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Existence and Multiplicity of Positive Solutions For a Class of Elliptic Boundary Value Problems in the Half-Space

Tsing-San Hsu

Center for General Education
Chang Gung University, Kwei-Shan, Tao-Yuan 333, Taiwan
e-mail: tshsu@mail.cgu.edu.tw

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Abstract

In this paper, we prove that if $b(x)$ satisfies some suitable conditions, then $-\Delta u + u = b(x)u^p$ in $\mathbb{R}^N_+$ with the boundary condition $u(x',0) = \lambda g(x')$ has at least two positive solutions if $0 < \lambda < \lambda^*$, a minimal positive solution if $\lambda = \lambda^*$ and no positive solution if $\lambda > \lambda^*$.

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1 Introduction

In this paper, we consider the following elliptic boundary value problem

$$\begin{cases} -\Delta u + u = b(x)u^p \text{ in } \mathbb{R}^N_+, \\ u \in H^1\left(\mathbb{R}^N_+\right),\ u > 0 \text{ in } \mathbb{R}^N_+, \\ u(x',0) = \lambda g(x'), \end{cases} \tag{1.1$_\lambda$}$$

where $\lambda > 0$, $1 < p < 2^* - 1$, $2^* = 2N/(N-2)$ when $N \geq 3$, and $2^* = \infty$ when $N = 2$, $x' = (x_1, ..., x_{N-1}) \in \mathbb{R}^{N-1}$, $\mathbb{R}^N_+ = \left\{x = (x', x_N) \in \mathbb{R}^N \mid x_N > 0\right\}$ is the upper half-space on $\mathbb{R}^N$ and $b(x)$ is a positive, bounded and continuous function on $\mathbb{R}^N_+$. Moreover, $b(x)$ satisfies the assumption $(H1)$ below

$(H1)$ $b(x) \geq b_\infty > 0$ in $\mathbb{R}^N_+$ and $\lim_{|x|\to\infty,\, x\in\mathbb{R}^N_+} b(x) = b_\infty$.

We define

$$\begin{aligned} \|u\| &= \left(\int_{\mathbb{R}^N_+} (|\nabla u|^2 + |u|^2)dx\right)^{1/2}, \\ \|u\|_q &= \left(\int_{\mathbb{R}^N_+} |u|^q dx\right)^{1/q} \text{ for } 2 \leq q < \infty, \\ \|u\|_\infty &= \sup_{x\in\mathbb{R}^N_+} |u(x)|, \\ M &= \inf\{\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx : \int_{\mathbb{R}^N} |u|^{p+1}dx = 1\}. \end{aligned}$$

It is well-known that the following problem

$$\begin{cases} -\Delta u + u = b_\infty u^p \text{ in } \Omega, \\ u \in H^1_0(\Omega),\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.2}$$

has a unique radial positive solution w in the whole space $\mathbb{R}^N$ (see Berestycki-Lions [5] or Gidas-Ni-Nirenberg [11]). Esteban-Lions [10] asserted that (1.2) does not admit any nontrivial solution in an Esteban-Lions domain. The definition of an Esteban-Lions domain is that for a proper unbounded domain Ω in $\mathbb{R}^N$, there is a unit vector $\chi \in \mathbb{R}^N$ such that $n(x)\cdot\chi \geq 0$ and $n(x)\cdot\chi \not\equiv 0$ on $\partial\Omega$, where $n(x)$ is the unit outward normal vector to $\partial\Omega$ at the point x. A typical example is the half-space $\mathbb{R}^N_+$. Hence, it is interesting to study existence of solutions of $(1.1)_\lambda$ in the half-space $\mathbb{R}^N_+$.

In [3], Ai-Zhu considered $(1.1)_\lambda$ with $b(x) \equiv 1$. They proved that there exists $\lambda^* > 0$ such that $(1.1)_\lambda$ has at least two positive solutions if $0 < \lambda < \lambda^*$, a minimal positive solution if $\lambda = \lambda^*$, and no positive solution if $\lambda > \lambda^*$.

In this paper, motivated by [3], we extend and improve the paper by Ai-Zhu [3]. First, we deal with the more general function $b(x)$ instead of $b(x) \equiv 1$, and second, we provide lower and upper bound for λ^*, and third, we also prove the uniqueness of positive solution if $\lambda = \lambda^*$. Our main results are as follows.

Theorem 1.1 *Assume that $g \in H^{1/2}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $g(x') \geq 0$, $g \not\equiv 0$ and $b(x)$ satisfies the assumption $(H1)$. Then there exists a $\lambda^* \in (0,\infty)$ such that*
(i) $(1.1)_\lambda$ has at least two positive solutions u_λ, U_λ and $u_\lambda < U_\lambda$ if $0 < \lambda < \lambda^*$;
(ii) $(1.1)_\lambda$ has a unique positive solution u_{λ^*} if $\lambda = \lambda^*$;
(iii) $(1.1)_\lambda$ has no positive solution if $\lambda > \lambda^*$.
Furthermore, u_λ is strictly increasing with respect to λ and

$$\begin{aligned} \lambda_1 \equiv\ & \frac{(p-1)^{1/(p+1)} M^{(p+1)/(2p-2)}}{2^{(p^2+3p)/(p^2-1)} \|b\|_\infty^{1/(p-1)} \|v_0\|} \\ \leq\ & \lambda^* \\ \leq\ & \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\|v\|^2}{p\int_{\mathbb{R}^N} b(x) v_0^{p-1} v^2 dx}\right)^{1/(p-1)} \equiv \lambda_2, \end{aligned} \tag{1.3}$$

where u_λ is the unique minimal solution of $(1.1)_\lambda$, U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 4 and v_0 is the unique positive solution of (2.1).

2 Asymptotic behavior of the solutions

In order to solve $(1.1)_\lambda$, we first consider the Dirichlet problem for the linear equation

$$\begin{cases} -\Delta v_0 + v_0 = 0 \text{ in } \mathbb{R}^N_+, \\ v_0(x', 0) = g(x'). \end{cases} \tag{2.1}$$

Then if $g \in H^{1/2}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, we know that (2.1) has a unique solution $v_0 \in H^1\left(\mathbb{R}^N_+\right) \cap L^\infty(\mathbb{R}^N_+)$. Since $g(x') \geq 0$, $g \not\equiv 0$, by the maximum principle, we have $v_0 > 0$ in $\mathbb{R}^N_+$. It is easy to verify that if $v \in H^1_0\left(\mathbb{R}^N_+\right)$ satisfies the following equation

$$\begin{cases} -\Delta v + v = \lambda^{p-1} b(x)(v + v_0)^p, \\ v \in H^1_0\left(\mathbb{R}^N_+\right),\ v > 0 \text{ in } \mathbb{R}^N_+, \\ v(x', 0) = 0 \text{ in } \mathbb{R}^{N-1}, \end{cases} \tag*{$(2.2)_\lambda$}$$

then $u_\lambda = \lambda(v + v_0)$ is a solution of $(1.1)_\lambda$. The energy functional corresponding to $(2.2)_\lambda$ is defined by

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N_+} \left(|\nabla v|^2 + v^2\right) dx - \frac{\lambda^{p-1}}{p+1}\int_{\mathbb{R}^N_+} b(x)(v^+ + v_0)^{p+1} dx.$$

By the strong maximum principle, we know that the critical points of the functional I_λ are the positive solutions of $(2.2)_\lambda$.

Lemma 2.1 *Suppose* $g \in H^{1/2}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $g(x') \geq 0$ *and* $g \not\equiv 0$. *If* v_0 *is the unique positive solution of* (2.1), *then* $v_0 \in L^\infty(\mathbb{R}^N_+)$ *and* $\lim_{x_N \to \infty} v_0(x) = 0$.

Proof. By section 42, chapter VI of Zofia [16], we have that

$$v_0(x) = P_{x_N}(x') \star g(x'),$$

where $P_{x_N}(x') = \dfrac{\pi^{-N/2} x_N}{(x_N^2 + |x'|^2)^{N/2}} \displaystyle\int_0^\infty t^{\frac{N-1}{2}} e^{-t} e^{-\frac{x_N^2 + |x'|^2}{4t}} dt.$

We note that the inequality

$$t^{\frac{N-1}{2}} e^{-t} \leq C_1 \frac{e^{-(1-\varepsilon/4)t}}{\sqrt{\pi(1-\varepsilon/4)t}} \tag{2.3}$$

holds for any fixed $0 < \varepsilon < 1$, for all $t \geq 0$ and some constant C_1 independent of t. From (2.3), the following formula (see Zofia [16])

$$e^{-\beta} = \int_0^\infty \frac{e^{-s}}{\sqrt{\pi s}} e^{-(\frac{\beta^2}{4s})} ds \text{ for any } \beta > 0,$$

and the definition of the convolution, we have

$$\begin{aligned}
& v_0(x) \\
= \ & P_{x_N}(x') \star g(x') \\
\leq \ & C_1 \int_{\mathbb{R}^{N-1}} \frac{\pi^{-N/2} x_N}{(x_N^2 + |x'-z|^2)^{N/2}} \int_0^\infty \frac{e^{-(1-\varepsilon/4)t}}{\sqrt{\pi(1-\varepsilon/4)t}} e^{-\frac{(1-\varepsilon/4)(x_N^2+|x'-z|^2)}{4(1-\varepsilon/4)t}} g(z)dtdz \\
= \ & \frac{C_1}{1-\varepsilon/4} \int_{\mathbb{R}^{N-1}} \frac{\pi^{-N/2} x_N}{(x_N^2 + |x'-z|^2)^{N/2}} e^{-\sqrt{1-\varepsilon/4}\sqrt{x_N^2+|x'-z|^2}} g(z)dz \\
\leq \ & C_2 \int_{\mathbb{R}^{N-1}} e^{-\sqrt{1-\varepsilon/2}\sqrt{x_N^2+|x'-z|^2}} g(z)dz.
\end{aligned}$$

It is easy to see that the following inequality

$$\sqrt{|a|^2 + |b|^2)} \geq \vartheta a + \sqrt{1-\vartheta^2} b$$

holds for any $a, b > 0$ and $0 \leq \vartheta \leq 1$. Let $a = x_N$, $b = |x' - z|$, and $\vartheta = \sqrt{\frac{1-\varepsilon}{1-\varepsilon/2}}$. Then we have

$$\begin{aligned}
v_0(x) \leq \ & C_2 e^{-\sqrt{1-\varepsilon} x_N} \int_{\mathbb{R}^{N-1}} e^{-\sqrt{\varepsilon/2}|x'-z|} g(z)dz \\
\leq \ & C_3 e^{-\sqrt{1-\varepsilon} x_N}
\end{aligned}$$

for some constants C_2, C_3 and any $x_N \geq 0$. For $\delta = 1 - \sqrt{1-\varepsilon} \in (0,1)$, we get

$$v_0(x) \leq c e^{-(1-\delta)x_N}, \quad \text{for all } x_N \geq 0.$$

Hence, $\lim_{x_N \to \infty} v_0(x) = 0$.

Lemma 2.2 *If $v \in H_0^1(\mathbb{R}_+^N)$ is a weak solution of the equation $-\Delta v + v = g$ and $g \in L^q(\mathbb{R}_+^N)$ for some $q \in [2, \infty)$, then $v \in W^{2,q}(\mathbb{R}_+^N)$ and $\|v\|_{W^{2,q}(\mathbb{R}_+^N)} \leq C(N,q) \|g\|_q$.*

Proof. We extend u and g to all $\mathbb{R}^N$ by odd reflection, that is, by setting

$$\widetilde{u}(x', x_N) = -u(x', -x_N), \ \widetilde{g}(x', x_N) = -g(x', -x_N)$$

for $x_N < 0$, where $x' = (x_1, x_2, \cdots, x_{N-1})$. It follows that the extended functions satisfy $-\Delta\widetilde{u} + \widetilde{u} = \widetilde{g}$ weakly in $\mathbb{R}^N$. To show this, we take an arbitrary test function $\varphi \in C_0^1(\mathbb{R}^N)$, and for $\varepsilon > 0$, let η be an even function in $C^1(\mathbb{R})$ such that $\eta(x_N) = 0$ for $|x_N| \leq \varepsilon$, $\eta(x_N) = 1$ for $|x_N| \geq 2\varepsilon$ and $|\eta'| \leq 2/\varepsilon$. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} \eta \widetilde{g} \varphi dx &= \int_{\mathbb{R}^N} \nabla\widetilde{u} \cdot \nabla(\eta\varphi) dx + \int_{\mathbb{R}^N} \widetilde{u}\eta\varphi dx \\
&= \int_{\mathbb{R}^N} \eta \nabla\widetilde{u} \cdot \nabla\varphi dx + \int_{\mathbb{R}^N} \eta' \varphi \frac{\partial \widetilde{u}}{\partial x_N} dx + \int_{\mathbb{R}^N} \widetilde{u}\eta\varphi dx.
\end{aligned}$$

Now, we obtain that

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} \eta' \varphi \frac{\partial \widetilde{u}}{\partial x_N} dx \right| &= \left| \int_{0<x_N<2\varepsilon} (\varphi(x', x_N) - \varphi(x', -x_N)) \eta' \frac{\partial \widetilde{u}}{\partial x_N} dx \right| \\
&< 8 \max |\nabla\varphi| \int_{0<x_N<2\varepsilon} \left| \frac{\partial \widetilde{u}}{\partial x_N} \right| dx \\
&\longrightarrow 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Consequently, letting $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^N} \widetilde{g}\varphi dx = \int_{\mathbb{R}^N} \nabla\widetilde{u} \cdot \nabla\varphi dx + \int_{\mathbb{R}^N} \widetilde{u}\varphi dx,$$

so that $\widetilde{u} \in H^1(\mathbb{R}^N)$ is a weak solution of $-\Delta\widetilde{u} + \widetilde{u} = \widetilde{g}$. By the Calderon-Zygmund inequality and [7], Chap. II, section 8, Proposition 27 (or [14], Proposition 4.3), we have $\widetilde{u} \in W^{2,q}(\mathbb{R}^N)$ and $\|\widetilde{u}\|_{W^{2,q}(\mathbb{R}^N)} \le C(N,q)\,\|\widetilde{g}\|_q$. Hence, we have $u \in W^{2,q}(\mathbb{R}^N_+)$ and $\|u\|_{W^{2,q}(\mathbb{R}^N_+)} \le C(N,q)\,\|g\|_q$.

Lemma 2.3 *Suppose $g \in H^{1/2}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $g(x') \ge 0$ and $g \not\equiv 0$. If $u \in H^1\left(\mathbb{R}^N_+\right)$ is a solution of $(1.1)_\lambda$, then $u \in L^\infty(\mathbb{R}^N_+)$ and $\lim_{x_N \to \infty} u(x) = 0$.*

Proof. Suppose u is a solution of $(1.1)_\lambda$. Then there exists a solution v of $(2.2)_\lambda$ such that

$$u = \lambda(v + v_0).$$

From $b(x)$ is bounded on $\mathbb{R}^N_+$, we deduce that

$$|\lambda^{p-1} b(x)(v + v_0)^p| \le C_1(|v|^p + |v_0|^p), \tag{2.4}$$

where C_1 is some constant. By Lemma 2.1, we obtain $v_0 \in L^q(\mathbb{R}^N_+)$ for all $q \in [2,\infty]$. Since v satisfies

$$-\Delta v + v = \lambda^{p-1} b(x)(v + v_0)^p \qquad \text{in } H^{-1}(\mathbb{R}^N_+),$$

and by (2.4) and Hsu-Lin [12, Lemma 3.5], we have

$$v \in L^q(\mathbb{R}^N_+) \text{ for } q \in [2,\infty).$$

Hence

$$\lambda^{p-1} b(x)(v + v_0)^p \in L^2(\mathbb{R}^N_+) \cap L^q(\mathbb{R}^N_+) \quad \text{ for all } q > \frac{N}{2}.$$

Then by Lemma 2.2, we have $v \in W^{2,q}(\mathbb{R}^N_+)$ for all $q > \frac{N}{2}$. By the Sobolev embedding theorem, $v \in C^{1,\alpha}(\overline{\mathbb{R}^N_+})$ for all $0 < \alpha < 1$ and there exists $C > 0$ such that for any $r > 1$,

$$\|v\|_{L^\infty(\overline{B}^c_r)} \le C\|v\|_{W^{2,q}(\overline{B}^c_r)}$$

where

$$\overline{B}^c_r = \{x \in \mathbb{R}^N_+ \mid |x| > r\}.$$

This implies $\lim_{|x|\to\infty} v(x) = 0$. By Lemma 2.1, we obtain that $u \in L^\infty(\mathbb{R}^N_+)$ and $\lim_{x_N\to\infty} u(x) = 0$.

Remark 2.1 From the proof of the above lemma, we also have that every solution $v \in H^1_0(\mathbb{R}^N_+)$ of $(2.2)_\lambda$ belongs to $L^\infty(\mathbb{R}^N_+) \cap C^{1,\alpha}(\overline{\mathbb{R}^N_+})$ for all $0 < \alpha < 1$ and $\lim_{x\in\mathbb{R}^N_+, |x|\to\infty} v(x) = 0$.

Lemma 2.4 *Suppose $g \in H^{1/2}(\mathbb{R}^{N-1}) \cap L^{\infty}(\mathbb{R}^{N-1})$, $g(x') \geq 0$ and $g \not\equiv 0$. If u_λ is a solution of $(1.1)_\lambda$, then there exist positive constants C and $0 < \delta < 1$ such that*

$$u_\lambda(x) \geq Ce^{-(1+\delta)x_N} \text{ for } x = (x', x_N) \text{ with } x_N \geq 1 \text{ and } |x'| \leq 1. \tag{2.5}$$

Proof. First we consider the following Dirichlet problem

$$\begin{cases} -\Delta u + u = 0 \text{ in } \mathbb{R}^N_+, \\ u(x', 0) = \lambda g(x'). \end{cases} \tag{2.6}$$

Then we get (see Ai-Zhu [3, Lemma 2.4])

$$u(x) \geq Ce^{-(1+\delta)x_N} \text{ for } x_N \geq 1 \text{ and } |x'| \leq 1.$$

It is clear that any solution of $(1.1)_\lambda$ is a supersolution of (2.6). Hence, we obtain (2.5).

3 Existence of the minimal solution

In this section, we shall prove the existence of the minimal solution of $(1.1)_\lambda$ by using the mountain pass theorem and the standard barrier method.

Lemma 3.1 *If $b(x)$ satisfies the assumption $(H1)$, then $(1.1)_\lambda$ has a solution u_λ if $0 < \lambda < \lambda_1$, where λ_1 is given by (1.3).*

Proof. Assume that $\lambda \in (0, \lambda_1)$, $B_\rho = \left\{u \in H_0^1\left(\mathbb{R}^N_+\right) \mid \|u\| < \rho\right\}$, and also let $S_\rho = \left\{u \in H_0^1\left(\mathbb{R}^N_+\right) \mid \|u\| = \rho\right\}$. Clearly, $I_\lambda(0) < 0$ for any $\lambda > 0$. Using the Sobolev embedding theorem and by the assumption $(H1)$, we get that for any $v \in S_\rho$

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\rho^2 - \frac{\lambda^{p-1}}{p+1}\int_{\mathbb{R}^N_+} b(x)(v^+ + v_0)^{p+1}dx \\ &\geq \frac{1}{2}\rho^2 - \frac{2^{p+1}\lambda^{p-1}}{p+1}\|b\|_\infty \left(\|v\|_{p+1}^{p+1} + \|v_0\|_{p+1}^{p+1}\right) \\ &\geq \frac{1}{2}\rho^2 - \frac{2^{p+1}\lambda^{p-1}}{p+1}\|b\|_\infty M^{-\frac{p+1}{2}} \left(\rho^{p+1} + \|v_0\|^{p+1}\right), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} M &= \inf\{\int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2)dx \mid \int_{\mathbb{R}^N} |v|^{p+1}dx = 1\} \\ &= \inf\{\int_{\mathbb{R}^N_+} (|\nabla v|^2 + |v|^2)dx \mid \int_{\mathbb{R}^N_+} |v|^{p+1}dx = 1\} \end{aligned}$$

(see Wang [15, Proposition 14]). Set

$$f(\rho) = \frac{1}{2}\rho^2 - \lambda^{p-1}C_1(\rho^{p+1} + C_2),$$

where $C_1 = \frac{2^{p+1}}{p+1}\|b\|_\infty M^{-\frac{p+1}{2}}$ and $C_2 = \|v_0\|^{p+1}$. It then follows that $f(\rho)$ achieves a maximum at $\rho_\lambda = [C_1(p+1)]^{-(p-1)^{-1}}\lambda^{-1}$ and

$$\begin{aligned} f(\rho_\lambda) = &\ \frac{1}{2}\rho_\lambda^2 - \lambda^{p-1}C_1(\rho_\lambda^{p+1} + C_2) \\ = &\ \frac{1}{2}[(C_1(p+1)]^{-\frac{2}{p-1}}\lambda^{-2} - C_1[C_1(p+1)]^{-\frac{p+1}{p-1}}\lambda^{-2} - C_1C_2\lambda^{p-1} \\ = &\ \left[\frac{p-1}{2}(p+1)^{-\frac{p+1}{p-1}}C_1^{-\frac{2}{p-1}} - C_1C_2\lambda^{p+1}\right]\lambda^{-2}. \end{aligned} \tag{3.2}$$

For

$$0 < \lambda < \lambda_1 = \frac{(p-1)^{1/(p+1)}M^{(p+1)/(2p-2)}}{2^{(p^2+3p)/(p^2-1)}\|b\|_\infty^{1/(p-1)}\|v_0\|}$$

we can deduce that

$$\lambda < (\frac{p-1}{2})^{1/(p+1)}(p+1)^{-1/(p-1)}C_1^{-2/(p^2-1)}(C_1C_2)^{-1/(p+1)}. \tag{3.3}$$

Then, by $(3.1)-(3.3)$, for all $v \in S_{\rho_\lambda}$, we obtain

$$I_\lambda(v) = f(\rho_\lambda) > 0 \text{ for all } 0 < \lambda < \lambda_1.$$

Let $\beta = \inf\{I_\lambda(v) \mid v \in \overline{B_\rho}\}$. Then, clearly $\beta > -\infty$. By the Ekeland variational principle [9], there exists a $(PS)_\beta$-sequence $\{v_k\} \subset \overline{B_\rho}$, that is, $I_\lambda(v_k) = \beta + o(1)$ and $I'_\lambda(v_k) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N_+)$ as $k \to \infty$. Then there exists a subsequence, still denote by $\{v_k\}$, and $v \in H^1_0(\mathbb{R}^N_+)$ such that $v_k \rightharpoonup v$ weakly in $H^1_0(\mathbb{R}^N_+)$, $v_k \to v$ strongly in $L^q_{loc}(\mathbb{R}^N_+)$ for $2 \le q < 2N/(N-2)$ and $v_k \to v$ almost everywhere in $\mathbb{R}^N_+$. Since $I'_\lambda(v_k) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N_+)$ as $k \to \infty$, we have $I'_\lambda(v) = 0$ in $H^{-1}(\mathbb{R}^N_+)$ and $v > 0$ in $\mathbb{R}^N_+$. Thus, it follows that $(1.1)_\lambda$ is solvable.

Lemma 3.2 *If $b(x)$ satisfies the assumption $(H1)$, then there exists a $\lambda^* > 0$ such that*
(i) $(1.1)_\lambda$ has a minimal positive solution u_λ if $0 < \lambda < \lambda^*$ and u_λ is strictly increasing in λ;
(ii) $(1.1)_\lambda$ has no positive solution if $\lambda > \lambda^*$.

Proof. Denote

$$Q = \{\lambda \in (0,\infty) \mid (1.1)_\lambda \text{ is solvable}\}.$$

By lemma 3.1, $Q \neq \varnothing$. Next, we claim that $(1.1)_\lambda$ has at least one solution for any $\lambda \in (0, \lambda^*)$ where $\lambda^* = \sup Q > 0$. It is easy to verify that 0 is a subsolution of $(1.1)_\lambda$ for any $\lambda > 0$. Actually, by the definition of λ^*, for any $\lambda \in (0,\lambda^*)$, there exists a $\lambda' \in (\lambda, \lambda^*)$ such that $(1.1)_{\lambda'}$ has a solution $u_{\lambda'} > 0$, that is,

$$\begin{cases} -\Delta u_{\lambda'} + u_{\lambda'} = b(x)u^p_{\lambda'} \text{ in } \mathbb{R}^N_+, \\ u_{\lambda'}(x', 0) = \lambda' g(x') \ge \lambda g(x'). \end{cases}$$

By the standard barrier method [1], there exists a solution $u_\lambda > 0$ of $(1.1)_\lambda$ such that $0 \le u_\lambda \le u_{\lambda'}$. Since 0 is not a solution of $(1.1)_\lambda$ and $\lambda' > \lambda$, applying the maximum principle, we obtain $0 < u_\lambda < u_{\lambda'}$. Using the result of Amann [1, Theorem 9.4] again, we can choose a minimal positive solution u_λ of $(1.1)_\lambda$.

Remark 3.1 Let us denote $E = \{\lambda \in (0, \infty) \mid (2.2)_\lambda \text{ is solvable}\}$. By adopting the argument in Lemma 3.2 , we obtain that $\sup E = \sup Q = \lambda^* > 0$ and
(i) $(2.2)_\lambda$ has a minimal positive solution v_λ if $0 < \lambda < \lambda^*$. Moreover, $u_\lambda = \lambda(v_\lambda + v_0)$ and v_λ is strictly increasing in λ;
(ii) $(2.2)_\lambda$ has no positive solution if $\lambda > \lambda^*$.

Let u_λ be the minimal positive solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$, we study the eigenvalue problem

$$\begin{cases} -\Delta v + v = \sigma_\lambda p b(x) u_\lambda^{p-1} v \text{ in } \mathbb{R}^N_+, \\ v \in H_0^1\left(\mathbb{R}^N_+\right),\ v > 0 \text{ in } \mathbb{R}^N_+. \end{cases} \tag{3.4}$$

Then we have the following lemma.

Lemma 3.3 *Suppose $b(x)$ satisfies the assumption $(H1)$ and the first eigenvalue σ_λ of (3.4) is defined by*

$$\sigma_\lambda = \inf\left\{\int_{\mathbb{R}^N_+}\left(|\nabla v|^2 + v^2\right)dx \mid v \in H_0^1\left(\mathbb{R}^N_+\right) \text{ and } \int_{\mathbb{R}^N_+} p b(x) u_\lambda^{p-1} v^2 dx = 1\right\}.$$

Then
(i) σ_λ is achieved;
(ii) $\sigma_\lambda > 1$ and σ_λ is strictly decreasing in λ for $\lambda \in (0, \lambda^*)$.

Proof. (i) By the definition of σ_λ, we know that $0 < \sigma_\lambda < \infty$. Let $\{v_k\} \subset H_0^1\left(\mathbb{R}^N_+\right)$ be a minimizing sequence of σ_λ, that is,

$$\int_{\mathbb{R}^N_+}\left(|\nabla v_k|^2 + v_k^2\right)dx = \sigma_\lambda + o(1) \text{ as } k \to \infty \text{ and } \int_{\mathbb{R}^N_+} p b(x) u_\lambda^{p-1} v_k^2 dx = 1 \text{ for all } k.$$

This implies that $\{v_k\}$ is bounded in $H_0^1\left(\mathbb{R}^N_+\right)$. Then there exist a subsequence, still denoted by $\{v_k\}$, and $v \in H_0^1\left(\mathbb{R}^N_+\right)$ such that $v_k \rightharpoonup v$ weakly in $H_0^1\left(\mathbb{R}^N_+\right)$, and $v_k \to v$ almost everywhere in $\mathbb{R}^N_+$. Thus,

$$\int_{\mathbb{R}^N_+}\left(|\nabla v|^2 + v^2\right)dx \le \liminf_{k\to\infty}\int_{\mathbb{R}^N_+}\left(|\nabla v_k|^2 + v_k^2\right)dx = \sigma_\lambda.$$

For any fixed $R > 0$, let $B_{R_+} = \left\{x \in \mathbb{R}^N_+ \mid |x| < R\right\}$. Since $b(x)$ is bounded on $\mathbb{R}^N_+$ and by the Hölder inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N_+} p b(x) u_\lambda^{p-1}|v_k - v|^2 dx\right| \\
\le& \int_{B_{R_+}} p b(x) u_\lambda^{p-1}|v_k - v|^2 dx + \int_{\mathbb{R}^N_+\setminus B_{R_+}} p b(x) u_\lambda^{p-1}|v_k - v|^2 dx \\
\le& C_1\left(\int_{B_{R_+}} u_\lambda^{p+1} dx\right)^{(p-1)/(p+1)}\left(\int_{B_{R_+}}|v_k - v|^{p+1} dx\right)^{2/(p+1)} \\
&+ C_1\left(\int_{\mathbb{R}^N_+\setminus B_{R_+}} u_\lambda^{p+1} dx\right)^{(p-1)/(p+1)}\left(\int_{\mathbb{R}^N_+\setminus B_{R_+}}|v_k - v|^{p+1} dx\right)^{2/(p+1)},
\end{aligned}$$

where C_1 is independent of k and R. Since $v_k \to v$ strongly in $L^s(B_{R_+})$ for $2 \leq s < 2N/(N-2)$, $\{v_k\}$ is bounded sequence in $H_0^1(\mathbb{R}_+^N)$, taking $k \to \infty$, then $R \to \infty$, and we obtain

$$\int_{\mathbb{R}_+^N} pb(x)u_\lambda^{p-1}v^2dx = 1.$$

Therefore, v achieves σ_λ. Clearly, $|v|$ also achieves σ_λ. By the maximum principle, we may assume $v > 0$ in $\mathbb{R}_+^N$.

(ii) From the argument of the above Lemma, we note that, if u_λ and $u_{\lambda'}$ are positive solutions of $(1.1)_\lambda$ with $0 < \lambda < \lambda' < \lambda^*$, then $u_{\lambda'} > u_\lambda$. Applying the Taylor expansion, we obtain

$$\begin{aligned} -\Delta\left(u_{\lambda'} - u_\lambda\right) + \left(u_{\lambda'} - u_\lambda\right) &= b(x)(u_{\lambda'}^p - u_\lambda^p) \\ &> pb(x)u_\lambda^{p-1}\left(u_{\lambda'} - u_\lambda\right). \end{aligned} \tag{3.5}$$

By (i), v achieves σ_λ, that is, v is a positive solution of (3.4). Multiplying (3.5) by v, we get

$$\sigma_\lambda \int_{\mathbb{R}_+^N} pb(x)u_\lambda^{p-1}v\left(u_{\lambda'} - u_\lambda\right)dx > \int_{\mathbb{R}_+^N} pb(x)u_\lambda^{p-1}v\left(u_{\lambda'} - u_\lambda\right)dx.$$

Thus, $\sigma_\lambda > 1$. Letting v_λ be a minimizer of σ_λ, we have

$$\int_{\mathbb{R}_+^N} pb(x)u_{\lambda'}^{p-1}v_\lambda^2dx > \int_{\mathbb{R}_+^N} pb(x)u_\lambda^{p-1}v_\lambda^2dx = 1.$$

Then there is a constant $t \in (0,1)$ such that

$$\int_{\mathbb{R}_+^N} pb(x)u_{\lambda'}^{p-1}\left(tv_\lambda\right)^2dx = 1.$$

Therefore, $\sigma_{\lambda'} \leq t^2\left\|v_\lambda\right\|^2 < \left\|v_\lambda\right\|^2 = \sigma_\lambda$, that is, σ_λ is strictly decreasing in λ for $\lambda \in (0, \lambda^*)$.

Lemma 3.4 *If $b(x)$ satisfies the assumption $(H1)$, then λ^* is finite and $\lambda_1 \leq \lambda^* \leq \lambda_2$, where λ_1 and λ_2 are given by* (1.3).

Proof. Since $\sigma_\lambda > 1$ and $u_\lambda \geq \lambda v_0$ for $0 < \lambda < \lambda^*$, we select $v > 0$ (independent of λ) in $H_0^1(\mathbb{R}_+^N)$. Then by the definition of σ_λ and the assumption $(H1)$, we have

$$\begin{aligned} \int_{\mathbb{R}_+^N}\left(|\nabla v|^2 + v^2\right)dx &\geq \sigma_\lambda \int_{\mathbb{R}_+^N} pb(x)u_\lambda^{p-1}v^2dx \\ &> \lambda^{p-1}\int_{\mathbb{R}_+^N} pb(x)v_0^{p-1}v^2dx. \end{aligned}$$

Thus,

$$\lambda \leq \inf_{v \in H_0^1(\mathbb{R}_+^N)\setminus\{0\}}\left(\frac{\|v\|^2}{p\int_{\mathbb{R}_+^N} b(x)v_0^{p-1}v^2dx}\right)^{1/(p-1)} \quad \text{for all } \lambda \in (0, \lambda^*).$$

This implies that λ^* is finite and

$$\lambda^* \leq \inf_{v \in H_0^1(\mathbb{R}_+^N)\setminus\{0\}} \left(\frac{\|v\|^2}{p \int_{\mathbb{R}_+^N} b(x) v_0^{p-1} v^2 dx} \right)^{1/(p-1)}. \tag{3.6}$$

By (3.6), Lemma 3.1 and the definition of λ^*, we have that $\lambda_1 \leq \lambda^* \leq \lambda_2$.

In order to establish the existence of a solution u_{λ^*} for $(1.1)_{\lambda^*}$, we need an a priori $L^q(\mathbb{R}_+^N)$ bound for the minimal solution v_λ.

Lemma 3.5 *Let $\lambda \in (0, \lambda^*)$. Suppose $b(x)$ satisfies the assumption $(H1)$. Then for any $q \in [2, +\infty)$, there exists a positive constant C_q, independent of v_λ, such that*

$$\int_{\mathbb{R}_+^N} v_\lambda^q dx \leq C_q, \tag{3.7}$$

where v_λ is the minimal solution of $(2.2)_\lambda$.

Proof. Our method is a combination of ideas found in papers of Egnell [8] and Brezis-Kato [6]. For $j \geq 1$, $t \geq 0$, define $\phi_j(t) = t^j$, and $\psi_j(t) = \int_0^t (\phi_j'(s))^2 ds = \frac{j^2}{2j-1} t^{2j-1}$. For $\lambda \in (0, \lambda^*)$, we denote that u_λ and v_λ are the corresponding minimal solutions of $(1.1)_\lambda$ and $(2.2)_\lambda$, respectively. Then we have $u_\lambda = \lambda(v_\lambda + v_0)$, where v_0 is the unique solution of (2.1). From the definition of σ_λ and since $\sigma_\lambda > 1$, we have

$$\int_{\mathbb{R}_+^N} (|\nabla v|^2 + v^2) dx > p \int_{\mathbb{R}_+^N} b(x) u_\lambda^{p-1} v^2 dx \tag{3.8}$$

for all $v \in H_0^1(\mathbb{R}_+^N)$. Since $\phi_j(v_\lambda) \in H_0^1(\mathbb{R}_+^N)$, by the asymptotic property of v_λ (see Remark 2.1), we may choose $v = \phi_j(v_\lambda)$ in (3.8) to obtain

$$\int_{\mathbb{R}_+^N} (\phi_j'(v_\lambda))^2 |\nabla v_\lambda|^2 dx + \int_{\mathbb{R}_+^N} {\phi_j}^2(v_\lambda) dx > p \int_{\mathbb{R}_+^N} b(x) u_\lambda^{p-1} {\phi_j}^2(v_\lambda) dx. \tag{3.9}$$

Since $u_\lambda = \lambda(v_\lambda + v_0)$, v_λ is a solution of $(2.2)_\lambda$, and $\psi_j(v_\lambda) \in H_0^1(\mathbb{R}_+^N)$, we also have

$$\begin{aligned} \int_{\mathbb{R}_+^N} \psi_j'(v_\lambda) |\nabla v_\lambda|^2 dx + \int_{\mathbb{R}_+^N} \psi_j(v_\lambda) v_\lambda dx = \;& \lambda^{p-1} \int_{\mathbb{R}_+^N} b(x)(v_\lambda + v_0)^p \psi_j(v_\lambda) dx \\ = \;& \int_{\mathbb{R}_+^N} b(x) u_\lambda^{p-1} (v_\lambda + v_0) \psi_j(v_\lambda) dx. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$(p - \frac{j^2}{2j-1}) \int_{\mathbb{R}_+^N} b(x) u_\lambda^{p-1} v_\lambda^{2j} dx \leq \frac{j^2}{2j-1} \int_{\mathbb{R}_+^N} b(x) u_\lambda^{p-1} v_0 v_\lambda^{2j-1} dx. \tag{3.11}$$

Since $\frac{j^2}{2j-1} \geq 1$ and is increasing in j, we may choose $j_0 > 1$ sufficiently close to 1 such that $\frac{j^2}{2j-1} < p$ for $j \leq j_0$. Set $\alpha(j,p) = p - \frac{j^2}{2j-1}$. By (3.11), the fact $u_\lambda \geq \lambda v_\lambda$ and the hypothesis $b(x) \geq b_\infty > 0$ for $x \in \mathbb{R}^N_+$, we obtain

$$\lambda^{p-1}\alpha(j,p)b_\infty \int_{\mathbb{R}^N_+} v_\lambda^{p+2j-1}dx \leq p\int_{\mathbb{R}^N_+} b(x)u_\lambda^{p-1}v_0v_\lambda^{2j-1}dx \tag{3.12}$$

for $1 \leq j \leq j_0$.

Notice that $\frac{p+2j-1}{p} \geq \frac{2N}{N+2}$ and $v_0 \in L^q(\mathbb{R}^N_+)$ for all $q \in [2,\infty]$. Therefore, the boundedness of $b(x)$ and Hölder inequality imply that

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} b(x)u_\lambda^{p-1}v_0v_\lambda^{2j-1}dx\\
\leq\ &\lambda^{p-1}C_1\int_{\mathbb{R}^N_+}(v_\lambda^{p+2j-2}v_0 + v_\lambda^{2j-1}v_0^p)dx\\
\leq\ &\lambda^{p-1}C_1\Big[\Big(\int_{\mathbb{R}^N_+} v_\lambda^{p+2j-1}dx\Big)^{\frac{p+2j-2}{p+2j-1}}\Big(\int_{\mathbb{R}^N_+} v_0^{p+2j-1}dx\Big)^{\frac{1}{p+2j-1}}\\
&+\Big(\int_{\mathbb{R}^N_+} v_\lambda^{p+2j-1}dx\Big)^{\frac{2j-1}{p+2j-1}}\Big(\int_{\mathbb{R}^N_+} v_0^{p+2j-1}dx\Big)^{\frac{p}{p+2j-1}}\Big]\\
\leq\ &\lambda^{p-1}C_2\Big(\|v_\lambda\|_{p+2j-1}^{p+2j-2} + \|v_\lambda\|_{p+2j-1}^{2j-1}\Big).
\end{aligned} \tag{3.13}$$

for some positive constants C_1, C_2, depending only on $\|v_0\|_2$ and $\|v_0\|_\infty$. From (3.13) and Young's inequality, it follows that for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}^N_+} b(x)u_\lambda^{p-1}v_0v_\lambda^{2j-1}dx \leq \lambda^{p-1}C_\epsilon(C_2^{p+2j-1} + C_2^{\frac{p+2j-1}{p}}) + \lambda^{p-1}\epsilon\int_{\mathbb{R}^N_+} u_\lambda^{p+2j-1}dx. \tag{3.14}$$

By choosing $\epsilon > 0$ sufficiently small and using (3.12) and (3.14), we obtain

$$\int_{\mathbb{R}^N_+} v_\lambda^{p+2j-1}dx \leq C_3,\ j \leq j_0, \text{ for some } j_0 > 1, \text{ for all } 0 < \lambda < \lambda^*, \tag{3.15}$$

where $C_3 > 0$ is a constant depending only on p, j_0, ϵ, $\|v_0\|_2$ and $\|v_0\|_\infty$. This shows that (3.7) holds for all $q \in [p+1, p+2j_0-1]$. To establish (3.7) for all $q \in [2, p+1]$, we only need to prove the case $q = 2$. Multiplying $(2.2)_\lambda$ by v_λ, and using integration by parts, Hölder's inequality and (3.15), we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^N_+} v_\lambda^2 dx &\leq \lambda^{p-1}\int_{\mathbb{R}^N_+} b(x)(v_\lambda + v_0)^p v_\lambda dx\\
&\leq C_4\int_{\mathbb{R}^N_+}(v_\lambda^{p+1} + v_\lambda v_0^p)dx\\
&\text{(by using } b(x) \text{ is bounded and } 0 < \lambda < \lambda^* < +\infty)\\
&\leq C_5 + C_4\Big(\int_{\mathbb{R}^N_+} v_0^{p+1}dx\Big)^{\frac{p}{p+1}}\Big(\int_{\mathbb{R}^N_+} v_\lambda^{p+1}dx\Big)^{\frac{1}{p+1}}\\
&\leq C_5 + C_6,
\end{aligned}$$

where C_4, C_5 and C_6 are positive constants, independent of v_λ. Therefore, (3.7) holds for all $q \in [2, p+2j_0-1]$.

Now, we use ideas in Brezis-Kato [6] to prove that (3.7) holds for all $q \geq p+1$. Since v_λ is a solution of $(2.2)_\lambda$, and $\phi_{2s+1}(v_\lambda) \in H_0^1(\mathbb{R}_+^N)$, for $s \geq 0$, then we have

$$\begin{aligned} & \int_{\mathbb{R}_+^N} |\nabla v_\lambda|^2 \phi'_{2s+1}(v_\lambda)dx + \int_{\mathbb{R}_+^N} v_\lambda \phi_{2s+1}(v_\lambda)dx \\ = \quad & \lambda^{p-1} \int_{\mathbb{R}_+^N} b(x)(v_\lambda + v_0)^p \phi_{2s+1}(v_\lambda)dx. \end{aligned} \tag{3.16}$$

Now, set $2^* = \frac{2N}{N+2}$, $s_0 = j_0 - 1 > 0$ and $q_0 = 2s_0 + p + 1 = p + 2j_0 - 1$. Therefore, it follows from Hölder's inequality, (3.15) and (3.16) and the fact $v_0 \in L^q(\mathbb{R}_+^N)$ for all $q \in [2, \infty]$ and the boundedness of $b(x)$, that

$$\begin{aligned} \int_{\mathbb{R}_+^N} |\nabla(v_\lambda^{s_0+1})|^2 dx &= (s_0+1)^2 \int_{\mathbb{R}_+^N} |\nabla v_\lambda|^2 v_\lambda^{2s_0} dx \\ &\leq C_7 \int_{\mathbb{R}_+^N} v_\lambda^{2s_0+1}(v_\lambda + v_0)^p dx \\ &\leq C_8 \int_{\mathbb{R}_+^N} (v_\lambda^{p+2s_0+1} + v_\lambda^{2s_0+1} v_0^p) dx \\ &\leq C_8 \|v_\lambda\|_{p+2s_0+1}^{p+2s_0+1} + C_8 \|v_0\|_{p+2s_0+1}^p \|v_\lambda|_{p+2s_0+1}^{2s_0+1} \\ &\leq C_9, \end{aligned}$$

where C_7, C_8 and C_9 are positive constants, independent of v_λ. By the Sobolev's inequality, there exists $C > 0$, independent of v_λ, such that

$$\int_{\mathbb{R}_+^N} v_\lambda^{2^*(s_0+1)} dx \leq C.$$

The desired inequality (3.7) then follows easily by iteration: set $2s_i + p + 1 = 2^*(s_{i-1}+1)$ and $q_i = 2s_i + p + 1$, for $i = 1, 2, \cdots$.

Proposition 3.1 *If $b(x)$ satisfies the assumption $(H1)$, then there exists a $\lambda^* < \infty$ such that $(2.2)_\lambda$ has, for each $\lambda \in (0, \lambda^*]$, a minimal positive solution $v_\lambda \in C^{1,\alpha}(\overline{\mathbb{R}_+^N}) \cap H_0^1(\mathbb{R}_+^N)$, for all $0 < \alpha < 1$. Thus, $(1.1)_{\lambda^*}$ has a minimal positive solution u_{λ^*}.*

Proof. The conclusions of Proposition 3.1 for $\lambda \in (0, \lambda^*)$ follow from Remark 2.1 and Remark 3.1. We then consider the case $\lambda = \lambda^*$. By the Sobolev Embedding Theorem, Lemma 2.2 and Lemma 3.5, for $0 < \alpha < 1$, we have

$$\begin{aligned} \|v_\lambda\|_{C^{1,\alpha}(\overline{\mathbb{R}_+^N})} &\leq C_1 \|v_\lambda\|_{W^{2,q_\alpha}(\mathbb{R}_+^N)} \\ &\leq C_2 \|\lambda^{p-1} b(x)(v_\lambda + v_0)^p\|_{L^{q_\alpha}(\mathbb{R}_+^N)} \\ &\leq C_3, \end{aligned}$$

where $q_\alpha = \frac{N}{1-\alpha}$ and C_1, C_2, C_3 are positive constants independent of v_λ. A simple diagonalization argument and the Arzela-Ascoli theorem may be employed to show that there exist a subsequence $\lambda_k \to \lambda^*$ and a function $v_{\lambda^*} \in H_0^1(\mathbb{R}_+^N)$ such that $v_{\lambda_k} \to v_{\lambda^*}$ and $|\nabla v_{\lambda_k}| \to |\nabla v_{\lambda^*}|$ uniformly on each compact subset of $\mathbb{R}_+^N$. Hence, we obtain that

$$\int_{\mathbb{R}_+^N} \nabla v_{\lambda^*} \nabla v dx + \int_{\mathbb{R}_+^N} v_{\lambda^*} v dx = \int_{\mathbb{R}_+^N} (\lambda^*)^{p-1} b(x)(v_{\lambda^*} + v_0)^p v dx$$

for all $v \in H_0^1(\mathbb{R}_+^N)$. Therefore, it is easy to see that v_{λ^*} is a positive minimal solution of $(2.2)_{\lambda^*}$. By Remark 2.1, we also have $v_{\lambda^*} \in C^{1,\alpha}(\overline{\mathbb{R}_+^N})$. Let $u_{\lambda^*} = \lambda^*(v_{\lambda^*} + v_0)$. Then by Remark 3.1, we can deduce u_{λ^*} is a positive minimal solution of $(1.1)_{\lambda^*}$.

4 Existence of the second solution

When $\lambda \in (0, \lambda^*)$, we have known that $(1.1)_\lambda$ has a minimal solution u_λ by Lemma 3.2. Then we need only to prove that $(1.1)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + \overline{v}$, where $\overline{v}$ is a solution of the following equation

$$\begin{cases} -\Delta v + v = b(x)[(v + u_\lambda)^p - u_\lambda^p] \text{ in } \mathbb{R}_+^N, \\ v \in H_0^1(\mathbb{R}_+^N),\ v > 0 \text{ in } \mathbb{R}_+^N. \end{cases} \tag{4.1}$$

Associated with (4.1), we define the energy functional $J : H_0^1(\mathbb{R}_+^N) \to \mathbb{R}$ as follows:

$$J(v) = \frac{1}{2} \int_{\mathbb{R}_+^N} \left(|\nabla v|^2 + v^2 \right) dx - \int_{\mathbb{R}_+^N} \int_0^{v^+} b(x)[(s + u_\lambda)^p - u_\lambda^p] ds dx.$$

Using the maximum principle, we know that the nontrivial critical points of energy functional J are the positive solutions of (4.1).

Lemma 4.1 *For any $\epsilon > 0$, there is a positive constant C_ϵ such that*

$$(\xi + s)^p - \xi^p - p\xi^{p-1}s \le \epsilon \xi^{p-1} s + C_\epsilon s^p \quad \textit{for all } s \ge 0,\ \xi > 0.$$

Proof. From the fact

$$\lim_{t \to 0^+} \frac{(1+t)^p - 1 - pt}{t} = 0 \text{ and } \lim_{s \to \infty} \frac{(1+t)^p - 1 - pt}{t^p} = 1,$$

we obtain that for any $\epsilon > 0$, there is a positive constant C_ϵ such that

$$(1+t)^p - 1 - pt \le \epsilon t + C_\epsilon t^p \quad \text{for all } t \ge 0.$$

Letting $\xi > 0$, $s \ge 0$, and taking $t = s/\xi$ in the above inequality, we deduce that

$$(\xi + s)^p - \xi^p - p\xi^{p-1}s \le \epsilon \xi^{p-1} s + C_\epsilon s^p.$$

Lemma 4.2 *If $b(x)$ satisfies the assumption $(H1)$, then there exist $\rho > 0$ and $\gamma > 0$ such that*

$$J(v)|_{S_\rho} \ge \gamma > 0, \tag{4.2}$$

where $S_\rho = \left\{ u \in H_0^1\left(\mathbb{R}_+^N\right) \mid \|u\| = \rho \right\}$.

Proof. For any $\varepsilon > 0$, by Lemma 4.1 (with $\xi = u_\lambda$), there exists a positive constant C_ε such that

$$\begin{aligned} J(v) = &\ \frac{1}{2}\int_{\mathbb{R}^N_+}(|\nabla v|^2+v^2)dx - \frac{1}{2}p\int_{\mathbb{R}^N_+} b(x)u_\lambda^{p-1}(v^+)^2dx \\ &-\int_{\mathbb{R}^N_+}\int_0^{v^+} b(x)[(u_\lambda+s)^p - u_\lambda^p - pu_\lambda^{p-1}s]dsdx \\ \geq &\ \frac{1}{2}\Big[\int_{\mathbb{R}^N_+}(|\nabla v|^2+v^2)dx - (p+\varepsilon)\int_{\mathbb{R}^N_+} b(x)u_\lambda^{p-1}(v^+)^2dx\Big] \\ &-\frac{C_\varepsilon}{p+1}\int_{\mathbb{R}^N_+} b(x)(v^+)^{p+1}dx. \end{aligned} \tag{4.3}$$

Furthermore, from the definition of σ_λ, for any $v \in H_0^1\left(\mathbb{R}^N_+\right)$,

$$\int_{\mathbb{R}^N_+}\left(|\nabla v|^2+v^2\right)dx \geq \sigma_\lambda \int_{\mathbb{R}^N_+} pb(x)u_\lambda^{p-1}v^2dx.$$

Therefore, by (4.3), the boundedness of $b(x)$ and the Sobolev inequality, we obtain

$$J(v) \geq \frac{1}{2\sigma_\lambda}(\sigma_\lambda - 1 - \frac{\varepsilon}{p})\|v\|^2 - C'_\varepsilon\|v\|^{p+1}$$

for some $C'_\varepsilon > 0$. Since $\sigma_\lambda > 1$, we may choose $\varepsilon > 0$ small enough such that $\sigma_\lambda - 1 - \varepsilon/p > 0$. If we take $\varepsilon = p\left(\sigma_\lambda - 1\right)/2$, then

$$J(v) \geq \frac{1}{4\sigma_\lambda}\left(\sigma_\lambda - 1\right)\|v\|^2 - C'_\varepsilon\|v\|^{p+1}.$$

Hence, there exist $\rho > 0$ and $\gamma > 0$ such that $J(v)|_{S_\rho} \geq \gamma > 0$.

Now, we introduce the following elliptic equation in the whole space.

$$\begin{cases} -\Delta u + u = b_\infty u^p \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N),\ N \geq 2. \end{cases} \tag{4.4}$$

The corresponding variational functional of (4.4) is

$$I^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+u^2\right)dx - \frac{1}{p+1}\int_{\mathbb{R}^N} b_\infty(u^+)^{p+1}dx,\ u \in H^1(\mathbb{R}^N).$$

By Bahri-Lions [4] and Kwong [13], we know that (4.4) has a unique radial symmetric ground state solution $\varpi(x) > 0$ in $\mathbb{R}^N$ such that

$$M^\infty = I^\infty(\varpi) = \sup_{t\geq 0} I^\infty(t\varpi), \tag{4.5}$$

and there is a constant $C > 0$ such that

$$\begin{cases} \varpi(x)|x|^{(N-1)/2}\exp(|x|) \to C, \\ \varpi'(x)|x|^{(N-1)/2}\exp(|x|) \to -C, \end{cases} \text{ as } |x| \to \infty. \tag{4.6}$$

Moreover, we have the following decomposition Lemma.

Lemma 4.3 *Let $\{v_k\}$ be a $(PS)_\beta$-sequence of J, that is, $J(v_k) \to \beta$ and $J'(v_k) \to 0$ strongly in $H^{-1}(\mathbb{R}^N_+)$ as $k \to \infty$. If $b(x)$ satisfies the assumption $(H1)$, then there exist a subsequence, still denote by $\{v_k\}$, an integer $l \geq 0$, sequences $\{x^i_k\} \subset \mathbb{R}^N_+ (1 \leq i \leq l)$, and a solution $\overline{v}$ of (4.1) such that*

$$v_k \rightharpoonup \overline{v} \text{ weakly in } H^1_0\left(\mathbb{R}^N_+\right),$$
$$v_k - \left(\overline{v} + \textstyle\sum_{i=1}^{l} \varpi(\cdot - x^i_k)\right) \to 0 \text{ strongly in } H^1_0\left(\mathbb{R}^N_+\right),$$
$$\beta = J(\overline{v}) + lM^\infty,$$

and that for each $1 \leq i \leq l$, the sequence of the last component of $\{x_k\}$ approaches $+\infty$ as $k \to \infty$.

Thus we agree that the above results hold without u^i and x^i_k for $l = 0$.

Proof. See Ai-Zhu [3, Lemma 3.2].

Let $\eta(x)$ be a function in $C^\infty_0(\mathbb{R}^N)$ satisfying: $0 \leq \eta(x) \leq 1$ for $x \in \mathbb{R}^N$, $\eta(x) \equiv 1$ for $x_N > 1$ and $\eta(x) \equiv 0$ for $x_N < \frac{1}{2}$. Set $e_N = (0, 0, \cdots, 0, 1)$ and denote $\eta_\tau(x) = \eta(x + \tau e_N)$, $\varpi_\tau = \varpi(x - \tau e_N)$ for $\tau \in (0, \infty)$. Clearly, $\eta\varpi_\tau \in H^1_0(\mathbb{R}^N_+)$.

Lemma 4.4 *Let $\lambda \in (0, \lambda^*)$ be fixed. If $b(x)$ satisfies the assumption $(H1)$, then*
(i) there exists $t_0 > 0$ such that $J(t\eta\varpi_\tau) < 0$ for $t \geq t_0$, $\tau \geq 2$,
(ii) there exists $\tau_0 > 0$ such that the following inequalities hold for $\tau \geq \tau_0$:

$$0 < \sup_{t \geq 0} J(t\eta\varpi_\tau) < M^\infty. \tag{4.7}$$

Proof. (i) Applying the Green's formula, realizing $\eta \in [0, 1]$ and by (4.4), we obtain

$$\begin{aligned} J(t\eta\varpi_\tau) = \ & \frac{1}{2}t^2 \int_{\mathbb{R}^N_+} (|\nabla(\eta\varpi_\tau)|^2 + |\eta\varpi_\tau|^2)dx \\ & - \frac{1}{p+1}t^{p+1} \int_{\mathbb{R}^N_+} b(x)(\eta\varpi_\tau)^{p+1}dx \\ & - \int_{\mathbb{R}^N_+} \int_0^{t\eta\varpi_\tau} b(x)[(s + u_\lambda)^p - u^p_\lambda - s^p]dsdx \\ \leq \ & \frac{1}{2}t^2 \int_{\mathbb{R}^N} (-\Delta\varpi + \varpi)(\eta^2_\tau\varpi)dx + \frac{1}{2}t^2 \int_{\mathbb{R}^N} |\nabla\eta_\tau|^2|\varpi|^2dx \\ & - \frac{1}{p+1}t^{p+1} \int_{\mathbb{R}^N} b(x)\eta(x)\varpi^{p+1}(x - \tau e_N)dx \\ \leq \ & \frac{1}{2}t^2 \int_{\mathbb{R}^N} b_\infty\varpi^{p+1}dx + \frac{1}{2}t^2(\max_{x \in \mathbb{R}^N} |\nabla\eta|^2) \int_{\mathbb{R}^N} |\varpi|^2dx \\ & - \frac{t^{p+1}}{p+1} \int_{\{(x', x_N) | x_N > 1\}} b(x)\varpi^{p+1}(x - \tau e_N)dx. \end{aligned} \tag{4.8}$$

Set $B_1(\tau e_N) = \{x \in \mathbb{R}^N \mid |x - \tau e_N| < 1\}$. From the assumption $(H1)$ and the positiveness of ϖ, as $\tau \geq 2$, we deduce that

$$\begin{aligned} & \int_{\{(x' x_N)|x_N>1\}} b(x)\varpi^{p+1}(x-\tau e_N)dx \\ \geq & \int_{B_1(\tau e_N)} b_\infty \varpi^{p+1}(x-\tau e_N)dx \\ = & \int_{\{x\in\mathbb{R}^N||x|\leq 1\}} b_\infty \varpi^{p+1}(x)dx \\ = & \ C > 0 \end{aligned} \tag{4.9}$$

where C is independent of τ. Combining (4.8) and (4.9), there exist some positive constants C_1, C_2, independent of τ, such that

$$J(t\eta\varpi_\tau) \leq C_1 t^2 - C_2 t^{p+1} \text{ for all } \tau \geq 2. \tag{4.10}$$

From (4.10) we obtain the result of (i).

(ii) Due to Lemma 4.2 we can find $t > 0$ such that $\|t\eta\varpi_\tau\| = \rho$. The left inequality in (4.7) follows from (4.2). To show the right inequality, we have by (i) that $J(t\eta\varpi_\tau) < 0$ for $t \geq t_0$, $\tau \geq 2$, there is $t_2 \in (0, t_0)$ such that

$$\sup_{t\geq 0} J(t\eta\varpi_\tau) = \sup_{0\leq t\leq t_2} J(t\eta\varpi_\tau), \text{ for any } \tau \geq 2.$$

Since J is continuous in $H_0^1(\mathbb{R}_+^N)$ and $J(0) = 0$, there exists a constant $t_1 > 0$ such that

$$J(t\eta\varpi_\tau) < M^\infty, \text{ for any } \tau \in (0,\infty) \text{ and } 0 \leq t < t_1.$$

Then, to prove (4.7), we now only to prove the following inequality

$$\sup_{t_1\leq t\leq t_2} J(t\eta\varpi_\tau) < M^\infty \text{ for } \tau \text{ large enough .}$$

By the definition of J, we get

$$\begin{aligned} J(t\eta\varpi_\tau) = & \frac{t^2}{2}\int_{\mathbb{R}_+^N}(|\nabla(\eta\varpi_\tau)|^2 + |\eta\varpi_\tau|^2)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}_+^N} b_\infty(\eta\varpi_\tau)^{p+1}dx \\ & + \frac{t^{p+1}}{p+1}\int_{\mathbb{R}_+^N}(b_\infty - b(x))(\eta\varpi_\tau)^{p+1}dx \\ & - \int_{\mathbb{R}_+^N}\int_0^{t\eta\varpi_\tau} b(x)[(s+u_\lambda)^p - u_\lambda^p - s^p]dsdx. \end{aligned}$$

Since ϖ is the ground state solution of (4.4) and by the assumption $(H1)$, then we have

$$\begin{aligned} \sup_{t\leq t_2} J(t\eta\varpi_\tau) \leq & \frac{t^2}{2}\int_{\mathbb{R}^N}(-\Delta\varpi + \varpi)(\eta_\tau^2\varpi)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} b_\infty\varpi^{p+1}dx \\ & + \frac{t_2^2}{2}\int_{\mathbb{R}^N}|\nabla\eta|^2|\varpi_\tau|^2dx \\ & + \frac{t_2^{p+1}}{p+1}\int_{\mathbb{R}^N} b_\infty(1-\eta^{p+1})\varpi_\tau^{p+1}dx \\ & - \int_{\mathbb{R}_+^N}\int_0^{t\eta\varpi_\tau} b(x)[(s+u_\lambda)^p - u_\lambda^p - s^p]dsdx. \end{aligned} \tag{4.11}$$

By (4.5) and (4.6), we have that there exists a constant $\tau_1 \geq 2$ such that, for all $\tau \geq \tau_1$,

$$\begin{aligned} & \frac{t^2}{2}\int_{\mathbb{R}^N}(-\Delta\varpi+\varpi)(\eta_\tau^2\varpi)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} b_\infty \varpi^{p+1}dx \\ \leq \ & I^\infty(t\varpi) \\ \leq \ & \sup_{t\geq 0} I^\infty(t\varpi) = M^\infty, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \frac{t_2^2}{2}\int_{\mathbb{R}^N}|\nabla\eta|^2|\varpi_\tau|^2dx + \ & \frac{t_2^{p+1}}{p+1}\int_{\mathbb{R}^N} b_\infty(1-\eta^{p+1})\varpi_\tau^{p+1}dx \\ \leq \ & C_1\int_{\{(x'x_N)|x_N<1\})}(|\varpi_\tau|^2+|\varpi_\tau|^{p+1})dx \\ \leq \ & C_1\int_{\{(x'x_N)|x_N<1\}}\exp(-2|x-\tau e_N|)|x-\tau e_N|^{-N+1}dx \\ \leq \ & C_1\int_{\tau-1}^{\infty}\exp(-2r)dr \\ = \ & C_2\exp(-2\tau), \end{aligned} \tag{4.13}$$

where $C_1,\ C_2 > 0$ are independent of τ.

Set $B_1(\tau e_N) = \{x \in \mathbb{R}^N \mid |x-\tau e_N| < 1\}$. For $\tau \geq \tau_2 = 2+\tau_1$, we have that $B_1(\tau e_N) \subset \{(x', x_N) \in \mathbb{R}_+^N \mid |x'| < 1,\ x_N > 1\}$. Note that $(a+b)^p \geq a^p + b^p$, for all $a \geq 0$, $b \geq 0$, $p > 1$. Then for $\tau \geq \tau_2$, we have $\eta(x) = 1$ on $B_1(\tau e_N)$ and

$$\begin{aligned} & \int_{\mathbb{R}_+^N}\int_0^{t\eta\varpi_\tau} b(x)[(s+u_\lambda)^p - u_\lambda^p - s^p]dsdx \\ \geq \ & \int_{B_1(\tau e_N)}\int_0^{t\varpi_\tau} b(x)[(s+u_\lambda)^p - u_\lambda^p - s^p]dsdx \\ = \ & \int_{B_1(\tau e_N)}\int_0^{t\varpi_\tau} b(x)\Big([(s+u_\lambda)^{p-1} - s^{p-1}]s + [(s+u_\lambda)^{p-1} - u_\lambda^{p-1}]u_\lambda\Big)dsdx \\ \geq \ & \int_{B_1(\tau e_N)}\int_0^{t\varpi_\tau} b(x)[(s+u_\lambda)^{p-1} - u_\lambda^{p-1}]u_\lambda dsdx \\ = \ & \int_{B_1(\tau e_N)} b(x)\Big[\frac{(t\varpi_\tau+u_\lambda)^p - u_\lambda^p}{p\varpi_\tau} - tu_\lambda^{p-1}\Big]\varpi_\tau u_\lambda dx. \end{aligned} \tag{4.14}$$

Since $\lim_{x_N\to\infty} u_\lambda(x) = 0$, there exist $\tau_3 \geq \tau_2$ and $\alpha > 0$, such that

$$\frac{(t\varpi_\tau+u_\lambda)^p - u_\lambda^p}{p\varpi_\tau} - tu_\lambda^{p-1} \geq \alpha,\ \text{for } \tau \geq \tau_3,\ x \in B_1(\tau e_N),\ t \in [t_1, t_2].$$

By the assumption $(H1)$ and Lemma 2.4, we deduce from (4.14) that

$$\begin{aligned} & \int_{\mathbb{R}^N_+} \int_0^{t\eta\varpi_\tau} b(x)[(s+u_\lambda)^p - u_\lambda^p - s^p] ds dx \\ \geq \;& \alpha b_\infty \int_{B_1(\tau e_N)} \varpi(x-\tau e_N) u_\lambda(x) dx \\ \geq \;& C \int_{B_1(\tau e_N)} \varpi(x-\tau e_N) \exp[-(\tau+1)(1+\delta)] dx \\ \geq \;& C_2 \exp(-(1+\delta)\tau), \end{aligned} \tag{4.15}$$

for some $0 < \delta < 1$ and for all $\tau \geq \tau_3$, $t \in [t_1, t_2]$ where $C_2 > 0$ is independent of τ.

If $\delta = 1/2$, by $(4.11) - (4.15)$ and $b(x) \geq b_\infty$, we get, for $\tau \geq \tau_3$ and $t \in [t_1, t_2]$,

$$J(t\eta\varpi_\tau) \leq M^\infty + C_1 \exp(-2\tau) - C_2 \exp(-\frac{3}{2}\tau)$$

where C_1 and C_2 are independent of τ. Hence, we can find some $\tau_0 \geq \tau_3$ large enough such that

$$C_1 \exp(-2\tau) - C_2 \exp(-\frac{3}{2}\tau) < 0$$

for all $\tau \geq \tau_0$ and (4.7) is proved.

Proposition 4.1 *If $b(x)$ satisfies the assumption $(H1)$, then (4.1) has at least one solution for $\lambda \in (0, \lambda^*)$.*

Proof. By Lemma 4.4, we know that there are positive constants τ_0 and t_0 such that $J(t_0\eta\varpi_{\tau_0}) < 0$. We set

$$\Gamma = \{h \in C([0,1], H_0^1(\mathbb{R}^N_+)) \mid h(0) = 0,\ h(1) = t_0\eta\varpi_{\tau_0}\}.$$

Then, by (4.2) and (4.7) we get

$$0 < \gamma \leq \alpha = \inf_{h\in\Gamma} \max_{0\leq s\leq 1} J(h(s)) < M^\infty. \tag{4.16}$$

Applying the mountain pass lemma of Ambrosetti-Rabinowitz [2], there exists a $(PS)_\alpha$-sequence $\{v_k\}$ such that

$$J(v_k) \to \alpha \text{ and } J'(v_k) \to 0 \text{ in } H^{-1}(\mathbb{R}^N_+).$$

By Lemma 4.3, there exist a subsequence, still denoted by $\{v_k\}$, an integer $l \geq 0$, sequence $\{x_k^i\}$ in $\mathbb{R}^N_+$, $1 \leq i \leq l$, with the last component tend to $+\infty$, and a solution $\overline{v}$ of (4.1) such that

$$\begin{cases} v_k \rightharpoonup \overline{v} \text{ weakly in } H_0^1(\mathbb{R}^N_+), \\ v_k - [\overline{v} + \sum_{i=1}^{l} \varpi(\cdot - x_k^i)] \to 0 \text{ strongly in } H^1(\mathbb{R}^N), \\ \alpha = J(\overline{v}) + lM^\infty. \end{cases} \tag{4.17}$$

We only need to prove $\overline{v} \not\equiv 0$ in $\mathbb{R}^N_+$. Indeed, if $\overline{v} \equiv 0$, then by (4.17), we have

$$\alpha = 0 \text{ if } l = 0, \quad \alpha \geq M^\infty \text{ if } l \geq 1.$$

This contradicts (4.16). So $\overline{v} \not\equiv 0$ and we know that $\overline{v} > 0$ in $\mathbb{R}^N_+$ by the strong maximum principle.

Corollary 4.1 *If $b(x)$ satisfies the assumption $(H1)$, then $(1.1)_\lambda$ has another positive solution U_λ if $0 < \lambda < \lambda^*$.*

Proof. Applying Proposition 4.1, then $(1.1)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + \overline{v}$, where u_λ is a minimal solution of $(1.1)_\lambda$ and $\overline{v}$ is a solution of (4.1).

5 Uniqueness of minimal solutions

Let us denote by u_λ the minimal solution of $(1.1)_\lambda$ and U_λ is the second solution of $(1.1)_\lambda$ constructed in Corollary 4.1. For each solution u of $(1.1)_\lambda$, let $\sigma_\lambda(u)$ denote the number defined by

$$\sigma_\lambda(u) = \inf\left\{ \int_{\mathbb{R}^N_+} \left(|\nabla w|^2 + w^2\right) dx \mid w \in H^1_0(\mathbb{R}^N_+), \int_{\mathbb{R}^N_+} pb(x)u^{p-1}w^2 dx = 1 \right\},$$

which is the smallest eigenvalue of the following problem

$$\begin{cases} -\Delta w + w^2 = \sigma_\lambda(u)pb(x)u^{p-1}w \text{ in } \mathbb{R}^N_+, \\ w > 0, \ w \in H^1_0(\mathbb{R}^N_+). \end{cases}$$

Lemma 5.1 (i) *Let u be a solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$. Then $\sigma_\lambda(u) > 1$ if and only if $u = u_\lambda$. Moreover, u_λ is unique for $\lambda \in (0, \lambda^*)$.*
(ii) $\sigma_\lambda(U_\lambda) < 1$.

Proof. (i) Now, let $\psi \geq 0$ and $\psi \in H^1_0(\mathbb{R}^N_+)$. Since u and u_λ are the solution of $(1.1)_\lambda$, then

$$\begin{aligned} & \int_{\mathbb{R}^N_+} \nabla\psi \cdot \nabla(u_\lambda - u)dx + \int_{\mathbb{R}^N_+} \psi(u_\lambda - u)dx \\ = & \int_{\mathbb{R}^N_+} b(x)(u^p_\lambda - u^p)\psi dx \\ = & \int_{\mathbb{R}^N_+} b(x)\Big(\int_u^{u_\lambda} pt^{p-1}dt\Big)\psi dx \\ \geq & \int_{\mathbb{R}^N_+} pb(x)u^{p-1}(u_\lambda - u)\psi dx. \end{aligned} \tag{5.1}$$

Let $\psi = (u - u_\lambda)^+ \geq 0$ and $\psi \in H^1_0(\mathbb{R}^N_+)$. If $\psi \not\equiv 0$, then (5.1) implies

$$-\int_{\mathbb{R}^N_+} (|\nabla\psi|^2 + \psi^2)dx \geq -\int_{\mathbb{R}^N_+} pb(x)u^{p-1}\psi^2 dx.$$

Therefore, by $\sigma_\lambda(u) > 1$ and the definition of $\sigma_\lambda(u)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N_+}(|\nabla\psi|^2+\psi^2)dx &\le \int_{\mathbb{R}^N_+} pb(x)u^{p-1}\psi^2dx\\ &< \sigma_\lambda(u)\int_{\mathbb{R}^N_+} pb(x)u^{p-1}\psi^2dx\\ &\le \int_{\mathbb{R}^N_+}(|\nabla\psi|^2+\psi^2)dx,\end{aligned}$$

and we get a contradiction. Hence $\psi \equiv 0$, and $u = u_\lambda$ in $\mathbb{R}^N_+$. On the other hand, by Lemma 3.3, we also have that $\sigma_\lambda(u_\lambda) > 1$. This completes the proof of (i).

(ii) By (i), we get that $\sigma_\lambda(U_\lambda) \le 1$ for $\lambda \in (0, \lambda^*)$. We claim that $\sigma_\lambda(U_\lambda) = 1$ can not occur. We proceed by contradiction. Set $w = U_\lambda - u_\lambda$; we have $w > 0$ in $\mathbb{R}^N_+$ and

$$-\Delta w + w = b(x)[U_\lambda^p - (U_\lambda - w)^p], \quad w \in H^1_0(\mathbb{R}^N_+). \tag{5.2}$$

By $\sigma_\lambda(U_\lambda) = 1$, we have that the problem

$$-\Delta\phi + \phi = pb(x)U_\lambda^{p-1}\phi, \quad \phi \in H^1_0(\mathbb{R}^N_+) \tag{5.3}$$

possesses a positive solution ϕ_1.

Multiplying (5.2) by ϕ_1 and (5.3) by w, integrating and subtracting we deduce that

$$\begin{aligned}0 = &\int_{\mathbb{R}^N_+} b(x)[U_\lambda^p - (U_\lambda - w)^p - pU_\lambda^{p-1}w]\phi_1dx\\ = &-\frac{1}{2}\int_{\mathbb{R}^N_+} p(p-1)b(x)\xi_\lambda^{p-2}w^2\phi_1dx,\end{aligned}$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w \equiv 0$, that is $U_\lambda = u_\lambda$ for $\lambda \in (0, \lambda^*)$. This is a contradiction. Hence, we have that $\sigma_\lambda(U_\lambda) < 1$ for $\lambda \in (0, \lambda^*)$.

Lemma 5.2 *Suppose that u_λ is the minimal solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*]$ and $\sigma_\lambda(u_\lambda) > 1$. Then for any $h(x) \in H^{-1}(\mathbb{R}^N_+)$, problem*

$$-\Delta w + w = pb(x)u_\lambda^{p-1}w + h(x),\ w \in H^1_0(\mathbb{R}^N_+) \tag{5.4$_\lambda$}$$

has a solution.

Proof. Consider the functional

$$\Phi(w) = \frac{1}{2}\int_{\mathbb{R}^N_+}(|\nabla w|^2 + w^2)dx - \frac{1}{2}\int_{\mathbb{R}^N_+} pb(x)u_\lambda^{p-1}w^2dx - \int_{\mathbb{R}^N_+} h(x)wdx,$$

where $w \in H^1_0(\mathbb{R}^N_+)$. From Hölder inequality and Young′s inequality, we have, for any $\varepsilon > 0$, that

$$\begin{aligned}\Phi(w) \ge\ & \frac{1}{2}(1 - \sigma_\lambda(u_\lambda)^{-1})\|w\|^2 - \frac{1}{2}\varepsilon\|w\|^2 - \frac{C_\varepsilon}{2}\|g\|^2_{H^{-1}(\mathbb{R}^N_+)}\\ \ge\ & -C\|g\|^2_{H^{-1}(\mathbb{R}^N_+)}\end{aligned} \tag{5.5}$$

if we choose ε small.

Now, let $\{w_k\} \subset H_0^1(\mathbb{R}_+^N)$ be the minimizing sequence of variational problem

$$d = \inf\{\Phi(w) | w \in H_0^1(\mathbb{R}_+^N)\}.$$

From (5.5) and $\sigma_\lambda(u_\lambda) > 1$, we can also deduce that $\{w_k\}$ is bounded in $H_0^1(\mathbb{R}_+^N)$, if we choose ε small. So we may suppose that

$$\begin{aligned} &w_k \rightharpoonup w \text{ weakly in } H_0^1(\mathbb{R}_+^N) \text{ as } k \to \infty, \\ &w_k \to w \text{ almost everywhere in } \mathbb{R}_+^N \text{ as } k \to \infty. \end{aligned}$$

By Fatou′s Lemma,

$$\|w\|^2 \leq \liminf \|w_k\|^2.$$

By the weak convergence and repeating the same arguments used in the proof of Lemma 3.3, we have, as $k \to \infty$,

$$\begin{aligned} \int_{\mathbb{R}_+^N} g w_k dx &\to \int_{\mathbb{R}_+^N} g w dx, \\ \int_{\mathbb{R}_+^N} b(x) u_\lambda^{p-1} w_k^2 dx &\to \int_{\mathbb{R}_+^N} b(x) u_\lambda^{p-1} w^2 dx. \end{aligned}$$

Therefore

$$\Phi(w) \leq \lim_{k\to\infty} \Phi(w_k) = d,$$

and hence $\Phi(w) = d$, which gives that w is a solution of $(5.4)_\lambda$.

Lemma 5.3 *Suppose u_{λ^*} is a minimal solution of $(1.1)_{\lambda^*}$, then $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution of $(1.1)_{\lambda^*}$ is unique.*

Proof. Let $v_\lambda = \frac{u_\lambda}{\lambda} - v_0$ be a solution of $(2.2)_\lambda$ for $\lambda \in (0, \lambda^*]$. Define $G : \mathbb{R} \times H_0^1(\mathbb{R}_+^N) \longrightarrow H^{-1}(\mathbb{R}_+^N)$ by

$$G(\lambda, v) = \Delta v - v + \lambda^{p-1} b(x)(v + v_0)^p.$$

Since $\sigma_\lambda(u_\lambda) > 1$ for $\lambda \in (0, \lambda^*)$, so $\sigma_{\lambda^*}(u_{\lambda^*}) \geq 1$. If $\sigma_{\lambda^*}(u_{\lambda^*}) > 1$, the equation $G_v(\lambda^*, v_{\lambda^*})\phi = 0$ has only trivial solution. From Lemma 5.2, G_v maps $\mathbb{R} \times H_0^1(\mathbb{R}_+^N)$ onto $H^{-1}(\mathbb{R}_+^N)$. Applying the implicit function theorem to G, we can find a neighborhood $(\lambda^* - \delta, \lambda^* + \delta)$ of λ^* such that $(2.2)_\lambda$ possesses a solution v_λ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This implies that $(1.1)_\lambda$ possesses a solution u_λ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This is contradictory to the definition of λ^*. Hence, we obtain that $\sigma_{\lambda^*}(u_{\lambda^*}) = 1$.

Next, we are going to prove that u_{λ^*} is unique. In fact, suppose $(1.1)_{\lambda^*}$ has another solution $U_{\lambda^*} \geq u_{\lambda^*}$. Setting $w = U_{\lambda^*} - u_{\lambda^*}$, we have

$$-\Delta w + w = b(x)[(w + u_{\lambda^*})^p - u_{\lambda^*}^p], \quad w \in H_0^1(\mathbb{R}_+^N). \tag{5.6}$$

By $\sigma_{\lambda^*}(u_{\lambda^*}) = 1$, we have that the problem

$$-\Delta \phi + \phi = p b(x) u_{\lambda^*}^{p-1} \phi, \quad \phi \in H_0^1(\mathbb{R}_+^N) \tag{5.7}$$

possesses a positive solution ϕ_1.

Multiplying (5.6) by ϕ_1 and (5.7) by w, integrating and subtracting we deduce that

$$\begin{aligned} 0 = & \int_{\mathbb{R}^N_+} b(x)[(w + u_{\lambda^*})^p - u^p_{\lambda^*} - p u^{p-1}_{\lambda^*} w]\phi_1 dx \\ = & \frac{1}{2}\int_{\mathbb{R}^N_+} p(p-1)\xi^{p-2}_{\lambda^*} w^2 \phi_1 dx, \end{aligned}$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w)$. Thus $w \equiv 0$.

Proof of Theorem 1.1. By Lemmas 3.2, 3.4, 5.1, 5.3, Proposition 3.1 and Corollary 4.1, Theorem 1.1 holds.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review **18** (1976), 620–709.

[2] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theorem and applications*, J. Functional Anal. **14** (1973), 349–381.

[3] J. Ai and X.P. Zhu, *Positive solutions of inhomogeneous elliptic boundary value problems in the half space*, Comm. in P.D.E. **15** (1990), 1421–1446.

[4] A. Bahri and P.L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. I. H. P. Analyse non lineaire **14** (1997), 365–413.

[5] H. Berestycki and P.L. Lions, *Nonlinear scalar-field equations, I, Existence of a ground state; II, Existence of infinitely many solutions*, Arch. Rational Mech. Anal. **82** (1983), 313–375.

[6] H. Brezis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. **58** (1979), 137–151.

[7] R. Dautray and J.L. Lions, Analyse mathématique et calcul numérique pour les sciences et les techniques, Vol 1, Masson, Pairs, 1984.

[8] H. Egnell, *Asymptotic results for finite energy solutions of semilinear elliptic equations*, J. Differential Equation **98** (1992), 34–56.

[9] I. Ekeland, *Non-convex minimization problems*, Bull. Amer. Math. Soc. **1** (1979), 443–474.

[10] M.J. Esteban and P.L. Lions, *Existence and non-existence results for semilinear elliptic problems in unbounded domains*, Proc. Roy. Soc. Edin. **93A** (1982), 1–12.

[11] B. Giads, W.M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[12] T. S. Hsu and H.L. Lin, *Bifurcation of positive entire solutions for a semilinear elliptic equation*, Bull. Austral. Math. Soc. **72** (2005), 349–370.

[13] M.K. Kwong, *Uniqueness of positive solution of $\Delta u - u + u^p = 0$ in R^N*, Arch. Rational Mech. Anal. **105** (1989), 243–266.

[14] W. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

[15] H.C. Wang, *A Palais-Smale approach to problems in Esteban-Lions domains with hole*, Trans. Math. Soc. **352** (2000), 4237–4256.

[16] S. Zofia, Fourier transformation and linear differential equations, Polish Scientific Publishers, Warszawa, 1977.