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Perturbation of Symmetry and Multiplicity of Solutions For Strongly Indefinite Elliptic Systems

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Abstract

We consider the following elliptic system:

$$\begin{cases} -\Delta u = |v|^{p-1}v + h(x) & x \in \Omega \\ -\Delta v = |u|^{q-1}u + k(x) & x \in \Omega \\ u = v = 0 & x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a smooth bounded domain. If $h(x) \equiv k(x) \equiv 0$, the system presents a natural $\mathbb{Z}_2$ symmetry, which guarantees the existence of infinitely many solutions. In this paper we show that the multiplicity structure can be maintained if (p, q) lies below a suitable curve in $\mathbb{R}^2$.

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1 Introduction

Recently, there has been active research conducted on the study of semilinear elliptic systems: see for example [15] for a survey on the argument. Such systems are called variational if solutions can be viewed as critical points of an associated functional defined on a suitable function space. Restricting our attention to second order elliptic systems with two unknowns, whose principal part is given by the differential operator $-\Delta$, we consider systems of the form

$$\begin{cases} -\Delta u = f(x,u,v) & x \in \Omega \\ -\Delta v = g(x,u,v) & x \in \Omega \\ u = v = 0 & x \in \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$.

We say that (1.1) is a potential system if there exists a function $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class C^1 such that

$$\frac{\partial F}{\partial u} = f, \qquad -\frac{\partial F}{\partial v} = g,$$

that is,

$$\begin{cases} -\Delta u = \partial_u F & x \in \Omega \\ +\Delta v = \partial_v F & x \in \Omega \\ u = v = 0 & x \in \partial\Omega. \end{cases}$$

These are the Euler-Lagrange equations of the functional

$$\Phi(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega F(x,u,v)$$

whose critical points are the weak solutions of equations (1.1). If F satisfies suitable growth conditions, this functional is well defined in the cartesian product $E = H_0^1(\Omega) \times H_0^1(\Omega)$, by virtue of the Sobolev embedding theorem; note that Φ has a strongly indefinite quadratic part. Systems of this type have been studied, for example, in [9], [13], [18]; recently existence and multiplicity results have been obtained also for indefinite systems with critical growth (see e.g. [16], [11]).

We say that (1.1) is a Hamiltonian system if there exists a function $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of class C^1 such that

$$\frac{\partial H}{\partial v} = f, \qquad \frac{\partial H}{\partial u} = g,$$

that is,

$$\begin{cases} -\Delta u = \partial_v H & x \in \Omega \\ -\Delta v = \partial_u H & x \in \Omega \\ u = v = 0 & x \in \partial\Omega. \end{cases}$$

By analogy with the scalar case one would guess that the subcritical case occurs if the growths of H with respect to u and v are both less than $2^* = (N+2)/(N-2)$: in this case one could search the weak solutions of the Hamiltonian system as critical points of the functional

$$\Phi(u,v) = \int_\Omega \nabla u \nabla v - \int_\Omega H(x,u,v)$$

which is well defined on $E = H_0^1(\Omega) \times H_0^1(\Omega)$. Nevertheless, the coupling now also occurs in the quadratic part of Φ, and therefore is much stronger than in the potential case. An immediate consequence is that this approach is too restrictive: there is no longer one appropriate choice of function spaces, and for the notion of criticality one has to take into consideration the fact that the system is coupled. In [12], [18] and [21] appeared the notion of Critical Hyperbola, which replaces the notion of critical exponent of the scalar case when $N \geq 3$,

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N} \tag{1.2}$$

that is associated to Hamiltonian system when $\partial_v H$ grows like v^p as $v \to +\infty$ and $\partial_u H$ grows like u^q as $u \to +\infty$, and the dependance on the other variables is of some lower orders. It is known that for any point $(p,q) \in \mathbb{R}^2$ below the critical hyperbola the Hamiltonian system has a nontrivial solution (see [12], [18], [21], [20] and [19]), whereas for points (p,q) on the critical hyperbola one finds the typical problems of non-compactness and non-existence of solutions (see [30] and [23]).

If $F(x,u,v)$ or $H(x,u,v)$ is even in (u,v), the potential, respectively Hamiltonian, system possesses a natural $\mathbb{Z}_2$-symmetry: by analogy with the scalar case, one would expect the existence of infinitely many solutions. In the scalar case, the standard variational method for dealing with even equations is based on the symmetric version of the Mountain Pass Theorem of Ambrosetti-Rabinowitz (see [27]); this theorem is no longer applicable in the case of an elliptic system, since the functional associated is strongly indefinite. Nevertheless, by means of a Galerkin type approximation, one can reduce the strongly indefinite functional to a semidefinite situation (see [10], [6], [14], [7], [16] and others). A different approach to the problem of symmetric indefinite functional was given by Angenent and van der Vorst in [2], who applied Floer's version of Morse theory to Hamiltonian elliptic systems, in the spirit of [8]; see also [3].

As in the scalar case, one could ask if the multiplicity structure can be maintained by adding a perturbation term of lower order. This problem has been extensively investigated in the case of a single equation: a partial answer was independently obtained by Struwe [28], Bahri-Berestycki [5], Rabinowitz [24], Bahri-Lions [8], who showed in important works that the multiplicity structure can be maintained restricting the growth range of the nonlinearity with suitable bounds depending on N.

The problem of perturbed symmetry of elliptic systems have been treated, to our knowledge, only by Clapp, Ding and Hernandez-Linares in [11]. Here the authors obtain a multiplicity result only for perturbed symmetric potential systems, where the perturbation terms can depend also on the unknowns (u,v) (with suitable limitations on the growth in u, v). The proof, as mentioned before, is based on a Galerkin type approximation, which reduces the study of the strongly indefinite functional associated to the potential system to a semidefinite situation, thus allowing the use of the Morse theory methods as in [8].

The aim of this paper is to obtain a multiplicity result for Hamiltonian systems with perturbed symmetries of the type:

$$\begin{cases} -\Delta u = |v|^{p-1}v + h(x) & x \in \Omega \\ -\Delta v = |u|^{q-1}u + k(x) & x \in \Omega \\ u = v = 0 & x \in \partial\Omega \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain. In this case, as observed, there is no longer a one appropriate choice of the function spaces. In [21] and [18] the authors propose the use of Sobolev spaces of fractional order, obtaining the critical hyperbola, whereas in [17] the authors choose a different approach, based on an Orlicz space setting, which yields the same result when the hypotheses overlap. In this paper we follow the idea of de Figueiredo-Felmer [18] and van der Vorst [21], defining the variational setting of (1.3) on a cartesian product of suitable fractional Sobolev spaces: roughly speaking, these spaces, denoted by $H^s(\Omega)$, $s > 0$, consist of functions whose derivative of order s is in $L^2(\Omega)$ (they can be defined by means of interpolation or Fourier expansion). Therefore, even if we reduce to a semidefinite situation by means of Galerkin type approximation, the classical Morse theory methods, as in [8] and in [11], are not applicable. For this reason our approach to the problem of perturbation from symmetry follows the first one proposed by Struwe [28], Bahri-Berestycki [5] and Rabinowitz [24]. This yields our main result:

Theorem 1.1 *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, and let $p, q > 1$ satisfying the following conditions:*

$$\begin{aligned} \frac{1}{p+1} + \frac{1}{q+1} + \frac{p+1}{p(q+1)} &> \frac{2N-2}{N} \quad \text{if } q \geq p \\ \frac{1}{p+1} + \frac{1}{q+1} + \frac{q+1}{q(p+1)} &> \frac{2N-2}{N} \quad \text{if } q \leq p. \end{aligned} \tag{1.4}$$

Then, for any $h, k \in L^2(\Omega)$, problem (1.3) has infinitely many solutions.

Conditions (1.4) define a region in the (p, q) plane which is strictly contained in the subcritical one, as shown in Fig. 1 below.

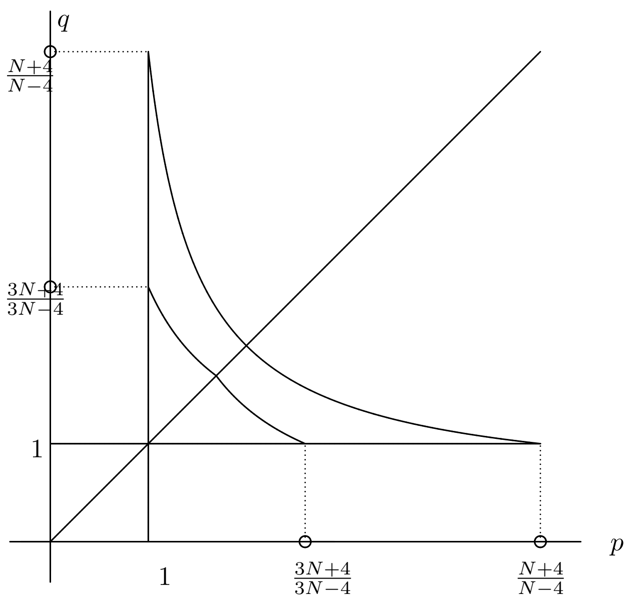

Figure 1: Critical Hyperbola and Theorem (1.1), $N > 4$.

The paper is organized as follows. In Section 2 we describe the variational formulation of the perturbed system (1.3). In Section 3 we consider the symmetric system which arises from (1.3) when $h(x) \equiv k(x) \equiv 0$, exhibiting the unbounded sequence of critical values of the functional associated to the symmetric problem. In Section 4 we define a suitable modified functional J associated to the perturbed problem, in the spirit of [26], whose critical points are solutions of (1.3). In Section 5 we construct minimax sequences, strictly related to the existence of critical points of J, by means of Galerkin type approximation. In Sections 6 and 7, finally, we prove Theorem 1.1 comparing upper and lower bounds of the minimax sequences constructed before.

Remark 1.1 Another possible approach to the perturbed Hamiltonian system (1.3) might be to apply Floer's version of Morse theory as done by Angenent and van der Vorst in [2], in the spirit of [8]. However, in the scalar case the proof deeply depends on the relation between the Morse index of a critical point and the number of non positive eigenvalues of the operator $-\Delta + V(x)$ (see [8] or [29]), whereas in the hamiltonian case this relation makes no sense. Nevertheless, Angenent and van der Vorst give an alternative description of the index of a critical point $z = (u, v)$ in terms of the spectrum of an integral operator associated with the matrix

$$P(x) = \begin{pmatrix} H_{uu}(x, z(x)) & H_{vu}(x, z(x)) \\ H_{uv}(x, z(x)) & H_{vv}(x, z(x)) \end{pmatrix}$$

(see [3], Section 3). We don't known if this variational description of the index could be somehow used to obtain estimates on the growth of the minmax sequences associated to the functional J, as in [8].

2 Variational formulation.

In this section we establish the functional analytic framework needed to study problem (1.3) from the variational point of view, and we give the variational formulation for (1.3). We begin with the spaces $\Theta^r(\Omega)$, which are defined in terms of the domains of fractional powers of the Laplacian in $L^2(\Omega)$ with zero Dirichlet boundary conditions, i.e.

$$-\Delta : H^2(\Omega) \cap H^1_0(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$

where $H^2(\Omega), H^1_0(\Omega)$ are the usual Sobolev spaces; namely $\Theta^r(\Omega) = D((-\Delta)^{r/2})$ for $0 \le r \le 2$, and the corresponding operator is denoted by A^r

$$A^r = (-\Delta)^{r/2} : \Theta^r(\Omega) \to L^2(\Omega).$$

The spaces Θ^r are Hilbert spaces with inner product and associated norm

$$\begin{aligned} (u, v)_{\Theta^r} &= \int_\Omega A^r u A^r v dx = ((-\Delta)^{r/2} u, (-\Delta)^{r/2} v)_{L^2}, \\ \|u\|^2_{\Theta^r} &= \int_\Omega |A^r u|^2 dx = \|(-\Delta)^{r/2} u\|^2_{L^2}, \end{aligned}$$

see Lions and Magenes [22]. Let us fix in $H_0^1(\Omega)$ a system of orthogonal and L^2-normalized eigenfunctions $\varphi_1, \varphi_2, \varphi_3, ...,$ of $-\Delta$, $\varphi_1 > 0$, corresponding to positive eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \uparrow +\infty$, counted with their multiplicity. Then, writing

$$u = \sum_{k=1}^{\infty} \xi_k \varphi_k, \quad \text{with } \xi_k = \int_\Omega u \varphi_k dx,$$

it is well known that

$$A^r u = (-\Delta)^{r/2} u = \sum_{k=1}^{\infty} \lambda_k^{r/2} \xi_k \varphi_k, \tag{2.1}$$

with domain

$$\Theta^r(\Omega) = D((-\Delta)^{r/2}) = \{\sum_{k=1}^{\infty} \xi_k \varphi_k \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^r \xi_k^2 < \infty\}, \tag{2.2}$$

if $r \geq 0$. Then we can identify $\Theta^r(\Omega)$ with the space

$$\bar{\omega}^r = \{\xi = \{\xi_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} \lambda_k^r \xi_k^2 < \infty\}, \quad (\xi, \eta)_r = \sum_{k=1}^{\infty} \lambda_k^r \xi_k \eta_k, \tag{2.3}$$

and

$$(u, v)_{\Theta^r} = ((-\Delta)^{r/2} u, (-\Delta)^{r/2} v)_{L^2} = (\xi, \eta)_r, \qquad \|u\|_{\Theta^r} = |\xi|_r. \tag{2.4}$$

The spaces $\Theta^r(\Omega)$, with $r < 0$, can be introduced as a representation of the dual spaces $\Theta^r(\Omega)'$, using the Fourier characterization (2.2) of $\Theta^r(\Omega)$ (see [21]). The motivation to introduce these spaces is to extend $A(\mathbf{u}) = \int \nabla u \nabla v$ to functions u and v with different regularity properties, and to define an appropriate functional associated to (1.3): this approach has been introduced by Hulshof and van der Vorst in [21], and by de Figueiredo and Felmer in [18]; hence we will be brief. Let us first consider the quadratic part. Using the previous notations, the quadratic form $A(\mathbf{u})$ can also be written as $A(\mathbf{u}) = \int \nabla u \nabla v = \sum_{k=1}^{\infty} \lambda_k \xi_k \eta_k$, where $u = \sum_{k=1}^{\infty} \xi_k \varphi_k$ and $v = \sum_{k=1}^{\infty} \eta_k \varphi_k$. Hence, if we define the product Hilbert spaces

$$E^r(\Omega) = \Theta^r(\Omega) \times \Theta^{2-r}(\Omega), \qquad 0 < r < 2, \tag{2.5}$$

the quadratic form $A(\mathbf{u})$ uniquely extends to a selfadjoint bounded linear operator $L : E^r(\Omega) \to E^r(\Omega)$ as follows:

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k \xi_k \eta_k &= \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^r (\lambda_k^{1-r} \eta_k) \xi_k + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^{2-r} (\lambda_k^{r-1} \xi_k) \eta_k \\ &= \frac{1}{2} ((-\Delta)^{1-r} v, u)_{\Theta^r} + \frac{1}{2} ((-\Delta)^{r-1} v, u)_{\Theta^{2-r}} \\ &= \frac{1}{2} (L\mathbf{u}, \mathbf{u})_{E^r} \end{aligned}$$

where

$$L\mathbf{u} = ((-\Delta)^{1-r} v, (-\Delta)^{r-1} u) \qquad \mathbf{u} = (u, v) \in E^r(\Omega) \tag{2.6}$$

(see [21]). Next we consider the eigenvalue problem

$$L\mathbf{u} = \lambda\mathbf{u} \quad \text{in } E^r(\Omega).$$

Using (2.6) we can write equivalently

$$\begin{aligned} (-\Delta)^{1-r} v &= \lambda u \\ (-\Delta)^{r-1} u &= \lambda v \end{aligned}$$

which give directly

$$v = \lambda^2 v$$

so that $\lambda = \pm 1$. The associated eigenvectors are

$$\mathbf{u}^+ = (u, (-\Delta)^{r-1} u) \qquad \text{for } \lambda = 1 \tag{2.7}$$

and

$$\mathbf{u}^- = (u, -(-\Delta)^{r-1} u) \qquad \text{for } \lambda = -1. \tag{2.8}$$

We can define the eigenspaces

$$E^{\pm} = \{(u, \pm(-\Delta)^{r-1} u) : u \in \Theta^r(\Omega)\}; \tag{2.9}$$

orthonormal bases consisting of eigenvectors of $E^{\pm}$ are given by

$$\left\{ \mathbf{e}_k^{\pm} := \frac{1}{\sqrt{2}} (\lambda_k^{-r/2} \varphi_k, \pm \lambda_k^{r/2-1} \varphi_k) \right\}_{k \in \mathbb{N}}, \tag{2.10}$$

and we have

$$E^r(\Omega) = E^+ \oplus E^- = \left\{ \mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-, \mathbf{u}^{\pm} \in E^{\pm} \right\}. \tag{2.11}$$

We also find that, for $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$,

$$A(\mathbf{u}) = \frac{1}{2} (L\mathbf{u}, \mathbf{u})_{E^r} = A(\mathbf{u}^+) + A(\mathbf{u}^-),$$

and

$$A(\mathbf{u}^+) - A(\mathbf{u}^-) = \frac{1}{2} \|\mathbf{u}\|_{E^r}^2.$$

The derivative of $A(\mathbf{u})$ defines a bilinear form on $E^r(\Omega)$

$$B(\mathbf{u}, \mathbf{\Phi}) = A'(\mathbf{u})\mathbf{\Phi} = (L\mathbf{u}, \mathbf{\Phi})_{E^r}, \quad \mathbf{u}, \mathbf{\Phi} \in E^r(\Omega). \tag{2.12}$$

Next we define the Lagrangian $I(\mathbf{u}) : E^r(\Omega) \to \mathbb{R}$ associated to problem (1.3). First, we need the following Sobolev embedding theorem for fractional order spaces (see [22]):

Theorem 2.1 *If* $0 < 2r < N$, *the inclusions*

$$\Theta^r(\Omega) \hookrightarrow H^r(\Omega) \hookrightarrow L^p(\Omega) \quad \text{if} \quad 1 \le p \le \frac{2N}{N-2r} < \infty \tag{2.13}$$

are bounded; the second inclusion is compact if $1 \le p < 2N/(N-2r)$.
If $2r \ge N$, *then the inclusions are bounded and the second one is compact for any* $1 \le p < \infty$.

This theorem will allow us to define the Lagrangian associated to problem (1.3) in a consistent way. An immediate consequence of Theorem 2.1 is that, by definition of $E^r(\Omega)$,

$$E^r(\Omega) \hookrightarrow L^{p+1}(\Omega) \times L^{q+1}(\Omega)$$

if

$$1 \leq q+1 \leq \frac{2N}{N-2r}, \qquad 1 \leq p+1 \leq \frac{2N}{N+2r-4} \tag{2.14}$$

for $N > 2r$ and $N > 4-2r$, that is,

$$N\left[\frac{1}{2} - \frac{1}{q+1}\right] < r < 2 - N\left[\frac{1}{2} - \frac{1}{p+1}\right]. \tag{2.15}$$

This embedding is compact if both inequalities bounding p and q from above are strict. If $2r \geq N$, there is no restriction on p, whereas if $4-2r \geq N$ there is no restriction on q. Therefore, the Lagrangian I associated to problem (1.3) is well defined on $E^r(\Omega)$ if p and q satisfy inequality (2.14), while we only require that $0 < r < 2$: this restriction is motivated by the fact that we need the compactness of the inclusion $E^r(\Omega) \hookrightarrow L^2(\Omega) \times L^2(\Omega)$. The limiting values of p and q in (2.14) can be represented in the first quadrant of the (p,q)-plane as a section of the well known critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$$

which vanishes for $N \leq 2$.

Combining the extension L of the quadratic form $A(\mathbf{u}) = \int \nabla u \nabla v$ to $E^r(\Omega)$ defined in (2.6) with these inclusions, we can define the Lagrangian

$$\begin{aligned} I(\mathbf{u}) &= \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^r} - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{p+1}\int_\Omega |v|^{p+1}dx \\ &\quad - \int_\Omega ku\,dx - \int_\Omega hv\,dx \end{aligned} \tag{2.16}$$

associated to the perturbed system (1.3), which is well defined for $\mathbf{u} = (u,v) \in E^r(\Omega)$ if p, q satisfy (2.14) and $0 < r < 2$. We remark that critical points of $I(\mathbf{u})$ are classical solutions of problem (1.3): see for example [21]. Hence, to prove Theorem 1.1 it suffices to show that $I(\mathbf{u})$ has an unbounded sequence of critical values. To do so, we require an estimate on the deviation from symmetry of I of the form

$$|I(\mathbf{u}) - I(-\mathbf{u})| \leq \beta(|I(\mathbf{u})|^{1/\mu} + 1) \tag{2.17}$$

for $\mathbf{u}$ in $E^r(\Omega)$ and some $\beta > 0$. Unfortunately I does not satisfy (2.17); however, it can be modified in such a way that the new functional J satisfy (2.17) and large critical values of J are also critical values of I.

3 The symmetric case.

In this section we consider the symmetric problem

$$\begin{cases} -\Delta u = |v|^{p-1}v & x \in \Omega \\ -\Delta v = |u|^{q-1}u & x \in \Omega \\ u = v = 0 & x \in \partial\Omega \end{cases} \tag{3.1}$$

that arises from (1.3) if $k(x) \equiv h(x) \equiv 0$. System (3.1) possesses a natural symmetry, which guarantees the existence of infinitely many solutions. The aim of this section is to exhibit these symmetrical critical values, which will be used later on to construct the critical values of the perturbed system (1.3). The infinitely many solutions of problem (3.1) can be found as critical points of the corresponding functional I by means of a version of the symmetric Mountain Pass Theorem of Ambrosetti-Rabinowitz, valid for strongly indefinite functionals.

Let E be a Banach space with norm $\|\cdot\|$. Suppose that E has a direct sum decomposition $E = E^1 \oplus E^2$ with both E^1, E^2 being infinite dimensional. Let P^i denote the projections from E onto E^i. Assume $\{e_n^1\}$, $\{e_n^2\}$ are basis for E^1 and E^2 respectively. Set

$$X_n = \langle e_1^1, ..., e_n^1 \rangle \oplus E^2, \qquad X^k = E^1 \oplus \langle e_1^2, ..., e_k^2 \rangle, \tag{3.2}$$

and let $(X^k)^\perp$ denote the complement of X^k in E. For a functional $I \in \mathcal{C}^1(E, \mathbb{R})$ set $I_n := I|_{X_n}$ the restriction of I on X_n. Denote the upper and lower level sets, respectively, by $I_a = \{z \in E : I(z) \geq a\}$, $I^b = \{z \in E : I(z) \leq b\}$ and $I_a^b = I_a \cap I^b$. Then we have the following theorem (see [16]).

Theorem 3.1 *Let E as above and let $I \in \mathcal{C}^1(E, \mathbb{R})$ be even with $I(0) = 0$. In addition, suppose, for each $k \in \mathbb{N}$, the conditions below hold:*

(I$_1$) *there exists $R_k > 0$ such that $I(\mathbf{z}) \leq 0$ for all $\mathbf{z} \in X^k$ with $\|\mathbf{z}\| \geq R_k$;*

(I$_2$) *there exist $r_k > 0$ and $a_k \to +\infty$ such that $I(\mathbf{z}) \geq a_k$ for all $\mathbf{z} \in (X^{k-1})^\perp$ with $\|\mathbf{z}\| = r_k$;*

(I$_3$) *I is bounded from above on bounded sets of X^m;*

(I$_4$) *I satisfies the $(PS)_c^*$ condition for any $c \geq 0$: that is, any sequence $\{\mathbf{z}_n\} \subset E$ such that $\mathbf{z}_n \in X_n$ for any $n \in \mathbb{N}$, $I(\mathbf{z}_n) \to c$ and $I_n'(\mathbf{z}_n) \equiv \nabla(I|_{X_n})(\mathbf{z}_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence.*

Then the functional I possesses an unbounded sequence $\{c_k\}$ of critical values.

Remark 3.1 This theorem is a version of the Mountain Pass Theorem of Ambrosetti and Rabinowitz for strongly indefinite symmetric functionals, due to de Figueiredo and Ding (see [16]). Other versions of the same theorem are known, where the $(PS)_c^*$ condition is replaced by other variants, or by the usual (PS) (cf. [4], [9], [14] and references therein).

The sequence of critical values can be constructed by means of certain Galerkin approximations (see [6], [10], [16]), as we briefly recall. Using the previous notations, set

$$B_k := \left\{u \in X^k : \|\mathbf{u}\| \le R_k\right\}, \tag{3.3}$$

the ball of radius R_k in X^k,

$$B_k^n := B_k \cap X_n = \left\{\mathbf{u} \in X^k \cap X_n : \|\mathbf{u}\| \le R_k\right\}, \tag{3.4}$$

and define the following sets of continuous maps

$$\Gamma_k^n := \{h \in \mathcal{C}(B_k^n, X_n) : h(-\mathbf{u}) = -h(\mathbf{u}),\ h(\mathbf{u}) = \mathbf{u} \text{ on } \partial B_k^n\}; \tag{3.5}$$

finally define

$$c_k^n := \inf_{h\in\Gamma_k^n} \sup_{\mathbf{u}\in B_k^n} I(h(\mathbf{u})). \tag{3.6}$$

Then, it can be proved that for each $k \in \mathbb{N}$ fixed (large enough, if necessary), the sequences c_k^n converge to critical values c_k of the functional I, as n tends to $+\infty$; that is, the limits

$$c_k := \lim_{n\to+\infty} c_k^n \tag{3.7}$$

define critical values of the symmetric functional I.

The functional $I(\mathbf{u})$ associated to the symmetric problem (3.1) obviously satisfies the hypotheses of Theorem 3.1, with $E = E^r(\Omega)$, $E^1 = E^-$, $E^2 = E^+$, $e_j^1 := \mathbf{e}_j^-$, $e_j^2 := \mathbf{e}_j^+$, as we briefly prove in the following.

- As regards hypothesis (I_1), observe that $\mathbf{u} \in X^k$ can be decomposed into the orthogonal sum $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$ with $\mathbf{u}^- = (u_1, v_1) \in E^-$ and $\mathbf{u}^+ = (u_2, v_2)$ belonging to the finite dimensional space $\langle \mathbf{e}_1^+, ..., \mathbf{e}_k^+\rangle \equiv X^k \cap E^+$; furthermore, by definition of the eigenvectors (2.10), u_2 and v_2 belong to the finite dimensional spaces $E_k = \langle\varphi_1, ...\varphi_k\rangle$. By definition of I, and recalling the embedding Theorem 2.1, we have, for $\mathbf{u} \in X^k$, $\|\mathbf{u}\|_{E^r} = R$

$$\begin{aligned}
I(\mathbf{u}) &= \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^r} - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{p+1}\int_\Omega |v|^{p+1}dx \\
&\le -\frac{1}{2}\|\mathbf{u}^-\|_{E^r}^2 + \frac{1}{2}\|\mathbf{u}^+\|_{E^r}^2 - c_q\int_\Omega \left(|u_1|^{q+1} + |u_2|^{q+1}\right)dx \\
&\quad -c_p\int_\Omega \left(|v_1|^{p+1} + |v_2|^{p+1}\right)dx \\
&\le -\frac{1}{2}\|\mathbf{u}^-\|_{E^r}^2 - c_q\|u_1\|_{q+1}^{q+1} - c_p\|v_1\|_{p+1}^{p+1} \\
&\quad +\frac{1}{2}\|\mathbf{u}^+\|_{E^r}^2 - c_q\|u_2\|_{\Theta^r}^{q+1} \inf_{w\in E_k,\ \|w\|_{\Theta^r}=1}\int_\Omega |w|^{q+1}dx \\
&\quad -c_p\|v_2\|_{\Theta^{2-r}}^{p+1} \inf_{w\in E_k,\ \|w\|_{\Theta^{2-r}}=1}\int_\Omega |w|^{p+1}dx \\
&\le \frac{1}{2}\|\mathbf{u}\|_{E^r}^2 - c_q(k,r)\|u_2\|_{\Theta^r}^{q+1} - c_p(k,r)\|v_2\|_{\Theta^{2-r}}^{p+1} \\
&\le \frac{1}{2}R^2 - c_{p,q}(k,r)R^{\min(p+1,q+1)}
\end{aligned}$$

which tends to $-\infty$ as $R \to +\infty$.

• The verification of hypothesis (I_2) follows from the classical interpolation inequality in the L^p spaces:

$$\|f\|_{L^{s_0}} \leq \|f\|_{L^{s_1}}^{\alpha} \|f\|_{L^{s_2}}^{1-\alpha}, \qquad \frac{1}{s_0} = \frac{\alpha}{s_1} + \frac{1-\alpha}{s_2}, 0 \leq \alpha \leq 1. \tag{3.8}$$

Indeed, if $\mathbf{u} \in (X^{k-1})^{\perp}$, $(L\mathbf{u}, \mathbf{u})_{E^r} = \|\mathbf{u}\|_{E^r}^2$; combining the interpolation inequality (3.8) with the Sobolev embedding (2.1) yields

$$\begin{aligned} I(\mathbf{u}) &= \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^r} - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{p+1}\int_\Omega |v|^{p+1}dx \\ &\geq \frac{1}{2}\|\mathbf{u}\|_{E^r}^2 - \left(\|u\|_2^{\alpha}\|u\|_{\frac{2N}{N-2r}}^{1-\alpha}\right)^{q+1} - \left(\|v\|_2^{\beta}\|v\|_{\frac{2N}{N+2r-4}}^{1-\beta}\right)^{p+1} \end{aligned}$$

where $\alpha = 1 - \dfrac{N(q-1)}{2r(q+1)}$ and $\beta = 1 - \dfrac{N(p-1)}{(4-2r)(p+1)}$. Now, let us observe that $(X^{k-1})^{\perp} = \langle \mathbf{e}_k^+, \mathbf{e}_{k+1}^+, ...\rangle$: hence, if $\mathbf{u} = (u, v) \in (X^{k-1})^{\perp}$ it is easy to verify, using definitions (2.3), (2.4) that

$$\|u\|_2 \leq \frac{1}{\lambda_k^{r/2}}\|u\|_{\Theta^r}, \tag{3.9}$$

$$\|v\|_2 \leq \frac{1}{\lambda_k^{1-r/2}}\|v\|_{\Theta^{2-r}}. \tag{3.10}$$

Combining (3.9), (3.10) with the Sobolev embedding Theorem 2.1 in the left hand side of the previous inequality yields

$$\begin{aligned} I(\mathbf{u}) &\geq \frac{1}{2}\|\mathbf{u}\|_{E^r}^2 - \frac{C}{\lambda_k^{\alpha(q+1)r/2}}\|u\|_{\Theta^r}^{q+1} - \frac{C}{\lambda_k^{\beta(p+1)(1-r/2)}}\|v\|_{\Theta^{2-r}}^{p+1} \\ &= \frac{1}{2}\|u\|_{\Theta^r}^2 - C\left(\lambda_k^{-\frac{2r(q+1)-N(q-1)}{4(q+1)}}\|u\|_{\Theta^r}\right)^{q+1} \\ &\quad + \frac{1}{2}\|v\|_{\Theta^{2-r}}^2 - C\left(\lambda_k^{-\frac{(4-2r)(p+1)-N(p-1)}{4(p+1)}}\|v\|_{\Theta^{2-r}}\right)^{p+1} \\ &= \|u\|_{\Theta^r}^2\left(\frac{1}{2} - C\lambda_k^{-\frac{2r(q+1)-N(q-1)}{4}}\|u\|_{\Theta^r}^{q-1}\right) \\ &\quad + \|v\|_{\Theta^{2-r}}^2\left(\frac{1}{2} - C\lambda_k^{-\frac{(4-2r)(p+1)-N(p-1)}{4}}\|v\|_{\Theta^{2-r}}^{p-1}\right). \end{aligned}$$

On the other hand, since $(X^{k-1})^{\perp} = \langle \mathbf{e}_k^+, \mathbf{e}_{k+1}^+, ...\rangle \subseteq E^+$, by definition (2.9) of the eigenspace E^+ if $\mathbf{u} = (u, v) \in (X^{k-1})^{\perp}$, then $\mathbf{u} = (u, v) = (u, (-\Delta)^{r-1}u)$, and

$$\|v\|_{\Theta^{2-r}} = \|(-\Delta)^{r-1}u\|_{\Theta^{2-r}} = \|u\|_{\Theta^r}. \tag{3.11}$$

Hence

$$\begin{aligned} I(\mathbf{u}) \quad \geq \quad & \|u\|^2_{\Theta^r} \left(1 - C\lambda_k^{-\frac{2r(q+1)-N(q-1)}{4}} \|u\|^{q-1}_{\Theta^r} \right. \\ & \left. -C\lambda_k^{-\frac{(4-2r)(p+1)-N(p-1)}{4}} \|u\|^{p-1}_{\Theta^r} \right). \end{aligned}$$

By (2.14), the exponents of λ_k in the last expression are strictly positive (recall that we choose p, q below the critical hyperbola); therefore, recalling that $\lambda_k \geq C \cdot k^{2/N}$ for $k \to +\infty$, the verification of (I_2) can be easily concluded.

- Hypothesis (I_3) is clearly verified, whereas the verification of $(PS)^*_c$ is standard, and follows the one which will be given in the proof of (**3**) of Proposition 4.2, recalling that that $I(\mathbf{u}) \equiv J(\mathbf{u})$, so it is omitted here (see also [21] or [14]).

Hence we can conclude that the symmetric problem (3.1) possesses an unbounded sequence of critical values, defined by (3.7) and (3.6).

4 A modified functional.

The aim of this section is to define a suitable modified functional $J(\mathbf{u})$, satisfying (2.17), and whose critical points are solutions of the original perturbed problem (1.3). We first need the following proposition.

Proposition 4.1 *There exists a constant A depending on $\|h\|_{L^2(\Omega)}$, $\|k\|_{L^2(\Omega)}$ such that if $I'(\mathbf{u})\mathbf{u} = 0$, then*

$$\frac{1}{q+1} \int_\Omega |u|^{q+1} dx + \frac{1}{p+1} \int_\Omega |v|^{p+1} dx \leq A\sqrt{I^2(\mathbf{u}) + 1}. \tag{4.1}$$

Proof. We follow the proof in [24]. Suppose that $I'(\mathbf{u})\mathbf{u} = 0$. Then, by simple estimates,

$$\begin{aligned} I(\mathbf{u}) \quad &= \quad I(\mathbf{u}) - \frac{1}{2} I'(\mathbf{u})\mathbf{u} \\ &= \quad (\frac{1}{2} - \frac{1}{q+1}) \int_\Omega |u|^{q+1} dx + (\frac{1}{2} - \frac{1}{p+1}) \int_\Omega |v|^{p+1} dx \\ & \quad\ -\frac{1}{2} \int_\Omega hu dx - \frac{1}{2} \int_\Omega kv dx \\ &\geq \quad C_1 \int_\Omega |u|^{q+1} dx + C_2 \int_\Omega |v|^{p+1} dx - C_h \|u\|_2 - C_k \|v\|_2 \\ &\geq \quad C_3 \int_\Omega |u|^{q+1} dx + C_4 \int_\Omega |v|^{p+1} dx - C_5 \\ &\geq \quad C_6 \left\{ \int_\Omega |u|^{q+1} dx + \int_\Omega |v|^{p+1} dx \right\} - C_7 \end{aligned}$$

where we have used the following inequality: for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|f\|_2 \leq \varepsilon \|f\|^r_r + C_\varepsilon$$

which is valid for any $f \in L^r(\Omega)$, $r > 2$.

Hence (4.1) follows immediately.

The idea underlying the construction of the modified functional J is, roughly speaking, to preserve the perturbation only where $\int |u|^{q+1} + \int |v|^{p+1}$ is bounded from above by $C|I(\mathbf{u})|$, and to eliminate it where not.
To do so, let $\chi \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R})$ be a function satisfying $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t \geq 2$ and $-2 < \chi'(t) < 0$ for $1 < t < 2$. Set

$$Q(\mathbf{u}) = Q(u, v) = 2A\sqrt{I^2(\mathbf{u}) + 1}$$

and

$$\psi(\mathbf{u}) = \psi(u, v) = \chi\left(\frac{1}{Q(\mathbf{u})}\left[\frac{1}{q+1}\int_\Omega |u|^{q+1}dx + \frac{1}{p+1}\int_\Omega |v|^{p+1}dx\right]\right).$$

Note that if $\mathbf{u}$ is a critical point of I, then the argument of χ lies in $[0, \frac{1}{2}]$ by Proposition (4.1) and therefore $\psi(\mathbf{u}) = 1$. Finally we set

$$\begin{aligned} J(\mathbf{u}) \quad = \quad & \frac{1}{2}(L\mathbf{u}, \mathbf{u})_{E^r} - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx - \frac{1}{p+1}\int_\Omega |v|^{p+1}dx \\ & -\psi(\mathbf{u})\left(\int_\Omega kudx + \int_\Omega hvdx\right) \end{aligned} \tag{4.2}$$

for $\mathbf{u} = (u, v)$ in $E^r(\Omega)$. It is easily seen that $J \in \mathcal{C}^1(E^r(\Omega), \mathbb{R})$; furthermore, if $\mathbf{u}$ is a critical point of I, then $J(\mathbf{u}) = I(\mathbf{u})$. the following proposition contains the properties of J which we need.

Proposition 4.2 *Let $f \in L^2(\Omega)$. Then*

(1) *There is a constant $\beta > 0$, depending on $\|h\|_2$ and $\|k\|_2$, such that*

$$|J(\mathbf{u}) - J(-\mathbf{u})| \leq \beta\left(|J(\mathbf{u})|^{\frac{1}{q+1}} + |J(\mathbf{u})|^{\frac{1}{p+1}} + 1\right) \tag{4.3}$$

for $\mathbf{u} \in E^r(\Omega)$.

(2) *There is a constant $M_0 > 0$, depending on $\|h\|_2$, $\|k\|_2$, such that if $J(\mathbf{u}) \geq M_0$ and $J'(\mathbf{u}) = 0$ then $J(\mathbf{u}) = I(\mathbf{u})$ and $I'(\mathbf{u}) = 0$.*

(3) *There is a constant $M_1 \geq M_0$ such that for any $c > M_1$, J satisfies $(PS)_c$ and $(PS)^*_c$.*

Proof. We follow the proof in [24], Proposition 10.16.

- To prove (1), note first that if $\mathbf{u} \notin$ supp $\psi(\cdot) \cup$ supp $\psi(-\cdot)$, then $\psi(\mathbf{u}) = \psi(-\mathbf{u}) = 0$ and $J(\mathbf{u}) = J(-\mathbf{u})$, so that (4.3) is valid. Hence, let us suppose that $\mathbf{u} \in$ supp $\psi(\cdot) \cup$ supp $\psi(-\cdot)$. If $\mathbf{u} \in$ supp ψ, then

$$\left|\int_\Omega kudx + \int_\Omega hvdx\right| \leq \alpha_{q,p}\left(|I(\mathbf{u})|^{\frac{1}{q+1}} + |I(\mathbf{u})|^{\frac{1}{p+1}} + 1\right), \tag{4.4}$$

where $\alpha_{q,p}$ depends on q, p, $\|k\|_2$ and $\|h\|_2$. Indeed, by Schwartz and Hölder inequalities and by definition of $\psi(\mathbf{u})$,

$$\begin{aligned}\left|\int_\Omega kudx + \int_\Omega hvdx\right| &\leq \|k\|_2\|u\|_2 + \|h\|_2\|v\|_2 \leq C\left(\|u\|_{q+1} + \|v\|_{p+1}\right)\\ &\leq C\left(\int_\Omega |u|^{q+1}dx + \int_\Omega |v|^{p+1}dx\right)^{\frac{1}{q+1}} +\\ &\quad C\left(\int_\Omega |u|^{q+1}dx + \int_\Omega |v|^{p+1}dx\right)^{\frac{1}{p+1}}\\ &\leq C\left[4A\left(I^2(\mathbf{u})+1\right)^{\frac{1}{2(q+1)}} + 4A\left(I^2(\mathbf{u})+1\right)^{\frac{1}{2(p+1)}} + 1\right]\end{aligned}$$

which implies directly (4.4). Now, by definition,

$$|J(\mathbf{u}) - J(-\mathbf{u})| \leq (\psi(\mathbf{u}) + \psi(-\mathbf{u}))\left|\int_\Omega kudx + \int_\Omega hvdx\right|;$$

and combining this inequality with (4.4) yields

$$\begin{aligned}|J(\mathbf{u}) - J(-\mathbf{u})| &\leq \alpha_{q,p}(\psi(\mathbf{u}) + \psi(-\mathbf{u}))\left(|I(\mathbf{u})|^{\frac{1}{q+1}} + |I(\mathbf{u})|^{\frac{1}{p+1}} + 1\right)\\ &\leq c(\psi(\mathbf{u}) + \psi(-\mathbf{u}))\left(|J(\mathbf{u})|^{\frac{1}{q+1}} + \left|\int_\Omega kudx\right|^{\frac{1}{q+1}}\right.\\ &\quad \left. + |J(\mathbf{u})|^{\frac{1}{p+1}} + \left|\int_\Omega hvdx\right|^{\frac{1}{p+1}} + 1\right)\\ &\leq 2c\left(|J(\mathbf{u})|^{\frac{1}{q+1}} + \left|\int_\Omega kudx\right|^{\frac{1}{q+1}} + |J(\mathbf{u})|^{\frac{1}{p+1}} + \left|\int_\Omega hvdx\right|^{\frac{1}{p+1}} + 1\right).\end{aligned}$$

Since the exponents are smaller than 1, the k and h terms on the right-hand side can be absorbed into the left-hand side, yielding (4.3). A similar estimate is valid for $\mathbf{u} \in \operatorname{supp} \psi(-\cdot)$.

- To prove (**2**), it suffices to show that if M_0 is large and $\mathbf{u}$ is a critical point of J with $J(\mathbf{u}) \geq M_0$, then

$$Q(\mathbf{u})^{-1}\left(\frac{1}{q+1}\int_\Omega |u|^{q+1}dx + \frac{1}{p+1}\int_\Omega |v|^{p+1}dx\right) < 1; \tag{4.5}$$

indeed, by definition of ψ, (4.5) implies $\psi(\mathbf{v}) \equiv 1$ for $\mathbf{v}$ near $\mathbf{u}$. Hence $\psi'(\mathbf{u}) = 0$, so $J(\mathbf{u}) = I(\mathbf{u})$, $J'(\mathbf{u}) = I'(\mathbf{u})$ and (**2**) follows. Therefore we will prove that (4.5) holds. Let $\mathbf{u} = (u, v)$ and $\mathbf{w} = (w, z)$ be in $E^r(\Omega)$. Then, by definition of J,

$$\begin{aligned}J'(\mathbf{u})\mathbf{w} &= (L\mathbf{u}, \mathbf{w})_{E^r} - \int_\Omega |u|^{q-1}uwdx - \int_\Omega |v|^{p-1}vzdx \\ &\quad -\psi(\mathbf{u})\left(\int_\Omega kwdx + \int_\Omega hzdx\right) - \psi'(\mathbf{u})\mathbf{w}\left(\int_\Omega kudx + \int_\Omega hvdx\right),\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}\psi'(\mathbf{u})\mathbf{w} &= \chi'(\theta(\mathbf{u}))\theta'(\mathbf{u})\mathbf{w} \\ &= \chi'(\theta(\mathbf{u}))Q(\mathbf{u})^{-2}\left\{Q(\mathbf{u})\left[\int_\Omega |u|^{q-1}uwdx + \int_\Omega |v|^{p-1}vzdx\right]\right. \\ &\quad \left. -(2A)^2\theta(\mathbf{u})I(\mathbf{u})I'(\mathbf{u})\mathbf{w}\right\}\end{aligned}$$

and

$$\theta(\mathbf{u}) = Q(\mathbf{u})^{-1}\left(\frac{1}{q+1}\int_\Omega |u|^{q+1}dx + \frac{1}{p+1}\int_\Omega |v|^{p+1}dx\right).$$

Regrouping terms in (4.6) yields

$$\begin{aligned}J'(\mathbf{u})\mathbf{w} &= (1+T_1(\mathbf{u}))(L\mathbf{u},\mathbf{w})_{E^r} \\ &\quad -(1+T_2(\mathbf{u}))\left(\int_\Omega |u|^{q-1}uwdx + \int_\Omega |v|^{p-1}vzdx\right) \\ &\quad -(\psi(\mathbf{u})+T_1(\mathbf{u}))\left(\int_\Omega kwdx + \int_\Omega hzdx\right),\end{aligned} \tag{4.7}$$

where

$$T_1(\mathbf{u}) = \chi'(\theta(\mathbf{u}))(2A)^2\theta(\mathbf{u})Q(\mathbf{u})^{-2}I(\mathbf{u})\left(\int_\Omega kudx + \int_\Omega hvdx\right) \tag{4.8}$$

and

$$T_2(\mathbf{u}) = T_1(\mathbf{u}) + \chi'(\theta(\mathbf{u}))Q(\mathbf{u})^{-1}\left(\int_\Omega kudx + \int_\Omega hvdx\right). \tag{4.9}$$

Let us now consider the term $J'(\mathbf{u})\mathbf{u}$: from (4.7) we have

$$\begin{aligned}J'(\mathbf{u})\mathbf{u} &= (1+T_1(\mathbf{u}))(L\mathbf{u},\mathbf{u})_{E^r} \\ &\quad -(1+T_2(\mathbf{u}))\left(\int_\Omega |u|^{q+1}dx + \int_\Omega |v|^{p+1}dx\right) \\ &\quad -(\psi(\mathbf{u})+T_1(\mathbf{u}))\left(\int_\Omega kudx + \int_\Omega hvdx\right).\end{aligned} \tag{4.10}$$

Therefore, if $\psi(\mathbf{u}) = 1$ and $T_1(\mathbf{u}) = T_2(\mathbf{u}) = 0$, we obtain $J'(\mathbf{u})\mathbf{u} = I'(\mathbf{u})\mathbf{u}$ and $J(\mathbf{u}) = I(\mathbf{u})$, so that (4.5) follows from (4.1). Otherwise, consider

$$I(\mathbf{u}) - \frac{1}{2(1+T_1(\mathbf{u}))}J'(\mathbf{u})\mathbf{u} \tag{4.11}$$

and suppose that $\mathbf{u}$ is a critical point for J. Since $0 \le \psi(\mathbf{u}) \le 1$, if $T_1(\mathbf{u})$ and $T_2(\mathbf{u})$ are both small enough, the calculation made in the proof of Proposition 4.1, when carried out for (4.11), leads to (4.1) with A replaced by a larger constant which is smaller than $2A$. But then (4.5) holds. Therefore, it suffices to show that $T_1(\mathbf{u}), T_2(\mathbf{u}) \to 0$ as $M_0 \to \infty$. If $u \notin \text{supp } \psi$ then $T_1(\mathbf{u}) = T_2(\mathbf{u}) = 0$; hence we assume that $u \in \text{supp } \psi$. Observe first that, by definition (4.8) of T_1 and (4.4)

$$|T_1(\mathbf{u})| \le 4\alpha_{p.q}(|I(\mathbf{u})|^{\frac{1}{q+1}} + |I(\mathbf{u})|^{\frac{1}{p+1}} + 1)|I(\mathbf{u})|^{-1}, \tag{4.12}$$

where we have used the properties $|\chi'| < 2$ and $\theta(\mathbf{u}) < 2$ if $\mathbf{u} \in \text{supp}\ \psi$. Therefore, to conclude we need an estimate relating $I(\mathbf{u})$ and $J(\mathbf{u})$ for $u \in \text{supp}\ \psi$. By definition,

$$I(\mathbf{u}) \geq J(\mathbf{u}) - \left| \int_\Omega kudx + \int_\Omega hvdx \right|;$$

thus, by (4.4),

$$I(\mathbf{u}) + \alpha_{q,p} \left(|I(\mathbf{u})|^{\frac{1}{q+1}} + |I(\mathbf{u})|^{\frac{1}{p+1}} \right) \geq J(\mathbf{u}) - \alpha_{p,q} \geq M_0/2 \qquad (4.13)$$

for M_0 large enough. If $I(\mathbf{u}) \leq 0$, estimate (4.13) implies that

$$\frac{\alpha_{p,q}^{(q+1)'}}{(q+1)'} + \frac{\alpha_{p,q}^{(p+1)'}}{(p+1)'} + \frac{1}{q+1}|I(\mathbf{u})| + \frac{1}{p+1}|I(\mathbf{u})| \geq M_0/2 + |I(\mathbf{u})|,$$

where $(q+1)'$, $(p+1)'$ are, respectively, the conjugate exponents of $q+1$, $p+1$. But this is impossible for $p, q \geq 1$ and M_0 large enough: therefore, we can assume $I(\mathbf{u}) > 0$. In this case, (4.13) implies that $I(\mathbf{u}) \to +\infty$ as $M_0 \to +\infty$, which shows, together with (4.12), that $T_1(\mathbf{u}) \to 0$ as $M_0 \to +\infty$. Analogous estimates yield $T_2(\mathbf{u}) \to 0$ as $M_0 \to +\infty$, and (**2**) holds.

- Let us now verify (**3**). We have to show that there is a constant $M_1 \geq M_0$ such that for any sequence $\{\mathbf{u}_n\}$ in $E^r(\Omega)$ satisfying

$$M_1 < J(\mathbf{u}_n) < K \text{ for } n \text{ large}, \qquad J'(\mathbf{u}_n) \to 0 \text{ as } n \to \infty \qquad (4.14)$$

has a convergent subsequence. The key point here is to prove that such a sequence is necessarily bounded in $E^r(\Omega)$. Indeed, by (4.7) and (2.12),

$$J'(\mathbf{u}_n) = (1 + T_1(\mathbf{u}_n))A'(\mathbf{u}_n) - \mathcal{H}(\mathbf{u}_n)$$

where $\mathcal{H}$ is compact and $|T_1(\mathbf{u}_n)| \leq 1/2$ for M_1 large enough. Therefore, since $J'(\mathbf{u}_n)$ converges in $(E^r(\Omega))' = E^{-r}(\Omega)$, the compactness of $\mathcal{H}$ implies that a subsequence of $A'(\mathbf{u}_n)$ also converges. In view of (2.12), we also have that $L\mathbf{u}_n$ and $\mathbf{u}_n$ converge in $E^r(\Omega)$, because L is invertible. To prove that $\mathbf{u}_n$ is bounded we proceed as follows. By (4.14), for any $\varepsilon > 0$ there is a $n(\varepsilon)$ such that for $n \geq n(\varepsilon)$

$$\begin{aligned}
K + \varepsilon \|\mathbf{u}_n\|_{E^r} &\geq J(\mathbf{u}_n) - \frac{1}{2(1 + T_1(\mathbf{u}_n))} J'(\mathbf{u}_n)\mathbf{u}_n \\
&= \left(\frac{1 + T_2(\mathbf{u}_n)}{1 + T_1(\mathbf{u}_n)} \frac{1}{2} - \frac{1}{q+1} \right) \int_\Omega |u_n|^{q+1} dx \\
&\quad + \left(\frac{1 + T_2(\mathbf{u}_n)}{1 + T_1(\mathbf{u}_n)} \frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega |v_n|^{p+1} dx \\
&\quad + \left(\frac{\psi(\mathbf{u}_n) + T_1(\mathbf{u}_n)}{1 + T_2(\mathbf{u}_n)} \frac{1}{2} - \psi(\mathbf{u}_n) \right) \int_\Omega (ku_n + hv_n) dx;
\end{aligned}$$

recalling that $T_1(\mathbf{u}), T_2(\mathbf{u}) \to 0$ as $M_1 \to +\infty$, we can choose M_1 sufficiently large such that the coefficients of the integral terms $\int |u_n|^{q+1}$, $\int |v_n|^{p+1}$ are strictly positive, that is (remember that $0 \le \psi(\mathbf{u}_n) \le 1$),

$$\begin{aligned} K + \varepsilon \|\mathbf{u}_n\|_{E^r} &\ge C_q \int_\Omega |u_n|^{q+1} dx + C_p \int_\Omega |v_n|^{p+1} dx - C \int_\Omega |ku_n + hv_n| \, dx \\ &\ge C_q \int_\Omega |u_n|^{q+1} dx + C_p \int_\Omega |v_n|^{p+1} dx - C \left(\|k\|_2 \|u_n\|_2 + \|h\|_2 \|v_n\|_2 \right) \\ &\ge C_q \int_\Omega |u_n|^{q+1} dx + C_p \int_\Omega |v_n|^{p+1} dx - C' \left(\|u_n\|_{q+1} + \|v_n\|_{p+1} \right) \\ &\ge C'_q \int_\Omega |u_n|^{q+1} dx + C'_p \int_\Omega |v_n|^{p+1} dx - C'' \end{aligned}$$

where the constants appearing in the previous inequalities depend only on M_1, q, p and not on n. Therefore we can conclude that, for some new constants $K, \varepsilon > 0$,

$$K + \varepsilon \|\mathbf{u}_n\|_{E^r} \ge \int_\Omega |u_n|^{q+1} dx + \int_\Omega |v_n|^{p+1} dx. \tag{4.15}$$

Decompose $\mathbf{u}_n = \mathbf{u}_n^+ + \mathbf{u}_n^-$, where $\mathbf{u}_n^+ \in E^+$, $\mathbf{u}_n^- \in E^-$. Writing $\mathbf{u}_n^\pm = (u_n^\pm, v_n^\pm)$, we also have, by (2.7), (2.8) and (2.11) (the value of the constant C can possibly change)

$$\begin{aligned} \|\mathbf{u}_n^\pm\|_{E^r}^2 - \varepsilon \|\mathbf{u}_n^\pm\|_{E^r} &\le \left| \left(L\mathbf{u}_n, \mathbf{u}_n^\pm \right)_{E^r} - \frac{1}{1 + T_1(\mathbf{u}_n)} J'(\mathbf{u}_n)\mathbf{u}_n^\pm \right| \\ &\le \frac{1 + T_2(\mathbf{u}_n)}{1 + T_1(\mathbf{u}_n)} \int_\Omega \left(|u_n|^q |u_n^\pm| + |v_n|^p |v_n^\pm| \right) dx \\ &\quad + \frac{\psi(\mathbf{u}_n) + T_1(\mathbf{u}_n)}{1 + T_1(\mathbf{u}_n)} \int_\Omega \left(|k||u_n^\pm| + |h||v_n^\pm| \right) dx \\ &\le C \|u_n\|_{q+1}^q \|u_n^\pm\|_{q+1} + C \|v_n\|_{p+1}^p \|v_n^\pm\|_{p+1} \\ &\quad + C \left(\|u_n^\pm\|_2 + |v_n^\pm\|_2 \right) \\ &\le C \|u_n\|_{q+1}^q \|u_n^\pm\|_{\Theta^r} + C \|v_n\|_{p+1}^p \|v_n^\pm\|_{\Theta^{2-r}} + C \|\mathbf{u}_n^\pm\|_{E^r} \\ &\le C \left(\|u_n\|_{q+1}^q + \|v_n\|_{p+1}^p + 1 \right) \|\mathbf{u}_n^\pm\|_{E^r}. \end{aligned}$$

Dividing the first and the last expressions by $\|\mathbf{u}_n^\pm\|_{E^r}$ we obtain

$$\|\mathbf{u}_n^\pm\|_{E^r} - \varepsilon \le C \left(\|u_n\|_{q+1}^q + \|v_n\|_{p+1}^p + 1 \right). \tag{4.16}$$

Combining (4.16) for $\mathbf{u}_n = \mathbf{u}_n^+ + \mathbf{u}_n^-$, together with (4.15), it follows that, possibly for some new constants,

$$\|\mathbf{u}_n\|_{E^r} \le C\{1 + \{K + \varepsilon \|\mathbf{u}_n\|_{E^r}\}^{q/(q+1)} + \{K + \varepsilon \|\mathbf{u}_n\|_{E^r}\}^{p/(p+1)}\}$$

which keeps $\|\mathbf{u}_n\|_{E^r}$ away from infinity. This implies that the Palais-Smale condition is satisfied for M_1 sufficiently large. The verification of $(PS)_c^*$ follows in the same way, and thereby the proof is concluded.

Property $(\mathbf{2})$ of Proposition 3.2 guarantees that large critical values of J are also critical values of I. Hence, in what follows we shall seek large critical values of J.

5 Minimax methods.

The aim of this section is to construct suitable minimax sequences which are strictly related to the existence of critical values of the modified functional J, applying the method developed by Rabinowitz to deal with perturbation of symmetry (see [24]) in this case. The idea is to construct suitable minimax sequences d_k^n, "perturbing" the ones defining the symmetric critical values (in a sense that will be specified): comparison arguments between the values of the two sequences will yield our thesis.

Observe first that, following the same lines as for the verification of (I_1) of Theorem 3.1, it is not hard to prove that for any $k \in \mathbb{N}$ there is a R_k such that $J(\mathbf{u}) \leq 0$ if $\mathbf{u} \in (B_k)^{\complement}$, where B_k is the sphere of radius R_k in X^k defined in (3.3). Hence let us define the minimax sequences c_k^n as in (3.6), with $J(\mathbf{u})$ instead of $I(\mathbf{u})$, that is,

$$c_k^n = \inf_{h \in \Gamma_k^n} \sup_{\mathbf{u} \in B_k^n} J(h(\mathbf{u})).$$

It is easy to verify that there is a sequence b_k, independent on n, such that for any k large enough

$$c_k^n \leq b_k \qquad \text{for any } n \in \mathbb{N}. \tag{5.1}$$

Indeed, by definition (since $\mathrm{id} \in \Gamma_k^n$) and applying Theorem 2.1,

$$\begin{aligned}
c_k^n &\leq \sup_{\mathbf{u} \in B_k^n} J(\mathbf{u}) \leq \sup_{\mathbf{u} \in B_k} J(\mathbf{u}) \\
&\leq \sup_{\mathbf{u} \in B_k} \left[\frac{1}{2}\|\mathbf{u}^+\|_{E^r}^2 - \frac{1}{2}\|\mathbf{u}^-\|_{E^r}^2 + C\|\mathbf{u}\|_2 \right] \\
&\leq \sup_{\mathbf{u} \in B_k} \left[\frac{1}{2}\|\mathbf{u}^+\|_{E^r}^2 - \frac{1}{2}\|\mathbf{u}^-\|_{E^r}^2 + C\|\mathbf{u}\|_{E^r} \right] \\
&\leq \sup_{\mathbf{u} \in B_k} \left[\frac{1}{2}\|\mathbf{u}\|_{E^r}^2 + C\|\mathbf{u}\|_2 \right] \leq C R_k^2
\end{aligned}$$

for each $n \in \mathbb{N}$, and for $k \to +\infty$. Following the idea in [14], [10], [16], it is also possible to prove that the sequence c_k^n is bounded from below by a sequence a_k which is independent of n: that is, for any k large enough

$$a_k \leq c_k^n \qquad \text{for any } n \in \mathbb{N}. \tag{5.2}$$

This fact, together with (5.1), will be used to prove the existence of the limit sequence c_k, as in the symmetric case. To prove (5.2), we need first the following version of the Intersection Lemma:

Lemma 5.1 *Let us assume B_k^n, Γ_k^n and $(X^{k-1})^\perp$ as before. Then, for any $h \in \Gamma_k^n$, $0 < R < R_k$ there holds*

$$h(B_k^n) \cap \partial B_R \cap (X^{k-1})^\perp \neq \emptyset \qquad \textit{for any } n \in \mathbb{N}. \tag{5.3}$$

Proof. We follow the proof given by Rabinowitz (Proposition 9.23 in [24]). Hence we will be brief. Let $\hat{O}_k^n := \{x \in B_k^n \mid h(x) \in B_R\}$. Since h is odd, $0 \in \hat{O}_k^n$. Let O_k^n denote the component of $\hat{O}_k^n$ containing 0. Since B_k^n is bounded, O_k^n is a symmetric bounded neighborhood of 0 in $X^k \cap X_n$; therefore $\gamma(\partial O_k^n) = k + n$, where γ denotes the Krasnoleskii genus. We claim that

$$h(\partial O_k^n) \subset \partial B_R. \tag{5.4}$$

Assuming (5.4) for the moment, set $W = \{\mathbf{x} \in B_k^n \mid h(\mathbf{x}) \in \partial B_R\}$. Then (5.4) implies $\partial O_k^n \subset W$; hence, by the monotonicity property of Krasnoleskii genus, $\gamma(W) = k + n$, so that $\gamma(h(W)) \geq k + n$. Therefore, recalling that $\mathrm{codim}(X^{k-1})^\perp = k - 1$, $h(W) \cap (X^{k-1})^\perp \neq \emptyset$. On the other hand, by definition, $h(W) \subset h(B_k^n) \cap \partial B_R$; consequently (5.3) holds.

It remains to prove (5.4). Suppose $\mathbf{x} \in \partial O_k^n$ and $h(\mathbf{x}) \in \overset{\circ}{B}_R$. If $\mathbf{x} \in \overset{\circ}{B}{}_k^n$, there is a neighborhood N of $\mathbf{x}$ such that $h(N) \in \overset{\circ}{B}_R$. But then $\mathbf{x} \notin \partial O_k^n$. Thus $\mathbf{x} \in \partial B_k^n$, with ∂ relative to $X^n \cap X_k$. But on ∂B_k^n, $h = id$. Consequently, if $\mathbf{x} \in \partial B_k^n$ and $h(\mathbf{x}) \in \overset{\circ}{B}_R$, $R > \|h(\mathbf{x})\| = \|\mathbf{x}\| = R_k$, contrary to the hypothesis. Thus (5.4) must hold.

Applying the Intersection Lemma 5.1 we are now able to prove (5.2). Let us fix k large enough. Then, for any $n \in \mathbb{N}$, for any $h \in \Gamma_k^n$ and $0 < R < R_k$ there is a $\mathbf{w}_n \in h(B_k^n) \cap \partial B_R \cap (X^{k-1})^\perp$, so that by definition of c_k^n,

$$\begin{aligned} c_k^n &= \inf_{h \in \Gamma_k^n} \sup_{\mathbf{u} \in B_k^n} J(h(\mathbf{u})) \\ &\geq \inf_{h \in \Gamma_k^n} J(\mathbf{w}_n) \\ &\geq \inf_{h \in \Gamma_k^n} \sup_{0<R<R_k} \inf_{\mathbf{u} \in \partial B_R \cap (X^{k-1})^\perp} J(\mathbf{u}) \\ &= \sup_{0<R<R_k} \inf_{\mathbf{u} \in \partial B_R \cap (X^{k-1})^\perp} J(\mathbf{u}). \end{aligned}$$

Observe now that the last term of the previous inequality does not depend on n, so that (5.2) is proved. Combining (5.2) with (5.1) yields, for k large enough,

$$a_k \leq c_k^n \leq b_k \qquad \text{for any } n \in \mathbb{N}$$

so that it is possible passing to the limit as n tends to $+\infty$ (up to a subsequence, if necessary), defining

$$c_k = \lim_{n \to +\infty} c_k^n$$

as in the symmetric case. This new minimax sequence c_k, constructed for $J(\mathbf{u})$, is not in general a sequence of critical values for J, unless $k(x) \equiv h(x) \equiv 0$.

Let us now construct new sequences d_k^n, d_k appropriately "perturbing" c_k^n. First of all, we define a new sequence of sets

$$U_k^n := \left\{ \mathbf{u} = t\mathbf{e}_{k+1}^+ + \mathbf{w} \mid t \in [0, R_{k+1}], \mathbf{w} \in B_{k+1}^n \cap X^k, \|u\| \le R_{k+1} \right\}; \tag{5.5}$$

then define the new classes of functions

$$\begin{aligned} \Lambda_k^n : \;\; = \;\; & \left\{ H \in \mathcal{C}(U_k^n, X_n) : H\mid_{B_k^n} \in \Gamma_k^n,\ H(\mathbf{u}) = \mathbf{u} \right. \\ & \left. \text{on } Q_k^n = (\partial B_{k+1}^n \cap X^{k+1}) \cup \left((B_{R_{k+1}} \setminus B_{R_k}) \cap X^k \right) \right\}. \end{aligned} \tag{5.6}$$

As one can easily observe, the new set U_k^n is nothing that an half of the sphere B_{k+1}^n, and the new class of functions Λ_k^n are defined such that any $H \in \Lambda_k^n$, suitably symmetrized, belongs also to Γ_{k+1}^n. Combining these facts with the estimate on the deviation from symmetry of J, (4.3), will be the key ingredient to obtain an upper bound on the minimax sequences c_k^n (and then also on c_k). Now set

$$d_k^n := \inf_{H \in \Lambda_k^n} \sup_{\mathbf{u} \in U_k^n} J(H(\mathbf{u})). \tag{5.7}$$

Comparing the definition of d_k^n with the one of c_k^n, (3.6) shows that $d_k^n \ge c_k^n$. Furthermore, we can easily prove that d_k^n is bounded from above independently on n, as in (5.1); indeed,

$$\begin{aligned} d_k^n \;\; & \le \sup_{\mathbf{u} \in U_k^n} J(\mathbf{u}) \le \sup_{\mathbf{u} \in U_k} J(\mathbf{u}) \\ & \le \sup_{\mathbf{u} \in B_{k+1}} \left[\frac{1}{2} \|\mathbf{u}^+\|_{E^r}^2 - \frac{1}{2} \|\mathbf{u}^-\|_{E^r}^2 + C\|\mathbf{u}\|_2 \right] \\ & \le \sup_{\mathbf{u} \in B_{k+1}} \left[\frac{1}{2} \|\mathbf{u}^+\|_{E^r}^2 - \frac{1}{2} \|\mathbf{u}^-\|_{E^r}^2 + C\|\mathbf{u}\|_{E^r} \right] \\ & \le \sup_{\mathbf{u} \in B_{k+1}} \left[\frac{1}{2} \|\mathbf{u}\|_{E^r}^2 + C\|\mathbf{u}\|_2 \right] \le C R_{k+1}^2 \end{aligned}$$

for each $n \in \mathbb{N}$, and for $k \to +\infty$. Therefore,

$$a_k \le c_k^n \le d_k^n \le \tilde{b}_k \quad \text{ for any } n \in \mathbb{N}$$

and it is possible to define (up to a subsequence)

$$d_k = \lim_{n \to +\infty} d_k^n.$$

Clearly, $d_k \ge c_k$. Furthermore, we have the following fundamental proposition.

Proposition 5.1 *Assume $d_k > c_k \ge M_1$. For $\delta \in (0, d_k - c_k)$, define*

$$\Lambda_k^n(\delta) := \{ H \in \Lambda_k^n \mid J(H(\mathbf{u})) \le c_k^n + \delta \textit{ for } \mathbf{u} \in B_k^n \}$$

and

$$d_k^n(\delta) := \inf_{H \in \Lambda_k^n(\delta)} \sup_{\mathbf{u} \in U_k^n} J(H(\mathbf{u})). \tag{5.8}$$

Then (eventually up to a subsequence) the limit

$$d_k(\delta) := \lim_{n\to+\infty} d_k^n(\delta) \tag{5.9}$$

exists for any $k \in \mathbb{N}$ large enough, and it is a critical value of J.

Proof. The proof of Proposition 5.1 is based on the following standard "deformation lemma" (see, e.g., [1]).

Lemma 5.2 *Let E be a real Banach space, let $I \in \mathcal{C}^1(E,\mathbb{R})$ and assume that I satisfies $(PS)_c$. For $s \in \mathbb{R}$ set $A_s = \{u \in E \mid I(u) \le s\}$. If c is not a critical value of I, given an $\bar\varepsilon > 0$ there exists an $\varepsilon \in (0,\bar\varepsilon)$ and $\eta \in \mathcal{C}([0,1]\times E, E)$ such that:*

1° *$\eta(t,u) = u$ for all $t \in [0,1]$, if $I(u) \notin [c-\bar\varepsilon, c+\bar\varepsilon]$*

2° *$\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.*

The proof of Proposition 5.1 follows along the same lines as in [24], adapted to this sort of Galerkin approximation inspired by [14], [10] and others. If $d_k > c_k$, for any $\delta \in (0, d_k - c_k)$ there is a $n_k \in \mathbb{N}$ (which depends on k,δ) such that

$$0 < \delta < d_k^n - c_k^n \qquad \text{for any } n \ge n_k.$$

Consider now, for $n > n_k$, $d_k^n(\delta)$ as defined in (5.8), and assume that it is not a critical value of $J_n = J|_{X_n}$. Set $\bar\varepsilon = \frac{1}{2}(d_k^n - c_k^n - \delta) > 0$. Then there exist ε and η as in the deformation lemma 5.2. Choose $H \in \Lambda_k^n(\delta)$ such that

$$\max_{\mathbf{u}\in U_k^n} J(H(\mathbf{u})) \le d_k^n(\delta) + \varepsilon. \tag{5.10}$$

Consider $\eta(1, H(\cdot))$: clearly this function belongs to $\mathcal{C}(U_k^n, X_n)$; if $\mathbf{u} \in Q_k^n$, $H(\mathbf{u}) = \mathbf{u}$ since $H \in \Lambda_k^n$. Therefore, $J(H(\mathbf{u})) = J(\mathbf{u}) \le 0$ via the definition of R_k and R_{k+1} (which do not depend on n). Moreover, by the choice of $\bar\varepsilon$ and the assumption $c_k > M_1 > 0$, $J(H(\mathbf{u})) = J(\mathbf{u}) \le 0 < c_k^n + \bar\varepsilon < d_k^n - \bar\varepsilon \le c_k^n(\delta) - \bar\varepsilon$. Hence, by 1° of the deformation lemma 5.2, we have

$$\eta(1, H(\mathbf{u})) = H(\mathbf{u}) = \mathbf{u} \qquad \text{for } \mathbf{u} \in Q_k^n.$$

Further, since $H \in \Lambda_k^n(\delta)$, if $\mathbf{u} \in B_k^n$,

$$J(H(\mathbf{u})) \le c_k^n + \delta < d_k^n - \bar\varepsilon \le d_k^n(\delta) - \bar\varepsilon$$

by the choice of δ and $\bar\varepsilon$. Therefore, again by 1° of the deformation lemma 5.2,

$$\eta(1, H(\mathbf{u})) = H(\mathbf{u}) \qquad \text{for } \mathbf{u} \in B_k^n,$$

so that we can conclude that $\eta(1, H(\cdot)) \in \Lambda_k^n(\delta)$. Thus, by definition of $d_k^n(\delta)$, we get

$$d_k^n(\delta) \le \max_{\mathbf{u}\in U_k^n} J(\eta(1, H(\mathbf{u}))). \tag{5.11}$$

On the other hand, (5.10) and 2° of Lemma 5.1 yields

$$\max_{\mathbf{u}\in U_k^n} J(\eta(1,H(\mathbf{u}))) \le d_k^n(\delta) - \varepsilon,$$

contrary to (5.11). Hence $d_k^n(\delta)$ is a critical value of J_n. Now, let us apply the $(PS)_c^*$ condition, which is satisfied by $J(\mathbf{u})$. Indeed, we have just proved, for any k large enough, the existence of a sequence $\{\mathbf{z}_k^n\} \subset E^r$ such that for each $n \ge n_k$, $\mathbf{z}_k^n \in X_n$, $J_n'(\mathbf{z}_k^n) = 0$ and $J(\mathbf{z}_k^n) = d_k^n(\delta) \to d_k(\delta) > M_1$ as $n \to +\infty$ (the existence of the limit $d_k(\delta)$, up to a subsequence, can be easily proved, since $c_k^n \le d_k^n(\delta) \le d_k^n$). Hence, $\mathbf{z}_k^n$ is a $(PS)_c^*$ sequence (with $c = d_k(\delta) > M_1$), and by property $(\mathbf{I}_4)$ of Proposition 4.2 we can conclude that, along a subsequence, $\mathbf{z}_k^n \to \mathbf{z}_k$ as $n \to +\infty$, with $J(\mathbf{z}_k) = d_k(\delta)$ and $J'(\mathbf{z}_k) = 0$. Hence $d_k(\delta) = \lim_{n\to+\infty} d_k^n(\delta)$ (up to a subsequence) is a critical value of $J(\mathbf{u})$, and the proof is completed.

On the basis of Proposition 5.1, to prove the existence of infinitely many critical values for $J(\mathbf{u})$ it suffices to show that, up to a subsequence,

$$d_k > c_k \ge M_1 \qquad \text{for } k \in \mathbb{N}, \text{ and } c_k \to +\infty \text{ as } k \to +\infty.$$

This will be done in the following sections, estimating the growth of c_k.

6 A lower bound for c_k

The aim of this section is to obtain an estimate from below on the growth of the minimax sequence c_k. First of all, we will obtain lower bounds for the minimax values c_k^n, and then also for the sequence c_k: we recall that this sequence will in general not consist of critical values of J, unless $k(x) \equiv h(x) \equiv 0$.

To estimate from below the growth of c_k^n we follow the same argument used to prove (5.2), based on the Intersection Lemma 5.1 combined with the classical interpolation inequality, in the same spirit of [24].

Proposition 6.1 *Let* $\frac{1}{q+1} + \frac{1}{p+1} > \frac{N-2}{N}$ *and*

$$N\left[\frac{1}{2} - \frac{1}{q+1}\right] < r < 2 - N\left[\frac{1}{2} - \frac{1}{p+1}\right]. \tag{6.1}$$

Then there are $\gamma > 0$ *and* $\tilde{k} \in \mathbb{N}$ *such that for all* $k \ge \tilde{k}$,

$$c_k \ge \gamma k^{2\alpha_r} \tag{6.2}$$

where

$$\alpha_r = \min(q_1, p_1), \tag{6.3}$$

$$q_1 = \frac{q+1}{q-1}\frac{r}{N} - \frac{1}{2}, \tag{6.4}$$

$$p_1 = \frac{p+1}{p-1}\frac{2-r}{N} - \frac{1}{2}. \tag{6.5}$$

Proof. We remark here that the pair (p, q) lies below the critical hyperbola; for any fixed (p, q), the value of r, which identifies the space E^r, is not fixed, but can be chosen in the range defined by (6.1) (see Theorem 2.1). The aim of this proposition is to obtain a lower bound for c_k that depends only on r, with r unknown in (6.1) . The "optimal" choice of r, in dependance of p, q, will be a fundamental argument of the next Section.

First of all, we will prove a lower bound for the minimax sequences c_k^n. Let $k \in \mathbb{N}$ be fixed. Let $h \in \Gamma_k^n$ and $R < R_k$. By the Intersection Lemma 5.1, for any $n \in \mathbb{N}$ there exists a $\mathbf{w}_n \in h(B_k^n) \cap \partial B_R \cap (X^{k-1})^\perp$ so that

$$\max_{\mathbf{u} \in B_k^n} J(h(\mathbf{u})) \geq J(\mathbf{w}_n) \geq \inf_{\mathbf{u} \in \partial B_R \cap (X^{k-1})^\perp} J(\mathbf{u}). \tag{6.6}$$

Therefore, to obtain a lower bound for c_k^n we have to estimate $J(\mathbf{u})$, where $\mathbf{u} \in \partial B_R \cap (X^{k-1})^\perp$ and $0 < R < R_k$. As remarked in the symmetric case (Section 3), if $\mathbf{u} \in \partial B_R \cap (X^{k-1})^\perp \subset E^+$ then

$$\mathbf{u} = (u, v) = (u, (-\Delta)^{r-1} u)$$

and

$$\|v\|_{\Theta^{2-r}} = \|(-\Delta)^{r-1} u\|_{\Theta^{2-r}} = \|u\|_{\Theta^r}. \tag{6.7}$$

Suitably combining the classical interpolation inequality (3.8) with the Sobolev embedding Theorem 2.1, as in the symmetric case, yields the following inequalities (that coincides with the classical Gagliardo-Nirenberg inequalities when r is an integer)

$$\|u\|_{q+1} \leq C\|u\|_2^\theta \|u\|_{\Theta^r}^{1-\theta} \quad \text{with} \quad \theta = 1 - \frac{N}{r}\left(\frac{1}{2} - \frac{1}{q+1}\right), \tag{6.8}$$

$$\|v\|_{q+1} \leq C\|v\|_2^\zeta \|v\|_{\Theta^{2-r}}^{1-\zeta} \quad \text{with} \quad \zeta = 1 - \frac{N}{2-r}\left(\frac{1}{2} - \frac{1}{p+1}\right). \tag{6.9}$$

Since $\mathbf{u} \in (X^{k-1})^\perp$, then $(L\mathbf{u}, \mathbf{u})_{E^r} = \|\mathbf{u}\|_{E^r}^2 = 2\|u\|_{\Theta^r}^2$ by (6.7). Combining (6.8), (6.9) and the estimates (3.9), (3.10) we obtain (we use the same letter C for different constants)

$$\begin{aligned}
J(\mathbf{u}) &\geq \frac{1}{2}\|\mathbf{u}\|_{E^r}^2 - \frac{1}{q+1}\|u\|_{q+1}^{q+1} - \frac{1}{p+1}\|v\|_{p+1}^{p+1} - \|k\|_2\|u\|_2 - \|h\|_2\|v\|_2 \\
&\geq \|u\|_{\Theta^r}^2 - C\|u\|_{q+1}^{q+1} - C\|v\|_{p+1}^{p+1} dx - C \\
&\geq \|u\|_{\Theta^r}^2 - C\left(\|u\|_2^\theta \|u\|_{\Theta^r}^{1-\theta}\right)^{q+1} - C\left(\|v\|_2^\zeta \|v\|_{\Theta^{2-r}}^{1-\zeta}\right)^{p+1} - C \\
&\geq \|u\|_{\Theta^r}^2 - C\lambda_k^{-\frac{r}{2}\theta(q+1)}\|u\|_{\Theta^r}^{q+1} - C\lambda_k^{-\frac{2-r}{2}\zeta(p+1)}\|u\|_{\Theta^r}^{p+1} - C \\
&\geq \|u\|_{\Theta^r}^2 - Ck^{-\frac{r}{N}\theta(q+1)}\|u\|_{\Theta^r}^{q+1} - Ck^{-\frac{2-r}{N}\zeta(p+1)}\|u\|_{\Theta^r}^{p+1} - C
\end{aligned} \tag{6.10}$$

since $\lambda_k \geq Ck^{2/N}$ for $k \to +\infty$, where θ, ζ satisfy conditions (6.8), (6.9). Inserting these values of θ, ζ in the right hand side of (6.10) we obtain

$$J(\mathbf{u}) \geq \|u\|_{\Theta_r}^2 - C\frac{\|u\|_{\Theta_r}^{q+1}}{k^{q_r}} - C\frac{\|u\|_{\Theta_r}^{p+1}}{k^{p_r}} - C. \tag{6.11}$$

where

$$\begin{aligned} q_r &= \frac{r}{N}(q+1)\left[1-\frac{N}{r}\left(\frac{1}{2}-\frac{1}{q+1}\right)\right] \\ p_r &= \frac{2-r}{N}(p+1)\left[1-\frac{N}{2-r}\left(\frac{1}{2}-\frac{1}{p+1}\right)\right]. \end{aligned}$$

To maximize the righthand side in (6.11), let us choose

$$\|u\|_{\Theta^r} \asymp k^\alpha,$$

where α is unknown. Then

$$\begin{aligned} J(\mathbf{u}) &\geq \|u\|^2_{\Theta_r} - C\frac{\|u\|^{q+1}_{\Theta_r}}{k^{q_r}} - C\frac{\|u\|^{p+1}_{\Theta_r}}{k^{p_r}} - C \\ &\asymp k^{2\alpha} - k^{\alpha(q+1)-q_r} - k^{\alpha(p+1)-p_r} - C. \end{aligned}$$

It is easy to verify that the optimal choice of α is

$$\alpha = \alpha_r = \min\left(\frac{q_r}{q-1}, \frac{p_r}{p-1}\right) = \min(q_1, p_1);$$

that is, for any r in (6.1) and for any $\mathbf{u} \in \partial B_R \cap (X^{k-1})^\perp$ with $R = 2\sqrt{\gamma}k^{\alpha_r}$,

$$\begin{aligned} J(\mathbf{u}) &\geq 4\gamma k^{2\alpha_r} - C\gamma^{q+1}k^{\frac{2q_r}{q-1}} - C\gamma p + 1 k^{\frac{2p_r}{p-1}} - C \\ &\geq \gamma k^{2\alpha_r} \end{aligned}$$

for γ small enough. We remark that the condition $R = 2\sqrt{\gamma}k^{\alpha_r} < R_k$ is satisfied since $J(\mathbf{u}) < 0$ if $\|\mathbf{u}\|_{E^r} \geq R_k$, by definition of R_k, and this contradicts the last inequality.

We are now ready to complete the proof. By (3.6) and (6.6), for any $0 < R < R_k$

$$c_k^n = \inf_{h\in\Gamma_k^n} \sup_{\mathbf{u}\in B_k^n} J(h(\mathbf{u})) \geq \inf_{\mathbf{u}\in\partial B_R\cap(X^{k-1})^\perp} J(\mathbf{u});$$

and choosing $R = R(k) = 2\sqrt{\gamma}k^{\alpha_r}$ as before yields

$$c_k^n \geq \gamma k^{2\alpha_r}$$

for any $n \in \mathbb{N}$. Since the constants appearing in the last estimate do not depend on n, as we have just remarked, we can pass to the limit for $n \to \infty$, obtaining the thesis.

7 Proof of Theorem 1.1

In this final section we shall complete the proof of Theorem 1.1. The idea of the proof is a reduction to the absurd: basing on Proposition 5.1, we will assume that $c_k = d_k$ for k large, obtaining an upper bound on the growth of c_k which is in contrast with the lower bound proved in Proposition 6.1. Hence we will conclude that $d_k > c_k$ for k large, which yields the existence of an unbounded sequence of critical values $d_k(\delta)$ for J, then also for I. Therefore, we need first an estimate from above on the growth of c_k (under the assumptions that $c_k = d_k$).

Proposition 7.1 *If $c_k = d_k$ for all $k \geq k_1$, there exist two constants α_1, $\alpha_2 > 0$ and $k_2 \geq k_1$ such that*

$$c_k \leq \alpha_1 k^{\frac{q+1}{q}} + \alpha_2 k^{\frac{p+1}{p}} \tag{7.1}$$

for all $k \geq k_2$.

Proof. We follow the proof in [24]. Let $k > k_1$; then, there is a sequence $\{\varepsilon_n\}$ (depending on k) such that

$$d_k^n \leq c_k^n + \varepsilon_n \quad \text{and} \quad \varepsilon_n \to 0 \quad \text{as } n \to +\infty.$$

Let $\varepsilon > 0$ and choose n large enough such that $\varepsilon_n < \varepsilon$. Then, choose $H \in \Lambda_k^n$ such that

$$\max_{\mathbf{u} \in U_k^n} J(H(\mathbf{u})) \leq d_k^n + \varepsilon \leq c_k^n + 2\varepsilon. \tag{7.2}$$

Since $B_{k+1}^n = U_k^n \cup (-U_k^n)$, H can be continuously extended to B_{k+1}^n as an odd function, still denoted with H. Therefore, $H \in \Gamma_{k+1}^n$ and

$$c_{k+1}^n = \inf_{h \in \Gamma_{k+1}^n} \max_{\mathbf{u} \in B_{k+1}^n} J(h(\mathbf{u})) \leq \max_{\mathbf{u} \in B_{k+1}^n} J(H(\mathbf{u})) = J(H(\mathbf{w}_k^n)) \tag{7.3}$$

for some $\mathbf{w}_k^n \in B_{k+1}^n$. If $\mathbf{w}_k^n \in U_k^n$, by (7.2) and (7.3),

$$c_{k+1}^n \leq J(H(\mathbf{w}_k^n)) \leq \max_{\mathbf{u} \in U_k^n} J(H(\mathbf{u})) \leq c_k^n + 2\varepsilon. \tag{7.4}$$

If $\mathbf{w}_k^n \in -U_k^n$, by the oddness of H and the estimate on the deviation from symmetry (4.3), we obtain

$$\begin{aligned} J(H(-\mathbf{w}_k^n)) &= J(-H(\mathbf{w}_k^n)) \\ &\geq J(H(\mathbf{w}_k^n)) - \beta\left(|J(H(\mathbf{w}_k^n))|^{\frac{1}{q+1}} + |J(H(\mathbf{w}_k^n))|^{\frac{1}{p+1}} + 1\right). \end{aligned}$$

Since $c_k \to +\infty$ as $k + \infty$, (7.3) and the previous inequality imply that $J(-H(\mathbf{w}_k^n)) > 0$ for n and k large enough. Then, combining (7.3) with (4.3) yields

$$\begin{aligned} c_{k+1}^n &\leq J(H(\mathbf{w}_k^n)) = J(-H(-\mathbf{w}_k^n)) \\ &\leq J(H(-\mathbf{w}_k^n)) + \beta\left(|J(H(-\mathbf{w}_k^n))|^{\frac{1}{q+1}} + |J(H(-\mathbf{w}_k^n))|^{\frac{1}{p+1}} + 1\right) \\ &\leq c_k^n + 2\varepsilon + \beta\left(|c_k^n + 2\varepsilon|^{\frac{1}{q+1}} + |c_k^n + 2\varepsilon|^{\frac{1}{p+1}} + 1\right), \end{aligned} \tag{7.5}$$

where we have used the fact that if $\mathbf{w}_k^n \in -U_k^n$, then $-\mathbf{w}_k^n \in U_k^n$ and $J(H(-\mathbf{w}_k^n)) \leq c_k^n + 2\varepsilon$, by (7.4). Since ε is arbitrary (recalling that $\varepsilon_n \to 0$ as $n \to +\infty$), (7.4) and (7.5) imply

$$c_{k+1} \leq c_k + \beta\left(c_k^{\frac{1}{q+1}} + c_k^{\frac{1}{p+1}} + 1\right) \tag{7.6}$$

for all k large enough. Applying standard arguments, inequality (7.6) implies directly our thesis: see e.g. [24].

Proof. By Proposition 5.1, if $d_k > c_k$ for k large (up to a subsequence, if necessary) there is a sequence of unbounded critical values $d_k(\delta)$ for J, and then also for I, by Proposition 4.2. Therefore, it suffices to show that $d_k > c_k$ for k large. On the contrary, let us assume that $c_k = d_k$ as $k \to +\infty$. Then the last Proposition 7.1 assures the estimate from above (7.1) on the growth of c_k, which depends only on p, q; on the other hand, we have the estimate from below (6.2) proved in Proposition 6.1, which depends on $r \in \left(N\left(\frac{1}{2} - \frac{1}{q+1}\right), 2 - N(\frac{1}{2} - \frac{1}{p+1})\right)$, for any pair (p, q) below the critical hyperbola. Theorem 1.1 will be proved if we choose $r \in \left(N\left(\frac{1}{2} - \frac{1}{q+1}\right), 2 - N(\frac{1}{2} - \frac{1}{p+1})\right)$ such that

$$\begin{aligned} \min(2q_1, 2p_1) &= \min\left(2\frac{q+1}{q-1}\frac{r}{N} - 1,\ 2\frac{p+1}{p-1}\frac{2-r}{N} - 1\right) \\ &> \max\left\{\frac{q+1}{q}, \frac{p+1}{p}\right\}. \end{aligned} \tag{7.7}$$

Let us now consider the case

$$q \geq p \tag{7.8}$$

that is,

$$\max\left\{\frac{q+1}{q}, \frac{p+1}{p}\right\} = \frac{p+1}{p}.$$

Our aim now is to discuss the value of $\max(2q_1, 2p_1)$.

Let us fix (p, q) below the critical hyperbola $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. By definition of p_1 and q_1,

$$\min(2q_1, 2p_1) = 2q_1$$

if and only if

$$2\frac{q+1}{q-1}\frac{r}{N} - 1 \leq 2\frac{p+1}{p-1}\frac{2-r}{N} - 1,$$

that is, as one can easily verify, if and only if

$$r \leq \frac{(p+1)(q-1)}{pq-1} := r_{p,q}. \tag{7.9}$$

First of all, we observe that this limiting value of r is consistent with the condition (6.1) on r. Indeed,

$$N\left[\frac{1}{2} - \frac{1}{q+1}\right] < \frac{(p+1)(q-1)}{pq-1} < 2 - N\left[\frac{1}{2} - \frac{1}{p+1}\right]$$

for any pair (p, q) such that

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$$

as one can verify (we have used the following decomposition $pq - 1 = (p+1)(q+1) - (p+1) - (q+1)$). We have now two possible choices for r:

(**i**) if $r \leq r_{p,q}$, then $\min(2q_1, 2p_1) = 2q_1$;

(**ii**) otherwise, if we choose $r > r_{p,q}$, then $\min(2q_1, 2p_1) = 2p_1$.

Hence we discuss separately the two cases.

(i) Consider first the choice

$$N\left[\frac{1}{2} - \frac{1}{q+1}\right] < r \leq r_{p,q} = \frac{(p+1)(q-1)}{pq-1}.$$

In this case, recalling that we are assuming $q \geq p$, (7.7) yields

$$2q_1 = \frac{2r}{N}\frac{q+1}{q-1} - 1 > \frac{p+1}{p},$$

that is,

$$r > \frac{N}{2}\frac{q-1}{q+1}\frac{2p+1}{p} := r^L_{p,q}. \tag{7.10}$$

Observe first that $r^L_{p,q} > N[\frac{1}{2} - \frac{1}{q+1}] = \frac{N}{2}\frac{q-1}{q+1}$ for any $p, q > 1$, so that (7.10) is a condition effectively stronger than (6.1). Hence, condition (7.10) can be satisfied for certain r if and only if

$$r^L_{p,q} < r_{p,q},$$

that is,

$$\frac{N}{2}\frac{q-1}{q+1}\frac{2p+1}{p} < \frac{(p+1)(q-1)}{pq-1}.$$

Recalling that $pq - 1 = (p+1)(q+1) - (p+1) - (q+1)$, the last inequality can be written as

$$2 - \frac{2}{N} < \frac{2}{p+1} + \frac{2}{q+1} + \frac{1}{p}\left(\frac{1}{p+1} + \frac{1}{q+1} - 1\right),$$

that gives the following condition on (p, q):

$$\frac{1}{p+1} + \frac{1}{q+1} + \frac{p+1}{p(q+1)} > \frac{2N-2}{N}. \tag{7.11}$$

Therefore, for any pair (p, q) verifying (7.11) we can choose $r \leq r_{p,q}$ such that conditions (6.1), (7.10) are satisfied. Condition (7.11) defines a new region in the (p, q) plane which is contained in the subcritical region delimited by the critical hyperbola.

(ii) Consider now the other possible choice of r,

$$r_{p,q} = \frac{(p+1)(q-1)}{pq-1} < r < 2 - N\left[\frac{1}{2} - \frac{1}{p+1}\right].$$

In this case, recalling that we are assuming $q \geq p$, (7.7) yields

$$2p_1 = 2\frac{2-r}{N}\frac{p+1}{p-1} - 1 > \frac{p+1}{p},$$

that is,

$$r < r^U_{p,q} := 2 - \frac{N}{2}\frac{p-1}{p+1}\frac{2p+1}{p}. \tag{7.12}$$

Observe that $r^U_{p,q} < 2 - N(\frac{1}{2} - \frac{1}{p+1})$ for any $p > 0$, so that (7.12) is a condition effectively stronger than (6.1). Hence, condition (7.12) can be satisfied for certain r if and only if

$$r_{p,q} < r^U_{p,q},$$

that is,

$$\frac{(p+1)(q-1)}{pq-1} < 2 - \frac{N}{2}\frac{p-1}{p+1}\frac{2p+1}{p}.$$

This inequality is equivalent to

$$\frac{(q+1)(p-1)}{pq-1} > \frac{N}{2}\frac{p-1}{p+1}\frac{2p+1}{p};$$

using the decomposition $pq-1 = (p+1)(q+1)-(p+1)-(q+1)$ the last inequality yields

$$2 - \frac{2}{N} < \frac{2}{p+1} + \frac{2}{q+1} + \frac{1}{p}\left(\frac{1}{p+1} + \frac{1}{q+1} - 1\right),$$

that is equal to condition (7.11) found in case (i).

From cases (i) and (ii), we can conclude that there are values of r (satisfying (6.1)) such that condition (7.7) holds if and only if (p,q) verify (7.11) (assuming $q \geq p$). By symmetry arguments, we immediately conclude that if $p \geq q$, (7.7) holds for values of (p,q) satisfying the corresponding condition

$$\frac{1}{p+1} + \frac{1}{q+1} + \frac{q+1}{q(p+1)} > \frac{2N-2}{N}. \tag{7.13}$$

Combining (7.11) with (7.13) yields the thesis.

Remark 7.1 We note that for $p = q$ the limiting curve (1.4) obtained in Theorem 1.1 assumes the same (limiting) value independently obtained by Struwe and Rabinowitz in [28], [26] in the case of a single perturbed equation. Furthermore, we observe that the subcritical region in the (p,q) plane obtained in Theorem 1.1 do not include supercritical values (in the sense of Sobolev embedding) of p and q, so that one could ask if the variational setting introduced in Section 2 is meaningful. Nevertheless, the optimal choice of the exponent r (that is, the optimal choice of the space E^r) is obtained not for $r = 1$ (the case $E^1 = H^1_0 \times H^1_0$): that is, the regularity allowed for the functions u, v is strictly related to the pair (p,q), and choosing a priori the space $H^1_0 \times H^1_0$ would be too restrictive also for perturbed systems with Sobolev-subcritical nonlinear terms.

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