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Elliptic Obstacle Problems With Natural Growth on the Gradient and Singular Nonlinear Terms*

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Abstract

Given a bounded, open set Ω in $\mathbb{R}^N$ $(N \geq 3)$, $\psi \in W^{1,p}(\Omega)$ $(p > N)$ such that $\psi^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and a suitable strictly positive (see (1.4)) function $a \in L^q(\Omega)$ with $q > N/2$, we prove the existence of positive solution $w \in H_0^1(\Omega)$ of some variational inequality with a singular nonlinearity whose typical model is

$$\left.\begin{array}{r} w(x) \geq \psi(x) \text{ a.e. } x \in \Omega \\ \dfrac{|\nabla w|^2}{w} \in L^1_{\text{loc}}(\Omega),\ \dfrac{|\nabla w|^2}{w}(w - \psi^+) \in L^1(\Omega) \\ \displaystyle\int_\Omega \nabla w \nabla (v - w) + \int_\Omega \frac{|\nabla w|^2}{w}(v - w) \geq \int_\Omega a(x)(v - w),\ \forall v \in K_1. \end{array}\right\}$$

where the set of test functions K_1 consists of all functions $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $v(x) \geq \psi(x)$ a.e. $x \in \Omega$ and supp $(v - \psi^+) \subset\subset \Omega$. Bigger classes of test functions are also studied. We also recover the case in which the variational inequality reduces to an equation.

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1 Introduction

We study the existence of bounded and positive solutions of obstacle problems involving the quasilinear differential operator

$$Aw = -\Delta w + g(w)|\nabla w|^2$$

in an open and bounded subset Ω in $\mathbb{R}^{\mathbb{N}}$ with $N \geq 3$ and g a real continuous function. One of the main difficulties in proving the existence of solution stems from the quadratic growth in ∇w. Indeed, this implies that A does not map the Sobolev space $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$.

The obstacle problem for quadratic nonlinear operators appears, for instance, in stochastic control problems where simultaneously a continuous and an impulse control are considered (see [3]).

In [2] and [9], the authors consider more general operators than A. Specifically, the Laplace operator is replaced by a Leray-Lions operator and a general operator $g(x, w, \nabla w)$ substitutes $g(w)|\nabla w|^2$. In the particular case of the operator A, their results can be formulated in the following way. Consider a function $a \in L^1(\Omega)$ and assume that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying the sign condition

$$g(s)s \geq 0,$$

for every $s \in \mathbb{R}$. Under these conditions, it is proved in [2] that if $\psi : \Omega \longrightarrow [-\infty, +\infty)$ is a measurable function such that the convex set

$$K \equiv \{v \in H_0^1(\Omega) \cap L^\infty(\Omega) :\ v(x) \geq \psi(x) \text{ a.e. } x \in \Omega\} \not\equiv \emptyset \tag{1.1}$$

then there exists a solution $w \in H_0^1(\Omega)$ of the obstacle problem

$$\left.\begin{array}{r} w(x) \geq \psi(x) \text{ a.e. } x \in \Omega, \quad g(w)|\nabla w|^2,\ g(w)|\nabla w|^2 w \in L^1(\Omega) \\ \displaystyle\int_\Omega \nabla w \nabla (v-w) + \int_\Omega g(w)|\nabla w|^2 (v-w) \geq \int_\Omega a(x)(v-w) \quad \forall v \in K \end{array}\right\} \tag{1.2}$$

On the other hand, in [9] it is also proved that if the obstacle $\psi \in L^\infty(\Omega)$ satisfies

$$\{v \in H_0^1(\Omega) \,/\, v(x) \geq \psi(x) \text{ a.e. } x \in \Omega\} \neq \emptyset,$$

then there exists a solution $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.2). If, in addition to the boundedness of the obstacle, $\psi \in W^{1,p}(\Omega)$ and $a \in L^q(\Omega)$ with $p > N$ and $q > N/2$, then the solution w is locally Hölder continuous in Ω.

More existence results are given in [4, 5]. In this case, the term $a_0(x)w$ is added to the operator A, where $a_0(x)$ is bounded from below and above by positive constants.

To the best knowledge of the authors, the case of nonlinearities g having a singularity at $s = 0$ has not been treated in the literature. Our purpose is to study the existence of bounded and positive solutions w provided that the continuous function $g : (0, +\infty) \longrightarrow [0, +\infty)$ satisfies that

$$\limsup_{s \to 0} s g(s) < +\infty \tag{1.3}$$

and the function a is strictly positive in the sense that

$$\text{ess inf}\,\{a(x) \,/\, x \in \omega\} > 0, \quad \forall \omega \subset\subset \Omega. \tag{1.4}$$

Thus, this paper can be considered as an extension of the previous results in [1] for the boundary value problem associated to the operator A with zero Dirichlet boundary conditions. Roughly speaking, condition (1.3) implies that the term $g(w)|\nabla w|^2$ could blow-up as $w(x)$ tends to zero. This is the reason why the techniques in the previous papers does not work in our setting. Even more, the convex set K of test functions has to be modified in order to guarantee that the term $\int_\Omega g(w)|\nabla w|^2(v-w)$ is well-defined. To do this, let us take the convex set

$$K_1 \quad \equiv \quad \left\{ v \in H_0^1(\Omega) \cap L^\infty(\Omega) : \begin{array}{l} v(x) \geq \psi(x) \text{ a.e. } x \in \Omega \\ \text{supp } (v - \psi^+) \subset\subset \Omega \end{array} \right\}.$$

Observe that if $\psi : \Omega \longrightarrow \mathbb{R}$ is a measurable function such that $\psi^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $K_1 \neq \emptyset$. Hence, we are considering the existence of $w \in H_0^1(\Omega)$ satisfying

$$\left.\begin{array}{r} w(x) \geq \psi(x) \text{ a.e. } x \in \Omega \\ g(w)|\nabla w|^2 \in L^1_{\text{loc}}(\Omega),\ g(w)|\nabla w|^2(w - \psi^+) \in L^1(\Omega) \\ \displaystyle\int_\Omega \nabla w \nabla (v-w) + \int_\Omega g(w)|\nabla w|^2 (v-w) \geq \int_\Omega a(x)(v-w),\ \forall v \in K_1. \end{array}\right\} \tag{1.5}$$

To solve this problem, we introduce a sequence of continuous functions g in $\mathbb{R}$ which approximate g in $(0, +\infty)$ and the sequence of approximated operators $A_n w = -\Delta w +$

$g_n(w)n|\nabla w|^2/(n + |\nabla w|^2)$ for $n \in \mathbb{N}$. The solutions $w_n \in H_0^1(\Omega)$ of the variational inequalities associated to them are given by the Leray-Lions theorem [8, Téorème 8.2]. Then, one of the key points is to prove that w_n is away from zero at every compactly embedded subset Ω_0 in Ω (see Proposition 3.2 below). In order to do accomplish this, we need the continuity of w_n, and thus we impose the natural assumptions to have locally Hölder solutions w_n, i.e. we suppose that $a \in L^q(\Omega)$ for some $q > N/2$ and that $\psi \in W^{1,p}(\Omega)$, with[1] $p > N$. Finally, passing to the limit $n \to \infty$, we obtain the existence of solution of (1.5). In conclusion, we prove the following theorem.

Theorem 1.1 *Assume that $\psi \in W^{1,p}(\Omega)$ with $p > N$ and $\psi^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Let $a \in L^q(\Omega)$ with $q > N/2$ and satisfying (1.4). Suppose also that $g : (0, +\infty) \longrightarrow [0, +\infty)$ is a continuous function verifying (1.3). Then, there exists $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ that solves (1.5). Moreover, if $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ then w also solves*

$$\left.\begin{aligned} w(x) \geq \psi(x) \ a.e.\ x \in \Omega \\ g(w)|\nabla w|^2 \in L^1_{loc}(\Omega),\ g(w)|\nabla w|^2(w-\psi) \in L^1(\Omega) \\ \int_\Omega \nabla w \nabla (v-w) + \int_\Omega g(w)|\nabla w|^2 (v-w) \geq \int_\Omega a(x)(v-w),\ \forall v \in K_2 \end{aligned}\right\} \tag{1.6}$$

where the nonempty set K_2 is defined as

$$K_2 \equiv \left\{ v \in H_0^1(\Omega) \cap L^\infty(\Omega) : \begin{array}{l} v(x) \geq \psi(x) \ a.e.\ x \in \Omega \\ supp\ (v-\psi) \subset\subset \Omega \end{array} \right\}.$$

Furthermore, if $\psi \in W_0^{1,p}(\Omega)$ with $p > N$ and supp $\psi \subset\subset \Omega$, then w also solves the problem (1.2) and we extend to singular nonlinearities g the results of [2] (for the case $\psi \in W_0^{1,p}(\Omega)$, $a \in L^q(\Omega)$ with $p > N$, $q > N/2$ and supp $\psi \subset\subset \Omega$). Specifically, we prove the following theorem.

Theorem 1.2 *Assume that $\psi \in W^{1,p}(\Omega)$ with $p > N$ and supp $\psi^+ \subset\subset \Omega$. For $q > N/2$, let $a \in L^q(\Omega)$ be a function satisfying (1.4). Suppose also that $g : (0, +\infty) \longrightarrow [0, +\infty)$ is a continuous function verifying (1.3). Then the solution w given by Theorem 1.1 is such that $g(w)|\nabla w|^2 \in L^1(\Omega)$. In particular, if supp $\psi \subset\subset \Omega$, then w solves (1.2).*

As a by-product, we improve a previous result of [1], where the existence of solution of the boundary value problem

$$\left.\begin{array}{rcll} -\Delta w + g(w)|\nabla w|^2 &=& a(x) & x \in \Omega \\ w &=& 0 & x \in \partial\Omega \end{array}\right\} \tag{1.7}$$

is studied in the particular case when $a \in L^\infty(\Omega)$.

[1]We remark explicitly that since we are not assuming the smoothness of the boundary $\partial\Omega$, we can not assure that $\psi \in L^\infty(\Omega)$ from the inequality $p > N$.

Theorem 1.3 *Assume that $a \in L^q(\Omega)$ with $q > N/2$ and satisfies (1.4). Suppose also that $g : (0, +\infty) \longrightarrow [0, +\infty)$ is a continuous function verifying (1.3). Then, there exists $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution of (1.7) in the sense*

$$\int_\Omega \nabla w \nabla \varphi + \int_\Omega g(w)|\nabla w|^2 \varphi = \int_\Omega a(x)\varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

The paper is organized as follows: in the second section we recall the proof of the C^α-regularity of solutions for some linear obstacle problems that will be useful in the third section, in which we prove the existence results.

2 A regularity result

In this section, we include, for the convenience of the reader, the proof of the well-known C^α-regularity of solutions for some linear obstacle problems (see [6] and references therein).

Theorem 2.1 *Let ψ be a function in $W^{1,p}(\Omega)$ with $p > N$ and $\psi^+ \in H_0^1(\Omega)$. Suppose also that $a_1 \in L^q(\Omega)$, where $q > \frac{N}{2}$. If $\alpha \in (0, \min\{1 - N/p, 2 - N/q\})$, then every solution $w \in H_0^1(\Omega)$ of*

$$\begin{gathered}\int_\Omega \nabla w \nabla (v - w) \geq \int_\Omega a_1(x)(v - w), \quad \forall v \in H_0^1(\Omega) : v(x) \geq \psi(x) \text{ a.e. } x \in \Omega \\ w(x) \geq \psi(x) \text{ a.e. } x \in \Omega,\end{gathered} \tag{2.1}$$

is locally Hölder continuous with exponent α in Ω.

Remark 2.1 In the case when $\partial\Omega$ is smooth (for instance, Lipschitz) then it is possible to show that $w \in C^\alpha(\overline{\Omega})$.

Proof. Fix $0 < \alpha < \min\{1 - N/p, 2 - N/q\}$. Let Ω_0 be an open subset with smooth boundary which is compactly embedded in Ω. We are going to prove that every solution $w \in H_0^1(\Omega)$ of (2.1) belongs to $C^\alpha(\overline{\Omega_0})$. To do that, if we denote by $B(x_0, r)$ the ball in $\mathbb{R}^N$ of centre x_0 and radius $r > 0$, it suffices to show for some positive constants C_1 and C_2 and $\sigma := \min\{N - 2N/p, N + 2 - 2N/q\}$ that

$$\int_{B(x_0,\rho)} |\nabla w|^2 \leq C_1 \left(\frac{\rho}{r}\right)^N \int_{B(x_0,r)} |\nabla w|^2 + C_2 r^\sigma, \tag{2.2}$$

for every $x_0 \in \Omega_0$ and $0 < \rho < r < \operatorname{dist}(x_0, \partial\Omega)$.

Indeed, by [7, A useful lemma, p. 44], this implies that for a new positive constant C_3,

$$\int_{B(x_0,\rho)} |\nabla w|^2 \leq C_3 \left(\frac{\rho}{r}\right)^N \int_{B(x_0,r)} |\nabla w|^2 + C_2 \rho^\sigma,$$

for every $x_0 \in \Omega_0$ and $0 < \rho < r < \operatorname{dist}(x_0, \partial\Omega)$. Therefore, taking $\lambda = N + 2\alpha$, we have

$$\rho^{-(\lambda-2)} \int_{\Omega_0 \cap B(x_0,\rho)} |\nabla w|^2 \leq C_3 \frac{\rho^{N+2-\lambda}}{r^N} \int_{B(x_0,r)} |\nabla w|^2 + C_2 \rho^{\sigma+2-\lambda}$$

and using that $\sigma + 2 > \lambda$, we deduce that ∇w belongs to the Morrey space $L^{2,\lambda-2}(\Omega_0)$ of all functions $u \in L^2(\Omega_0)$ such that

$$\sup_{\substack{x\in\Omega_0 \\ 0<\rho<\operatorname{diam}\,\Omega_0}} \rho^{-(\lambda-2)} \int_{\Omega_0 \cap B(x,\rho)} |u|^2 < \infty.$$

In particular (see [7, Conclusion 5, p. 44]), w is in the Campanato space $\mathcal{L}^{2,\lambda}(\Omega_0)$ given by

$$\mathcal{L}^{2,\lambda}(\Omega_0) = \left\{ u \in L^2(\Omega_0) : \sup_{\substack{x\in\Omega_0 \\ 0<\rho<\operatorname{diam}\,\Omega_0}} \rho^{-\lambda} \int_{\Omega_0 \cap B(x,\rho)} \left| u - \fint_{\Omega_0 \cap B(x,\rho)} u \right|^2 < \infty \right\},$$

where $\fint_{\Omega_0 \cap B(x,\rho)} u = [1/\operatorname{meas}(\Omega_0 \cap B(x,\rho))] \int_{\Omega_0 \cap B(x,\rho)} u$.

Since $N < \lambda < N+2$ and $\partial\Omega_0$ is smooth (for instance Lipschitz), by Campanato theorem [7, Theorem 3.1] we know that $\mathcal{L}^{2,\lambda}(\Omega_0)$ is isomorphic to $C^{(\lambda-N)/2}(\overline{\Omega_0})$ and thus $w \in C^{(\lambda-N)/2}(\overline{\Omega_0})$. The proof will be concluded by observing that $\alpha = \frac{\lambda-N}{2}$.

To prove (2.2), we fix $x_0 \in \Omega_0$ and $0 < r < \operatorname{dist}(x_0, \partial\Omega)$, and we take $\overline{w}$, $\overline{\overline{w}} \in H^1(B(x_0,r))$ satisfying

$$\left.\begin{array}{rcll} -\Delta(\overline{w} - \psi) & = & 0 & x \in B(x_0,r) \\ \overline{w} & = & w & x \in \partial B(x_0,r) \end{array}\right\} \tag{2.3}$$

and

$$\left.\begin{array}{rcll} -\Delta\overline{\overline{w}} & = & 0 & x \in B(x_0,r) \\ \overline{\overline{w}} & = & \overline{w} & x \in \partial B(x_0,r). \end{array}\right\} \tag{2.4}$$

For every $\rho \in (0,r)$ we have

$$\begin{aligned} \int_{B(x_0,\rho)} |\nabla w|^2 &\leq 2\int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 + 2\int_{B(x_0,\rho)} |\nabla\overline{w}|^2 \\ &\leq 2\int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 \\ &\quad +4\int_{B(x_0,r)} |\nabla(\overline{w}-\overline{\overline{w}})|^2 + 4\int_{B(x_0,\rho)} |\nabla\overline{\overline{w}}|^2. \end{aligned}$$

Using the weak Harnack inequality (see [7]) for the derivatives of the harmonic function $\overline{\overline{w}}$, we have

$$\int_{B(x_0,\rho)} |\nabla\overline{\overline{w}}|^2 \leq c\left(\frac{\rho}{r}\right)^N \int_{B(x_0,r)} |\nabla\overline{\overline{w}}|^2,$$

and consequently

$$\int_{B(x_0,\rho)} |\nabla w|^2 \leq 2\int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 + 4\int_{B(x_0,r)} |\nabla(\overline{w}-\overline{\overline{w}})|^2 + 4c\left(\frac{\rho}{r}\right)^N \int_{B(x_0,r)} |\nabla\overline{\overline{w}}|^2. \tag{2.5}$$

The three integrals on the right hand side of this inequality will be studied in the following three steps respectively. We denote by $C_1, C_2, C_3, ...$ positive constants independent from w.

Step 1: Study of $\int_{B(x_0,r)} |\nabla(w-\overline{w})|^2$.

We observe that $w-\overline{w} \in H_0^1(B(x_0,r))$ and the maximum principle assures that $\overline{w} \geq \psi$ in $B(x_0,r)$.

In particular,

$$v = \begin{cases} \overline{w} & x \in B(x_0,r), \\ w & x \in \Omega \setminus B(x_0,r) \end{cases}$$

satisfies that $v - w \in H_0^1(\Omega)$ and $v(x) \geq \psi(x)$. Now we can take $v \in H_0^1(\Omega)$ as test function in (2.1) to obtain

$$\int_{B(x_0,r)} \nabla w \nabla(\overline{w}-w) \geq \int_{B(x_0,r)} a_1(x)(\overline{w}-w).$$

Thus,

$$\begin{aligned} \int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 &= \int_{B(x_0,r)} \nabla w\nabla(w-\overline{w}) - \int_{B(x_0,r)} \nabla\overline{w}\nabla(w-\overline{w}) \\ &\leq \int_{B(x_0,r)} a_1(x)(w-\overline{w}) - \int_{B(x_0,r)} \nabla(\overline{w}-\psi)\nabla(w-\overline{w}) \\ &\quad - \int_{B(x_0,r)} \nabla\psi\nabla(w-\overline{w}) \\ &= \int_{B(x_0,r)} a_1(x)(w-\overline{w}) - \int_{B(x_0,r)} \nabla\psi\nabla(w-\overline{w}), \end{aligned}$$

where the last equality is consequence of taking $w-\overline{w}$ as test function in (2.3). Hölder and Sobolev inequalities (with $\mathcal{S}$ the best constant in the Sobolev imbedding) imply that

$$\begin{aligned} \int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 &\leq \|w-\overline{w}\|_{L^{2^*}(B(x_0,r))}\|a_1\|_{L^q(B(x_0,r))}|B(x_0,r)|^{\frac{1}{2}+\frac{1}{N}-\frac{1}{q}} \\ &\quad + \|\nabla(w-\overline{w})\|_{L^2(B(x_0,r))}\|\nabla\psi\|_{L^p(\Omega)}|B(x_0,r)|^{\frac{1}{2}-\frac{1}{p}} \\ &\leq \mathcal{S}\|\nabla(w-\overline{w})\|_{L^2(B(x_0,r))}\|a_1\|_{L^q(B(x_0,r))}|B(x_0,r)|^{\frac{1}{2}+\frac{1}{N}-\frac{1}{q}} \\ &\quad + \|\nabla(w-\overline{w})\|_{L^2(B(x_0,r))}\|\nabla\psi\|_{L^p(\Omega)}|B(x_0,r)|^{\frac{1}{2}-\frac{1}{p}} \\ &\leq \|\nabla(w-\overline{w})\|_{L^2(B(x_0,r))}C_3 r^{\min\{\frac{N}{2}-\frac{N}{p},\frac{N}{2}+1-\frac{N}{q}\}}, \end{aligned}$$

for $r > 0$ small enough. Applying the Young inequality, we deduce that

$$\int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 \quad \leq \quad \frac{1}{2}\int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 + \frac{C_3^2}{2} r^\sigma,$$

i.e.,

$$\int_{B(x_0,r)} |\nabla(w-\overline{w})|^2 \quad \leq \quad C_4 r^\sigma. \tag{2.6}$$

Step 2: Study of $\displaystyle\int_{B(x_0,r)} |\nabla(\overline{\overline{w}}-\overline{w})|^2$.

Taking $\overline{\overline{w}}-\overline{w}$ as test function in the problems (2.3) and (2.4), we get

$$\begin{aligned}
\int_{B(x_0,r)} |\nabla(\overline{\overline{w}}-\overline{w})|^2 &= \int_{B(x_0,r)} \nabla\overline{\overline{w}}\nabla(\overline{\overline{w}}-\overline{w}) - \int_{B(x_0,r)} \nabla\overline{w}\nabla(\overline{\overline{w}}-\overline{w}) \\
&= \int_{B(x_0,r)} \nabla\overline{\overline{w}}\nabla(\overline{\overline{w}}-\overline{w}) - \int_{B(x_0,r)} \nabla(\overline{w}-\psi)\nabla(\overline{\overline{w}}-\overline{w}) \\
&\quad - \int_{B(x_0,r)} \nabla\psi\nabla(\overline{\overline{w}}-\overline{w}) \\
&= -\int_{B(x_0,r)} \nabla\psi\nabla(\overline{\overline{w}}-\overline{w}).
\end{aligned}$$

Using again the Hölder and Young inequalities,

$$\begin{aligned}
\int_{B(x_0,r)} |\nabla(\overline{\overline{w}}-\overline{w})|^2 &\leq \|\nabla(\overline{\overline{w}}-\overline{w})\|_{L^2(B(x_0,r))}\|\nabla\psi\|_{L^p(\Omega)}|B(x_0,r)|^{\frac{1}{2}-\frac{1}{p}} \\
&= \|\nabla(\overline{\overline{w}}-\overline{w})\|_{L^2(B(x_0,r))} C_2 r^{\frac{N}{2}-\frac{N}{p}} \\
&\leq \frac{1}{2}\int_{B(x_0,r)} |\nabla(\overline{\overline{w}}-\overline{w})|^2 + \frac{C_2^2}{2} r^{N-\frac{2N}{p}},
\end{aligned}$$

i.e.,

$$\int_{B(x_0,r)} |\nabla(\overline{\overline{w}}-\overline{w})|^2 \quad \leq \quad C_5 r^\sigma. \tag{2.7}$$

Step 3: Study of $\displaystyle\int_{B(x_0,r)} |\nabla\overline{\overline{w}}|^2$.

Choosing $\overline{\overline{w}}-\overline{w}$ as test function in (2.4), we obtain

$$\int_{B(x_0,r)} |\nabla\overline{\overline{w}}|^2 = \int_{B(x_0,r)} \nabla\overline{\overline{w}}\nabla\overline{w}.$$

Hence,

$$\begin{aligned}
0 &\leq \int_{B(x_0,r)} |\nabla(\overline{\overline{w}}-\overline{w})|^2 = \int_{B(x_0,r)} \nabla\overline{\overline{w}}\nabla(\overline{\overline{w}}-\overline{w}) - \int_{B(x_0,r)} \nabla\overline{w}\nabla(\overline{\overline{w}}-\overline{w}) \\
&= -\int_{B(x_0,r)} \nabla\overline{\overline{w}}\nabla\overline{w} + \int_{B(x_0,r)} |\nabla\overline{w}|^2 = -\int_{B(x_0,r)} |\nabla\overline{\overline{w}}|^2 + \int_{B(x_0,r)} |\nabla\overline{w}|^2,
\end{aligned}$$

or equivalently $\int_{B(x_0,r)} |\nabla \overline{\overline{w}}|^2 \leq \int_{B(x_0,r)} |\nabla \overline{w}|^2$.

On the other hand, taking $\overline{w} - w$ as test function in (2.3), we have

$$\int_{B(x_0,r)} |\nabla \overline{w}|^2 = \int_{B(x_0,r)} \nabla \overline{w} \nabla w + \int_{B(x_0,r)} \nabla \psi \nabla \overline{w} - \int_{B(x_0,r)} \nabla \psi \nabla w,$$

and applying the Young inequality in each term of the right hand side, we get

$$\int_{B(x_0,r)} |\nabla \overline{w}|^2 \leq \frac{1}{2} \int_{B(x_0,r)} |\nabla \overline{w}|^2 + \frac{3}{2} \int_{B(x_0,r)} |\nabla \psi|^2 + \frac{3}{2} \int_{B(x_0,r)} |\nabla w|^2,$$

that is,

$$\begin{aligned} \int_{B(x_0,r)} |\nabla \overline{w}|^2 &\leq 3 \int_{B(x_0,r)} |\nabla \psi|^2 + 3 \int_{B(x_0,r)} |\nabla w|^2 \\ &\leq \int_{B(x_0,r)} |\nabla w|^2 + 3 \|\nabla \psi\|^2_{L^p(\Omega)} \left| \frac{\omega_N}{N} r^N \right|^{1-\frac{2}{p}} \\ &= 3 \int_{B(x_0,r)} |\nabla w|^2 + C_6 r^{N-\frac{2N}{p}} \\ &\leq 3 \int_{B(x_0,r)} |\nabla w|^2 + C_6 r^\sigma. \end{aligned}$$

Consequently,

$$\int_{B(x_0,r)} |\nabla \overline{\overline{w}}|^2 \leq 3 \int_{B(x_0,r)} |\nabla w|^2 + C_6 r^\sigma. \tag{2.8}$$

Step 4: Conclusion. Using (2.6), (2.7) and (2.8) in (2.5) we derive that

$$\begin{aligned} \int_{B(x_0,\rho)} |\nabla w|^2 &\leq 2C_4 r^\sigma + 4C_5 r^\sigma + 4c \left(\frac{\rho}{r}\right)^N \left(3 \int_{B(x_0,r)} |\nabla w|^2 + C_6 r^\sigma \right) \\ &\leq C_7 \left(\frac{\rho}{r}\right)^N \int_{B(x_0,r)} |\nabla w|^2 + C_8 r^\sigma. \end{aligned}$$

which is the required estimate (2.2). □

3 Proof of the main results

We recall that $\mathcal{S}$ is denoting the Sobolev constant,

$$\mathcal{S} = \inf_{u \in H_0^1(\Omega) - \{0\}} \frac{\|u\|^2_{H_0^1(\Omega)}}{\|u\|^2_{L^{2^*}(\Omega)}},$$

where, as usual, $2^* = 2N/(N-2)$ and $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$. For $\beta = 2^* - 1 - 1/q$, we consider the constant

$$C_q = (\mathcal{S}^{-1})^{2^*} \|a\|_{L^q(\Omega)}^{2^*} |\Omega|^{\beta-1} 2^{2^* \frac{\beta}{\beta-1}}. \tag{3.1}$$

We consider the following approximated obstacle problems

$$\left.\begin{array}{r} w \in H_0^1(\Omega), \quad w(x) \geq \psi(x) \text{ a.e. } x \in \Omega \\ \displaystyle\int_\Omega \nabla w \nabla (v-w) + \int_\Omega g_n(w) \frac{|\nabla w|^2}{1+\frac{1}{n}|\nabla w|^2}(v-w) \geq \int_\Omega a(x)(v-w) \\ \forall v \in H_0^1(\Omega) \,:\, v(x) \geq \psi(x) \text{ a.e. } x \in \Omega \end{array}\right\} \tag{3.2}$$

where g_n is the continuous function given by

$$g_n(s) := \begin{cases} g(s), & s \geq \dfrac{1}{n}, \\ n^2 s^2 g(s), & 0 < s \leq \dfrac{1}{n}, \\ 0, & s = 0, \\ -g_n(-s), & s < 0. \end{cases}$$

We remark that g_n satisfies that

$$g_n(s) \overset{n\to+\infty}{\longrightarrow} g(s), \quad \forall s > 0$$

$$g_n(s) \leq g(s), \quad \forall s > 0.$$

$$g_n(s)s \geq 0, \quad \forall s \in \mathbb{R}.$$

We prove the existence of solution for the approximated obstacle problems and their basic properties.

Proposition 3.1 *Let us assume the hypotheses of Theorem 1.1. Then (3.2) admits a non-negative solution $w_n \in H_0^1(\Omega)$ satisfying*

1. *there exists $\tilde{C}_q > 0$ such that*

$$\|w_n\|_{H_0^1(\Omega)} \leq \tilde{C}_q, \ \forall n \in \mathbb{N}.$$

2. *$w_n \in L^\infty(\Omega)$ and $\|w_n\|_{L^\infty(\Omega)} \leq C_q$.*

3. *$w_n \in C^\alpha_{\mathrm{loc}}(\Omega)$, for every $0 < \alpha < \min\{1 - N/p, 2 - N/q\}$.*

4. *$w_n(x) > 0$ a.e. $x \in \Omega$.*

Proof. The existence of w_n may be deduced from the classical results in [8]. Consider $\widetilde{K} = \{w \in H_0^1(\Omega) : w \geq \psi\}$ and the operator $A : \widetilde{K} \longrightarrow H^{-1}(\Omega)$ given by

$$A(w)(v) = \int_\Omega \nabla w \nabla v + \int_\Omega g_n(w) \frac{|\nabla w|^2}{1 + \frac{1}{n}|\nabla w|^2} v, \quad \forall v \in H_0^1(\Omega), \, \forall w \in \widetilde{K}.$$

Note that $\widetilde{K}$ is a nonempty closed and convex set and A is pseudo-monotone. In addition, A is coercive in the sense that there exists $v_0 \in \widetilde{K}$ such that

$$\lim_{\|v\|_{H_0^1(\Omega)} \to \infty} \frac{A(v)(v - v_0)}{\|v\|_{H_0^1(\Omega)}} = +\infty.$$

Thus, by [8, Téorème 8.2] we deduce the existence of w_n.

Moreover, taking $v = w_n^+$ as test function in (3.2), we have

$$\int_\Omega \nabla w_n \nabla (w_n^+ - w_n) + \int_\Omega g_n(w_n) \frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2} (w_n^+ - w_n) \geq \int_\Omega a(x)(w_n^+ - w_n),$$

or equivalently

$$\int_\Omega |\nabla w_n^-|^2 + \int_\Omega g_n(w_n) \frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2} w_n^- \leq \int_\Omega a(x) w_n^-,$$

where $w_n^- = \min\{w_n, 0\}$. Thus, using the fact that the functions $g_n(s)s$ and $a(x)$ are non-negative, we get

$$\int_\Omega |\nabla w_n^-|^2 \leq \int_\Omega a(x) w_n^- \leq 0,$$

i.e. $w_n^- \equiv 0$ and hence $w_n \geq 0$.

Now we deal with the proof of case 1. First, observe that we can use $v = \psi^+$ as test function in (3.2) to obtain

$$\int_\Omega \nabla w_n \nabla (\psi^+ - w_n) + \int_\Omega g_n(w_n) \frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2} (\psi^+ - w_n) \geq \int_\Omega a(x)(\psi^+ - w_n).$$

Since $w_n \geq \psi^+$, the term $\displaystyle\int_\Omega g_n(w_n) \frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2} (\psi^+ - w_n)$ is non-positive and we deduce that

$$\begin{aligned}
\|w_n\|_{H_0^1(\Omega)}^2 &= \int_\Omega |\nabla w_n|^2 = \int_\Omega \nabla w_n \nabla (w_n - \psi^+) + \int_\Omega \nabla w_n \nabla \psi^+ \\
&\leq \int_\Omega a(x) w_n - \int_\Omega a(x) \psi^+ + \int_\Omega \nabla w_n \nabla \psi^+ \\
&\leq \int_\Omega a(x) w_n + \int_\Omega \nabla w_n \nabla \psi^+ \\
&\leq \|a\|_{L^q(\Omega)} \|w_n\|_{L^{q'}(\Omega)} + \|w_n\|_{H_0^1(\Omega)} \|\nabla \psi\|_{L^2(\Omega)}.
\end{aligned}$$

Using the Hölder inequality and the Sobolev embedding, we obtain

$$\|w_n\|^2_{H_0^1(\Omega)} \quad \leq \quad \left[\|a\|_{L^q(\Omega)}|\Omega|^{\widetilde{\beta}}\mathcal{S}^{-\frac{1}{2}} + \|\nabla\psi\|_{L^2(\Omega)}\right]\|w_n\|_{H_0^1(\Omega)},$$

with $\widetilde{\beta} = \frac{1}{q'} - \frac{1}{2^*}$, that is

$$\|w_n\|_{H_0^1(\Omega)} \leq \|a\|_{L^q(\Omega)}|\Omega|^{\widetilde{\beta}}\mathcal{S}^{-\frac{1}{2}} + \|\nabla\psi\|_{L^2(\Omega)} \equiv \widetilde{C}_q,$$

and the proof of case 1 is completed.

In order to prove case 2, we consider the real functions

$$T_k(s) = \begin{cases} -k & \text{if } s \leq -k, \\ s & \text{if } -k < s < k, \\ k & \text{if } k \leq s, \end{cases} \tag{3.3}$$

and $G_k(s) = s - T_k(s)$, for every $s \in \mathbb{R}$. For $k > 0$ large enough, we take $v = T_k(w_n)$ as test function in (3.2) to obtain

$$\int_\Omega \nabla w_n \nabla[-G_k(w_n)] + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}[-G_k(w_n)] \geq \int_\Omega a(x)[-G_k(w_n)].$$

In particular, taking into account that $g_n(w_n)G_k(w_n) \geq 0$, we have

$$\int_\Omega |\nabla G_k(w_n)|^2 \leq \int_\Omega a(x)G_k(w_n).$$

Using this inequality and the Stampacchia method [10], we deduce that

$$\|w_n\|_{L^\infty(\Omega)} \leq C_q$$

and case 2 is proved.

Choosing

$$a_1(x) = a(x) - g_n(w_n(x))\frac{|\nabla w_n(x)|^2}{1+\frac{1}{n}|\nabla w_n(x)|^2},$$

the solution w_n verifies (2.1) and thus, by Theorem 2.1 we conclude case 3.

To prove case 4, we use $w_n \in L^\infty(\Omega)$ to choose $k_n > 0$ such that

$$g_n(w_n(x)) \leq k_n w_n(x), \quad \forall x \in \Omega.$$

Let $\varphi \in H_0^1(\Omega)$ be a non-negative function. Taking $v = w_n + \varphi$ as test function in (3.2), we have

$$\int_\Omega \nabla w_n \nabla\varphi + \int_\Omega k_n n w_n \varphi \geq \int_\Omega \nabla w_n \nabla\varphi + \int_\Omega g_n(w_n)\frac{|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}\varphi \geq \int_\Omega a(x)\varphi.$$

In particular, the solution w_n satisfies (in the weak sense)

$$0 \lneqq a(x) \leq -\Delta w_n + k_n n w_n, \quad x \in \Omega,$$

$$w_n \geq 0, \quad x \in \partial\Omega.$$

Then the strong maximum principle assures that $w_n(x) > 0$, for every $x \in \Omega$ and the proof of case 4 is also concluded. □

The next proposition is the keystone to pass to the limit in the approximated obstacle problems.

Proposition 3.2 *Suppose that the hypotheses of Theorem 1.1 are fulfilled and let Ω_0 be a compactly embedded subset of Ω. Then there exists $c_{\Omega_0} > 0$ independent of n such that every solution w_n given by Proposition 3.1 satisfies*

$$w_n \geq c_{\Omega_0}, \quad \forall x \in \Omega_0.$$

Proof. Since g is a continuous function satisfying (1.3), there exists $\Lambda > 1$ such that

$$g(s) \leq \frac{\Lambda}{s}, \quad \forall s \in (0, C_q]. \tag{3.4}$$

The result follows from [1, Proposition 2.3], once we prove that w_n is a supersolution for the quasilinear problem

$$\begin{array}{c} -\Delta w + \frac{\Lambda}{w}|\nabla w|^2 = a(x), \quad x \in \Omega \\ w \in H_0^1(\Omega). \end{array}$$

In order to do that, let $\varphi \in H_0^1(\Omega)$ be a non-negative function. We use $v = w_n + \varphi$ as test function in (3.2) and, taking into account that $g_n(s) \leq g(s) \leq \Lambda/s$ for every $s \in (0, C_q]$, we deduce that $w_n \in H_0^1(\Omega) \cap C(\Omega)$ satisfies in the weak sense

$$0 \leq a(x) \leq -\Delta w_n + g_n(w_n)\frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2} \leq -\Delta w_n + \frac{\Lambda}{w_n}|\nabla w_n|^2, \quad x \in \Omega.$$

□

Proof of Theorem 1.1. We begin by showing the existence of solution of (1.5). The proof will be made in two steps. First we obtain that, up to a subsequence, w_n has a limit w in $H^1_{\text{loc}}(\Omega)$ and then we can pass to the limit in the approximated obstacle problems to prove that w is a solution of (1.5) (Step 2).

Step 1: Strongly convergence to w in $H^1(\Omega_0)$ for every compactly embedded open Ω_0 in Ω. Thanks to Proposition 3.1, passing to a subsequence, we can assume that the sequence $\{w_n\}$ is weakly converging to some w in $H_0^1(\Omega)$, strongly converging to w in $L^2(\Omega)$ and almost everywhere converging to $w(x)$ in Ω with $\psi \leq w \in L^\infty(\Omega)$ and $\|w\|_{L^\infty(\Omega)} \leq C_q$. To prove that w_n is strongly convergent to w in $H^1_{\text{loc}}(\Omega)$, it suffices to show that

$$\lim_{n\to+\infty} \int_\Omega |\nabla(w_n - w)^+|^2 \xi = 0, \tag{3.5}$$

and

$$\lim_{n\to+\infty}\int_\Omega |\nabla(w_n-w)^-|^2\xi = 0, \tag{3.6}$$

for every $\xi \in C_0^\infty(\Omega)$ with $\xi \geq 0$.

To verify (3.5), we take $v = w_n - [w_n - w]^+$ as test function in (3.2) to obtain

$$-\int_\Omega \nabla w_n \nabla [w_n-w]^+ - \int_\Omega g_n(w_n)\frac{|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}[w_n-w]^+ \geq -\int_\Omega a(x)[w_n-w]^+.$$

Using that $g_n(w_n) \geq 0$, this implies that

$$\begin{aligned}\int_\Omega |\nabla[w_n-w]^+|^2 &= \int_\Omega \nabla w_n \nabla[w_n-w]^+ - \int_\Omega \nabla w \nabla [w_n-w]^+\\ &\leq \int_\Omega a(x)[w_n-w]^+ - \int_\Omega \nabla w\nabla[w_n-w]^+. \end{aligned}\tag{3.7}$$

Using the almost everywhere convergence of $w_n(x)$ to $w(x)$, it is proved that 0 is the unique accumulation point of the sequence $\{[w_n-w]^+\}$ in the weak topology of $H_0^1(\Omega)$. By this and the boundedness of $\{[w_n-w]^+\}$, we derive the weak convergence of $\{[w_n-w]^+\}$ to 0 in $H_0^1(\Omega)$. Then, we conclude from (3.7) that

$$\lim_{n\to+\infty}\int_\Omega |\nabla\,[w_n-w]^+\,|^2 = 0,$$

i.e., the strong convergence of $(w_n-w)^+$ to zero in $H_0^1(\Omega)$. From this it is easy to show that (3.5) holds.

Now we deal with (3.6). Let $\xi\in C_0^\infty(\Omega)$ be a non-negative function. Choose $\Omega_0 \subset\subset \Omega$ such that supp $\xi\subset\Omega_0$. By Proposition 3.2, there exists $c_0>0$ such that

$$w_n(x)\geq c_0>0,\ \ \forall x\in\Omega_0.$$

Since $g_n(w_n)\leq g(w_n)\leq 1/c_0 := c$, we have

$$g_n(w_n)\frac{|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}\leq c|\nabla w_n|^2 \quad \forall x\in\Omega_0. \tag{3.8}$$

For $\gamma\geq\frac{c^2}{4}$, consider the real function $\varphi_\gamma(s) = se^{\gamma s^2}$ for every $s\in\mathbb{R}$. For $z_n\equiv[w_n-w]^-$, take $v=w_n-\varphi_\gamma(z_n)\xi\geq w_n\geq\psi$ as test function in (3.2) to deduce that

$$\begin{aligned}&-\int_{\Omega_0}\nabla w_n\nabla z_n\,\varphi_\gamma'(z_n)\xi - \int_{\Omega_0}\nabla w_n\nabla\xi\varphi_\gamma(z_n)\\ &-\int_{\Omega_0} g_n(w_n)\frac{|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}\varphi_\gamma(z_n)\xi \geq -\int_{\Omega_0} a(x)\varphi_\gamma(z_n)\xi.\end{aligned}$$

Hence, from (3.8) we obtain

$$\begin{aligned}&\int_{\Omega_0}|\nabla z_n|^2\varphi_\gamma'(z_n)\xi + \int_{\Omega_0} c|\nabla w_n|^2\varphi_\gamma(z_n)\xi\\ &\leq -\int_{\Omega_0}\nabla w\nabla z_n\varphi_\gamma'(z_n)\xi - \int_{\Omega_0}\nabla w_n\nabla\xi\varphi_\gamma(z_n) + \int_{\Omega_0} a\varphi_\gamma(z_n)\xi.\end{aligned}$$

By using this and the fact that $\varphi'_\gamma(s) + c\varphi_\gamma(s) = e^{\gamma s^2}\left[1 + 2\gamma s^2 + sc\right] \geq 1/2$, for all $s < 0$, we get

$$\begin{aligned}\int_\Omega |\nabla(w_n - w)^-|^2\xi &= \int_{\Omega_0} |\nabla z_n|^2 \xi \\ &\leq 2\int_{\Omega_0} |\nabla z_n|^2 \left\{\varphi'_\gamma(z_n) + c\varphi_\gamma(z_n)\right\}\xi \\ &\leq 2\int\limits_{w_n - w<0} c|\nabla w|^2\varphi_\gamma(z_n)\xi - 2c\int\limits_{w_n-w<0} \nabla w_n \nabla w \varphi_\gamma(z_n)\xi \\ &\quad -2\int_{\Omega_0} \nabla w \nabla z_n \varphi'_\gamma(z_n)\xi - 2\int_{\Omega_0} \nabla w_n \nabla \xi \varphi_\gamma(z_n) \\ &\quad +2\int_{\Omega_0} a\varphi_\gamma(z_n)\xi. \end{aligned} \tag{3.9}$$

Then, by Lebesgue dominated convergence theorem, we can observe that every term in the right hand side of (3.9) converges to zero and we conclude the proof of (3.5) and Step 1.

Step 2. We prove, in this step, that w is a solution of (1.5). To make it, choose $v \in K_1$ and take it as test function in (3.2) to obtain

$$\begin{aligned}\int_\Omega \nabla w_n \nabla v &+ \int_\Omega g_n(w_n)\frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2}(v - \psi^+) - \int_\Omega a(x)(v - w_n) \\ &\geq \int_\Omega \left[|\nabla w_n|^2 + \frac{g_n(w_n)|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2}(w_n - \psi^+)\right]. \end{aligned} \tag{3.10}$$

Since w_n is weakly convergent to w in $H_0^1(\Omega)$, the sequence $\int_\Omega \nabla w_n \nabla v$ tends to $\int_\Omega \nabla w \nabla v$. It also implies the strong convergence of w_n to w in $L^2(\Omega)$ and hence the convergence of $\int_\Omega a(x)(v - w_n)$ to $\int_\Omega a(x)(v - w)$. On the other hand, choosing an open subset Ω_0 compactly embedded in Ω, such that supp $(v - \psi^+) \subset \Omega_0$ and using Proposition 3.2, we have $w_n(x) \geq c_0 > 0$ for every $x \in \Omega_0$ for some $c_0 > 0$. Therefore, by using $g_n \leq g$, (3.4) and Step 1, we have

$$g_n(w_n)\frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2}(v-\psi^+) \leq g(w_n)\frac{|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2}|v-\psi^+| \leq \frac{\Lambda}{c_0}h^2_{\Omega_0}\|v-\psi^+\|_{L^\infty(\Omega)},$$

and we can apply the Lebesgue theorem to deduce that

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega \frac{g_n(w_n)|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2}(v - \psi^+) &= \lim_{n\to+\infty}\int_{\Omega_0} \frac{g_n(w_n)|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2}(v - \psi^+) \\ &= \int_{\Omega_0} \lim_{n\to+\infty} \frac{g_n(w_n)|\nabla w_n|^2}{1 + \frac{1}{n}|\nabla w_n|^2}(v - \psi^+) \\ &= \int_{\Omega_0} g(w)|\nabla w|^2(v - \psi^+) \\ &= \int_\Omega g(w)|\nabla w|^2(v - \psi^+) \end{aligned}$$

Consequently, taking limits in (3.10) as n tends to infinity, we conclude from Fatou's lemma that

$$\begin{aligned}\int_\Omega \nabla w \nabla v &+ \int_\Omega g(w)|\nabla w|^2(v-\psi^+) - \int_\Omega a(x)(v-w)\\ &\geq \liminf_{n\to\infty} \int_\Omega \left[|\nabla w_n|^2 + \frac{g_n(w_n)|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}(w_n-\psi^+)\right]\\ &\geq \int_\Omega \left[|\nabla w|^2 + g(w)|\nabla w|^2(w-\psi^+)\right],\end{aligned}$$

i.e.,

$$\int_\Omega \nabla w\nabla(v-w) + \int_\Omega g(w)|\nabla w|^2(v-w) \geq \int_\Omega a(x)(v-w),$$

for every $v \in K_1$, and hence that w is a solution of (1.5).

Finally, to prove that if, in addition, $\psi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, then w is also a solution of (1.6), we only have to repeat the argument used in the previous Step 2 by replacing ψ^+ by ψ and K_1 by K_2. □

Proof of Theorem 1.2. Let w and w_n be the solutions given respectively by Theorem 1.1 and Proposition 3.1. Observe that if supp $\psi^+ \subset\subset \Omega$, then there exists $\Omega_0 \subset\subset \Omega$, such that

$$\psi(x) \leq 0,\ \forall x \in \Omega\setminus\overline{\Omega}_0. \tag{3.11}$$

We take $\Omega_1 \subset\subset \Omega$, with $\Omega_0 \subset\subset \Omega_1$, and $\varphi \in C^\infty(\Omega)$ such that

$$\begin{aligned}\varphi(x) &= 0 \qquad x \in \overline{\Omega_0}.\\ 0 \leq \varphi(x) &\leq 1 \qquad x \in \Omega_1\setminus\overline{\Omega}_0\\ \varphi(x) &= 1 \qquad x \in \overline{\Omega\setminus\Omega_1}.\end{aligned}$$

Choosing $v = w_n - T_{\frac{1}{n}}(w_n)\varphi \in H_0^1(\Omega)$ as test function in (3.2), we obtain

$$\begin{aligned}\int_{\Omega\setminus\overline{\Omega_1}} \frac{g_n(w_n)|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}T_{\frac{1}{n}}(w_n) &\leq \int_\Omega g_n(w_n)\frac{|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}T_{\frac{1}{n}}(w_n)\varphi\\ &\leq \int_\Omega a(x)T_{\frac{1}{n}}(w_n)\varphi + \int_\Omega \nabla w_n \nabla\left[T_{\frac{1}{n}}(w_n)\varphi\right]\\ &\leq \int_\Omega a(x)T_{\frac{1}{n}}(w_n)\varphi - \int_\Omega \left[\nabla w_n \nabla T_{\frac{1}{n}}(w_n)\right]\varphi\\ &\quad - \int_\Omega \left[\nabla w_n\nabla\varphi\right]T_{\frac{1}{n}}(w_n)\\ &\leq \int_\Omega a(x)T_{\frac{1}{n}}(w_n)\varphi - \int_\Omega \left[\nabla w_n\nabla\varphi\right]T_{\frac{1}{n}}(w_n).\end{aligned}$$

Multiplying by n and using that $T_n(s) \le n$ for every $s > 0$, we get

$$\begin{aligned}\int_{\Omega\setminus\overline{\Omega_1}} \frac{g_n(w_n)|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}\frac{T_{\frac{1}{n}}(w_n)}{\frac{1}{n}} &\le \int_\Omega a(x)\frac{T_{\frac{1}{n}}(w_n)}{\frac{1}{n}}\varphi - \int_\Omega \nabla w_n \nabla\varphi\left[\frac{T_{\frac{1}{n}}(w_n)}{\frac{1}{n}}\right]\\ &\le \int_\Omega a(x)\varphi(x) + \|w_n\|_{H_0^1(\Omega)}\|\varphi\|_{H_0^1(\Omega)} \le c,\end{aligned}$$

for some positive constant c. Taking *lim inf* as n tends to infinity and using Fatou's Lemma we deduce from the Step 1 of Theorem 1.1 that

$$\int_{\Omega\setminus\overline{\Omega_1}} g(w)|\nabla w|^2 \le c,$$

that is, $g(w)|\nabla w|^2 \in L^1(\Omega\setminus\overline{\Omega_1})$. In addition, Proposition 3.2 implies that $g(w)$ is bounded in Ω_1 and thus $g(w)|\nabla w|^2 \in L^1(\Omega_1)$. In conclusion, we have shown that $g(w)|\nabla w|^2 \in L^1(\Omega)$.

Now we prove that in the particular case supp $\psi \subset\subset \Omega$, w solves (1.2). Since $\psi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, by Theorem 1.1, the function w also solves (1.6) and hence we have

$$\left.\begin{aligned} w(x) \ge \psi(x) \text{ a.e. } x\in\Omega\\ g(w)|\nabla w|^2 \in L^1_{\text{loc}}(\Omega),\ g(w)|\nabla w|^2(w-\psi)\in L^1(\Omega)\\ \int_\Omega \nabla w\nabla(\varphi+\psi-w) + \int_\Omega g(w)|\nabla w|^2(\varphi+\psi-w) \ge \int_\Omega a(x)(\varphi+\psi-w),\\ \forall\varphi\in C_0^\infty(\Omega),\quad \varphi(x)\ge 0 \text{ a.e. } x\in\Omega.\end{aligned}\right\}$$

Using that $g(w)|\nabla w|^2 \in L^1(\Omega)$, we deduce from the density of $C_0^\infty(\Omega)$ into $H_0^1(\Omega)$ that w satisfies

$$\left.\begin{aligned} w(x) \ge \psi(x) \text{ a.e. } x\in\Omega\\ g(w)|\nabla w|^2 \in L^1(\Omega),\ g(w)|\nabla w|^2 w\in L^1(\Omega)\\ \int_\Omega \nabla w\nabla(\varphi+\psi-w) + \int_\Omega g(w)|\nabla w|^2(\varphi+\psi-w) \ge \int_\Omega a(x)(\varphi+\psi-w),\\ \forall\varphi\in H_0^1(\Omega),\quad \varphi(x)\ge 0 \text{ a.e. } x\in\Omega\end{aligned}\right\}$$

Clearly, this is equivalent to (1.2). □

Proof of Theorem 1.3. We denote by w the solution of (1.6) given by Theorem 1.1 with $\psi = -1$, and let w_n be the solutions of the approximated problems (3.2) given by Proposition 3.1. Observe that Theorem 1.3 follows once we prove that

$$\int_\Omega \nabla w\nabla\varphi + \int_\Omega g(w)|\nabla w|^2\varphi = \int_\Omega a(x)\varphi,$$

for every $\varphi \in C_0^\infty(\Omega)$ such that $\|\varphi\|_{L^\infty(\Omega)} \leq 1$. Given such a φ, we have

$$\psi - w_n = -1 - w_n \leq -1 \leq \varphi \leq 1 \leq 1 + w_n = w_n - \psi.$$

Choosing $v = -\varphi + w_n$, respectively, $v = \varphi + w_n$, as test function in (3.2) and taking limits as n goes to infinity, we deduce from the strong convergence (up to a subsequence) of w_n to w in $H^1_{\text{loc}}(\Omega)$ (Step 1 of Theorem 1.1), that

$$\int_\Omega \nabla w \nabla(-\varphi) + \int_\Omega g(w)|\nabla w|^2(-\varphi) \geq \int_\Omega a(x)(-\varphi),$$

respectively,

$$\int_\Omega \nabla w \nabla \varphi + \int_\Omega g(w)|\nabla w|^2 \varphi \geq \int_\Omega a(x)\varphi.$$

Hence we conclude that

$$\int_\Omega \nabla w \nabla \varphi + \int_\Omega g(w)|\nabla w|^2 \varphi = \int_\Omega a(x)\varphi.$$

□

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