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Multiple Symmetric Brake Orbits in Bounded Convex Symmetric Domains

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Abstract

In this paper, we prove that there exist at least two geometrically distinct symmetric brake orbits in every bounded convex symmetric domain in $\mathbf{R}^n$ for $n \geq 2$.

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1 Introduction

Let $V \in C^2(\mathbf{R}^n, \mathbf{R})$ and h be a positive real number such that $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$ is nonempty, bounded, open, connected, and symmetric with respect to the origin. Consider the following given energy problem of the second order autonomous Hamiltonian system on (τ, q):

$$\ddot{q}(t) + V'(q(t)) = 0, \qquad \forall t \in \mathbf{R} \tag{1.1}$$

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h, \qquad \forall t \in \mathbf{R}, \tag{1.2}$$

$$\dot{q}(0) = \dot{q}(\frac{\tau}{2}) = 0, \tag{1.3}$$

$$q(\frac{\tau}{2} + t) = q(\frac{\tau}{2} - t), \quad \forall t \in \mathbf{R}. \tag{1.4}$$

A solution (τ, q) of (1.1)-(1.4) is called a *brake orbit* in Ω. By (1.4) we have $q(t+\tau) = q(t)$ for all $t \in \mathbf{R}$, i.e., q is τ-periodic. A brake orbit (τ, q) is called *symmetric* if for some positive real number λ it satisfies $q(\lambda+t)=-q(t),\ \forall t\in\mathbf{R}$. We call two brake orbits q_1 and $q_2:\mathbf{R}\to\mathbf{R}^n$ *geometrically distinct*, if $q_1(\mathbf{R})\neq q_2(\mathbf{R})$. Denote by $\mathcal{O}(\Omega)$ and $\tilde{\mathcal{O}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in Ω respectively, by $\mathcal{O}_s(\Omega)$ and $\tilde{\mathcal{O}}_s(\Omega)$ the sets of all symmetric brake orbits and geometrically distinct symmetric brake orbits in Ω respectively.

Let $J=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$ and $N=\begin{pmatrix}-I & 0\\ 0 & I\end{pmatrix}$ be two matrices with I being the identity matrix on $\mathbf{R}^n$. For $H\in C^2(\mathbf{R}^{2n}\setminus\{0\},\mathbf{R})\cap C^1(\mathbf{R}^{2n},\mathbf{R})$ satisfying

$$H(Nx)=H(x),\qquad \forall\, x\in\mathbf{R}^{2n}, \tag{1.5}$$

we consider the following given energy problem for all $t\in\mathbf{R}$:

$$\begin{aligned}\dot{x}(t) &= JH'(x(t)), &(1.6)\\ H(x(t)) &= h, &(1.7)\\ x(-t) &= Nx(t), &(1.8)\\ x(\tau+t) &= x(t). &(1.9)\end{aligned}$$

A solution (τ,x) of (1.6)-(1.9) is called a *brake orbit* on the hypersurface $\Sigma\equiv\{y\in\mathbf{R}^{2n}\,|\,H(y)=h\}$. A brake orbit (τ,x) is called *symmetric*, if for some positive real number λ it satisfies $x(\lambda+t)=-x(t),\ \forall t\in\mathbf{R}$. Two brake orbits x_1 and $x_2:\mathbf{R}\to\mathbf{R}^{2n}$ are called *geometrically distinct*, if $x_1(\mathbf{R})\neq x_2(\mathbf{R})$. Denote by $\mathcal{J}_b(\Sigma)$ and $\tilde{\mathcal{J}}_b(\Sigma)$ the sets of all brake orbits and geometrically distinct brake orbits on Σ respectively, by $\mathcal{J}_{sb}(\Sigma)$ and $\tilde{\mathcal{J}}_{sb}(\Sigma)$ the sets of all symmetric brake orbits and geometrically distinct symmetric brake orbits on Σ respectively.

It is well-known that elements in $\mathcal{O}(\{V<h\})$ and $\mathcal{J}_b(H^{-1}(h))$ are one to one correspondent, and elements in $\mathcal{O}_s(\{V<h\})$ and $\mathcal{J}_{sb}(H^{-1}(h))$ are one to one correspondent.

For the existence results the readers can refer to [2], [3], [4], [5], [10], [12], [15], and [16].

In a recent paper [13], the authors studied the multiplicity of brake orbits without any pinching conditions and proved that there exist at least two geometrically distinct brake orbits in every bounded convex symmetric domain in $\mathbf{R}^n$ for $n\geq 2$. In fact, the authors proved the the following stronger result: there exist at least two brake orbits x, y in every bounded convex symmetric domain in $\mathbf{R}^n$ for $n\geq 2$ such that $x(\mathbf{R})\neq y(\mathbf{R})$ and $x(\mathbf{R})\neq -y(\mathbf{R})$.

A natural question is the following: can we obtain at least two symmetric brake orbits under the same conditions as in [13]?

In this paper, we give a positive answer to this question. Our main result is the following theorem:

Theorem 1.1 *For $n\geq 2$, suppose the following conditions hold:*
(H1) (smoothness) $H\in C^2(\mathbf{R}^{2n}\setminus\{0\},\mathbf{R})\cap C^1(\mathbf{R}^{2n},\mathbf{R})$,
(H2) (reversibility) $H(Ny)=H(y)$ for all $y\in\mathbf{R}^{2n}$,
(H3) (convexity) $H''(y)$ is positive definite for all $y\in\mathbf{R}^{2n}\setminus\{0\}$,
(H4) (symmetry) $H(-y)=H(y)$ for all $y\in\mathbf{R}^{2n}$.

Then for any given $h > \min\{H(y)\,|\,y \in \mathbf{R}^{2n}\}$ *and* $\Sigma = H^{-1}(h)$*, there holds*

$$^{\#}\tilde{\mathcal{J}}_{sb}(\Sigma) \geq 2. \tag{1.10}$$

Without loss of generality, in the rest of this paper, we assume $H(0) = 0$ and take the energy level $h > 0$.

By the one to one correspondence between $\tilde{\mathcal{O}}_s(\{V < h\})$ and $\tilde{\mathcal{J}}_{sb}(\{H = h\})$, we get the following corollary.

Corollary 1.1 *For* $n \geq 2$*, suppose* $V(0) = 0$*,* $V(q) \geq 0$*,* $V(-q) = V(q)$ *and* $V''(q)$ *is positive definite for all* $q \in \mathbf{R}^n \setminus \{0\}$*. Then for any given* $h > 0$ *and* $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$*, there holds*

$$^{\#}\tilde{\mathcal{O}}_s(\Omega) \geq 2. \tag{1.11}$$

For a solution of (1.6)-(1.9) (τ, x), the Maslov-type index $\mu_1(x)$ and the mean Maslov-type index $\hat{\mu}_1(x)$ are defined by Definition 2.5 and (1.24) of [13] respectively. A direct corollary of Propositions A-C of [13] is the following proposition.

Proposition 1.1 *For* $n \geq 2$*, suppose* $(H1)$*-*$(H3)$ *hold. Then for any* τ*-periodic solution* x *of (1.6)-(1.9) there holds*

$$\hat{\mu}_1(x) > 1. \tag{1.12}$$

This paper is organized as follows. In Section 2, we take the dual variational method and critical point theory to find critical points with prescribed Morse index following the method of [7] and [13]. In Section 3 we establish the relationship between the Morse index defined in Section 2 and the Maslov-type index μ_1. Based on these results, we prove Theorem 1.1 in Section 4.

2 Variational set–up

In [6], I. Ekeland and H. Hofer used the dual variational method to study multiple periodic orbits on compact convex hypersurfaces in $\mathbf{R}^{2n}$. Recently in [13], Y. Long, D. Zhang and C. Zhu used this method to study multiple existence of brake orbits in bounded symmetric convex domains. Now we modify slightly the method in [13] to study symmetric brake orbits on Σ.

Define

$$H_\Sigma(x) = j_\Sigma(x)^2, \quad \forall x \in \mathbf{R}^{2n}, \tag{2.1}$$

where j_Σ is the gauge function of Σ. Then $H_\Sigma \in C^2(\mathbf{R}^{2n}\backslash\{0\}, \mathbf{R}) \cap C^{1,1}(\mathbf{R}^{2n}, \mathbf{R})$. Its Fenchel conjugate (cf.[6, 7]) is the function H_Σ^* defined by

$$H_\Sigma^*(y) = \max\{\langle x, y\rangle - H_\Sigma(x) |\, x \in \mathbf{R}^{2n}\}. \tag{2.2}$$

For $S_1 = \mathbf{R}/\mathbf{Z}$, we define a Hilbert space E_1 by

$$E_1 = \{x \in W^{1,2}(S_1, \mathbf{R}^{2n}) \mid x(-t) = Nx(t), \text{ and } x(t + \frac{1}{2}) = -x(t), \ \ \forall t \in \mathbf{R}\}. \tag{2.3}$$

The inner product on E_1 is given by

$$(x,y)_{E_1} = \int_0^1 \langle \dot{x}(t), \dot{y}(t)\rangle dt, \quad \forall x, y \in E_1. \tag{2.4}$$

The $C^{1,1}$ Hilbert manifold $M_\Sigma \subset E_1$ associated to Σ is defined by

$$M_\Sigma = \left\{ x \in E_1 \;\middle|\; \int_0^1 H_\Sigma^*(-J\dot{x}(t))dt = 1 \text{ and } \int_0^1 \langle J\dot{x}(t), x(t)\rangle dt < 0 \right\}. \tag{2.5}$$

Let $\mathbf{Z}_2 = \{-id, id\}$ be the usual $\mathbf{Z}_2$ group. We define the $\mathbf{Z}_2$-action on E_1 by

$$-id(x) = -x, \quad id(x) = x, \qquad \forall x \in E_1.$$

Note that the origin 0 is the only fixed point of this $\mathbf{Z}_2$-action on E_1. Since H_Σ^* is even, M_Σ is symmetric with respect to 0, i.e., $\mathbf{Z}_2$-invariant. There is an induced $\mathbf{Z}_2$-action on M_Σ. So M_Σ is a paracompact $\mathbf{Z}_2$-space. We define

$$A(x) = \frac{1}{2}\int_0^1 \langle J\dot{x}(t), x(t)\rangle dt. \tag{2.6}$$

Then $A \in C^\infty(E_1, \mathbf{R})$ and it is a $\mathbf{Z}_2$-invariant functional. Denote by A_Σ the restriction of A to M_Σ. For $d \in (0, \infty)$ we define

$$M_\Sigma^d \equiv A_\Sigma^{-1}((-\infty, d]). \tag{2.7}$$

Let

$$\Psi(x) = \int_0^1 H_\Sigma^*(-J\dot{x}(t))dt, \quad \forall x \in E_1. \tag{2.8}$$

Denote by $\alpha_\Sigma(d)$ the Fadell-Rabinowitz index $\alpha_\Sigma(d)$ for the G-space M_Σ^d (cf. [8]) in the same sense of section 7 of [13].

Let the Riemannian metric on M_Σ be the one induced from E_1. As in [7], A_Σ satisfies the following $(\mathrm{PS})_\mathrm{d}$ condition for $d \in (-\infty, 0)$.

$(\mathrm{PS})_\mathrm{d}$: $\| A_\Sigma'(x_k) \| \to 0$ *and* $A_\Sigma(x_k) \to d < 0$ *imply that* $\{x_k\}$ *is precompact in* M_Σ.

Define

$$\mathrm{Cr}(d) = \{x \in M_\Sigma | A_\Sigma'(x) = 0,\ A_\Sigma(x) = d\}.$$

Denote by $\mathrm{Cr}(\Sigma)$ the set of all critical points of A_Σ.

For $k \in \mathbf{N}$, $\theta \in \{0, 1/2\}$, and $z \in E$, we define

$$((k,\theta) * z)(t) = \frac{1}{k} z(kt + \theta). \tag{2.9}$$

Then

$$A_\Sigma((k,\theta) * z) = \frac{1}{k} A_\Sigma(z), \tag{2.10}$$

and $(k,\theta) * z \in \mathrm{Cr}(\Sigma)$, if $z \in \mathrm{Cr}(\Sigma)$. Denote by "$\sim$" the smallest equivalence relation containing the relation

$$z \mapsto (k,\theta) * z, \text{ for all } (k,\theta) \in \mathbf{N} \times \{0, 1/2\} \text{ and } z \in E_1, \tag{2.11}$$

and $[z]$ the equivalence class of $z \in E_1$.

Lemma 2.1 (i) *There is a bijection between* $\mathrm{Cr}(\Sigma)/\sim$ *and* $\tilde{\mathcal{J}}_{sb}(\Sigma)$.

(ii) $\alpha_\Sigma(d) < +\infty, \quad \forall d \in (-\infty, 0)$.

(iii) $d \to \alpha_\Sigma(d)$ *is non-decreasing.*

(iv) $\lim_{d\downarrow d_0} \alpha_\Sigma(d) = \alpha_\Sigma(d_0)$.

(v) $\mathrm{ind}(\mathrm{Cr}(d)) \geq \alpha_\Sigma(d) - \alpha_\Sigma(d^-), \quad \forall d \in (-\infty, 0)$, *where* ind *denotes the Fadell-Rabinowitz index. In particular, if* α_Σ *is discontinuous at* d, *then* $\mathrm{Cr}(d) \neq \emptyset$. *Moreover if* $\alpha_\Sigma(d) - \alpha_\Sigma(d^-) \geq 2$, *then* $\mathrm{Cr}(d)$ *contains infinitely many geometrically distinct symmetric brake orbits.*

(vi) $\lim_{d\uparrow 0} \alpha_\Sigma(d) = +\infty$.

Proof. The proof is similar to those of Lemmas 5 and 6 in [7]. We only prove (i) here. Let $z \in \mathrm{Cr}(\Sigma)$ be a critical point of A_Σ with period 1. Then there exists some real number $\delta \neq 0$, such that

$$A'(z) = \delta \Psi'(z). \tag{2.12}$$

Taking the inner product with every $h \in E_1$, we have

$$\int_0^1 \langle z(t), -J\dot{h}(t)\rangle dt = \int_0^1 \langle J\dot{z}(t), h(t)\rangle dt = \delta \int_0^1 \langle (H_\Sigma^*)'(-J\dot{z}(t)), -J\dot{h}(t)\rangle dt. \tag{2.13}$$

Let $h = z$. Using $\Psi(z) = 1$ and that A' and Ψ' are positively 1-homogeneous, we get

$$\delta = A_\Sigma(z) < 0. \tag{2.14}$$

By the facts $JN = -NJ$, $z \in E_1$, we get

$$(H_\Sigma^*)'(-J\dot{z}(-t)) = (H_\Sigma^*)'(-NJ\dot{z}(t)) = N(H_\Sigma^*)'(-J\dot{z}(t)).$$

By (2.13), we have

$$\int_0^1 \langle z(t) + \delta (H_\Sigma^*)'(-J\dot{z}(t)), -J\dot{h}(t)\rangle dt = 0, \quad \forall h \in E_1. \tag{2.15}$$

Since $(z + \delta(H_\Sigma^*)'(-J\dot{z}))(t + \frac{1}{2}) = (z + \delta(H_\Sigma^*)'(-J\dot{z}))(t), \ \ \forall t \in \mathbf{R}$, by (2.15), we have $z + \delta(H_\Sigma^*)'(-J\dot{z}) = 0$, i.e.,

$$z(t) = |\delta|(H_\Sigma^*)'(-J\dot{z}(t)). \tag{2.16}$$

Hence by the Legendre reciprocity formula (cf. page 92 of [6]), we have

$$|\delta|^{-1} H'_\Sigma(z(t) = -J\dot{z}(t). \tag{2.17}$$

Defining $z_1(t) = z(|\delta|t)$, we obtain

$$-J\dot{z}_1 = H'_\Sigma(z_1). \tag{2.18}$$

Hence

$$\frac{d}{dt} H_\Sigma(z_1(t)) = \langle -J\dot{z}_1(t), \dot{z}_1(t)\rangle = 0, \tag{2.19}$$

and $H_\Sigma(z_1(t))$ is independent of t. By (2.18) and the 2-homogeneity of H_Σ, we have

$$\int_0^{1/|\delta|}\langle -J\dot z_1(t), z_1(t)\rangle dt = \int_0^{1/|\delta|}\langle H'_\Sigma(z_1(t)), z_1(t)\rangle dt = 2|\delta| H_\Sigma((z_1(0)), \quad (2.20)$$

i.e.,

$$H_\Sigma(z_1(0)) = \frac{1}{2}|\delta|\int_0^{1/|\delta|}\langle -J\dot z_1(t), z_1(t)\rangle dt. \quad (2.21)$$

By (2.14), we have

$$\begin{aligned}\int_0^{1/|\delta|}\langle -J\dot z_1(t), z_1(t)\rangle dt &= |\delta|\int_0^{1/|\delta|}\langle -J\dot z(|\delta|t), z(|\delta|t)\rangle dt \\ &= \int_0^1 \langle -J\dot z(t), z(t)\rangle dt \\ &= -2A_\Sigma(z) \\ &= 2|\delta|. \end{aligned} \quad (2.22)$$

So by (2.21) and (2.22), we obtain

$$H_\Sigma(z(t)) = |\delta|^2. \quad (2.23)$$

Therefore

$$H_\Sigma(|\delta|^{-1}z(t)) = 1, \quad \forall t \in \mathbf{R}. \quad (2.24)$$

Define

$$x(t) = x(z)(t) = |\delta|^{-1}z(|\delta|t), \quad \forall t \in \mathbf{R}. \quad (2.25)$$

We have

$$-J\dot x = H'_\Sigma(x), \quad H_\Sigma(x(t)) = 1, \quad \forall t \in \mathbf{R}, \quad (2.26)$$

and x is $|\delta|^{-1}$-periodic.

We define a map $\Phi : \mathrm{Cr}(\Sigma) \to \tilde{\mathcal{J}}_{sb}(\Sigma)$ by

$$\Phi(z)(t) = x(z)(t) = |A(z)|^{-1}z(|A(z)|t). \quad (2.27)$$

For any $x \in \tilde{\mathcal{J}}_{sb}(\Sigma)$, doing the above procedure backwards we obtain a $z \in \mathrm{Cr}(\Sigma)$, such that $\Phi(z) = x$. This yields that Φ is surjective. By the definition of Φ, we have $\Phi(z_1)(\mathbf{R}) = \Phi(z_2)(\mathbf{R})$ if and only if $z_1 \sim z_2$. Thus Φ induces a bijection from $\mathrm{Cr}(\Sigma)/\sim$ to $\tilde{\mathcal{J}}_{sb}(\Sigma)$. This proves (i).

Remark 2.1 Moreover, from (2.25), we have $x(z)(t + \frac{|\delta|^{-1}}{2}) = -x(z)(t)$. So $x(z)$ can not be an even times iteration of some symmetric brake orbit.

Let z be a critical point of A_Σ. Then $z(t) \neq 0$ for all $t \in \mathbf{R}$. Similar to formula (49) of [7] and (7.36) of [13], we define a quadratic form $Q_{1,z}$ on E_1 by

$$\begin{aligned}Q_{1,z}(h) &= \frac{1}{2}\int_0^1 \langle J\dot h(t), h(t)\rangle dt \\ &- \frac{1}{2}A(z)\int_0^1 \langle (H^*_\Sigma)''(-J\dot z(t))J\dot h(t), J\dot h(t)\rangle dt, \quad \forall h \in E_1. \end{aligned} \quad (2.28)$$

Definition 2.1 $Q_{1,z}$ is called the *formal Hessian of* A_Σ *at* z. Denote by $m^-(Q_{1,z})$ the maximal dimension of linear subspaces in E_1 on which $Q_{1,z}$ is negative definite, and denote by $m^0(Q_{1,z})$ the dimension of kernel of $Q_{1,z}$ minus 1.

Remark 2.2 Since H_Σ is positively 2-homogeneous, z belongs to kernel of $Q_{1,z}$. Hence we have $Q_{1,z}$-orthogonal decomposition $E_1 = \mathbf{R}z \bigoplus T_z M_\Sigma$. Since $A(z) < 0$ and $H^*_\Sigma(-J\dot{z}(t))$ is positive definite, by the same discussion in [13], both $m^-(Q_{1,z})$ and $m^0(Q_{1,z})$ are finite.

By the same proof of Corollary 7.10 of [13], we have

Lemma 2.2 *For each* $k \in \mathbf{N}$, *there exists a critical point* z_k *of* A_Σ *such that the sequence* $\{A_\Sigma(x_k)\}_{k\in N}$ *increases strictly to zero and there holds*

$$m^-(Q_{1,z_k}) \le k-1 \le m^-(Q_{1,z_k}) + m^0(Q_{1,z_k}). \tag{2.29}$$

3 The relationship between the formal Morse index and the Maslov-type index μ_1

In Section 2, we defined the formal Hessian $Q_{1,z}$ for critical point z of A_Σ on M_Σ. Now we define a dual formal Hessian $Q_{2,z}$ for z. Then we consider the relationship between these two Hessians and the one defined by (7.36) in [13].

In [13] the Hilbert space E is defined by

$$E = \{x \in W^{1,2}(S_1, \mathbf{R}^{2n}) \mid x(-t) = Nx(t), \quad \forall t \in \mathbf{R}, \text{ and } \int_0^1 x(t)dt = 0\}. \tag{3.1}$$

The inner product on E is given by

$$(x,y)_E = \int_0^1 \langle \dot{x}(t), \dot{y}(t)\rangle dt, \quad \forall x,y \in E. \tag{3.2}$$

We define a subspace of E by

$$E_2 = \{x \in W^{1,2}(S_1, \mathbf{R}^{2n}) \mid x(-t) = Nx(t) \text{ and } x(t+\frac{1}{2}) = x(t), \ \forall t \in \mathbf{R}\}. \tag{3.3}$$

And we define the Hessian $Q_{2,z}$ on E_2 as follows:

$$\begin{aligned} Q_{2,z}(h) &= \frac{1}{2}\int_0^1 \langle J\dot{h}(t), h(t)\rangle dt \\ &- \frac{1}{2}A(z)\int_0^1 \langle (H^*_\Sigma)''(-J\dot{z}(t))J\dot{h}(t), J\dot{h}(t)\rangle dt, \quad \forall h \in E_2. \end{aligned} \tag{3.4}$$

In [13], the formal Hessian in the brake orbit sense for z is defined by

$$\begin{aligned} Q_z(h) &= \frac{1}{2}\int_0^1 \langle J\dot{h}(t), h(t)\rangle dt \\ &- \frac{1}{2}A(z)\int_0^1 \langle (H^*_\Sigma)''(-J\dot{z}(t))J\dot{h}(t), J\dot{h}(t)\rangle dt, \quad \forall h \in E. \end{aligned} \tag{3.5}$$

Similarly we can define the formal Morse indices $m^-(Q_z)$ and $m^-(Q_{2,z})$ on E and E_2 respectively. Since z belongs to the kernel of Q_z, the formal Morse index $m^-(Q_z)$ defined here coincides with the one defined by (7.36) in [13].

By direct computation we have the following Q_z-orthogonal decomposition

$$E = E_1 \bigoplus E_2. \tag{3.6}$$

Hence we have

$$m^-(Q_z) = m^-(Q_{2,z}) + m^-(Q_{1,z}). \tag{3.7}$$

Since $(H_\Sigma^*)''$ is even, for all $t \in S_1$ we have

$$(H_\Sigma^*)''(-J\dot{z}(t + \frac{1}{2})) = (H_\Sigma^*)''(-J(-\dot{z}(t))) = (H_\Sigma^*)''(-J\dot{z}(t)),$$

i.e., $(H_\Sigma^*)''(-J\dot{z})$ is $\frac{1}{2}$-periodic.

Let $x = x(z)$ defined by (2.25). Since $(H_\Sigma^*)''(-J\dot{z})$ is $\frac{1}{2}$-periodic, by Lemma 8.3 of [13], we have

$$\mu_1(\gamma_x, [0, \frac{T}{4}]) = m^-(Q_{2,z}) + n. \tag{3.8}$$

$$\mu_1(x, [0, T]) = \mu_1(\gamma_x, [0, \frac{T}{2}]) = m^-(Q_z) + n. \tag{3.9}$$

Hence by (3.7)-(3.9), we have proved the following lemma.

Lemma 3.1 *Let z be a critical point of A_Σ and $T = |A_\Sigma(z)|^{-1}$. Then $x = x(z)$ defined by (2.25) is a T-periodic symmetric brake orbit of H_Σ on Σ. We have:*

$$m^-(Q_{1,z}) = \mu_1(\gamma_x, [0, \frac{T}{2}]) - \mu_1(\gamma_x, [0, \frac{T}{4}]) \tag{3.10}$$

4 Proof of Theorem 1.1

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Assume the conclusion of Theorem 1.1 does not hold. By Lemma 2.1, the existence of one symmetric brake orbit is obtained. Then there is exactly one geometrically distinct symmetric brake orbit on Σ. Let H_Σ be the Hamiltonian function defined by (2.1) from Σ. Let z be a critical point of $A_\Sigma = A|_{M_\Sigma}$ with minimal period 1. Then $x_z \equiv x(z)$ defined by (2.25) is a brake orbit of H_Σ on Σ with minimal period $T = |A(z)|^{-1}$. Because of $^\#\tilde{\mathcal{J}}_b(\Sigma) = 1$, by an argument in [14], (T, x_z) must be the unique symmetric brake orbit on Σ for the Hamiltonian function H_Σ.

By Lemma 2.2, there exists a strictly increasing critical value sequence $\{c_i\}_{i\in\mathbf{N}}$ in $(-\infty, 0)$, and a critical point sequence z_is of A_Σ such that

$$A_\Sigma(z_i) = c_i, \quad \text{and} \quad m^-(Q_{1,(z_i)}) \le i - 1 \le m^-(Q_{1,(z_i)}) + m^0(Q_{1,(z_i)}). \tag{4.1}$$

By Lemma 2.1 and Remark 2.1, there exists odd natural number sequence k_i such that $z_i = ((k_i, 0) * z)$ for $i \in \mathbf{N}$. Then we have

$$m^-(Q_{1,(k_i,0)*z}) \le i - 1 \le m^-(Q_{1,(k_i,0)*z}) + m^0(Q_{1,(k_i,0)*z}). \tag{4.2}$$

Since $c_i = A((k_i, 0) * z) = \frac{1}{k_i} A(z) < 0$ is strictly increasing in i, so is k_i. Hence

$$k_i \ge 2i - 1. \tag{4.3}$$

By (2.9) and (2.25), $x((k_i, 0) * z)$ is just the k_i-th iteration of $x_z = x(z)$. This yields

$$x_z^{k_i} = x((k_i, 0) * z). \tag{4.4}$$

Thus by Lemma 3.1, the iteration $x_z^{k_i}$ has the property:

$$\mu_1(\gamma_{x_z^{k_i}}, [0, \frac{k_i T}{2}]) - \mu_1(\gamma_{x_z^{k_i}}, [0, \frac{k_i T}{4}]) \le i - 1 \tag{4.5}$$

By (4.4), (4.5), we have

$$\frac{\mu_1(\gamma_{x_z^{k_i}}, [0, \frac{k_i T}{2}])}{k_i} - \frac{\mu_1(\gamma_{x_z^{k_i}}, [0, \frac{k_i T}{4}])}{k_i} \le \frac{i-1}{k_i}. \tag{4.6}$$

By (4.3) and the definition of $\hat{\mu}_1(x_z, [0, T])$, if we let $i \to +\infty$, then we have

$$\hat{\mu}_1(x_z, [0, T]) - \frac{1}{2}\hat{\mu}_1(x_z, [0, T]) \le \frac{1}{2}. \tag{4.7}$$

Thus

$$\hat{\mu}_1(x_z, [0, T]) \le 1. \tag{4.8}$$

This contradicts Proposition 1.1, and Theorem 1.1 is proved.

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