Advanced Nonlinear Studies **6** (2006), 617–641

Isolated Singularities of Solutions of Quasilinear Anisotropic Elliptic Equations

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Received 14 June 2006
Communicated by Laurent Véron

Abstract

We consider a wide class of degenerate quasilinear second order elliptic equations with model representative $\sum_{i=1}^{n}\left(|u_{x_i}|^{p_i-2}u_{x_i}\right)_{x_i} = 0,\ 1 < p_1 \leq p_2 \leq \ldots \leq p_n$, whose solutions have singularity at a point. There are established sharp point-wise conditions for removable isolated singularity of solutions of such equations. For solutions with non-removable point singularity (source type or fundamental solution), precise upper and lower estimates near the singularity point are obtained.

1991 Mathematics Subject Classification. 35B40, 35J70.
Key words. Degenerate quasilinear anisotropic elliptic equation, removable isolated singularity, point-wise estimates

*Partially supported by grant INTAS Ref. Nr 03-51-5007 "Nonlinear Evolutions. Blow-up Phenomena. Stability and Instability."

1 Introduction

In this paper we study of the local behavior of an arbitrary generalized solution to the quasilinear anisotropic elliptic equation of general form

$$Lu := \sum_{i=1}^{n} \frac{d}{dx_i} a_i \left(x, u, \frac{\partial u}{\partial x} \right) - a_0 \left(x, u, \frac{\partial u}{\partial x} \right) = 0, \quad \forall x \in \Omega \backslash \{x_0\}, \tag{1.1}$$

near the isolated singularity point $x_0 \in \Omega \subset \mathbb{R}^n$, $n \geq 2$. Here Ω is a bounded domain. We assume that functions $a_i(x, s, \xi)$, $i = 0, 1, \ldots, n$, are defined in $\Omega \times \mathbb{R}^1 \times \mathbb{R}^n$, satisfy the Caratheodory conditions, and there exist constants K_1, K_2 such that for every $x \in \Omega$, $s \in R^1$, $\xi \in R^k$ the following inequalities hold:

$$\begin{gathered} \sum_{i=1}^{n} a_i(x, s, \xi)\xi_i \geq K_1 \sum_{i=1}^{n} |\xi_i|^{p_i} - g_1(x)|s|^p - f_1(x), \quad K_1 > 0, \\ |a_i(x, s, \xi)| \leq K_2 \left(\sum_{j=1}^{n} |\xi_j|^{p_j} \right)^{1-\frac{1}{p_i}} + g_2(x)|s|^{p\left(1-\frac{1}{p_i}\right)} + f_2(x),\ i = \overline{1, n},\ K_2 < \infty, \\ a_0(x, s, \xi) \leq \sum_{i=1}^{n} h_i(x)|\xi_i|^{p_i\left(1-\frac{1}{p}\right)} + g_3(x)|s|^{p-1} + f_3(x). \end{gathered} \tag{1.2}$$

Here the nonnegative functions $h_j(x), g_i(x), f_i(x)$, $i = 1, 2, 3$, $j = \overline{1, n}$, belong to certain Lebesgue classes over the domain Ω (see § 2),

$$1 < p_1 \leq p_2 \leq p_3 \leq \ldots \leq p_n < \infty, \quad \frac{1}{p} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}, \quad p \leq n. \tag{1.3}$$

In the sequel, all the constants in (1.2), (1.3) will be called the structural constants of equation (1.1). The equation

$$L_0 u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right) = 0 \ \text{ in } \ \Omega \backslash \{x_0\} \tag{1.4}$$

is the simplest example of equations from class (1.1). In the case

$$1 < p_1 = p_2 = \cdots = p_n = p \leq n \tag{1.5}$$

equation (1.4) has the source type (fundamental) solution of the form

$$u(x) = |x - x_0|^{-\frac{n-p}{p-1}}, \ \ p < n; \ \ u(x) = \ln |x - x_0|, \ \ p = n; \tag{1.6}$$

which exhibits the solution with the "minimal" singularity at $x = x_0$. J. Serrin [6] was the first who proved that even in the general case of the isotropic equation (1.1) with structural condition (1.2), (1.5) there is no solution singular at the point x_0 and satisfying the condition

$$u(x) = O\left(|x - x_0|^{-\frac{n-p}{p-1} + \delta} \right), \quad \delta > 0, \quad 1 < p < n, \tag{1.7}$$

$$u(x) = O\left(|\ln|x - x_0||^{1-\delta}\right), \quad \delta > 0, \quad p = n. \tag{1.8}$$

Further analysis of sufficient conditions for removability of singularities of solutions has been made by many authors for different classes of nonlinear elliptic and parabolic equations (cf., e.g. [8] and references therein). The precise point-wise sufficient condition for removability of the isolated singularity of the solution to general isotropic equation (1.1) (case (1.5)) was found in [5], which is in the form

$$\max_{r \leq |x-x_0| \leq R_0} |u(x)| = o\left(r^{-\frac{n-p}{p-1}}\right), \quad 1 < p < n, \tag{1.9}$$

$$\max_{r \leq |x-x_0| \leq R_0} |u(x)| = o\left(\left|\ln \frac{1}{r}\right|\right), \quad p = n. \tag{1.10}$$

For anisotropic equation (1.4), the explicit source type solution in the form similar to (1.6), is not known. Existence of the nonnegative fundamental solution to equation (1.4) was proved in [9] under the following additional restriction:

$$1 < p_1 \leq p_2 \leq \ldots \leq p_n < (n-1)(n-p)^{-1}p \ \text{ if } \ p < n. \tag{1.11}$$

Moreover, it was proved ([9]) that such a fundamental solution belongs to the anisotropic Sobolev space

$$W^1_{(\overline{q})}(\Omega) := \left\{ v \in W^{1,1}(\Omega) : \frac{\partial v}{\partial x_i} \in L^{q_i}(\Omega), \ i = \overline{1,n} \right\}, \tag{1.12}$$

where the real q_i satisfy the following conditions

$$1 < q_i < n(p-1)p^{-1}(n-1)^{-1}p_i, \quad i = \overline{1,n}. \tag{1.13}$$

It is easy to check that for an arbitrary $\delta > 0$ the function

$$u_\delta(x) := \left(\sum_{i=1}^{n} |x_i - x_{o,i}|^{b_i}\right)^{-\frac{n-p}{p-1}-\delta}, \quad b_i := \frac{p_i(p-1)}{p(p_i-1)+n(p-p_i)}, \ 1 < p < n, \tag{1.14}$$

satisfies (1.12) with some $q_i = q_i(\delta)$ from (1.13). Here $q_i(\delta) \to n(p-1)p^{-1}(n-1)^{-1}p_i$, $i = \overline{1,n}$, as $\delta \to 0$. This fact yields the following: In analogy with the isotropic case, one can guess that the function $u_0(x)$ ($u_\delta(x)$ from (1.14) with $\delta = 0$) determines the removability condition at the point x_0 in the sense that there is no solution singular at the point x_0 with asymptotic

$$|u(x)| = o(u_0(x)).$$

It is also natural to expect that $u_0(x)$ determines the asymptotic of the source type solution to the model equation as well as to the general anysotropic quasilinear equation.

The aim of this paper is to substantiate the above statements. The precise formulations will follow. The proofs are based on the extension of the method of local pointwise estimates developed by I. V. Skrypnik [7].

The paper is organized as follows. In Section 2 assumptions and main results are formulated. The auxiliary integral estimates for solutions with isolated singularities are established in Section 3. We prove point-wise estimates for solutions in Section 4. Section 5 contains the proof of the theorem on removability of singularities. In Section 6 we prove the upper and lower pointwise estimates on the fundamental solution. These estimates show the sharpness of the removability condition in Theorem 2.1 for fundamental solution of equation (1.4).

2 Framework and main results

First we formulate assumptions for the functions g_i, f_i, h_i from (1.2). Set

$$H_1(x) := \sum_{i=1}^{n} h_i^p(x)+g_1(x)+g_3(x)+f_1(x)+f_3(x), \quad H_2(x) := \sum_{i=1}^{n}(g_2(x)+f_2(x))^{\frac{p_i}{p_i-1}}$$

and suppose that there exists $\delta \in (0, \min((1, p_1-1)))$ such that

$$H_1 + H_2 \in L_{\frac{n}{p-\delta}}(\Omega), \quad 1 < p \leq n. \tag{2.1}$$

Definition 2.1 We say that the function u is a *weak solution* of equation (1.1) in $\Omega \backslash \{x_0\}$ if $u \in W^1_{(\overline{p})}(\Omega') := W^1_{p_1,p_2,\ldots,p_n}(\Omega')$ for an arbitrary subdomain $\Omega' \subset \Omega$ such that $x_0 \notin \overline{\Omega}'$ and the following integral identity holds

$$\sum_{i=1}^{n} \int_{\Omega} a_i\left(x, u, \frac{\partial u}{\partial x}\right) \frac{\partial}{\partial x_i}(\psi\varphi)dx + \int_{\Omega} a_0\left(x, u, \frac{\partial u}{\partial x}\right)\psi\varphi\, dx = 0, \tag{2.2}$$

where φ is an arbitrary element of $\overset{\circ}{W}{}^1_{(p)}(\Omega)$ and ψ is an arbitrary function from $C^1(\overline{\Omega})$, which is equal to zero in a neighborhood of x_0.

Now we introduce the family of subdomains. For $r > 0$, set

$$\Omega(r) := \left\{ x \in \Omega : l(x) := \left(\sum_{i=1}^{n} |x_i - x_{o,i}|^{\frac{b_i}{b_1}} \right)^{b_1} < r \right\},\ b_i \text{ is from } (1.14),\ 1 < p < n,$$

$$\Omega(r) := \left\{ x \in \Omega : l(x) := \left(\sum_{i=1}^{n} |x_i - x_{o,i}|^{\frac{p_i}{p_1}} \right)^{\frac{p_1}{n}} < r \right\}, \quad p = n. \tag{2.3}$$

Let

$$R_0 \in (0, \{\text{dist}\,(\{x_0\}, \partial\Omega)\}). \tag{2.4}$$

Next we define the function $M = M(r)$ characterizing local behaviour of the solution u in the neighbourhood of the point x_0:

$$M(r) := \operatorname{esssup}\{|u(x)| : x \in \Omega(R_0)\backslash\Omega(r)\}, \quad 0 < r < R_0. \tag{2.5}$$

The regularity result from [1] yields that $M(r) < \infty$ for an arbitrary $r > 0$. Now we are ready to formulate the main results.

Proposition 2.1 *Let u be a weak solution of equation (1.1) in $\Omega\backslash\{x_0\}$ (in the sense of Definition 2.1). Let parameters p_i satisfy conditions (1.3), (1.11). Suppose that the function $M(r)$, defined by u in (2.5), satisfies the following condition*

$$\lim_{r\to 0} M(r) r^{\frac{n-p}{p-1}} = 0 \quad \textit{if} \quad 1 < p < n, \quad \textit{or} \quad \lim_{r\to 0} M(r)\left(\ln\frac{1}{r}\right)^{-1} = 0 \quad \textit{if} \quad p = n. \tag{2.6}$$

Then there exist positive constants C_0, γ, which depend on the structural constants only, such that

$$M(r) \le C_0 r^{-\frac{n-p}{p-1}+\gamma} \quad \forall\, r : 0 < r < R_0, \quad 1 < p < n; \tag{2.7}$$

$$M(r) \le C_0 \left[\ln\frac{1}{r}\right]^{1-\gamma} \qquad \forall\, r : 0 < r < R_0, \quad p = n. \tag{2.8}$$

Theorem 2.1 *Let u be a solution of equation (1.1) in $\Omega\backslash\{x_0\}$. Let all conditions of Proposition 2.1 be satisfied, Additionally, suppose that*

$$g_1(x) = f_1(x) = 0, \qquad x \in \Omega. \tag{2.9}$$

Then the singularity of the solution $u(x)$ at the point $\{x_0\}$ is removable, and consequently integral identity (2.2) holds with $\psi = 1$.

Now we consider the source type (fundamental) solution $U(x)$ of equation (1.4), that is the weak solution of the boundary value problem

$$L_0 U = -\delta(x - x_0) \;\text{ in }\; \Omega, \quad U(x) = 0 \;\text{ on }\; \partial\Omega. \tag{2.10}$$

If condition (1.11) is satisfied, then the nonnegative solution to (2.10) exists ([9]) and satisfies (1.12). From (1.11) it follows that the inequalities $p_i - 1 < \dfrac{n(p-1)p_i}{p(n-1)}$ are valid, $i = \overline{1,n}$. Therefore (1.12) implies that

$$\frac{\partial U}{\partial x_i} \in L^{p_i - 1}(\Omega), \quad i = \overline{1,n}.$$

Hence we deduce from (2.10) that U satisfies the following integral identity

$$\sum_{i=1}^{n} \int\limits_{\Omega} \left|\frac{\partial U(x)}{\partial x_i}\right|^{p_i-2} \frac{\partial U(x)}{\partial x_i}\frac{\partial \varphi(x)}{\partial x_i} dx = \varphi(0) \tag{2.11}$$

for an arbitrary function $\varphi \in C_0^1(\Omega)$.

Now we introduce the function $m = m(\rho)$ characterizing the local behaviour of the fundamental solution U. Set

$$m(\rho) := \operatorname{ess\,inf}\, \{U(x) : x \in \partial\Omega(\rho)\}, \quad \forall \rho > 0, \quad 1 < p \le n. \tag{2.12}$$

The next theorem, the second main result of the paper, shows in particular that condition (2.6) is sharp.

Theorem 2.2 *Let U be the fundamental solution of equation (1.4), that is the solution of the boundary-value problem (2.10). Then there exist positive constants R_0, d_0, d_1, which depend on parameters $n, p_1, \dots, p_n$ only, such that the following estimates*

$$d_0\rho^{-\frac{n-p}{p-1}} \le M(\rho), \quad m(\rho) \le d_1\rho^{-\frac{n-p}{p-1}} \ \ \forall\rho < R_0, \ \ \text{if} \ \ 1 < p < n, \tag{2.13}$$

$$d_0 \ln\frac{1}{\rho} \le M(\rho), \quad m(\rho) \le d_1 \ln\frac{1}{\rho} \ \ \forall\rho < R_0, \ \ \text{if} \ \ p = n \tag{2.14}$$

hold.

3 Auxiliary integral estimates of the solution

We accomplish the analysis of the singularity of the solution in several steps. First, we deduce some integral estimates of the solution under consideration. Without loss of generality it can be assumed that the function $M(r)$ defined by (2.5) satisfies

$$\lim_{r\to 0} M(r) = \infty. \tag{3.1}$$

Suppose that the constant R_0 from (2.4) satisfies also the condition

$$M(R_0) \ge 1. \tag{3.2}$$

For an arbitrary $R \in (0, R_0)$, let the function u_R and the set $E(R)$ be defined by

$$u_R(x) = (u(x) - M(R), 0)_+,$$

$$E(R) = \{x \in \Omega(R)\backslash\{x_0\} : u(x) > M(R)\}. \tag{3.3}$$

Later on, by c, c_i, C, C_j we will denote different positive constants depending on structural constants only. Without loss of generality it can be assumed that $x_0 = 0$. Now we introduce the nonnegative cut-off function $\psi \in C^\infty(\mathbb{R}^1)$ satisfying conditions

$$\psi(t) = 0 \ \text{ for } \ t \le 1, \ \ \psi(t) = 1 \ \text{ for } \ t \ge 2, \ \ \left|\frac{d\psi(t)}{dt}\right| \le 2 \ \text{ for } \ t \in [1, 2]. \tag{3.4}$$

By ψ_r we denote our main cut-off function

$$\psi_r(x) := \psi\left(r^{-1}\left(\sum_{i=1}^n |x_i|^{\frac{b_i}{b_1}}\right)^{b_1}\right) \ \forall r > 0, \quad 1 < p < n; \tag{3.5}$$

$$\begin{gathered}\psi_r(x) := 2 - 2\,(\ln r)^{-1}\ln\left(\sum_{i=1}^n |x_i|^{\frac{p_i}{p_1}}\right)^{\frac{p_1}{n}} \ \text{ in } \ \Omega(\sqrt{r})\backslash\Omega(r), \\ \psi_r(x) = 0 \ \text{ in } \ \Omega(r), \quad \psi_r(x) = 1 \ \text{ outside } \ \Omega(\sqrt{r}), \quad \Omega(r) \ \text{ from } \ (2.3), \ \ p = n.\end{gathered} \tag{3.6}$$

It is easy to verify the following inequalities:

$$\begin{aligned} &\operatorname{mes}\Omega(r) \le cr^n; \\ &\int\limits_{\Omega(R_0)} \left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i} dx \le cr^{\frac{(n-p)(p_i-1)}{p-1}}, \qquad \forall r>0,\ \ 0<p<n; \\ &\int\limits_{\Omega(R_0)} \left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i} dx \le c\left(\ln r^{-1}\right)^{-(p_i-1)}, \qquad \forall r>0,\ \ p=n. \end{aligned} \tag{3.7}$$

Let us also introduce the function $\omega = \omega(r)$ by

$$\omega(r) := \left(M(r) r^{\frac{n-p}{p-1}}\right)^{p_1-1}, \qquad \forall r>0, \quad 1<p<n; \tag{3.8}$$

$$\omega(r) := \left(M(r)\left(\ln r^{-1}\right)^{-1}\right)^{p_1-1}, \qquad \forall r>0, \quad p=n. \tag{3.9}$$

Lemma 3.1 *Assume that conditions of Proposition 2.1 are satisfied. Then there exists a positive constants C_1 which depends on the structural constants only, such that*

$$\begin{gathered} \sum_{i=1}^n \int\limits_{E(R)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \psi_r^{p_n}(x)dx \le C_1\{M(r)\omega(r) + M^{p_n}(R)\}, \\ \forall r, R: 0<r<R<R_0, \quad 1<p\le n, \end{gathered} \tag{3.10}$$

where $\omega(r)$ is from (3.8) if $1<p<n$ and from (3.9) if $p=n$.

Proof. Test (2.2) by

$$\varphi(x) = u_R(x)\psi_r(x)^{p_n-1}, \quad \psi(x) = \psi_r(x),$$

where $\psi_r(x)$ is defined by (3.5) if $1<p<n$ and by (3.6) if $p=n$. After simple computations using structural conditions (1.2) and the Young inequality we deduce that

$$\begin{aligned} \sum_{i=1}^n \int\limits_{E(R)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \psi_r^{p_n}(x)dx \le c_1 \int\limits_{E(R)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx+ \\ + c_1 M^{p_n}(R) \int\limits_{E(R)\setminus\Omega(r)} \Big(g_1(x)+g_3(x)\Big)dx+ \\ + c_1\sum_{i=1}^n \int\limits_{K(r)\cap E(R)} \left[u_R^{p_i}(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i} + g_2^{\frac{p_i}{p_i-1}}(x)u^p(x)\right]dx+ \\ + c_1 \int\limits_{E(R)\setminus\Omega(r)} \left[f_1(x) + \sum_{i=1}^n [f_2(x)]^{\frac{p_i}{p_i-1}} + f_3(x)\right]dx, \end{aligned} \tag{3.11}$$

where $H_1(x)$ is from (2.1), $K(r) := \Omega(2r)\backslash\Omega(r)$ if $1 < p < n$, $K(r) := \Omega(\sqrt{r})\backslash\Omega(r)$ if $p = n$.

First we consider the case $1 < p < n$. We estimate the first term in the right-hand side of (3.11) using conditions (2.1), the Hölder inequality and inequality (2.1) from lemma 7.1 with $\alpha_1 = \cdots = \alpha_n = 0$. As a result we obtain

$$\int\limits_{E(R)\backslash\Omega(r)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx \le \int\limits_{E(R)\backslash\Omega(2r)} H_1(x)u_R^p(x)dx +$$

$$+\int\limits_{K(r)} H_1(x)u_R^p(x)dx \le c_2\left\{R^\delta\sum_{i=1}^n\int\limits_{E(R)\backslash\Omega(2r)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx + M^p(r)r^{n-p+\delta}\right\}. \tag{3.12}$$

Using conditions (2.1), property (3.7), and the Hölder inequality we obtain the following estimate

$$\sum_{i=1}^n\int\limits_{K(r)\cap E(R)}\left[u_R^{p_i}(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i} + [g_2(x)]^{\frac{p_i}{p_i-1}}u^p(x)\right]dx \le$$
$$\le c_3\sum_{i=1}^n M^{p_i}(r)r^{\frac{n-p}{p-1}(p_i-1)} + c_3M^p(r)r^{n-p+\delta}. \tag{3.13}$$

It follows from (2.1) that the second and fourth integral terms in the right-hand side of (3.11) are bound by a constant. Let us suppose that the constant R_0 satisfies also the following smallness condition

$$c_1c_2R_0^\delta \le 2^{-1}. \tag{3.14}$$

The next equality is evident:

$$p_1 - 1 = \min\left\{p-1,\ \min_{1\le i\le n}\{p_i-1\},\ \min_{1\le i\le n}\left\{\frac{p(p_i-1)}{p_i}\right\}\right\}. \tag{3.15}$$

Now from inequality (3.11) due to estimates (3.12), (3.13), assumption (3.14), property (3.15) and assumptions (2.6), (3.1) we obtain estimate (3.10).

For the case $p = n$ this statement can be proved in the same way. Instead of inequalities (3.12), (3.13) we have:

$$\int\limits_{E(R)\backslash\Omega(r)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx \le c_4\left\{R^{\frac{\delta}{2}}\sum_{i=1}^n\int\limits_{E(R)\backslash\Omega(2r)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx + M^p(r)r^{\frac{\delta}{2}}\right\}; \tag{3.16}$$

$$\sum_{i=1}^n\int\limits_{K(r)\cap E(R)}\left[u_R^{p_i}(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i} + [g_2(x)]^{\frac{p_i}{p_i-1}}u^p(x)\right]dx \le$$
$$\le c_5\sum_{i=1}^n M^{p_i}(r)\left(\ln r^{-1}\right)^{-(p_i-1)} + c_5M^n(r)r^{\frac{\delta}{2}}. \tag{3.17}$$

For an arbitrary $R < R_0$ we define the number $\rho_0 = \rho_0(R)$ by

$$M(\rho_0(R)) = \max\left\{2M(R),\ M\left(\frac{R}{2}\right)\right\}, \tag{3.18}$$

and the set $E(\rho, R)$ by

$$E(\rho, R) := \{x \in \Omega : 0 < u_R(x) \le M(\rho) - M(R)\} \text{ for all } \rho \le \rho_0(R). \tag{3.19}$$

Lemma 3.2 *Assume that all the conditions of Proposition 2.1 are satisfied. Then there exists a positive constant C_2, depending on structural constants only, such that the following estimate*

$$\begin{gathered}\sum_{i=1}^{n} \int\limits_{E(\rho,R)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \psi_r^{p_n}(x)dx \le C_2(M(\rho)\omega(r) + M^{p_n}(R)) + C_2 M(\rho)\times \\ \times \int\limits_{E(\rho)} \left\{\sum_{j=1}^{n} h_j(x)\left|\frac{\partial u}{\partial x_j}\right|^{p_j\left(1-\frac{1}{p}\right)} + g_3(x)u^{p-1}(x)\right\}\psi_r^{p_n}(x)dx, \quad 1 < p \le n,\end{gathered} \tag{3.20}$$

is true for all $0 < r < \rho < \rho_0(R)$.

Proof. Test integral identity (2.2) by

$$\varphi(x) = \min\left[u_R(x), M(\rho) - M(R)\right]\psi_r^{p_n-1}(x), \quad \psi(x) = \psi_r(x), \ \ 1 < p \le n. \tag{3.21}$$

Using assumptions (1.2) and the Young inequality we obtain the estimate

$$\begin{gathered}\sum_{i=1}^{n} \int\limits_{E(\rho,R)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \psi_r^{p_n}(x)dx \le \bar{c}_1 \int\limits_{E(\rho,R)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx + \bar{c}_1 M(\rho)\times \\ \times \sum_{i=1}^{n} \int\limits_{K(r)\cap E(R)} \left[\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}} + g_2(x)u^{p\left(1-\frac{1}{p_i}\right)} + f_2(x)\right]\left|\frac{\partial \psi_r}{\partial x_i}\right|\psi_r^{p_n-1}dx + \\ + \bar{c}_1 M(\rho) \int\limits_{E(\rho)} \left\{\sum_{j=1}^{n} h_j(x)\left|\frac{\partial u}{\partial x_j}\right|^{p_j\left(1-\frac{1}{p}\right)} + g_3(x)u^{p-1}(x) + f_3(x)\right\}\psi_r^{p_n}(x)dx + \\ + \bar{c}_1 M^p(R) \int\limits_{E(\rho,R)} (f_3(x) + g_3(x))\,dx.\end{gathered} \tag{3.22}$$

Now consider the case $p < n$. The first term in the right-hand side is estimated analogously to (3.12)

$$\int\limits_{E(\rho,R)} H_1(x)u_R^p(x)\psi_r^{p_n}(x)dx \leq$$
$$\leq \bar{c}_2 \left\{ R^\delta \sum_{i=1}^n \int\limits_{E(\rho,R)\setminus\Omega(2r)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + M^p(r)r^{n-p+\delta} \right\}. \tag{3.23}$$

Now we consider the second integral term in the right-hand side of (3.22). From the Hölder inequality, Lemma 3.1, definition (3.8), and inequality (3.7), we obtain

$$\sum_{i=1}^n \int\limits_{K(r)\cap E(R)} \left(\sum_{j=1}^n \int \left|\frac{\partial u}{\partial x_j}\right|^{p_j} \right)^{1-\frac{1}{p_i}} \left|\frac{\partial \psi_r}{\partial x_i}\right| \psi_r^{p_n-1}(x)dx \leq$$
$$\leq \sum_{i=1}^n \left(\int\limits_{K(r)\cap E(R)} \sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}\right|^{p_j} \psi_r^{p_n}(x)dx \right)^{\frac{p_i-1}{p_i}} \left(\int\limits_{K(r)} \left|\frac{\partial \psi_r}{\partial x_i}\right|^{p_i} dx \right)^{\frac{1}{p_i}} \leq \tag{3.24}$$
$$\leq \bar{c}_3 \sum_{i=1}^n (M(r)\omega(r))^{\frac{p_i-1}{p_i}} r^{\frac{n-p}{p-1}\cdot\frac{p_i-1}{p_i}} \leq \bar{c}_4\omega(r).$$

By the same arguments, using assumption (2.1), we deduce the estimate

$$\sum_{i=1}^n \int\limits_{K(r)\cap E(R)} g_2(x)u^{p(1-\frac{1}{p_i})}(x) \left|\frac{\partial \psi_r}{\partial x_i}\right| \psi_r^{p_n}(x)dx \leq \bar{c}_5\omega(r). \tag{3.25}$$

Let us suppose that the constant R_0 satisfies additionally the following smallness condition

$$\bar{c}_1\bar{c}_2R_0^\delta \leq 2^{-1}. \tag{3.26}$$

Taking into account (3.23)–(3.25), from (3.22) we derive the necessary estimate (3.20). The proof of inequality (3.20) in the case $p = n$ is the same with $\psi_r(x)$ defined in (3.6). Similarly to the proof of inequalities (3.16), (3.17) it is not hard to justify the estimates analogous to (3.23)–(3.25). □

Lemma 3.3 *Let all the assumptions of Proposition 2.1 be satisfied. Then there exist a positive constant C_3, depending on structural constants only, such that the estimates*

$$\sum_{i=1}^n \int\limits_{E(\rho)} u_R^{-q}(x) \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \psi_r^{p_n}(x)dx \leq C_3 \left(\omega(r) + r^{p\frac{n-p}{(p-1)^2}\cdot\frac{p_1-1}{p_1}} + \rho^{\frac{\delta}{2}} \right), \quad 1 < p < n; \tag{3.27}$$

$$\sum_{i=1}^{n}\int\limits_{E(\rho)} u_R^{-q}(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le C_3\left(\omega(r)+\left[\ln\frac{1}{r}\right]^{-p\frac{p_1-1}{p_1}}+\rho^{\frac{\delta}{2n}}\right),\quad p=n; \tag{3.28}$$

hold for all $0<r<\rho<\rho_0(R)<R_0$*, where* $q=1+2^{-1}(n-p)^{-1}\delta$ *if* $1<p<n$*,* $q>1$ *if* $p=n$*.*

Proof. Test integral identity (2.2) by

$$\varphi(x):=\left\{[M(\rho)-M(R)]^{1-q}-[\max(u_R(x),M(\rho)-M(R))]^{1-q}\right\}\psi_r^{p_n-1}(x),$$

$$\psi(x):=\psi_r(x), \tag{3.29}$$

where $\psi_r(x)$ is from (3.5) if $1<p<n$ and from (3.6) if $p=n$.

Using structural conditions (1.2) and the Young inequality, after simple computations we obtain

$$\sum_{i=1}^{n}\int\limits_{E(\rho)} u_R^{-q}(x)\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le c_1\int\limits_{E(\rho)} u_R^{-q}(x)\Big[g_1(x)u^p(x)+$$
$$+f_1(x)\Big]\psi_r^{p_n}(x)dx+c_1M^{1-q}(\rho)\sum_{i=1}^{n}\int\limits_{K(r)\cap E(\rho)}\left\{\left(\sum_{j=1}^{n}\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\right)^{1-\frac{1}{p_i}}+\right.$$
$$\left.+g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}(x)+f_2(x)\right\}\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{p_n-1}(x)dx+$$
$$+c_1M^{1-q}(\rho)\int\limits_{E(\rho)}\left\{\sum_{j=1}^{n}h_j^p(x)u_R^{q(p-1)}(x)+g_3(x)u^{p-1}(x)+f_3(x)\right\}\psi_r^{p_n}(x)dx. \tag{3.30}$$

Set

$$A_1:=\sum_{j=1}^{n}\int\limits_{E(\rho)} h_j^p(x)u_R^{q(p-1)}(x)\psi_r^{p_n}(x)dx.$$

It is easy to see that $E(\rho)\subset\Omega(\rho)\ \forall\ \rho<\rho_0(R_0)$. Therefore due to condition (2.1), the

Hölder inequality, and assumption (2.6), we have

$$A_1 < c_2 \left(\int\limits_{E(\rho)\setminus\Omega(r)} u_R^{\frac{q(p-1)n}{n-p+\delta}} dx \right)^{\frac{n-p+\delta}{n}} \leq c_3 \left(\int\limits_{\Omega(\rho)} \left(\sum_{i=1}^{n} |x_i|^{b_i} \right)^{-\frac{qn}{n-p+\delta}(n-p)} dx \right)^{\frac{n-p+\delta}{n}},$$

where b_i are from (1.14), $1 < p < n$;

$$A_1 < c_4 \left(\int\limits_{E(\rho)\setminus\Omega(r)} u_R^{\frac{q(p-1)n}{\delta}}(x) dx \right)^{\frac{\delta}{n}} \leq c_5 \left(\int\limits_{\Omega(\rho)} \left(\ln \left(\sum_{i=1}^{n} |x_i|^{\frac{p_i}{p_1}} \right)^{-\frac{p_1}{n}} \right)^{\frac{qn(p-1)}{\delta}} dx \right)^{\frac{\delta}{n}},$$

$p = n.$

(3.31)

Here $0 < r < \rho < \rho_0(R) < R_0$. Let us introduce new independent variables

$$\begin{aligned} &x_i := y_i^{\frac{\theta}{b_i}} \operatorname{sign} y_i, \ \ i = \overline{1,n}, \ \ \theta = 2 \max_{1\leq i\leq n}(1, b_i), \ 1 < p < n; \\ &x_i := y_i^{\theta \frac{n}{p_i}} \operatorname{sign} y_i, \ \ i = \overline{1,n}, \ \ \theta = 2 \max_{1\leq i\leq n}(1, \frac{p_i}{n}), \ p = n. \end{aligned} \tag{3.32}$$

Then after simple computations we deduce from (3.31) that

$$\begin{aligned} A_1 &\leq c_6 \left(\int\limits_{|y|^\theta < \rho} \left(\sum_{i=1}^{n} |y_i|^\theta \right)^{-\frac{qn(n-p)}{n-p+\delta}} \prod_{i=1}^{n} |y_i|^{\left(\frac{\theta}{b_i}-1\right)} dy \right)^{\frac{n-p+\delta}{n}} \leq \\ &\leq c_7 \left(\int\limits_0^{\rho^{\frac{1}{\theta}}} |y|^{-\theta\frac{qn(n-p)}{n-p+\delta} + \sum\limits_{i=1}^{n} \frac{\theta}{b_i} - 1} d|y| \right)^{\frac{n-p+\delta}{n}} \leq c_8 \rho^{\frac{\delta}{2}}, \ 1 < p < n; \\ A_1 &\leq c_9 \rho^{\frac{\delta}{n}} (\ln \rho^{-1})^{q(p-1)} < c_{10} \rho^{\frac{\delta}{2n}}, \ p = n, \end{aligned} \tag{3.33}$$

where $0 < r < \rho < \rho_0(R) < R_0$. In the same way, we deduce

$$\int\limits_{E(\rho)} g_3(x) u^{p-1}(x) \psi_r^{p_n}(x) dx \leq c_{11} \rho^{\frac{\delta}{2n}}, \ 1 < p \leq n. \tag{3.34}$$

The first term in the right-hand side of (3.30) is estimated as follows:

$$\int\limits_{E(\rho)} g_1(x)u_R^{-q}(x)u^p(x)\psi_r^{p_n}(x)dx \le c_{12}\int\limits_{E(\rho)} g_1(x)u_R^{p-q}(x)\psi_r^{p_n}(x)dx+$$

$$+c_{12}M^{p-q}(\rho)\int\limits_{E(\rho)} g_1(x)dx \le c_{13}\left(M^{p-q}(\rho)\rho^{n-p+\delta}+\rho^{\delta\left(\frac{1}{2(p-1)}+1\right)}\right),\ 1<p<n;$$

$$\int\limits_{E(\rho)} g_1(x)u_R^{-q}(x)u^p(x)\psi_r^{p_n}(x)dx \le c_{14}\left(M^{p-q}(\rho)\rho^{\delta}+\left(\ln\rho^{-1}\right)^{p-q}\rho^{\frac{\delta}{n}}\right),\ p=n. \tag{3.35}$$

Using condition (2.1) we infer

$$\int\limits_{E(\rho)} [f_1(x)+f_3(x)]dx \le c_{15}\rho^{n-p+\frac{\delta}{2n}},\ 1<p\le n; \tag{3.36}$$

$$\sum_{i=1}^{n}\int\limits_{E(\rho)} f_2(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|dx \le c_{16}\sum_{i=1}^{n} r^{\frac{n-p}{(p-1)^2}\cdot\frac{p_i-1}{p_i}p} \le c_{17}r^{\frac{n-p}{(p-1)^2}\cdot\frac{p_1-1}{p_1}p},\ 1<p<n;$$

$$\sum_{i=1}^{n}\int\limits_{E(\rho)} f_2(x)\left|\frac{\partial\psi_r}{\partial x_i}\right|dx \le c_{18}\left(\ln r^{-1}\right)^{-p\frac{p_1-1}{p_1}},\ p=n. \tag{3.37}$$

Then from (3.30), due to estimates (3.33)–(3.37) and inequalities (3.24), (3.25) for the case $1<p<n$ and analogues of these inequalities for the case $p=n$, we obtain the necessary estimates (3.27), (3.28). □

Our next proposition contains the main a priory estimate of the section.

Proposition 3.1 *Let all the assumptions of Proposition 2.1 be satisfied. Then there exists a positive constant C_4, depending on structural constants only, such that the following estimates hold:*

$$\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le$$
$$\le C_4\left\{M(\rho)\left[\omega(r)+r^{p\frac{n-p}{(p-1)^2}\cdot\frac{p_1-1}{p_1}}+\rho^{\frac{\delta}{2}}\right]+1\right\},\quad \text{if}\ \ 1<p<n; \tag{3.38}$$

$$\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^{p_n}(x)dx \le$$
$$\le C_4\left\{M(\rho)\left[\omega(r)+\left(\ln\frac{1}{r}\right)^{-p\frac{p_1-1}{p_1}}+\rho^{\frac{\delta}{2n}}\right]+1\right\},\quad \text{if}\ \ p=n. \tag{3.39}$$

Proof. Let us estimate the second term in the right hand-side of (3.20)by means of inequality (3.27). Using the Hölder inequality and estimate (3.33), we obtain, particulary

$$M(\rho)\int\limits_{E(\rho)} h_j(x)\left|\frac{\partial u}{\partial x_j}\right|^{p_j\left(1-\frac{1}{p}\right)} u_R^{-q\left(1-\frac{1}{p}\right)}(x)u_R^{q\left(1-\frac{1}{p}\right)}(x)\psi_r^{p_n}(x)dx \le$$

$$\le M(\rho)\left(\int\limits_{E(\rho)}\left|\frac{\partial u}{\partial x_j}\right|^{p_j} u_R^{-q}(x)\psi_r^{p_n}(x)dx\right)^{\frac{p-1}{p}}\left(\int\limits_{E(\rho)} h_j^p(x)u_R^{q(p-1)}(x)\psi_r^{p_n}(x)dx\right)^{\frac{1}{p}} \le$$

$$\le cM(\rho)\left(\omega(\rho)+r^{\frac{p(n-p)(p_1-1)}{(p-1)^2p_i}}+\rho^{\frac{\delta}{2}}\right)^{\frac{p-1}{p}}\rho^{\frac{\delta}{2p}}.$$

From (3.20) using the above estimate and inequality (3.34) we deduce the necessary estimate (3.38). In the same way we derive upper bound (3.39). □

In virtue of assumption (2.6) we have that $\omega(r)\to 0$ as $r\to 0$. Then, passing to the limit in (3.38), (3.39) as $r\to 0$ we obtain

Corollary 3.1 *Let all the assumption of Proposition 2.1 be satisfied. Then the following estimate holds:*

$$\sum_{i=1}^{n}\int\limits_{E(\rho,R)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx \le C_4\left(M(\rho)\rho^{\frac{\delta}{2n}}+1\right),\quad i=\overline{1,n},\quad 1<p\le n. \tag{3.40}$$

4 Point-wise estimates of the solution

Our further study of properties of the solutions requires the new family of cut-off functions. Let us fix positive numbers a_i, $i=1,2,3,4$, $0<a_1<a_2<a_3<a_4$, and define the function $\chi_{a_1,a_2}^{a_3,a_4}\in C^1(\mathbb{R}^1)$ such that $0\le\chi_{a_1,a_2}^{a_3,a_4}(s)\le 1\ \forall\ s\in\mathbb{R}^1$, $\chi_{a_1,a_2}^{a_3,a_4}(s)=1$ in (a_2,a_3), $\chi_{a_1,a_2}^{a_3,a_4}(s)=0$ in $\mathbb{R}^1\backslash(a_1,a_4)$, $\left|\frac{d\chi_{a_1,a_2}^{a_3,a_4}(s)}{ds}\right|<C\max\left\{(a_2-a_1)^{-1},(a_4-a_3)^{-1}\right\}$. Now we introduce the family of cut-off functions $\chi_l(x):=\chi_{a_1,a_2}^{a_3,a_4}(l(x))$, where the function $l(x)$ is from (2.3), $1<p\le n$. It is easy to see that

$$\chi_l(x)=1\ \text{ in }\ \Omega(a_3)\backslash\Omega(a_2),$$

$$\chi_l(x)=0\ \text{ in }\ \mathbb{R}^n\backslash\{\Omega(a_4)\backslash\Omega(a_1)\},$$

$$\left|\frac{\partial\chi_l(x)}{\partial x_j}\right|\le c\max\left\{\frac{a_1^{1-\frac{1}{b_i}}+a_2^{1-\frac{1}{b_i}}}{a_2-a_1},\frac{a_3^{1-\frac{1}{b_i}}+a_4^{1-\frac{1}{b_i}}}{a_4-a_3}\right\},\ j=\overline{1,n}, \tag{4.1}$$

where the constant c does not depend on a_i, $i=1,2,3,4$.

Let u be a solution from Theorem 2.1 and u_R is defined by (3.3) for an arbitrary $R, 0 < R < R_0$. Then we consider the family of sets

$$A_k(s,t) := \{x \in \Omega : u_R(x) > k\} \cap (\Omega(t) \backslash \Omega(s)), \;\; 0 < s < t. \tag{4.2}$$

Let us also fix some constant $\sigma \in (0, 2^{-1})$.

Lemma 4.1 *Let all the assumptions of Proposition 2.1 be satisfied. Then there exists a positive constant C_5, which does not depend, particulary, on $k > 0$, $r > 0$, σ, such that the following estimate holds:*

$$\sum_{i=1}^{n} \int\limits_{A_k\left(\frac{r(1+\sigma)}{2}, \frac{3r(1-\sigma)}{2}\right)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \le C_5 \sigma^{-p_n} r^{-\frac{p(n-p)}{p-1} - \delta} \left(\text{mes } A_k\left(\frac{r}{2}, \frac{3r}{2}\right)\right)^{1 - \frac{p}{n} + \frac{\delta}{n}},$$
$$1 < p < n,$$
$$\sum_{i=1}^{n} \int\limits_{A_k\left(\frac{r(1+\sigma)}{2}, \frac{3r(1-\sigma)}{2}\right)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \le C_5 \sigma^{-p_n} r^{-\delta} (\ln r^{-1})^n \left(\text{mes } A_k\left(\frac{r}{2}, \frac{3r}{2}\right)\right)^{\frac{\delta}{n}}, \;\; p = n. \tag{4.3}$$

Proof. Test (2.2) by

$$\varphi(x) = (u_R(x) - k)_+ \chi_r^{p_n - 1}(x), \quad \psi(x) = \chi_r(x), \quad 1 < p < n,$$

where $\chi_r(x) := \chi_{a_1,a_2}^{a_3,a_4}(l(x))$ with $l(x)$ from (2.3) and

$$a_1 = \frac{r}{2}, \;\; a_2 = \frac{r}{2}(1+\sigma), \;\; a_3 = \frac{3r}{2}(1-\sigma), \;\; a_4 = \frac{3r}{2}. \tag{4.4}$$

Denoting $A_{k,r} = A_k\left(\frac{r}{2}, \frac{3r}{2}\right)$, $A_{k,r,\sigma} = A_k\left(\frac{r(1+\sigma)}{2}, \frac{3r(1-\sigma)}{2}\right)$, using structural conditions (1.2) and the Young inequality with ε, after standard computations we deduce that

$$\begin{aligned} \sum_{i=1}^{n} \int\limits_{A_{k,r}} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \chi_r^{p_n}(x) dx &\le c_1 \int\limits_{A_{k,r}} H_1(x) u^p(x) \chi_r^{p_n}(x) dx + \\ &+ c_1 \int\limits_{A_{k,r}} (f_1(x) + f_3(x)) dx + c_1 \sum_{i=1}^{n} \int\limits_{A_{k,r} \backslash A_{k,r,\sigma}} u^{p_i}(x) \left|\frac{\partial \chi_r(x)}{\partial x_i}\right|^{p_i} dx + \\ &+ c_1 \sum_{i=1}^{n} \int\limits_{A_{k,r} \backslash A_{k,r,\sigma}} \left(g_2(x) u^{p\left(1 - \frac{1}{p_i}\right)}(x) + f_2(x)\right) (u_R(x) - k) \left|\frac{\partial \chi_r(x)}{\partial x_i}\right| dx, \end{aligned} \tag{4.5}$$

where $H_1(x)$ is from (2.1). Using conditions (2.1) and property (4.1), we estimate integrals in the right-hand side of (4.5) as follows:

$$\int\limits_{A_{k,r}} H_1(x) u^p(x) \chi_r^{p_n}(x) dx \le c_2 M^p \left(\frac{r}{2}\right) (\text{mes } A_{k,r})^{1 - \frac{p}{n} + \frac{\delta}{n}}, \tag{4.6}$$

$$\int\limits_{A_{k,r}} (f_1(x)+f_2(x))dx \le c_3 \left(\text{mes } A_{k,r}\right)^{1-\frac{p}{n}+\frac{\delta}{n}}, \tag{4.7}$$

$$\int\limits_{A_{k,r}\setminus A_{k,r,\sigma}} u^{p_i}(x)\left|\frac{\partial \chi_r}{\partial x_i}\right|^{p_i} dx \le c_4\sigma^{-p_i} r^{-\frac{p_i}{b_i}} M^{p_i}\left(\frac{r}{2}\right) \text{mes } (A_{k,r}\setminus A_{k,r,\sigma}), \tag{4.8}$$

$$\begin{gathered}\int\limits_{A_{k,r}\setminus A_{k,r,\sigma}} \left(g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}+f_2\right)(u_R(x)-k)\left|\frac{\partial \chi_r}{\partial x_i}\right| dx \le c_5\sigma^{-1}r^{-\frac{1}{b_i}}\times \\ \times\left(M\left(\frac{r}{2}\right)+M\left(\frac{r}{2}\right)^{p\left(1-\frac{1}{p_i}\right)+1}\right)\left[\text{mes}(A_{k,r}\setminus A_{k,r,\sigma})\right]^{1-\frac{p}{n}+\frac{p}{p_i n}+\frac{\delta(p_i-1)}{p_i n}}.\end{gathered} \tag{4.9}$$

By (3.7), it follows from $A_{k,r}\subset\Omega(2r)$ that mes $A_{k,r}\le cr^n$. Using this fact, condition (2.6), inequality (4.5) and estimates (4.6)–(4.9), we obtain the necessary inequality (4.3). □

Let us fix constants $k_0>0$, $\sigma_0\in\left(0,2^{-1}\right)$, $\rho>0$, and introduce the sequences:

$$\begin{aligned} k_h &= k_0(2-2^{-h}),\ \ h=0,1,2,\dots \\ t_h &= \frac{3}{2}\rho(1-\sigma_0(1-2^{-h})),\ \ h=0,1,2,\dots \\ s_h &= \frac{\rho}{2}(1+\sigma_0(1-2^{-h})),\ \ h=0,1,2,\dots \end{aligned} \tag{4.10}$$

Set

$$A_h := A_{k_h}(s_h,t_h),\ \ h=0,1,\dots, \tag{4.11}$$

where $A_{k_h}(s_h,t_h)$ is defined by (4.2). Now we prove the following statement which is a generalization of Theorem 5.1 from [2] to the anisotropic case.

Lemma 4.2 *Let all the conditions of Proposition 2.1 be fulfilled. Then there exists positive constant C_6, such that*

$$\left(\operatorname*{ess\,sup}_{\Omega\left(\frac{3\rho}{2}(1-\sigma_0)\right)\setminus\Omega\left(\frac{\rho}{2}(1+\sigma_0)\right)} u(x)\right)^{p\left(1+\frac{n}{\delta}\right)} \le C_6\rho^{-\left(1+\frac{p(n-p)}{(p-1)\delta}\right)n}\int\limits_{\Omega\left(\frac{3}{2}\rho\right)\setminus\Omega\left(\frac{\rho}{2}\right)} u_R^p(x)dx,$$

$$1<p<n, \tag{4.12}$$

$$\left(\operatorname*{ess\,sup}_{\Omega\left(\frac{3\rho}{2}(1-\sigma_0)\right)\setminus\Omega\left(\frac{\rho}{2}(1+\sigma_0)\right)} u(x)\right)^{n\left(1+\frac{n}{\delta}\right)} \le C_6(\ln\rho^{-1})^{\frac{2n^2}{\delta}}\rho^{-2n}\int\limits_{\Omega\left(\frac{3}{2}\rho\right)\setminus\Omega\left(\frac{\rho}{2}\right)} u_R^n(x)dx,\ \ p=n. \tag{4.13}$$

Proof. Consider the case $1<p<n$. Let us introduce additional sequences of numbers:

$$\bar{t}_h := \frac{1}{2}(t_h+t_{h+1}) = \frac{3}{2}\rho\left(1-\sigma_0\left(1-\frac{3}{4}2^{-h}\right)\right),\ \ h=0,1,\dots$$

$$\overline{s}_h := \frac{1}{2}(s_h + s_{h+1}) = \frac{\rho}{2}\left(1+\sigma_0\left(1-\frac{3}{4}2^{-h}\right)\right), \quad h = 0,1,\dots$$

We define the family of cut-off functions

$$\zeta_{h+1}(x) := \chi^{a_3(h),a_4(h)}_{a_1(h),a_2(h)}(l(x)), \quad h = 0,1,2,\dots,$$

where $a_1(h) = \overline{s}_h$, $a_2(h) = s_{h+1}$, $a_3(h) = t_{h+1}$, $a_4(h) = \overline{t}_h$.

Due to the Hölder inequality and the embedding theorem (Lemma 7.1 in Appendix) we have

$$J_{h+1} := \int\limits_{A_{h+1}} (u_R(x) - k_{h+1})^p dx \le \int\limits_{A_{k_{h+1}}(\overline{s}_h,\overline{t}_h)} (u_R(x) - k_{h+1})^p \zeta^p_{h+1}(x) dx \le$$

$$\le \left(\text{mes}\, A_{k_{h+1}}(\overline{s}_h,\overline{t}_h)\right)^{\frac{p}{n}} \left(\int\limits_{A_{k_{h+1}}(\overline{s}_h,\overline{t}_h)} (u_R(x) - k_{h+1})^{\frac{np}{n-p}} \zeta^{\frac{pn}{n-p}}_{h+1}(x) dx\right)^{\frac{n-p}{n}} \le$$

$$\le \left(\text{mes}\, A_{k_{h+1}}(\overline{s}_h,\overline{t}_h)\right)^{\frac{p}{n}} \times$$

$$\times \prod_{i=1}^{n}\left(\int\limits_{A_{k_{h+1}}(\overline{s}_h,\overline{t}_h)} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i} \zeta^{p_i}_{h+1}(x) + (u_R(x) - k_{h+1})^{p_i}\left|\frac{\partial \zeta_{h+1}}{\partial x_i}\right|^{p_i}\right) dx\right)^{\frac{p}{np_i}}. \tag{4.14}$$

In order to obtain this estimate in the case $p = n$ one chooses $q = \frac{2n^2}{\delta}$ in the embedding theorem.

From Lemma 4.1 it follows that

$$\begin{aligned}
&\int\limits_{A_{k_{h+1}}(\overline{s}_h,\overline{t}_h)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \zeta^{p_i}_{h+1} dx \le \\
&\le c_1 \sigma_0^{-p_n} \rho^{-\frac{p(n-p)}{p-1}-\delta} 2^{hp_n} \left(\text{mes}\, A_{k_{h+1}}(s_h,t_h)\right)^{1-\frac{p}{n}+\frac{\delta}{n}}
\end{aligned} \tag{4.15}$$

By virtue of property (4.1) and assumption (2.6) we have

$$\begin{aligned}
&\int\limits_{A_{k_{h+1}}(\overline{s}_h,\overline{t}_h)} (u_R - k_{h+1})^{p_i}\left|\frac{\partial \zeta_{h+1}}{\partial x_i}\right|^{p_i} dx \le \\
&\le c_2 \sigma_0^{-p_n} 2^{p_i h} \rho^{-\frac{p_i}{b_i}} M^{p_i}(\overline{s}_h) \text{mes}\, A_{k_{h+1}}(s_h,t_h) \le \\
&\le c_3 \sigma_0^{-p_n} 2^{p_n h} \rho^{-\frac{p(n-p)}{p-1}-\delta} \left(\text{mes}\, A_{k_{h+1}}(s_h,t_h)\right)^{1-\frac{p}{n}+\frac{\delta}{n}}.
\end{aligned} \tag{4.16}$$

Using estimates (4.15), (4.16), we deduce from (4.14)

$$J_{h+1} \le c_4 2^{p_n h} \rho^{-\left(\frac{p(n-p)}{p-1}+\delta\right)} \left(\text{mes}\, A_{k_{h+1}}(s_h,t_h)\right)^{1+\frac{\delta}{n}}. \tag{4.17}$$

Let us estimate $\operatorname{mes} A_{k_{h+1}}(s_h, t_h)$ from above as follows:

$$J_h \geq \int_{A_{k_{h+1}}(s_h,t_h)} (u_R - k_h)^p dx \geq (k_{h+1} - k_h)^p \operatorname{mes}\left(A_{k_{h+1}}(s_h, t_h)\right) =$$

$$= 2^{-p(h+1)} k_0^p \operatorname{mes}\left(A_{k_{h+1}}(s_h, t_h)\right).$$

Due to the last estimate, inequality (4.17) implies that

$$J_{h+1} \leq c_5 2^{p\left(2+\frac{\delta}{n}\right)h} k_0^{-p\left(1+\frac{\delta}{n}\right)} \rho^{-\left(\frac{p(n-p)}{p-1}+\delta\right)} J_h^{1+\frac{\delta}{n}}, \quad h = 0, 1, \dots \tag{4.18}$$

Inequality (4.18) via Lemma 7.3 implies that

$$J_h \to 0 \text{ as } h \to \infty, \tag{4.19}$$

if J_0 satisfies the following smallness condition

$$J_0 \leq \left[c_5 k_0^{-p\left(1+\frac{\delta}{n}\right)} \rho^{-\frac{p(n-p)}{p-1}+\delta}\right]^{-\frac{n}{\delta}} 2^{-\frac{p(2n+\delta)n}{\delta^2}} \tag{4.20}$$

or, equivalently,

$$k_0^{p\left(1+\frac{n}{\delta}\right)} \geq c_5^{\frac{n}{\delta}} 2^{\frac{2(2n+\delta)n}{\delta^2}} \rho^{-\left(n+\frac{n(n-p)p}{(n-1)\delta}\right)} J_0. \tag{4.21}$$

Estimate (4.12) follows immediately from (4.19), (4.21). Estimate (4.13) is deduced the same way. We remark only that the smallness condition in the case $p = n$ is

$$J_0 \leq \left[c_5 k_0^{-n(1+\frac{\delta}{2n})} \rho^{-\delta} (\ln \rho^{-1})^n\right]^{-\frac{2n}{\delta}} 2^{-\frac{2n^2}{\delta^2}(4n+\delta)}.$$

□

Now we are ready to prove Proposition 2.1, which is the main step in the proof of Theorem 2.1.

Proof of Proposition 2.1. Using the Hölder inequality and the inclusion $\Omega\left(\frac{3}{2}\rho\right) \setminus \Omega\left(\frac{\rho}{2}\right) \subset E\left(\frac{\rho}{2}, R\right)$, we obtain

$$I_1 = \int\limits_{\Omega\left(\frac{3}{2}\rho\right)\setminus\Omega\left(\frac{\rho}{2}\right)} u_R^p(x) dx \leq c\rho^p \left(\int\limits_{E\left(\frac{\rho}{2},R\right)} u_R^{\frac{np}{n-p}}(x) dx\right)^{\frac{n-p}{n}}.$$

Applying the embedding theorem (Lemma 7.1) with $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, inequality (3.40) and assumption (2.6), we deduce from the last inequality that

$$I_1 \leq c_1 \rho^p \sum_{i=1}^{n} \int\limits_{E\left(\frac{\rho}{2},R\right)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \leq c_2 \left(M(\rho)\rho^{p+\frac{\delta}{2}} + \rho^p\right) \leq c_3 \rho^{-\frac{n-p}{p-1}+p+\frac{\delta}{2}}.$$

Using this estimate in inequality (4.13), after simple computations we obtain

$$\operatorname*{ess\,sup}_{\Omega\left(\frac{3\rho}{2}(1-\sigma_0)\right)\setminus\Omega\left(\frac{\rho}{2}(1+\sigma_0)\right)} u(x) \le c_4 \rho^{-\frac{n-p}{p-1}+\frac{\delta}{2p\left(1+\frac{np}{\delta q}\right)}}. \tag{4.22}$$

From this estimate inequality (2.7) follows immediately. This completes the proof in the case $1 < p < n$. By the same argument, using inequality (3.40) and Lemma 7.1 with $\alpha_1 = \cdots = \alpha_n = 0,\ q = n$ we get Proposition 2.1 in the case $p = n$. □

5 Proof of Theorem 2.1

Define the number λ by

$$\lambda = \min\left\{\frac{1}{2}\cdot\frac{(p_1-1)(p-1)}{n-p}\gamma, \frac{n-1}{n-p}p - p_n, p_1 - 1\right\}, \tag{5.1}$$

where γ is from (2.7).

Lemma 5.1 *Let all the conditions of Theorem 2.1 be satisfied. Then there exist positive constants C_8, $\tilde{k}$ depending on structural constants only, such that the inequality*

$$\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{\lambda-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \le C_8\left(k^{p+\lambda-1}+1\right)\ \text{mes}\ A_k^{1-\frac{p}{n}+\frac{\delta}{n}}, \quad 1<p<n, \tag{5.2}$$

holds for all $k > \tilde{k}$.

Proof. It is easy to check that

$$v_k(x) := (u-k)_+^{\lambda}\psi_r^{l-1}(x) \in \overset{\circ}{W}{}^1_{(\overline{p})}(\Omega(R_0)) \qquad \forall k > M\left(\frac{R_0}{2}\right),\ l \ge p_n, \tag{5.3}$$

where $\psi_r(x)$ is the cut-off function from (3.5). Let $\varphi(x) = v_k(x),\ \ \psi = \psi_r(x),\ \ k > M\left(\frac{R_0}{2}\right)$. Due to (5.3) we can use $\varphi\psi$ as a test function in (2.2). Using structural conditions (1.2), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{\lambda-1}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\psi_r^l(x)dx \le c_1\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{p_i-1+\lambda}\left|\frac{\partial \psi_r}{\partial x_i}\right|^{p_i}dx+\\
&+c_1\sum_{i=1}^{n}\int\limits_{A_k}(u-k)^{\lambda}\Big[g_2(x)u^{p\left(1-\frac{1}{p_i}\right)}(x)+f_2(x)\Big]\left|\frac{\partial \psi_r}{\partial x_i}\right|dx+\\
&+c_1\int\limits_{A_k}\Big[\sum_{i=1}^{n}h_i^p(x)(u-k)^{p-1+\lambda}+g_3(x)(u-k)_+^{\lambda}u^{p-1}+\\
&+f_3(x)(u-k)^{\lambda}\Big]\psi_r^l(x)dx = \sum_{j=1}^{3}I_j,
\end{aligned} \tag{5.4}$$

where $A_k = \{x \in \Omega(R_0) : u(x) > k\}$. Property (2.7), the Hölder inequality, and estimate (3.7) imply that

$$I_1 \le c_2 r^{\frac{\gamma}{2}(p_1-1)\left(1+\frac{2\lambda}{p_1-1}\right)}, \;\; I_2 \le c_3 r^{\frac{\gamma}{2}\left(1-\frac{1}{p_i}\right)\left(1+\frac{2\delta}{\gamma}\right)}, \tag{5.5}$$

$$\begin{aligned} I_3 \le c_4 \left(\int_{A_k} (u-k)^{\frac{n(p-1+\lambda}{n-p}} \psi_r(x)^{\frac{ln}{n-p}} dx \right)^{\frac{n-p}{n}} (\text{mes } A_k)^{\frac{\delta}{n}} + \\ + c_4 \left(k^{p-1+\lambda} + 1\right) (\text{mes } A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} . \end{aligned} \tag{5.6}$$

Applying Lemma 7.2 with $\alpha_i = 1 - \lambda$, $i = \overline{1,n}$, $q = \dfrac{np}{(n-p)} \left(1 + \frac{\lambda-1}{p}\right)$, we deduce that

$$\begin{aligned} &\left(\int\limits_{A_k} \left[(u-k)\psi_r(x)^{\frac{l}{p-1+\lambda}} \right]^{\frac{n(p-1+\lambda)}{n-p}} dx \right)^{\frac{n-p}{n}} \le \\ &\le c_5 \sum_{i=1}^{n} \int\limits_{A_k} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r(x)^{\frac{l(p_i+\lambda-1)}{p+\lambda-1}} dx + \\ &+ c_5 \sum_{i=1}^{n} \int\limits_{A_k} (u-k)^{p_i-1+\lambda} \psi_r(x)^{\frac{l(p_i+\lambda-1)}{p+\lambda-1}-p_i} \left| \frac{\partial \psi}{\partial x_i} \right|^{p_i} dx := I_3^{(1)} + I_3^{(2)}. \end{aligned} \tag{5.7}$$

The term $I_3^{(2)}$ is estimated in the same way as I_1 in (5.5). Therefore from inequality (5.4), due to (5.5)–(5.7), one infers

$$\begin{aligned} &\sum_{i=1}^{n} \int_{A_k} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r^l(x) dx \le \\ &\le c_6 \left(\sum_{i=1}^{n} \int_{A_k} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \psi_r^{\frac{l(p_i+\lambda-1)}{p+\lambda-1}} dx \right) \times \\ &\times (\text{mes} A_k)^{\frac{\delta}{n}} + c_6 r^{\frac{\gamma}{2}(p_1-1)\left(1+\frac{2\lambda}{p_1-1}\right)} + c_6 r^{\frac{\gamma}{2}\frac{p_1-1}{p_1}\left(1+\frac{2\delta}{\gamma}\right)} + \\ &+ c_6 \left(k^{p+\lambda-1} + 1\right) (\text{mes} A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} . \end{aligned} \tag{5.8}$$

This inequality yields

$$\begin{aligned} &I_k(2r) := \sum_{i=1}^{n} \int\limits_{A_k \cap (\Omega \setminus \Omega(2r))} (u-k)^{\lambda-1} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \le \\ &\le c_6 (\text{mes} A_k)^{\frac{\delta}{n}} I_k(r) + c_6 \mu(r) + c_6 g(k), \\ &g(k) := \left(k^{p+\lambda-1} + 1\right) (\text{mes} A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} , \\ &\mu(r) := r^{\frac{\gamma}{2}(p_1-1)\left(1+\frac{2\lambda}{p_1-1}\right)} + r^{\frac{\gamma}{2}\frac{(p_1-1)}{p_1}\left(1+\frac{2\delta}{\gamma}\right)} \to 0 \text{ as } r \to 0. \end{aligned} \tag{5.9}$$

Now we come back to the term I_3 from (5.4) and estimate it using inequality (2.7), proved in Proposition 2.1. As a result, using additional condition (2.1), we obtain

$$I_3 \leq c_7 M(r)^\lambda \left(1 + M(r)^{p-1}\right) (\text{mes}A_k)^{1-\frac{p}{n}+\frac{\delta}{n}} \leq \\ \leq c_8 \left(r^{-\left(\frac{n-p}{p-1}-\gamma\right)(p+\lambda-1)} + 1\right) (\text{mes}A_k)^{1-\frac{p}{n}+\frac{\delta}{n}}. \tag{5.10}$$

Using now estimates (5.5), (5.10), we derive from inequality (5.4)

$$I_k(2r) \leq c_9 \left(r^{-\nu} + 1\right) \ \forall r > 0, \ \ \nu = \left(\frac{n-p}{(p-1)} - \gamma\right)(p+\lambda-1) > 0. \tag{5.11}$$

Thus, the nondecreasing function $I_k(r)$ satisfies both inequalities (5.9) and (5.11). We claim that there exist a number $\tilde{k} > 0$ and a sequence $r_i \to 0$ as $i \to \infty$ such that the estimate

$$I_k(r_i) \leq \left(\text{mes}A_{\tilde{k}}\right)^{-\frac{\delta}{n}} (\mu(r_i) + g(k)) \qquad \forall k \geq \tilde{k} \tag{5.12}$$

is valid. The proof of this claim is based upon the method of analysis of nonhonogeneous functional inequalities, introduced in [10]. Let us suppose that (5.12) fails and, consequently,

$$\mu(r) + g(k) \leq \left(\text{mes}A_{\tilde{k}}\right)^{\frac{\delta}{n}} I_k(r) \ \forall r : 0 < r < r_0. \tag{5.13}$$

Using (5.13), inequality (5.9) yields

$$I_k(2r) \leq 2c_6 \left(\text{mes}A_{\tilde{k}}\right)^{\frac{\delta}{n}} I_k(r) \ \forall k > \tilde{k}, \ \forall r < r_0. \tag{5.14}$$

Iterating this relationship, we deduce easily the estimate

$$I_k(r) \geq (2c_6)^{-1} \left(\text{mes}A_{\tilde{k}}\right)^{-\frac{\delta}{n}} I_k(r_0) \left(\frac{r}{r_0}\right)^{-h(\tilde{k})} \quad \forall r \leq r_0, \tag{5.15}$$

where $h(\tilde{k}) = (\ln 2)^{-1} \left(-\ln(2c_6) - \frac{\delta}{n} \ln\left(\text{mes}A_{\tilde{k}}\right)\right)$. Let $\tilde{k}$ be a fixed number such that the following inequality holds

$$h(\tilde{k}) = (\ln 2)^{-1} \left(\frac{\delta}{n} \ln\left(\left(\text{mes}A_{\tilde{k}}\right)^{-1}\right) - \ln(2c_6)\right) > \nu, \tag{5.16}$$

where ν is from (5.11). It is clear that for $\tilde{k}$ satisfying (5.16) estimate (5.15) contradicts estimate (5.11). Therefore our assumption (5.13) is not true, and estimate (5.12) is proved for $\tilde{k}$, satisfying condition (5.16). Using estimate (5.12) in the right-hand side of (5.9) and passing to the limit as $r_i \to 0$, we obtain the required estimate (5.2). This completes the proof in the case $1 < p < n$. In the case $p = n$, estimate (5.2) follows by the similar computations with using of imbedding Lemma 7.2 with $q = \frac{n}{\delta}(p+\lambda-1)$. □

Proof of Theorem 2.1. Due to Lemma 7.2, with $\alpha = 1 - \lambda$, $q = \dfrac{n}{n-p}(p+\lambda-1)$, we deduce from (5.2) that

$$\left(\int\limits_{A_k} (u-k)_+^{\frac{n(p+\lambda-1)}{n-p}} dx\right)^{\frac{n-p}{n}} \leq c_1 k^{p+\lambda-1} (\text{mes}A_k)^{1-\frac{p}{n}+\frac{\delta}{n}}. \tag{5.17}$$

Using the Hölder inequality, we have additionally

$$\int\limits_{A_k} (u-k)_+ dx \le \left(\int\limits_{A_k} (u-k)_+^{\frac{n(p+\lambda-1)}{n-p}} dx \right)^{\frac{n-p}{n(p+\lambda-1)}} (\mathrm{mes} A_k)^{1-\frac{n-p}{n(p+\lambda-1)}} . \qquad (5.18)$$

Combining (5.17) and (5.18), we obtain

$$\int\limits_{A_k} (u-k)_+ dx \le k\,(\mathrm{mes} A_k)^{1+\frac{\delta}{n(p+\lambda-1)}} \;\; \forall k > \tilde{k}. \qquad (5.19)$$

From (5.19) it follows that the solution u is bounded in $\Omega(R_0)$ (see Lemma 5.1, [2]). This together with (3.7) imply that we can pass to the limit in inequality (3.11) as $r \to 0$, and obtain

$$\int\limits_{\Omega} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx < c. \qquad (5.20)$$

Let us test integral identity (2.2) by $\psi(x) = \psi_r(x)$. Then due to the boundedness of the solution and property (5.20), it is easy to pass to the limit as $r \to 0$. So we obtain the required integral identity with an arbitrary $\varphi \in \overset{\circ}{W}{}^1_{(\overline{p})}(\Omega)$ and $\psi(x) \equiv 1$. Thus Theorem 2.1 is proved in the case $1 < p < n$. All proofs of these results can be repeated for the case $p = n$, using appropriate estimates from Section 3. □

6 Point-wise estimates of source-type solution

Proof of Theorem 2.2. Let us introduce the family of cut-off functions

$$\overline{\psi}_r(x) = 1 - \psi_r(x) \qquad \forall r > 0, \qquad (6.1)$$

where $\psi_r(x)$ is function from (3.5). Now we fix an arbitrary $\rho > 0$ and test integral identity (2.11) by the function $\varphi = \overline{\psi}{}^l_{2\rho}(x)$, $l > np$. Using the Hölder inequality, we arrive at

$$1 = l \sum_{i=1}^{n} \int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)} \left| \frac{\partial U}{\partial x_i} \right|^{p_i-2} \frac{\partial U}{\partial x_i} \overline{\psi}{}^{l-1}_{2\rho}(x) \frac{\partial \overline{\psi}_{2\rho}(x)}{\partial x_i} dx \le$$

$$\le c_1 \sum_{i=1}^{n} \left(\int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)} \left| \frac{\partial U}{\partial x_i} \right|^{p_i} U^{-\lambda} dx \right)^{\frac{p_i-1}{p_i}} \left(\int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)} U^{\lambda(p_i-1)} \left| \frac{\partial \overline{\psi}_{2\rho}}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}}, \qquad (6.2)$$

where $0 < \lambda < 1$. Let $\chi_\rho(x)$ be the cut-off function from (2.3) with

$$a_1 = \rho, \;\; a_2 = 2\rho, \;\; a_3 = 4\rho, \;\; a_4 = 6\rho.$$

Testing integral identity (2.11) by $\varphi(x) := U^{1-\lambda}(x)\chi_\rho^l(x)$ yields

$$\sum_{j=1}^{n} \int\limits_{\Omega(4\rho)\setminus\Omega(2\rho)} \left|\frac{\partial U}{\partial x_j}\right|^{p_j} U^{-\lambda}(x)dx \leq c_2 \sum_{j=1}^{n} \int\limits_{\Omega(6\rho)\setminus\Omega(\rho)} U^{p_j-\lambda}(x) \left|\frac{\partial \chi_\rho(x)}{\partial x_j}\right|^{p_j} dx. \quad (6.3)$$

Estimating the first term in right-hand side of (6.2) by (6.3) and using property (3.7), we obtain

$$\begin{aligned} 1 &\leq c_3 \sum_{i=1}^{n} \left(\sum_{j=1}^{n} M(\rho)^{p_j-\lambda} \rho^{\frac{(n-p)(p_j-1)}{p-1}} \right)^{\frac{p_i-1}{p_i}} \left(M(2\rho)^{\lambda(p_i-1)} \rho^{\frac{(n-p)(p_i-1)}{p-1}} \right)^{\frac{1}{p_i}} \leq \\ &\leq c_4 \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(M(\rho) \rho^{\frac{n-p}{p-1}} \right)^{p_j} \right]^{\frac{p_i-1}{p_i}}. \end{aligned} \quad (6.4)$$

From (6.4) the first estimate in (2.13) follows immediately. Now we prove the second inequality from (2.13). Let us define the following function $v \in \overset{\circ}{W}{}^1_{(\overline{p})}(\Omega(R_0))$, which was introduced by J. Serrin [6]:

$$v(x) = \begin{cases} 0, & u(x) \leq M(R) \\ u(x) - M(R), & M(R) < u(x) \leq m(\rho) \\ m(\rho) - M(R), & u(x) > m(\rho). \end{cases}$$

Now let us test integral identity (2.11) by $\varphi(x) = v(x)$. As a result we obtain

$$\sum_{i=1}^{n} \int\limits_{M(R)<u(x)<m(\rho)} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx = m(\rho) - M(R). \quad (6.5)$$

Due to Lemma 7.1 and (6.5), we have:

$$\left(\int_{\{u(x)\geq M(R)\}} |v(x)|^q dx \right)^{\frac{1}{q}} \leq c(m(\rho) - M(\rho))^{\frac{1}{p}}, \;\; q = \frac{np}{n-p}. \quad (6.6)$$

It is easy to see that

$$\int\limits_{\{u(x)\geq M(R)\}} |v(x)|^q dx \geq \int\limits_{u(x)\geq m(\rho)} |v(x)|^q dx \geq (m(\rho) - M(\rho))^q \rho^n. \quad (6.7)$$

Combining (6.6) and (6.7) we obtain the required estimate (2.13). Theorem 2.2 is proved in the case $1 < p < n$. Using similar arguments, it is not hard to prove the result in the case $p = n$. □

7 Appendix

Lemma 7.1 ([1]) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, $v \in \overset{\circ}{W}{}_1^1(\Omega)$, and*

$$\sum_{i=1}^{n} \int_{\Omega} |v(x)|^{\alpha_i} \left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx < \infty, \ \alpha_i \geq 0, \quad p_i \geq 1.$$

If $1 < p < n$, p is defined by (1.3), then $v \in L_q(\Omega)$, $q = \dfrac{np}{n-p}\left(1 + \dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{\alpha_i}{p_i}\right)$ and the following inequality holds:

$$\|v\|_{L_q(\Omega)} \leq K_3 \prod_{i=1}^{n} \left(\int_{\Omega} |v(x)|^{\alpha_i} \left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{np_i(1+\frac{1}{n}\sum\limits_{k=1}^{n}\frac{\alpha_k}{p_k})}}, \tag{7.1}$$

where the constant K_3 depends on n, α_i, p_i, $i = 1, \ldots, n$ only. If $p = n$, then $v \in L_q(\Omega)$ for an arbitrary $q > 1$ and inequality (7.1) holds with the constant K_3 depending on n, α_i, p_i, q, $i = 1, \ldots, n$ only.

It is easy to prove the following statement by obvious computations using Lemma 7.1.

Lemma 7.2 *Let $v \in \overset{\circ}{W}{}_1^1(\Omega)$ and*

$$\sum_{i=1}^{n} \int_{\Omega} |v|^{-\alpha_i} \left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx < \infty, \quad \alpha_i \geq 0, \quad p_i \geq 1, \quad \alpha_i < p_i, \quad i = \overline{1,n}.$$

If $1 < p < n$, then $v \in L_q(\Omega)$, $q = \dfrac{np}{n-p}\left(1 - \dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{\alpha_i}{p_i}\right)$, and the following inequality holds

$$\|v\|_{L_q(\Omega)} \leq \overline{K}_3 \prod_{i=1}^{n} \left(\int_{\Omega} |v(x)|^{-\alpha_i} \left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{np_i(1-\frac{1}{n}\sum\limits_{k=1}^{n}\frac{\alpha_k}{p_k})}} \tag{7.2}$$

with some positive constant $\overline{K}_3$ depending on $n, \alpha_i, p_i, i = \overline{1,n}$ only.

If $p = n$, then $v \in L_q(\Omega)$ for any $q > 1$ and inequality (7.2) holds with the constant $\overline{K}_3$ depending on $n, \alpha_i, p_i, q, \Omega, i = \overline{1,n}$.

Lemma 7.3 ([2]) *Let sequence y_l, $l = 0, 1, 2, \ldots$ of nonnegative numbers satisfy the following relationship:*

$$y_{l+1} \leq c b^l y_l^{1+\varepsilon}, \quad l = 0, 1, \ldots,$$

where positive constants $c > 0, \varepsilon > 0, b > 1$ do not depend on l. Then the following estimate is true

$$y_l \leq c^{\frac{(1+\varepsilon)^l - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^l - 1}{\varepsilon^2} - \frac{1}{\varepsilon}}.$$

Particularly, if

$$y_0 \le \theta := c^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$$

then $y_l \le \theta b^{-\frac{1}{\varepsilon}}$, *and consequently* $y_l \to 0$ *as* $l \to \infty$.

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