N+2m}{N-2m} \frac{f^+(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x))}{|x|^{N-2m} w_1(x)} \frac{(w_1 - w_1^\lambda)}{|x|^{4m}} \\
&\quad + \frac{N+2m}{N-2m} \frac{f^+(|x|^{N-2m} w_1(x), |x|^{N-2m} z_1(x))}{|x|^{N-2m} z_1(x)} \frac{(z_1 - z_1^\lambda)}{|x|^{4m}} \\
&\leq \frac{C_\lambda}{|x|^{4m}} [(w_1 - w_1^\lambda) + (z_1 - z_1^\lambda)].
\end{aligned} \tag{3.12}$$

Hence

$$-\Delta(w_m - w_m^\lambda) \leq \frac{C_\lambda}{|x|^{4m}} [(w_1 - w_1^\lambda)_+ + (z_1 - z_1^\lambda)_+], \text{ for } z_1 \geq z_1^\lambda.$$

Testing the equations in (3.2), (3.3) with the function $\varphi = (w_m - w_m^\lambda)_+(|x|^{m-1}\eta_\epsilon)^2$, we have

$$\begin{aligned}
&\int_{\Sigma_\lambda} |\nabla (w_m - w_m^\lambda)_+ |x|^{m-1}\eta_\epsilon|^2 \\
&= \int_{\Sigma_\lambda} (\frac{1}{|x|^{N+2m}} f(|x|^{N-2m}w_1, |x|^{N-2m}z_1) \\
&- \frac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m}w_1^\lambda, |x_\lambda|^{N-2m}z_1^\lambda))\varphi + \int_{\Sigma_\lambda} (w_m - w_m^\lambda)_+^2 |\nabla |x|^{m-1}\eta_\epsilon|^2 \\
&\le \int_{\Sigma_\lambda} (\frac{C_\lambda}{|x|^{4m}}[(w_1 - w_1^\lambda)_+ + (z_1 - z_1^\lambda)_+])(w_m - w_m^\lambda)_+(|x|^{m-1}\eta_\epsilon)^2 \\
&+ \int_{\Sigma_\lambda} (w_m - w_m^\lambda)_+^2 |\nabla |x|^{m-1}\eta_\epsilon|^2 \\
&\le C_\lambda (\int_{A_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} (\int_{\Sigma_\lambda} ((w_1 - w_1^\lambda)_+ |x|^{m-1}\eta_\epsilon)^{\frac{2N}{N-2}})^{\frac{N-2}{N}} \\
&+ C_\lambda (\int_{A_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} (\int_{\Sigma_\lambda} ((z_1 - z_1^\lambda)_+ |x|^{m-1}\eta_\epsilon)^{\frac{2N}{N-2}})^{\frac{N-2}{2N}} \\
&\times (\int_{\Sigma_\lambda} ((w_1 - w_1^\lambda)_+ |x|^{m-1}\eta_\epsilon)^{\frac{2N}{N-2}})^{\frac{N-2}{2N}} \\
&+ C(\int_{\Sigma_\lambda^\epsilon} ((w_1 - w_1^\lambda)_+)^{\frac{2N}{N-2}})^{\frac{N-2}{N}},
\end{aligned} \tag{3.13}$$

where $\Sigma_\lambda^\epsilon = \{x \in \Sigma_\lambda, \epsilon \le |x| \le 2\epsilon, \text{or } \frac{1}{\epsilon} \le |x| \le \frac{2}{\epsilon}\}$.

By Sobolev inequality, let $\epsilon \to 0$ in (3.13), we get

$$\begin{aligned}
&\int_{\Sigma_\lambda} |\nabla (w_m - w_m^\lambda)_+ |x|^{m-1}\eta_\epsilon|^2 \\
&\le C_\lambda (\int_{A_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} \int_{\Sigma_\lambda} |\nabla (w_1 - w_1^\lambda)_+ |x|^{m-1}\eta_\epsilon|^2 \\
&+ C_\lambda (\int_{A_\lambda^1} \frac{1}{|x|^{(m+1)N}})^{\frac{2}{N}} (\int_{\Sigma_\lambda} |\nabla (w_1 - w_1^\lambda)_+ |x|^{m-1}\eta_\epsilon|^2)^{\frac{1}{2}} \\
&\times (\int_{\Sigma_\lambda} |\nabla (z_1 - z_1^\lambda)_+ |x|^{m-1}\eta_\epsilon|^2)^{\frac{1}{2}}.
\end{aligned}$$

We proceed the same as in the proof of Proposition 2.3 and finish the proof of Lemma 3.2.

Let $\Lambda = \inf\{\lambda > 0 | w_i \le w_i^\mu, z_i \le z_i^\mu, \text{ in } \tilde{\Sigma}_\mu, i = 1, ..., m, \mu > \lambda\}$.

Lemma 3.3 *If* $\Lambda > 0$ *then* $w_1^\Lambda = w_1, z_1^\Lambda = z_1$ *for all* $x \in \tilde{\Sigma}_\Lambda$.

Proof. Firstly, we claim that $w_1^\Lambda \equiv w_1$ implies $z_1^\Lambda \equiv z_1$. In fact, by the equations (3.2) and (3.3),

$$\begin{aligned} &\tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w_1(x), |x|^{N-2m}z_1(x)) \\ = &\tfrac{1}{|x_\Lambda|^{N+2m}} f(|x_\Lambda|^{N-2m}w_1^\Lambda(x), |x_\Lambda|^{N-2m}z_1^\Lambda(x)) \\ \geq &\tfrac{1}{|x_\Lambda|^{(N-2m)q_1}|x|^{(N-2m)p_1}} f(|x|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1^\Lambda(x)) \\ > &\tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1^\Lambda(x)). \end{aligned}$$

Since $f(s,t)$ is nondecreasing in t, we deduce from the above inequality that

$$|x|^{N-2m}z_1(x) > |x_\Lambda|^{N-2m}z_1^\Lambda(x). \tag{3.14}$$

On the other hand,

$$\begin{aligned} &\tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w_1(x), |x|^{N-2m}z_1(x)) \\ = &\tfrac{1}{|x_\Lambda|^{N+2m}} f(|x_\Lambda|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1^\Lambda(x)) \\ \geq &\tfrac{1}{|x_\Lambda|^{N+2m}} f(|x_\Lambda|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1(x)) \\ \geq &\tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w_1(x), |x|^{N-2m}z_1(x)). \end{aligned}$$

Thus

$$\begin{aligned} &\tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w_1(x), |x|^{N-2m}z_1(x)) \\ = &\tfrac{1}{|x_\Lambda|^{N+2m}} f(|x_\Lambda|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1^\Lambda(x)) \\ = &\tfrac{1}{|x_\Lambda|^{N+2m}} f(|x_\Lambda|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1(x)), \end{aligned} \tag{3.15}$$

and hence

$$\frac{f(|x|^{N-2m}w_1(x), |x|^{N-2m}z_1(x))}{(|x|^{N-2m}w_1(x))^{p_1}(|x|^{N-2m}z_1(x))^{q_1}} = \frac{f(|x_\Lambda|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1(x))}{(|x_\Lambda|^{N-2m}w_1(x))^{p_1}(|x_\Lambda|^{N-2m}z_1(x))^{q_1}}. \tag{3.16}$$

By (3.14), we have

$$|x|^{N-2m}w_1(x) \geq |x_\Lambda|^{N-2m}w_1(x) = |x_\Lambda|^{N-2m}w_1(x),$$

$$|x|^{N-2m}z_1(x) \geq |x_\Lambda|^{N-2m}z_1^\Lambda(x) \geq |x_\Lambda|^{N-2m}z_1(x).$$

By (3.16) and the assumptions of Lemma 3.2,

$$\frac{f(|x|^{N-2m}w_1(x), |x|^{N-2m}z_1(x))}{(|x|^{N-2m}w_1(x))^{p_1}(|x|^{N-2m}z_1(x))^{q_1}} = \frac{f(|x_\Lambda|^{N-2m}w_1(x), |x_\Lambda|^{N-2m}z_1^\Lambda(x))}{(|x_\Lambda|^{N-2m}w_1(x))^{p_1}(|x_\Lambda|^{N-2m}z_1^\Lambda(x))^{q_1}}. \tag{3.17}$$

it follows from (3.15) and (3.17) that $z_1^{q_1} = (z_1^\Lambda)^{q_1}$, and hence $z = z_1^\Lambda$ since $q_1 > 0$.

Suppose that $w_1 - w_1^\Lambda \not\equiv 0$ and $z_1 - z_1^\Lambda \not\equiv 0$ in $\tilde{\Sigma}_\Lambda$, then $w_1 < w_1^\Lambda$, $z_1 < z_1^\Lambda$ in $\tilde{\Sigma}_\Lambda$. Now let χ_S be the characteristic function of set S. Then above discuss shows that $\frac{1}{|x|^{(m+1)N}}\chi_{A_\lambda^1}$ converges pointwisely to zero as $\lambda \to \Lambda$ in $\mathbb{R}^N \setminus (T_\Lambda \cup \{e_\Lambda\})$. Thus if $0 < \Lambda - \delta < \Lambda$, then $\frac{1}{|x|^{(m+1)N}}\chi_{A_\lambda^1} \leq \frac{1}{|x|^{(m+1)N}}\chi_{\Sigma_{\Lambda-\delta}} \in L^1(\Sigma_\lambda)$, by dominate convergence, we get

$$\int_{A_\lambda^1} \frac{1}{|x|^{(m+1)N}} \to 0 \text{ as } \lambda \to \Lambda,$$

and hence

$$C_\lambda^1(\int_{A_\lambda^1}\frac{1}{|x|^{(m+1)N}})^{\frac{4}{N}} < 1 \text{ for } \lambda \in (\Lambda-\delta,\Lambda).$$

Also we have

$$C_\lambda^2(\int_{A_\lambda^2}\frac{1}{|x|^{(m+1)N}})^{\frac{4}{N}} < 1 \text{ for } \lambda \in (\Lambda-\delta,\Lambda).$$

Recalling the previous arguments, this implies that $w_1 \leq w_1^\lambda$ and $z_1 \leq z_1^\lambda$ in Σ_λ for $\lambda \in (\Lambda-\delta,\Lambda)$, which contradicts with the definition of Λ.

Proof of Theorem 3.1. Again for simple reason, we assume that the summing on the right hand of the nonlinearities in (3.1) has only one term. Suppose that (u,v) is a solution of (3.6). Make the Kelvin's transform around a point $p \in \mathbb{R}^N$ and define $\Lambda = \Lambda(p,n)$ for a direction $n \in \mathbb{R}^N$. If $\Lambda(p,n) = 0$ for all p and n, then (u,v) is radially symmetric with respect to all $p \in \mathbb{R}^N$, and must be constant. If $\Lambda(p,n) > 0$ for some p and n, then the corresponding Kelvin's transform (w,z) is radially symmetric with respect to a point q other than p and regular at the pole p, hence (u,v) is regular at infinity, that is $u(x) \sim \frac{w(0)}{|x|^{N-2m}}, v(x) \sim \frac{z(0)}{|x|^{N-2m}}$ as $|x| \to \infty$. Since $\frac{f(s,t)}{s^{p_1}t^{q_1}}$ is nonincreasing in (s,t), it follows that $\frac{f(s,t)}{s^{p_1}t^{q_1}}$ is constant in a neighborhood of $(|x|^{N-2m}w(x), |x|^{N-2m}z(x)) = (u(\frac{x}{|x|^2}), v(\frac{x}{|x|^2}))$, which implies that $\frac{f(s,t)}{s^{p_1}t^{q_1}}$ is constant in the range of $\{(u(x),v(x))|x \in \mathbb{R}^N\}$. The same result is true for the function g. So the problem (3.6) reduces to

$$\begin{cases} (-\Delta)^m u = m u^{p_1} v^{q_1}, & u>0 \\ (-\Delta)^m v = l u^{p_2} v^{q_2}, & v>0 \end{cases} \text{ in } \mathbb{R}^N \tag{3.18}$$

where m, l are positive constants. Moreover (u,v) is regular at infinity. We can apply the method of moving plane to (u,v) and conclude that (u,v) is radially symmetric. It remains to determine the form of u, v. Suppose that (u,v) is radially symmetric with respect to the origin:

$$u(x) = \tilde{u}(|x|), v(x) = \tilde{v}(|x|).$$

Let (w,z) be the Kelvin's transform of (u,v) with the pole $p \neq 0$, that is

$$w(x) = \frac{1}{|x-p|^{N-2m}}u(\frac{x-p}{|x-p|^2}+p), z(x) = \frac{1}{|x-p|^{N-2m}}v(\frac{x-p}{|x-p|^2}+p).$$

(w,z) is radially symmetric with respect to some point q.

If $q = p$, then (u,v) is radially symmetric with respect to p too, hence constant. If $q \neq p$, it follows from [1, Lemma 7] that for some $A, B, s = s(p,q) > 0$

$$\begin{cases} u(x) = \frac{A}{(s^2+|x|^2)^{\frac{N-2m}{2}}} \\ v(x) = \frac{B}{(s^2+|x|^2)^{\frac{N-2m}{2}}}. \end{cases}$$

Let $A = a(N(N-2m)s^2)^{\frac{N-2m}{4}}, B = b(N(N-2m)s^2)^{\frac{N-2m}{4}}$, then $u = aU, v = bU$. Where $U = \frac{(N(N-2m)s^2)^{\frac{N-2m}{4}}}{(s^2+|x|^2)^{\frac{N-2m}{2}}}$ satisfies $(-\Delta)^m U = U^{\frac{N+2m}{N-2m}}$ in $\mathbb{R}^N$ and a, b satisfy

$$\begin{cases} a = m a^{p_1} b^{q_1}, \\ b = l a^{p_2} b^{q_2}. \end{cases}$$

We observe that in above proof, we only assume that $f(s,t), g(s,t)$ are continuous functions. If $f(s,t), g(s,t)$ are Lipschitz continuous functions with respect to t and s respectively, then the same results of Theorem 3.1 hold but with simpler assumptions. For example we consider the simpler system

$$\begin{cases} (-\Delta)^m u = & f(u,v) \\ (-\Delta)^m v = & g(u,v) \\ u > 0, & v > 0. \end{cases} \tag{3.19}$$

Theorem 3.2 *Let $(u,v) \in E$ be a positive weak solution of problem (3.19). Suppose that $f, g : [0,\infty) \times [0,\infty) \to \mathbb{R}$ are C^0 functions satisfying*

(i) $\frac{f(\gamma s,\gamma t)}{\gamma^{\frac{N+2m}{N-2m}}}, \frac{g(\gamma s,\gamma t)}{\gamma^{\frac{N+2m}{N-2m}}}$ are nonincreasing in γ, and either

(ii) $f(s,t)$ is increasing and locally Lipschitz continuous in t, that is

$$0 < f(s,t) - f(s,t') \leq L(m)(t - t')$$

provided that $m \geq t \geq t' > 0, m \geq s \geq 0$. And $g(s,t)$ is increasing and locally Lipschitz continuous in s in the sense that

$$0 < g(s,t) - g(s,t') \leq L_1(m)(s - s'),$$

or

$(ii)'$ $f(s,t)$ is increasing in t and nondecreasing in s ; $g(s,t)$ is increasing in s and nondecreasing in t.

(iii) there are $p \geq 1, q \geq 1$ but equal to 1 at the same time and $T > 0, S > 0$ such that $f(s,t) > t^p$, for $t > T$ and all s and $g(s,t) > s^q$ for $s > S$ and all t.

So, $f(s,t), g(s,t)$ are homogeneous of $\frac{N+2m}{N-2m}$ order in the range $\{(u(x), v(x)),\ x \in \mathbb{R}^N$. Moreover u, v are radially symmetric and regular at infinity and $u = aU, v = bU$, where a, b satisfy $f(\lambda_0 a, \lambda_0 b) = \lambda_0^{\frac{N+2m}{N-2m}} a, g(\lambda_0 a, \lambda_0 b) = \lambda_0^{\frac{N+2m}{N-2m}} b$, for $\lambda_0 = \max U$. And $U > 0$ satisfies $(-\Delta)^m U = U^{\frac{N+2m}{N-2m}}, U(0) = \lambda_0$.

Proof. We only address that how the main estimates in the proof of Lemma 3.4 hold. Indeed,

(1) The condition (ii)' holds. Since for any fixed $\lambda > 0$, $|x| \geq |x_\lambda|$, it follows that if $z \leq z^\lambda$,

$$\begin{aligned} & -\Delta(w_m - w_m^\lambda) \\ = & \tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m} w, |x|^{N-2m} z) - \tfrac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m} w^\lambda, |x_\lambda|^{N-2m} z^\lambda) \\ \leq & \tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m} w, |x|^{N-2m} z) - \tfrac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m} w^\lambda, |x_\lambda|^{N-2m} z \tfrac{w^\lambda}{w}) \\ \leq & \tfrac{1}{|x|^{N+2m}} (f(|x|^{N-2m} w, |x|^{N-2m} z)(1 - (\tfrac{w^\lambda}{w})^{\frac{N+2m}{N-2m}}) \\ \leq & \tfrac{C_\lambda}{|x|^{4m}} (w - w^\lambda). \end{aligned}$$

Where we apply the nonincreasing condition with $\gamma = \frac{w_\lambda}{w} (\frac{|x_\lambda|}{|x|})^{N-2} \leq 1$.

If $z \geq z^\lambda$,

$$\begin{aligned}
&-\Delta(w_m - w_m^\lambda)\\
&= \tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w, |x|^{N-2m}z) - \tfrac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m}w^\lambda, |x_\lambda|^{N-2m}z_\lambda)\\
&\leq \tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w, |x|^{N-2m}z) - \tfrac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m}\tfrac{w^\lambda z_\lambda}{z}, |x_\lambda|^{N-2m}\tfrac{w^\lambda z^\lambda}{w})\\
&\leq \tfrac{1}{|x|^{N+2m}} (f(|x|^{N-2m}w, |x|^{N-2m}z)(1 - (\tfrac{w^\lambda z^\lambda}{wz})^{\frac{N+2m}{N-2m}})\\
&\leq \tfrac{C_\lambda}{|x|^{4m}}[(w - w^\lambda) + (z - z^\lambda)],
\end{aligned}$$

where we apply the nonincreasing condition with $\gamma = \frac{w_\lambda}{w}\frac{z^\lambda}{z}(\frac{|x_\lambda|}{|x|})^{N-2m} \leq 1$.

(2) The condition (ii) holds. Again, for any fixed $\lambda > 0$, $|x| \geq |x_\lambda|$, if $z \leq z^\lambda$, the proof is the same as (ii)'.

If $z \geq z^\lambda$,

$$\begin{aligned}
&-\Delta(w_m - w_m^\lambda)\\
&= \tfrac{1}{|x|^{N+2m}} f(|x|^{N-2m}w, |x|^{N-2m}z) - \tfrac{1}{|x_\lambda|^{N+2m}} f(|x_\lambda|^{N-2m}w^\lambda, |x_\lambda|^{N-2m}z^\lambda)\\
&\leq \tfrac{1}{|x|^{N+2m}} (f(|x|^{N-2m}w, |x|^{N-2m}z) - f(|x|^{N-2m}w^\lambda, |x|^{N-2m}z^\lambda))\\
&= \tfrac{1}{|x|^{N+2m}} [(f(|x|^{N-2m}w, |x|^{N-2m}z) - f(|x|^{N-2m}w, |x|^{N-2m}z^\lambda))]\\
&\quad + \tfrac{1}{|x|^{N+2m}} [(f(|x|^{N-2m}w, |x|^{N-2m}z^\lambda) - f(|x|^{N-2m}w^\lambda, |x|^{N-m}z^\lambda))]\\
&\leq \tfrac{1}{|x|^{N+2m}} [L(m)|x|^{N-2m}(z - z^\lambda) + f(|x|^{N-2m}w, |x|^{N-2m}z^\lambda)(1 - (\tfrac{w^\lambda}{w})^{\frac{N+2m}{N-2m}})]\\
&\leq \tfrac{1}{|x|^{N+2m}} [L(m)|x|^{N-2m}(z - z^\lambda) + f(|x|^{N-2m}w, |x|^{N-2m}z)(1 - (\tfrac{w^\lambda}{w})^{\frac{N+2m}{N-2m}})]\\
&\leq \tfrac{C_\lambda}{|x|^{4m}}[(z - z^\lambda) + (w - w^\lambda)].
\end{aligned}$$

Proceeding the same arguments as in Lemma 3.2, we show that the integral inequality we need is true. The rest of the proof is the same as in Theorem 3.1 with necessary modifications.

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