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Concentration Phenomena in a Biharmonic Equation Involving the Critical Sobolev Exponent

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Abstract

In this paper, we consider the problem

$$\Delta^2 u = |u|^{8/(n-4)}u + \varepsilon f(x,u)$$

in Ω, $u = \Delta u = 0$ on $\partial\Omega$, where Ω is a bounded and smooth domain in $\mathbb{R}^n$, $n \geq 6$, ε is a small positive parameter, and f is a smooth function. Our main purpose is to construct families of solutions which concentrate around some well defined points depending on f.

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1 Introduction

In this paper, we deal with the concentration phenomena in a biharmonic equation under the Navier boundary condition. In the last decades, there has been a wide range of activity in the study of concentration phenomena for second order elliptic equations involving critical Sobolev exponent, see for instance [1], [4], [7], [10], [12], [13], [14], [16], [18], [19], [20], [21], [22], [23], [24] and the references therein. In sharp contrast to this, very little is known for equations involving the biharmonic operator. In the following, we will consider the following problem

$$(P_\varepsilon) \qquad \begin{cases} \Delta^2 u = u^p + \varepsilon f(x,u) & u > 0 \text{ in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\varepsilon > 0$, Ω is a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 5$, f is a smooth function, and $p+1 = 2n/(n-4)$ is the critical Sobolev exponent of the embedding $H^2 \cap H_0^1(\Omega)$ into $L^{p+1}(\Omega)$.

The existence and multiplicity of solutions of (P_ε) for ε small has been proved for some special f in [15], [26]. When $\varepsilon = 0$, the situation is more complex, Van Der Vorst showed in [26] that if Ω is starshaped (P_0) has no solution. As a consequence of this result, we see that solutions of (P_ε) may disappear for $\varepsilon = 0$, either vanishing uniformly, or blowing up at some points of the domain. In the case where $f(x,u) = u$, for instance, El Mehdi and Selmi [15] showed that if u_ε are solutions of (P_ε) which concentrate around a point x_0 as ε goes to zero, then x_0 is a critical point of the Robin's function φ (see (1.1)). Conversely they proved that any nondegenerate critical point x_0 of φ generates a family of solutions of (P_ε) concentrating around x_0 as ε goes to zero. They also proved that for ε small enough (P_ε) has at least as many solutions as the Ljusternik-Schnirelman category of Ω.

Here we will establish what happens in the case

$$f(x,u) = f(x), \quad f \not\equiv 0,$$

and we will not impose on the solution to be positive. We are thus reduced to finding solutions of the following problem

$$(Q_\varepsilon) \qquad \begin{cases} \Delta^2 u = |u|^{8/(n-4)} u + \varepsilon f(x) & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases}$$

We will also study the case where

$$f(x,u) = \varepsilon u + \varepsilon f(x).$$

Our results extend to fourth order equations some results of Olivier Rey for second order elliptic equations [24]. The proof is inspired by the work [24] and [25] of Rey. Compared with the second order case, further technical difficulties have to be solved by means of delicate and careful estimates. To overcome these difficulties, we use the method developed in [6], [9] and [15]. It involves a careful expansion of the Euler-Lagrange functional associated to (P_ε), and its gradient in a small neighborhood of highly concentrated functions. Such expansions use the techniques developed by Bahri [2] and Rey [20].

To state our results, we need to fix some notations. Let us define on Ω the following Robin's function

$$\varphi(x) = H(x,x), \quad \text{with} \quad H(x,y) = |x-y|^{4-n} - G(x,y), \text{ for } (x,y) \in \Omega \times \Omega, \quad (1.1)$$

where G is the Green's function of Δ^2, that is,

$$\forall x \in \Omega, \quad \begin{cases} \Delta^2 G(x,.) = c_n \delta_x & \text{in } \Omega \\ \Delta G(x,.) = G(x,.) = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_x denotes the Dirac mass at x and $c_n = (n-4)(n-2)|S^{n-1}|$.

For $\lambda > 0$ and $x \in \mathbb{R}^n$ we consider the functions

$$\delta_{x,\lambda}(y) = \frac{\lambda^{(n-4)/2}}{(1+\lambda^2|y-x|^2)^{(n-4)/2}}.$$

It is well known (see [17]) that $\delta_{x,\lambda}$ are the only solutions of

$$\Delta^2 u = c_0^{8/(n-4)} u^{(n+4)/(n-4)}, \ u > 0 \text{ in } \mathbb{R}^n, \tag{1.2}$$

with $u \in L^{p+1}(\mathbb{R}^n)$, $\Delta u \in L^2(\mathbb{R}^n)$ and $c_0 = [(n-4)(n-2)n(n+2)]^{(n-4)/8}$, and are also the only minimizers of the Sobolev inequality on the whole space, that is the only functions which achieve

$$S = \inf\left\{ |\Delta u|^2_{L^2(\mathbb{R}^n)} |u|^{-2}_{L^{2n/(n-4)}(\mathbb{R}^n)}, \ s.t. \ \Delta u \in L^2, \ u \in L^{2n/(n-4)}, u \neq 0 \right\}. \tag{1.3}$$

In the case of a bounded domain Ω, the function $\delta_{x,\lambda}$ does not satisfy the boundary conditions of (Q_ε). Hence we need to consider their projections $P\delta_{x,\lambda}$ on $H^2(\Omega) \cap H^1_0(\Omega)$, defined by

$$\begin{cases} \Delta^2 P\delta_{x,\lambda} = \Delta^2 \delta_{x,\lambda} & \text{in } \Omega \\ \Delta P\delta_{x,\lambda} = P\delta_{x,\lambda} = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that for $x \in \Omega$ and $\lambda d(x, \partial\Omega)$ large, $P\delta_{x,\lambda}$ the projection of $\delta_{x,\lambda}$, is an almost solution of (Q_ε) and our aim is to construct a family of solutions of the form $\alpha P\delta_{x,\lambda} + v$ where v is a function which goes to zero as ε tends to zero.

Let $\tilde{f}$ be the function defined by

$$\begin{cases} \Delta^2 \tilde{f} = f & \text{in } \Omega \\ \Delta \tilde{f} = \tilde{f} = 0 & \text{on } \partial\Omega. \end{cases}$$

The first result is the following:

Theorem 1.1 *Let $n \geq 6$. Assume that $\tilde{f} \in C^4(\Omega)$ and let $x_0 \in \Omega$ be such that*

(i) $\tilde{f}(x_0) > 0$;

(ii) *x_0 is a nondegenerate critical point of the function: $x \mapsto \dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$.*

Then, there exists a family (u_ε) of solutions of (Q_ε) which concentrate at x_0 as $\varepsilon \to 0$, that is,

$$|\Delta u_\varepsilon|^2 \to S^{n/2}\delta_{x_0}, \quad |u_\varepsilon|^{p+1} \to S^{n/2}\delta_{x_0}$$

in the sense of measures, where S is the best Sobolev constant defined by (1.3). Moreover, if $f \geq 0$ in Ω, (i) is satisfied and $u_\varepsilon > 0$ on Ω.

Next, we want to state a multiplicity result for problem (Q_ε). For this purpose, given a subset $F \subset \Omega$ we say that the category of F in Ω is k, denoted by $Cat(F, \Omega)$, if F may be covered by k closed sets, each contractible to a point in Ω, but not by $(k-1)$ such sets. We call category of Ω the positive integer $Cat(\Omega, \Omega)$ (see Chapter 2 of [11]).

Theorem 1.2 *Let $n \geq 6$. Assume that $f \geq 0$ and $\tilde{f} \in C^3(\Omega)$. Then, for ε small, (Q_ε) has at least as many positive solutions in $H^2(\Omega) \cap H_0^1(\Omega)$ as the category of Ω; each one of them concentrates at a critical point of the function $x \mapsto \frac{\tilde{f}(x)}{\varphi(x)^{1/2}}$.*

To study problem (Q_ε), we make the following change of variable

$$u = \tilde{u} + \varepsilon \tilde{f},$$

and we are led to consider the equivalent problem

$$(\tilde{Q}_\varepsilon) \qquad \begin{cases} \Delta^2 \tilde{u} = |\tilde{u} + \varepsilon \tilde{f}|^{p-1}(\tilde{u} + \varepsilon \tilde{f}) & \text{in } \Omega \\ \Delta \tilde{u} = \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that the study of $(\tilde{Q}_\varepsilon)$ allows us to state the same results concerning the following problem

$$(Q'_\varepsilon) \qquad \begin{cases} \Delta^2 u = |u|^{p-1} u & \text{in } \Omega \\ \Delta u = u = \varepsilon g & \text{on } \partial\Omega, \end{cases}$$

where $g \not\equiv 0$. Indeed, if we change the variable in $({Q'}_\varepsilon)$, writing

$$u = \tilde{u} + \varepsilon \tilde{g},$$

where $\tilde{g}$ is the function defined by

$$\begin{cases} \Delta^2 \tilde{g} = 0 & \text{in } \Omega \\ \Delta \tilde{g} = \tilde{g} = g & \text{on } \partial\Omega, \end{cases}$$

the problem (Q'_ε) becomes equivalent to the following

$$\begin{cases} \Delta^2 \tilde{u} = |\tilde{u} + \varepsilon \tilde{g}|^{p-1}(\tilde{u} + \varepsilon \tilde{g}) & \text{in } \Omega \\ \Delta \tilde{u} = \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

which is exactly $(\tilde{Q}_\varepsilon)$ with $\tilde{f}$ replaced by $\tilde{g}$. Hence the following result follows immediately from those for (Q_ε).

Theorem 1.3 *The results of Theorems 1.1, and 1.2 are valid for (Q'_ε), provided that in all statements f is replaced by g and $\tilde{f}$ by $\tilde{g}$.*

Our last result concerns the following problem

$$(R_\varepsilon) \qquad \begin{cases} \Delta^2 u = |u|^{p-1} u + \varepsilon u + \varepsilon f(x) & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 1.4 *For $\tilde{f} \in C^4(\Omega)$, assume that one of the following conditions holds:*
(i) $8 < n < 12$ and $x_0 \in \Omega$ is a nondegenerate critical point of the function $x \mapsto \frac{\tilde{f}(x)}{\varphi(x)^{1/2}}$ such that $\tilde{f}(x_0) > 0$.

(ii) $9 < n < 12$ and $x_0 \in \Omega$ is a nondegenerate critical point of the function $\tilde{f}$ such that $\tilde{f}(x_0) = 0$.

(iii) $n = 12$ *and* $x_0 \in \Omega$ *is a nondegenerate critical point of the function* $x \mapsto \frac{\tilde{f}(x)+1/2}{\varphi(x)^{1/2}}$ *such that* $\tilde{f}(x_0) > -1/2$.

(iv) $n > 12$ *and* $x_0 \in \Omega$ *is a nondegenerate critical point of the function* φ.
Then there exists a family (u_ε) *of solutions of* (R_ε) *which concentrate at* x_0 *as* ε *goes to* 0. *Moreover, if* $f \geq 0$, *these solutions are positive.*

Our paper is organized as follows. The next section will be devoted to some useful facts and some estimates for later use. Theorems 1.1, 1.2, and 1.4 are proved in Sections 3, 4, and 5 respectively. Finally, we collect in the appendix some results needed in the proof of our Theorems.

2 Useful facts and some estimates

We introduce the functional

$$J(\tilde{u}) = \frac{1}{2}\int_\Omega |\Delta \tilde{u}|^2 - \frac{1}{p+1}\int_\Omega |\tilde{u} + \varepsilon \tilde{f}|^{p+1} \tag{2.4}$$

defined on $\mathcal{H}(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$ whose critical points are solutions of $(\tilde{Q}_\varepsilon)$. Recall that, if $(\tilde{u}_\varepsilon)$ is a family of minimizing solutions of $(\tilde{Q}_\varepsilon)$, then $\tilde{u}_\varepsilon$ may disappear for $\varepsilon = 0$, either vanishing uniformly, or blowing up at some points of the domain. In the last case, for η_0 as small as desired, the function $\tilde{u}_\varepsilon$ belongs to the potential set

$$V(1, \eta_0) := \{\alpha P\delta_{x,\lambda} + v : x \in \Omega,\ |\alpha - c_0| < \eta_0,\ \lambda d(x, \partial\Omega) > \eta_0^{-1},\ \|v\| < \eta_0\}, \tag{2.5}$$

for ε small enough and where

$$\|u\| = \left(\int_\Omega |\Delta u|^2\right)^{1/2}, \qquad u \in \mathcal{H}(\Omega) := H^2(\Omega) \cap H_0^1(\Omega).$$

Therefore, to prove our results we need to focus on the functions of the form $\alpha P\delta_{x,\lambda} + v$ and to show that we can choose α, λ and x so that we have a critical point of J. It is proved in [3], that if $\tilde{u} \in V(1, \eta_0)$, the problem

$$\text{Minimize } \|\tilde{u} - \alpha P\delta_{x,\lambda}\| \text{ with respect to } \alpha > 0,\ x \in \Omega,\ \lambda > 0$$

has a unique solution $(\overline{\alpha}, \overline{x}, \overline{\lambda})$. Hence, denoting

$$v := u - \overline{\alpha} P\delta_{\overline{x},\overline{\lambda}},$$

it is easy to see that v belongs to

$$E_{\lambda,x} := \{v \in \mathcal{H}(\Omega) |\, \langle v, P\delta_{x,\lambda}\rangle = \langle v, \partial P\delta_{x,\lambda}/\partial\lambda\rangle = \langle v, \partial P\delta_{x,\lambda}/\partial x_i\rangle = 0, \text{ for } i \leq n\},$$

where

$$\langle u, v\rangle = \int_\Omega \Delta u \Delta v, \qquad u, v \in \mathcal{H}(\Omega). \tag{2.6}$$

Furthermore, the set $V(1,\eta_0)$, defined by (2.5), can be parametrized by

$$M := \{(\alpha,\lambda,x,v) \in \mathbb{R}\times\mathbb{R}_+^*\times\Omega\times\mathcal{H}(\Omega)/v\in E_{\lambda,x},\ |\alpha-c_0|<\eta_0,\\ \lambda d(x,\partial\Omega)>\eta_0^{-1},\ \| v \| < \eta_0\}, \qquad (2.7)$$

with η_0 is a positive constant.

Now, as in [20], we can see that $\tilde{u}=\alpha P\delta_{x,\lambda}+v$ is a critical point of J if and only if $(\alpha,\lambda,x,v)\in M$ is a critical point of the function K defined by

$$\begin{array}{rcl} K: M & \longrightarrow & \mathbb{R} \\ (\alpha,\lambda,x,v) & \longmapsto & J(\alpha P\delta_{x,\lambda}+v); \end{array}$$

that means there exists $(A,B,C_1\dots,C_n)\in\mathbb{R}^{n+2}$ such that

$$\begin{cases} \frac{\partial K}{\partial\alpha}(\alpha,\lambda,x,v)=0 & (E_\alpha)\\ \frac{\partial K}{\partial\lambda}(\alpha,\lambda,x,v)=\int_\Omega\Delta\big(B\frac{\partial^2P\delta_{x,\lambda}}{\partial\lambda^2}+\sum_{j=1}^nC_j\frac{\partial^2P\delta_{x,\lambda}}{\partial\lambda\partial x_j}\big)\Delta v & (E_\lambda)\\ \frac{\partial K}{\partial x_i}(\alpha,\lambda,x,v)=\int_\Omega\Delta\big(B\frac{\partial^2P\delta_{x,\lambda}}{\partial\lambda\partial x_i}+\sum_{j=1}^nC_j\frac{\partial^2P\delta_{x,\lambda}}{\partial x_j\partial x_i}\big)\Delta v,\ 1\le i\le n & (E_{x_i})\\ \frac{\partial K}{\partial v}(\alpha,\lambda,x,v)=AP\delta_{x,\lambda}+B\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+\sum_{j=1}^nC_j\frac{\partial P\delta_{x,\lambda}}{\partial x_j}. & (E_v)\end{cases} \qquad (2.8)$$

In order to prove Theorem 1.1, we show first that for a given (α,λ,x), $\lambda d(x,\partial\Omega)$ large enough, there exist $v_{\varepsilon,\alpha,\lambda,x}\in E_{\lambda,x}$ and $(A,B,C_1,\dots C_n)\in\mathbb{R}^{n+2}$ such that the equation (E_v) of (2.8) is satisfied. Moreover, $v_{\varepsilon,\alpha,\lambda,x}$ goes to zero as ε tends to zero. Then, the problem of finding (α,λ,x) such that the equations $(E_\alpha),(E_\lambda),(E_{x_i}),1\le i\le n$, are satisfied will turn out to be equivalent to finding a fixed point of a certain continuous map which we will estimate, and Brouwer's theorem will allow us to conclude.

Concerning Theorem 1.2, we proceed by successive optimizations of K, with respect to the different parameters v, α and λ on M. Then we get a new function

$$\mathcal{K}: x \longmapsto K(\alpha_x,\lambda_x,x,\overline{v}_x)=J(\alpha_xP\delta_{x,\lambda_x}+\overline{v}_x)$$

and we shall show that its gradient is pointing outward on the boundary of $\mathcal{K}_c$, where $\mathcal{K}_c$ is a subset of Ω with the same category as Ω. So by the Ljusternik-Schnirelman's theory, $\mathcal{K}$ has at least $p=\mathrm{Cat}(\Omega)$ many critical points $x_1\dots x_p$ of Ω. Then the $\alpha_{x_i}P\delta_{x_i,\lambda_{x_i}}+\overline{v}_{x_i}$, $1\le i\le p$ are critical points of J.

To prove theorem 1.4, we combine the estimates involved in the proof of theorem 1.1 with other in [15] and we use the same idea as in the proof of theorem 1.1.

Usually, in this type of problems, we first deal with the v-part of $\tilde{u}_\varepsilon$, in order to show that it is negligible with respect to the concentration phenomenon. We have

Proposition 2.1 *There exist $\varepsilon_1>0$, $T_1>0$ and a C^1 map, which to any*

$$(\varepsilon,\alpha,\lambda,x)\in(0,\varepsilon_1)\times\mathbb{R}\times\mathbb{R}_+^*\times\Omega,\ \textit{with } \lambda d>T_1,$$

associates $v_\varepsilon = v_{\varepsilon,\alpha,\lambda,x}$ belonging to $E_{\lambda,x}$ such that (E_v) is satisfied for some $(A,B,C_1,\dots,C_n)\in\mathbb{R}^{n+2}$. Such a v_ε is unique and it minimizes $K(\alpha,\lambda,x,v)$ with respect to $v\in E_{\lambda,x}$ and we have the estimate

$$\|v_\varepsilon\|\le C\begin{cases}\frac{1}{(\lambda d)^{n-4}}+\frac{\varepsilon}{\lambda^{(n-4)/2}}+\varepsilon^p & \text{if } n<12\\ \frac{(\log(\lambda d))^{2/3}}{(\lambda d)^8}+\frac{\varepsilon(\log\lambda)^{2/3}}{\lambda^4}+\varepsilon^p & \text{if } n=12\\ \frac{1}{(\lambda d)^{(n+4)/2}}+\frac{\varepsilon}{\lambda^4}+\varepsilon^p & \text{if } n>12\end{cases}$$

where C is a positive constant independent of ε and $d=d(x,\partial\Omega)$.

Proof. In order to simplify the notation, in the remainder we write δ instead of $\delta_{x,\lambda}$. Expanding $K(\alpha,\lambda,x,v)$ in a neighbourhood of $v=0$, we obtain

$$K(\alpha,\lambda,x,v)=K(\alpha,\lambda,x,0)-F_{\alpha,\lambda,x}(v)+Q_{\alpha,\lambda,x}(v)+R_{\alpha,\lambda,x}(v) \tag{2.9}$$

where

$$F_{\alpha,\lambda,x}(v)=\int_\Omega|\alpha P\delta+\varepsilon\tilde f|^{p-1}(\alpha P\delta+\varepsilon\tilde f)v, \tag{2.10}$$

$$Q_{\alpha,\lambda,x}(v)=\frac12\int_\Omega|\Delta v|^2-\frac p2\int_\Omega|\alpha P\delta+\varepsilon\tilde f|^{p-1}v^2 \tag{2.11}$$

and $R_{\alpha,\lambda,x}(v)$ satisfies

$$\begin{aligned}R_{\alpha,\lambda,x}(v)&=O\big(\parallel v\parallel^{\min(3,p+1)}\big)\\ R'_{\alpha,\lambda,x}(v)&=O\big(\parallel v\parallel^{\min(2,p)}\big)\\ R''_{\alpha,\lambda,x}(v)&=O\big(\parallel v\parallel^{\min(1,p-1)}\big).\end{aligned} \tag{2.12}$$

$F_{\alpha,\lambda,x}$ is a continuous linear form on $E_{\lambda,x}$ equipped with the $\mathcal{H}(\Omega)$-scalar product defined by (2.6). Therefore there exists a unique $f_{\alpha,\lambda,x}\in E_{\lambda,x}$ such that

$$F_{\alpha,\lambda,x}(v)=\langle f_{\alpha,\lambda,x},v\rangle. \tag{2.13}$$

$Q_{\alpha,\lambda,x}(v)$ is a continuous quadratic form over $E_{\lambda,x}$. Thus, according to [5] there exists $\beta_0>0$, independent of x, such that

$$Q_{\alpha,\lambda,x}(v)\ge\beta_0\|v\|^2. \tag{2.14}$$

Using (2.14) and the implicit function theorem, we derive the existence of a C^1 map which to $(\varepsilon,\alpha,\lambda,x)$ such that $|\alpha-c_0|<\eta_0$, $\lambda d(x,\partial\Omega)>1/\eta_0$, $\varepsilon<\varepsilon_0$ (η_0 and ε_0 small enough) associates $v_\varepsilon\in E_{x,\lambda}$ such that v_ε minimizes $K(\alpha,\lambda,x,v)$ with respect to $v\in E_{\lambda,x}$ and (E_v) is satisfied for certain $(A,B,C_1,\dots,C_n)\in\mathbb{R}^{n+2}$. Furthermore

$$\|v_\varepsilon\|=O(\|f_{\alpha,\lambda,x}\|). \tag{2.15}$$

We need to estimate $\langle f_{\alpha,\lambda,x},v\rangle$ for $v\in E_{\lambda,x}$.

$$\begin{aligned}\langle f_{\alpha,\lambda,x},v\rangle&=\int_\Omega|\alpha P\delta+\varepsilon\tilde f|^{p-1}(\alpha P\delta+\varepsilon\tilde f)v\\ &=\int_{B_d}|\alpha P\delta+\varepsilon\tilde f|^{p-1}(\alpha P\delta+\varepsilon\tilde f)v+\int_{\Omega\setminus B_d}|\alpha P\delta+\varepsilon\tilde f|^{p-1}(\alpha P\delta+\varepsilon\tilde f)v,\end{aligned} \tag{2.16}$$

where B_d denotes the ball of center x and radius d. We have

$$\int_{\Omega\setminus B_d} |\alpha P\delta+\varepsilon\tilde f|^{p-1}\big(\alpha P\delta+\varepsilon\tilde f\big)v = O\left(\int_{\Omega\setminus B_d}\big((P\delta)^p+\varepsilon^p|\tilde f|^p\big)|v|\right). \tag{2.17}$$

Using Holder's inequality and Sobolev embedding theorem with the fact that $|f|_\infty$ is bounded, we derive

$$\int_\Omega \varepsilon^p|\tilde f|^p|v| = O\left(\varepsilon^p\left(\int_\Omega|\tilde f|^{p+1}\right)^{\frac{p}{p+1}}\left(\int_\Omega|v|^{p+1}\right)^{\frac{1}{p+1}}\right)=O\big(\varepsilon^p\|v\|\big) \tag{2.18}$$

and

$$\int_{\Omega\setminus B_d}(P\delta)^p\,|v| = O\left(\Big(\int_{\Omega\setminus B_d}\delta^{p+1}\Big)^{\frac{p}{p+1}}\Big(\int_\Omega|v|^{p+1}\Big)^{\frac{1}{p+1}}\right)=O\left(\frac{\|v\|}{(\lambda d)^{\frac{n+4}{2}}}\right). \tag{2.19}$$

Therefore, (2.17), (2.18) and (2.19) imply

$$\int_{\Omega\setminus B_d}|\alpha P\delta+\varepsilon\tilde f|^{p-1}\big(\alpha P\delta+\varepsilon\tilde f\big)|v| = O\Big(\Big(\varepsilon^p+\frac{1}{(\lambda d)^{\frac{n+4}{2}}}\Big)\|v\|\Big). \tag{2.20}$$

We also have

$$\int_{B_d}|\alpha P\delta+\varepsilon\tilde f|^{p-1}\big(\alpha P\delta+\varepsilon\tilde f\big)v=\int_{B_d}(\alpha P\delta)^p v+O\left(\int_{B_d}P\delta^{p-1}|\varepsilon\tilde f||v|+(\varepsilon|\tilde f|)^p|v|\right). \tag{2.21}$$

Using Proposition 6.1, Holder's inequality and Sobolev embedding theorem we obtain

$$\int_{B_d}(P\delta)^p v = O\left(\frac{\|v\|}{(\lambda d)^{\min(n-4,\frac{n+4}{2})}}+(\text{if}\, n=12)\,\frac{(\log\lambda d)^{2/3}\|v\|}{(\lambda d)^8}\right), \tag{2.22}$$

$$\int_{B_d}P\delta^{p-1}|\varepsilon\tilde f||v| = O\left(\frac{\varepsilon\|v\|}{\lambda^{\min(4,\frac{n-4}{2})}}\,(\text{if}\, n\neq 12)\,,\,\frac{\varepsilon(\log\lambda)^{\frac23}\|v\|}{\lambda^4}\,(\text{if}\, n=12)\right). \tag{2.23}$$

Using (2.16), (2.18), (2.20), ..., (2.23), we get

$$\|f_{\alpha,\lambda,x}\| = O\begin{cases}\frac{1}{(\lambda d)^{n-4}}+\frac{\varepsilon}{\lambda^{(n-4)/2}}+\varepsilon^p & \text{if } n<12\\ \frac{(\log(\lambda d))^{2/3}}{(\lambda d)^8}+\frac{\varepsilon(\log\lambda)^{2/3}}{\lambda^4}+\varepsilon^2 & \text{if } n=12\\ \frac{1}{(\lambda d)^{(n+4)/2}}+\frac{\varepsilon}{\lambda^4}+\varepsilon^p & \text{if } n>12.\end{cases}$$

From (2.15), the same estimate holds for $\|v\|$. □

Next, we give useful estimates of the derivative of the function K with respect to the variables α, λ and x.

Proposition 2.2 *Let $(\alpha, \lambda, x, \overline{v}) \in M$, where $\overline{v} = v_\varepsilon$ is the function obtained in Proposition 2.1. Making the change of variable: $\alpha = c_0 - \rho$, we have*

$$\frac{\partial K}{\partial \alpha}(\alpha, \lambda, x, \overline{v}) = C_1 \rho + V_\alpha(\alpha,\ \lambda,\ x) \tag{2.24}$$

$$\frac{\partial K}{\partial \lambda}(\alpha, \lambda, x, \overline{v}) = C_2 \Big(\frac{\alpha \varepsilon \tilde{f}(x)}{\lambda^{(n-2)/2}} - \frac{\alpha^2 \varphi(x)}{\lambda^{n-3}} \Big) + V_\lambda(\alpha, \lambda, x) \tag{2.25}$$

$$\frac{\partial K}{\partial x}(\alpha, \lambda, x, \overline{v}) = C_3 \Big(\frac{\alpha^2 \varphi'(x)}{2\lambda^{n-4}} - \frac{\alpha \varepsilon \tilde{f}'(x)}{\lambda^{(n-4)/2}} \Big) + V_x(\alpha,\ \lambda,\ x) \tag{2.26}$$

where C_1, C_2 and C_3 are positive constants which depend on n only, V_α, V_λ and V_x are smooth functions which verify

$$V_\alpha(\alpha,\ \lambda,\ x) = O\left(\rho^2 + \frac{1}{(\lambda d)^{n-4}} + \frac{\varepsilon}{\lambda^{\frac{n-4}{2}}} + \varepsilon^{p+1} + \frac{\varepsilon^2}{\lambda^4} \right) \tag{2.27}$$

$$\begin{aligned} V_\lambda(\alpha,\ \lambda,\ x) = O\Bigg(&\frac{1}{\lambda^{n-1} d^{n-2}} + \frac{|\rho|}{\lambda^{n-3} d^{n-4}} + \frac{\varepsilon|\rho|}{\lambda^{\frac{n-2}{2}}} + \frac{\varepsilon}{\lambda^{\frac{n+2}{2}} d^2} \\ &+ \frac{\varepsilon^{p+1}}{\lambda} + \frac{\varepsilon^2 (\log \lambda)^\gamma}{\lambda^{\inf(5, n-3)}} \Bigg) \end{aligned} \tag{2.28}$$

$$\begin{aligned} V_x(\alpha,\ \lambda,\ x) = O\Bigg(&\frac{1}{(\lambda d)^{\inf(2n-9, n-2)} d} + \frac{|\rho|}{\lambda^{n-4} d^{n-3}} + \frac{\varepsilon|\rho|}{\lambda^{\frac{n-4}{2}}} + \frac{\varepsilon}{\lambda^{\frac{n-2}{2}}} + \frac{\varepsilon \log \lambda d}{\lambda^{\frac{n+2}{2}} d^4} \\ &+ \lambda \varepsilon^{p+1} + \frac{\varepsilon^2 (\log \lambda)^\gamma}{\lambda^{\inf(3, n-5)}} \Bigg), \end{aligned} \tag{2.29}$$

where $\gamma = 1$ if $n = 8$ and $\gamma = 0$ if $n \neq 8$.

Proof. To prove claim (2.24), we write

$$\begin{aligned} &\frac{\partial K}{\partial \alpha}(\alpha,\ \lambda,\ x,\ \overline{v}) \\ &= \int_\Omega \Delta(\alpha P\delta + \overline{v}) \Delta P\delta - \int_\Omega |\alpha P\delta + \overline{v} + \varepsilon \tilde{f}|^{p-1} (\alpha P\delta + \overline{v} + \varepsilon \tilde{f}) P\delta \\ &= \alpha \int_\Omega |\Delta P\delta|^2 - \int_\Omega (\alpha P\delta)^p P\delta - p \int_\Omega (\alpha P\delta)^{p-1} (\overline{v} + \varepsilon \tilde{f}) P\delta \\ &+ O\left(\int_{|\overline{v} + \varepsilon \tilde{f}| < P\delta} (\alpha P\delta)^{p-2} (\overline{v} + \varepsilon \tilde{f})^2 P\delta \right) + O\left(\int_{P\delta \leq |\overline{v} + \varepsilon \tilde{f}|} |\overline{v} + \varepsilon \tilde{f}|^p P\delta \right). \end{aligned} \tag{2.30}$$

Using Proposition 6.1, Holder's inequality and Sobolev embedding theorem, we obtain

$$\int_\Omega |\Delta P\delta|^2 = c_0^{p-1} S_n + O\left(\frac{1}{(\lambda d)^{n-4}} \right) \quad ; \quad \int_\Omega (P\delta)^{p+1} = S_n + O\left(\frac{1}{(\lambda d)^{n-4}} \right), \tag{2.31}$$

where $S_n = \int_{\mathbb{R}^n} \delta_{0,1}^{2n/(n-4)}$. Furthermore, as in (2.19) and (2.22), we get

$$\int_\Omega (P\delta)^p(\overline{v} + \varepsilon\tilde{f}) = O\left(\frac{\varepsilon}{\lambda^{\frac{n-4}{2}}} + \|\overline{v}\| \frac{(\log \lambda d)^\nu}{(\lambda d)^{\inf(n-4,(n+4)/2)}}\right), \tag{2.32}$$

where $\nu = 2/3$ if $n = 12$ and $\nu = 0$ if $n \neq 12$.

For the fourth integral in (2.30), using Holder's inequality and the fact that $|\tilde{f}|_\infty$ is bounded, we derive

$$\int_{|\overline{v}+\varepsilon\tilde{f}|\leq P\delta} (\alpha P\delta)^{p-1}(\overline{v} + \varepsilon\tilde{f})^2 \leq c\|\overline{v}\|^2 + c\varepsilon^2 \int_\Omega \delta^{p-1} = O\biggl(\|\overline{v}\|^2 + \frac{\varepsilon^2(\log \lambda)^\gamma}{\lambda^{\inf(4,n-4)}}\biggr), \tag{2.33}$$

where $\gamma = 1$ if $n = 8$ and $\gamma = 0$ if $n \neq 8$.

It remains to compute the last integral in (2.30). It is easy to obtain

$$\int_{P\delta\leq|\overline{v}+\varepsilon\tilde{f}|} |\overline{v} + \varepsilon\tilde{f}|^p P\delta \leq \int_\Omega |\overline{v} + \varepsilon\tilde{f}|^{p+1} = O\bigl(\|\overline{v}\|^{p+1} + \varepsilon^{p+1}\bigr). \tag{2.34}$$

From (2.30),..., (2.34) and Proposition 2.1, claim (2.24) follows.

Now we turn to the proof of our claim (2.25). As in claim (2.24), we have

$$\begin{aligned}
&\frac{\partial K}{\partial \lambda}(\alpha,\, \lambda,\, x,\, \overline{v})\\
&= \int_\Omega \Delta(\alpha P\delta + \overline{v})\Delta\left(\alpha\frac{\partial P\delta}{\partial\lambda}\right) - \int_\Omega |\alpha P\delta + \overline{v} + \varepsilon\tilde{f}|^{p-1}(\alpha P\delta + \overline{v} + \varepsilon\tilde{f})\alpha\frac{\partial P\delta}{\partial\lambda}\\
&= \alpha^2 \int_\Omega \Delta P\delta\Delta\frac{\partial P\delta}{\partial\lambda} - \int_\Omega (\alpha P\delta)^p\alpha\frac{\partial P\delta}{\partial\lambda} - p\int_\Omega (\alpha P\delta)^{p-1}(\overline{v} + \varepsilon\tilde{f})\alpha\frac{\partial P\delta}{\partial\lambda}\\
&+ O\left(\int_{|\overline{v}+\varepsilon\tilde{f}|\leq P\delta} (\alpha P\delta)^{p-2}(\overline{v} + \varepsilon\tilde{f})^2\alpha\frac{\partial P\delta}{\partial\lambda}\right) + O\left(\int_{P\delta<|\overline{v}+\varepsilon\tilde{f}|} |\overline{v} + \varepsilon\tilde{f}|^p\alpha\frac{\partial P\delta}{\partial\lambda}\right).
\end{aligned} \tag{2.35}$$

Since $|\tilde{f}|_\infty$ is bounded and $\lambda\partial P\delta/\partial\lambda = O(P\delta)$, using Holder's inequality and Sobolev embedding theorem, we derive

$$\int_{P\delta<|\overline{v}+\varepsilon\tilde{f}|} |\overline{v} + \varepsilon\tilde{f}|^p\alpha\frac{\partial P\delta}{\partial\lambda} = O\left(\frac{\|\overline{v}\|^{p+1}}{\lambda} + \frac{\varepsilon^{p+1}}{\lambda}\right), \tag{2.36}$$

$$\int_{|\overline{v}+\varepsilon\tilde{f}|\leq P\delta} (\alpha P\delta)^{p-2}(\overline{v} + \varepsilon\tilde{f})^2\frac{\partial P\delta}{\partial\lambda} = O\left(\frac{\|\overline{v}\|^2}{\lambda} + \frac{\varepsilon^2(\log \lambda)^\gamma}{\lambda^{\inf(5,n-3)}}\right), \tag{2.37}$$

where $\gamma = 1$ if $n = 8$ and $\gamma = 0$ if $n \neq 8$.

Now, using the fact that $\overline{v} \in E_{x,\lambda}$ and Proposition 6.1, one can check

$$\int_\Omega (P\delta)^{p-1}\frac{\partial P\delta}{\partial\lambda}\overline{v} = O\biggl(\frac{\|\overline{v}\|}{\lambda}\frac{(\log \lambda d)^\nu}{(\lambda d)^{\inf(n-4,(n+4)/2)}}\biggr), \tag{2.38}$$

where $\nu = 2/3$ if $n = 12$ and $\nu = 0$ if $n \neq 12$.

Claim (2.25) follows from Propositions 6.2, 6.3 and (2.35), ..., (2.38).

The proof of claim (2.26) is similar to the one of claim (2.25), so we will omit it. □

Proposition 2.3 *Let $n \geq 6$. Assume that $(\alpha,\, \lambda,\, x,\, \overline{v}) \in M$ where $\overline{v} = v_\varepsilon$ is the function obtained in proposition (2.1). Then we have*

$$K(\alpha, \lambda, x, \overline{v})$$
$$= \left(\frac{\alpha^2 c_0^{p-1}}{2} - \frac{n-4}{2n}\alpha^{p+1}\right) S_n + \left(\alpha^{p+1} - \frac{\alpha^2 c_0^{p+1}}{2}\right) J_n \frac{H(x,x)}{\lambda^{n-4}} - \alpha^p \varepsilon J_n \frac{\tilde{f}(x)}{\lambda^{(n-4)/2}}$$
$$+ O\left(\frac{1}{(\lambda d)^{n-2}} + \frac{\varepsilon}{\lambda^{(n-2)/2}} + \frac{\varepsilon \log(\lambda d)}{\lambda^{(n+4)/2} d^4} + \varepsilon^{p+1} + (if n > 12)\frac{\varepsilon^2 (\log \lambda)^{4/3}}{\lambda^8}\right)$$

where $S_n = \int_{\mathbb{R}^n} \delta_{o,1}^{2n/(n-4)}$, $J_n = \int_{\mathbb{R}^n} \delta_{o,1}^{(n+4)/(n-4)}$ and c_0 is defined by (1.2).

Proof. Using (2.9),..., (2.13) we obtain

$$\begin{aligned} K(\alpha,\, \lambda,\, x,\, \overline{v}) &= K(\alpha,\, \lambda,\, x,\, 0) + O\left(\|\overline{v}\|^2\right) + O\left(\|f_{\alpha,\lambda,x}\|\|\overline{v}\|\right) \\ &= \frac{\alpha^2}{2}\int_\Omega |\Delta P\delta|^2 - \frac{1}{p+1}\int_\Omega |\alpha P\delta + \varepsilon \tilde{f}|^{p+1} + O\left(\|v\|^2 + \|f_{\alpha,\lambda,x}\|\|v\|\right) \\ &= \frac{\alpha^2}{2}\int_\Omega |\Delta P\delta|^2 - \frac{1}{p+1}\int_\Omega (\alpha P\delta)^{p+1} - \int_\Omega (\alpha P\delta)^p \varepsilon \tilde{f} + O\left(\|\overline{v}\|^2\right) \\ &+ O\left(\int_{|\varepsilon\tilde{f}| \leq |\alpha P\delta|} (\alpha P\delta)^{P-1}(\varepsilon\tilde{f})^2 + \int_{|\varepsilon\tilde{f}| > |\alpha P\delta|} (\varepsilon\tilde{f})^{p+1} + \|f_{\alpha,\lambda,x}\|\|\overline{v}\|\right). \end{aligned} \tag{2.39}$$

Since $|\tilde{f}|_\infty$ is bounded, we derive

$$\int_{|\varepsilon\tilde{f}| > |\alpha P\delta|} (\varepsilon\tilde{f})^{p+1} = O\left(\varepsilon^{p+1}\right). \tag{2.40}$$

Now, observe that, if $n \geq 8$, then $p - 1 \leq 2$. Therefore

$$\int_{|\varepsilon\tilde{f}| < |\alpha P\delta|} P\delta^{P-1}(\varepsilon\tilde{f})^2 = O\left(\frac{1}{(\lambda d)^n} + \varepsilon^{\frac{n}{n-4}}\frac{\log(\lambda d)}{\lambda^{n/2}}(\text{if } n \geq 8) + \frac{\varepsilon^2}{\lambda^{n-4}}(\text{if } n < 8)\right). \tag{2.41}$$

Our proposition follows from (2.39), (2.40), (2.41) and Propositions 2.1, 6.2 and 6.3. □

3 Proof of Theorem 1.1

For the sake of simplicity, we assume that $x_0 = 0 \in \Omega,\ \tilde{f}(0) > 0$ and 0 is a nondegenerate critical point of the function $x \mapsto \frac{\tilde{f}(x)}{\varphi(x)^{1/2}}$. Then we can write

$$2\varphi(x)\tilde{f}'(x) - \tilde{f}(x)\varphi'(x) = Mx + o(|x|), \tag{3.42}$$

where M is an invertible matrix.

In the remainder, we will assume that x is restricted to a neighbourhood $\mathcal{O}$ of 0, such that

$$\forall x \in \overline{\mathcal{O}},\ \tilde{f}(x) > 0 \text{ and } d(\mathcal{O}, \partial\Omega) = d_0 > 0.$$

Taking the following change of variable

$$\frac{1}{\lambda^{(n-4)/2}} = \frac{\varepsilon \tilde{f}(x)}{\alpha\varphi(x)}(1+\xi), \qquad |\xi| < \frac{1}{2}, \tag{3.43}$$

then, (2.25) and (2.26) become

$$\frac{\partial K}{\partial \lambda}(\alpha, \lambda, x, \overline{v}) = -C_2 \frac{\alpha\varepsilon \tilde{f}(x)}{\lambda^{(n-2)/2}}\xi + V_\lambda(\alpha, \lambda, x) \tag{3.44}$$

$$\frac{\partial K}{\partial x}(\alpha, \lambda, x, \overline{v}) = -\frac{C_3}{2}\frac{\varepsilon^2 \tilde{f}(x)}{\varphi^2(x)}Mx + o\left(\varepsilon^2|x|\right) + O\left(\varepsilon^2|\xi|\right) + V_x(\alpha,\ \lambda,\ x). \tag{3.45}$$

Therefore, (2.24), (3.44) and (3.45) imply

$$\frac{\partial K}{\partial \alpha}(\alpha, \lambda, x, \overline{v}) = O\left(|\rho| + \varepsilon^2\right), \tag{3.46}$$

$$\frac{\partial K}{\partial \lambda}(\alpha, \lambda, x, \overline{v}) = O\left(\varepsilon^{\frac{2n-6}{n-4}}(|\rho| + |\xi|) + \varepsilon^{\frac{2n-2}{n-4}}\right), \tag{3.47}$$

$$\frac{\partial K}{\partial x}(\alpha, \lambda, x, \overline{v}) = O\left(\varepsilon^2(|\rho| + |\xi| + |x|) + \varepsilon^{\frac{2n-6}{n-4}}\right), \tag{3.48}$$

and the estimate of $\|v\|$ given in Proposition 2.1 becomes

$$\|v\| = O\left(\varepsilon^2 \text{ (if } n < 12),\ \varepsilon^2|\log\varepsilon|^{2/3} \text{ (if } n = 12),\ \varepsilon^p \text{ (if } n > 12)\right). \tag{3.49}$$

Now we need to estimate the numbers $A, B, C_1, \ldots, C_n$ which were determined by equation (E_v) defined in (2.8) . For this purpose we take the scalar product in $\mathcal{H}(\Omega)$ (see (2.6)) of equation (E_v) with respectively $P\delta$, $\frac{\partial P\delta}{\partial \lambda}$ and $\frac{\partial P\delta}{\partial x_i}$, $i = 1, \ldots, n$. Thus we obtain a quasi-diagonal system of linear equations in $A, B, C_1, \ldots, C_n$, whose coefficients are given by

$$\begin{cases} \|P\delta\|^2 = c_1 + O\left(\frac{1}{\lambda^{n-4}}\right), \ \left\langle \frac{\partial P\delta}{\partial \lambda}, \frac{\partial P\delta}{\partial x_i}\right\rangle = O\left(\frac{1}{\lambda^{n-3}}\right), \\ \left\langle P\delta, \frac{\partial P\delta}{\partial x_i}\right\rangle = O\left(\frac{1}{\lambda^{n-4}}\right), \ \left\langle \frac{\partial P\delta}{\partial x_i}, \frac{\partial P\delta}{\partial x_j}\right\rangle = c_2\lambda^2\delta_{ij} + O\left(\frac{1}{\lambda^{n-5}}\right), \\ \left\langle P\delta, \frac{\partial P\delta}{\partial \lambda}\right\rangle = O\left(\frac{1}{\lambda^{n-3}}\right), \ \left\| \frac{\partial P\delta}{\partial \lambda}\right\|^2 = \frac{c_3}{\lambda^2} + O\left(\frac{1}{\lambda^{n-2}}\right), \end{cases} \tag{3.50}$$

where c_i's are positive constants and δ_{ij} is the Kronecker symbol.

The right hand side is given by

$$\left\langle \frac{\partial K}{\partial v}, P\delta\right\rangle = \frac{\partial K}{\partial \alpha}; \quad \left\langle \frac{\partial K}{\partial v}, \frac{\partial P\delta}{\partial \lambda}\right\rangle = \frac{1}{\alpha}\frac{\partial K}{\partial \lambda}; \quad \left\langle \frac{\partial K}{\partial v}, \frac{\partial P\delta}{\partial x_i}\right\rangle = \frac{1}{\alpha}\frac{\partial K}{\partial x_i} \quad i = 1, \ldots, n. \tag{3.51}$$

Then using (3.46),..., (3.51) the solution of the system in $A, B, C_1, \ldots, C_n$ shows that

$$\begin{aligned} &A = O\left(|\rho| + \varepsilon^2\right) \\ &B = O\left(\varepsilon^{\frac{2n-10}{n-4}}(|\rho| + |\xi|) + \varepsilon^{\frac{2n-6}{n-4}})\right) \\ &C_i = O\left(\varepsilon^{\frac{2n-4}{n-4}}(|\rho| + |\xi| + |x|) + \varepsilon^{\frac{2n-2}{n-4}}\right), \quad \text{for } i = 1, \ldots, n. \end{aligned} \tag{3.52}$$

Now, a simple computation shows that

$$(i)\ \left\|\frac{\partial^2 P\delta}{\partial\lambda^2}\right\| = O\left(\frac{1}{\lambda^2}\right), \quad (ii)\ \left\|\frac{\partial^2 P\delta}{\partial\lambda\partial x_i}\right\| = O(1), \quad (iii)\ \left\|\frac{\partial^2 P\delta}{\partial x_i\partial x_j}\right\| = O(\lambda^2).$$

It follows that

$$\frac{\partial K}{\partial\lambda}(\alpha,\lambda,x,\overline{v}) = \int_\Omega \Delta\Big(B\frac{\partial^2 P\delta_{\lambda,x}}{\partial\lambda^2} + \sum_{j=1}^n C_j\frac{\partial^2 P\delta_{\lambda,x}}{\partial\lambda\partial x_j}\Big)\Delta\overline{v} = O\Big(\big(\frac{|B|}{\lambda^2} + \sum_{j=1}^n |C_j|\big)\|\overline{v}\|\Big) \tag{3.53}$$

and

$$\frac{\partial K}{\partial x_i}(\alpha,\lambda,x,\overline{v}) = \int_\Omega \Delta\Big(B\frac{\partial^2 P\delta_{\lambda,x}}{\partial\lambda\partial x_i} + \sum_{j=1}^n C_j\frac{\partial^2 P\delta_{\lambda,x}}{\partial x_j\partial x_i}\Big)\Delta\overline{v} = O\Big(\big(|B| + \lambda^2\sum_{j=1}^n |C_j|\big)\|\overline{v}\|\Big) \tag{3.54}$$

for $i = 1, \ldots, n$.

Combining (2.24), (3.44), (3.45), (3.49), (3.53) and (3.54) together with the estimates of $A, B, C_1, \ldots, C_n$, we conclude that the system of equations $(E_\alpha), (E_\lambda), (E_{x_i})$ given by (2.8) is equivalent to

$$\begin{cases} \rho = V_1(\varepsilon,\ \rho,\ \xi,\ x) \\ \xi = V_2(\varepsilon,\ \rho,\ \xi,\ x) \\ x = V_3(\varepsilon,\ \rho,\ \xi,\ x), \end{cases} \tag{3.55}$$

where V_1, V_2 and V_3 are some continuous functions satisfying the estimates

$$\begin{aligned} &V_1 = O\big(\rho^2 + \varepsilon^2\big) \\ &V_2 = O\big(|\rho| + \xi^2 + |x|^2 + \varepsilon^{4/(n-4)}\big) \\ &V_3 = O\big(|\rho| + |\xi| + |x|^2 + \varepsilon^{2/(n-4)}\big). \end{aligned}$$

Thus by the Brouwer's fixed point theorem, the system (3.55) has a solution $(\rho_\varepsilon,\ \xi_\varepsilon,\ x_\varepsilon)$ for ε small enough such that

$$\rho_\varepsilon = O(\varepsilon^2), \quad \xi_\varepsilon = O(\varepsilon^{4/(n-4)}), \quad x_\varepsilon = O(\varepsilon^{2/(n-4)}).$$

By construction

$$u_\varepsilon = \alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + \overline{v}_\varepsilon + \varepsilon\tilde{f}$$

with $\alpha_\varepsilon = c_0 - \rho_\varepsilon$, $\frac{1}{\lambda_\varepsilon^{(n-4)/2}} = \frac{\varepsilon \tilde{f}(x_\varepsilon)}{\alpha_\varepsilon \varphi(x_\varepsilon)}(1+\xi_\varepsilon)$ and $\overline{v}_\varepsilon = \overline{v}_{\varepsilon,\alpha_\varepsilon,\lambda_\varepsilon,x_\varepsilon}$ is a solution of (Q_ε). And one easily checks that

$$|\Delta u_\varepsilon|^2 \longrightarrow S^{n/2}\delta_0, \quad |u_\varepsilon|^{p+1} \longrightarrow S^{n/2}\delta_0, \text{ as } \varepsilon \longrightarrow 0.$$

To complete the proof of Theorem 1.1, it only remains to prove that if $f \geq 0$, then $u_\varepsilon > 0$ on Ω. For this purpose we will argue as in [8].

Since $f \geq 0$, using the maximum principle we derive $\tilde{f} \geq 0$, and therefore $|u_\varepsilon^-| \leq |v_\varepsilon|$, where $u_\varepsilon^- = max(0, -u_\varepsilon)$. It follows that $(u_\varepsilon^-)^{(n+4)/(n-4)} \in L^{2n/(n+4)}$. Let us introduce w satisfying

$$\Delta^2 w = -(u_\varepsilon^-)^{(n+4)/(n-4)}, \; w = \Delta w = 0. \tag{3.56}$$

Using a regularity argument, we derive that $w \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, by the maximum principle $w \leq 0$. Multiplying equation (3.56) by w and integrating on Ω, we obtain

$$\|w\|^2 = \int_\Omega \Delta^2 w.w = -\int_\Omega (u_\varepsilon^-)^{(n+4)/(n-4)} w \leq c\|w\||u_\varepsilon^-|_{L^{2n/(n-4)}}^{(n+4)/(n-4)}$$

so that we have either $\|w\| = 0$ and it follows $u_\varepsilon^- = 0$ or $\|w\| \neq 0$ and therefore

$$\|w\| \leq c|u_\varepsilon^-|_{L^{2n/(n-4)}}^{(n+4/(n-4)}. \tag{3.57}$$

Now, in view of the fact that $\tilde{u}_\varepsilon = \alpha_\varepsilon P\delta_{\lambda_\varepsilon,x_\varepsilon} + \overline{v}_\varepsilon$ is a solution of $(\tilde{Q}_\varepsilon)$ for ε small enough. We have

$$\begin{aligned} \int_\Omega \Delta^2 w \tilde{u}_\varepsilon = \int_\Omega w \Delta^2 \tilde{u}_\varepsilon &= \int_\Omega w|\tilde{u}_\varepsilon + \varepsilon \tilde{f}|^{8/(n-4)}(\tilde{u}_\varepsilon + \varepsilon \tilde{f}) \\ &= -\int_{u_\varepsilon \leq 0} (u_\varepsilon^-)^{(n+4)/(n-4)} w + \int_{u_\varepsilon \geq 0} (u_\varepsilon^+)^{(n+4)/(n-4)} w \\ &\leq -\int_{u_\varepsilon \leq 0} (u_\varepsilon^-)^{(n+4)/(n-4)} w, \quad (\text{since } w \leq 0) \\ &\leq -\int_\Omega (u_\varepsilon^-)^{(n+4)/(n-4)} w = \int_\Omega \Delta^2 w.w = \|w\|^2. \end{aligned} \tag{3.58}$$

On the other hand, since $\tilde{u}_\varepsilon \leq u_\varepsilon$, we derive

$$\int_\Omega \Delta^2 w \tilde{u}_\varepsilon = -\int_\Omega (u_\varepsilon^-)^{(n+4)/(n-4)} \tilde{u}_\varepsilon \geq \int_\Omega (u_\varepsilon^-)^{2n/(n-4)}. \tag{3.59}$$

From (3.57), (3.58) and (3.59), we deduce

$$|u_\varepsilon^-|_{L^{2n/(n-4)}}^{2n/(n-4)} \leq \|w\|^2 \leq c|u_\varepsilon^-|_{L^{2n/(n-4)}}^{2(n+4)/(n-4)}.$$

Now since, for ε sufficiently small, $|u_\varepsilon^-|_{L^{2n/(n-4)}}$ is small enough, we derive a contradiction, and the case $\|w\| \neq 0$ cannot occur. Therefore $u_\varepsilon^- = 0$ on Ω, and applying the strong maximum principle we derive $u_\varepsilon > 0$.

4 Proof of Theorem 1.2

Our aim in this section is to show that for ε small enough, problem (Q_ε) has at least p solutions, where p is the Ljusternik-Schnirelman category of Ω. We will use the same framework and notations as in Section 2. The method consists in looking for critical points of the functional J by successive optimizations of K, with respect to the different parameters α, λ, x and v. We assume that $x \in \Omega_d := \{y \in \Omega \,/ d(y, \partial\Omega) > d\}$, where d is a positive constant. We proceed in three steps:

Step 1. *Optimization with respect to* v. Arguing as in section 2 and using the implicit function theorem, we see that there exists a C^2-map which to each (α, λ, x), such that $|\alpha - c_0| \leq \eta_0$, $\lambda d(x, \partial\Omega) \geq \eta_0^{-1}$ with η_0 small, associates $\overline{v} = v_{\alpha,\lambda,x} \in E_{\lambda,x}$ such that (E_v) of (2.8) is satisfied. Moreover $\overline{v}$ satisfies the estimate of Proposition 2.1.

Step 2. *Optimization with respect to* α *and* λ. Given $x \in \Omega_d$, we have to find α and λ such that

$$\begin{cases} \frac{\partial K}{\partial \alpha} + \left\langle \frac{\partial K}{\partial v}, \frac{\partial \overline{v}}{\partial \alpha} \right\rangle = 0 \\ \\ \frac{\partial K}{\partial \lambda} + \left\langle \frac{\partial K}{\partial v}, \frac{\partial \overline{v}}{\partial \lambda} \right\rangle = 0. \end{cases} \tag{4.60}$$

Taking the derivatives, with respect to α and λ, of the equalities

$$\langle \overline{v}, P\delta \rangle = \left\langle \overline{v}, \frac{\partial P\delta}{\partial \lambda} \right\rangle = \left\langle \overline{v}, \frac{\partial P\delta}{\partial x_j} \right\rangle = 0, \;\; j = 1, \ldots, n$$

and using the fact that $\overline{v}$ satisfies (E_v), we see that system (4.60) is equivalent to

$$\begin{cases} \frac{\partial K}{\partial \alpha} = 0 \\ \\ \frac{\partial K}{\partial \lambda} = B\left\langle \frac{\partial^2 P\delta}{\partial \lambda^2}, \overline{v} \right\rangle + \sum_{j=1}^n C_j \left\langle \frac{\partial^2 P\delta}{\partial \lambda \partial x_j}, \overline{v} \right\rangle. \end{cases} \tag{4.61}$$

The expansions of $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \lambda}$ are given by (2.24) and (2.25). On the other hand, one can easily verify that

$$(i) \quad \left\| \frac{\partial^2 P\delta}{\partial \lambda^2} \right\| = O\left(\frac{1}{\lambda^2}\right); \quad (ii) \quad \left\| \frac{\partial^2 P\delta}{\partial \lambda \partial x_j} \right\| = O(1).$$

Then making the change of variable

$$\frac{1}{\lambda^{(n-4)/2}} = \frac{\varepsilon \tilde{f}(x)}{\alpha \varphi(x)}(1 + \xi), \;\; |\xi| < \frac{1}{2} \;\text{ and }\; \alpha = c_0 - \rho \tag{4.62}$$

and arguing as in Section 3, we see that system (4.61) is equivalent to

$$\begin{cases} \rho = V(\varepsilon, \rho, \xi, x) \\ \xi = W(\varepsilon, \rho, \xi, x), \end{cases} \tag{4.63}$$

where V and W are some C^1 functions, which satisfy the estimates

$$V = O(\rho^2 + \varepsilon^2)$$
$$W = O\big(|\rho| + \xi^2 + |x|^2 + \varepsilon^{4/(n-4)}\big).$$

We have

Proposition 4.1 *There exists a C^1 map, which to $x \in \Omega_d$ associates (α_x, λ_x), such that* (4.60) *is satisfied, where α_x and λ_x are defined by* (4.62). *Moreover, $\rho_x = O(\varepsilon^2)$ and $\xi_x = O(\varepsilon^{4/(n-4)})$.*

Proof. Using Brouwer's fixed point theorem, the existence of the map $x \mapsto (\alpha_x, \lambda_x)$ follows from (4.63). We need to prove that it is a C^1 map. By using the implicit function theorem, it suffices to show that the point (α_x, λ_x), for $x \in \Omega_d$ being fixed, is a nondegenerate critical point of the C^2 map

$$\begin{aligned} K_x : M_x &\longrightarrow \mathbb{R} \\ (\alpha, \lambda) &\longmapsto K(\alpha, \lambda, x, \overline{v}) \end{aligned}$$

where $M_x = \{(\alpha, \lambda) \in \mathbb{R} \times \mathbb{R}_+^*, |\alpha - c_0| \leq \eta_0,\ \lambda d(x, \partial\Omega) \geq \eta_0^{-1}\}$. Then, we have just to prove that the determinant of the matrix

$$\begin{pmatrix} \frac{\partial^2 K_x}{\partial \alpha^2} & \frac{\partial^2 K_x}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 K_x}{\partial \lambda \partial \alpha} & \frac{\partial^2 K_x}{\partial \lambda^2} \end{pmatrix} \tag{4.64}$$

is nonzero for $(\alpha, \lambda) = (\alpha_x, \lambda_x)$. We have

$$\frac{\partial K_x}{\partial \alpha} = \frac{\partial K}{\partial \alpha}, \quad \frac{\partial K_x}{\partial \lambda} = \frac{\partial K}{\partial \lambda} + \langle \frac{\partial K}{\partial v}, \frac{\partial \overline{v}}{\partial \lambda} \rangle.$$

Therefore

$$\begin{cases} \frac{\partial^2 K_x}{\partial \alpha^2} = \frac{\partial^2 K}{\partial \alpha^2} + \langle \frac{\partial^2 K}{\partial \alpha \partial v}, \frac{\partial \overline{v}}{\partial \alpha} \rangle \\ \frac{\partial^2 K_x}{\partial \alpha \partial \lambda} = \frac{\partial^2 K}{\partial \alpha \partial \lambda} + \langle \frac{\partial^2 K}{\partial \alpha \partial v}, \frac{\partial \overline{v}}{\partial \lambda} \rangle \\ \frac{\partial^2 K_x}{\partial \lambda^2} = \frac{\partial^2 K}{\partial \lambda^2} + 2 \langle \frac{\partial^2 K}{\partial \lambda \partial v}, \frac{\partial \overline{v}}{\partial \lambda} \rangle + \langle \frac{\partial K}{\partial v}, \frac{\partial^2 \overline{v}}{\partial \lambda^2} \rangle. \end{cases} \tag{4.65}$$

First let us estimate the scalar products in $\mathcal{H}(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$ which occur in (4.65). Since $\frac{\partial^2 K}{\partial \alpha \partial v}$ is orthogonal to $E_{\lambda,x}$ and $\frac{\partial \overline{v}}{\partial \alpha}$ is in $E_{\lambda,x}$, we have

$$\langle \frac{\partial^2 K}{\partial \alpha \partial v}, \frac{\partial \overline{v}}{\partial \alpha} \rangle = 0. \tag{4.66}$$

We write

$$\frac{\partial \overline{v}}{\partial \lambda} = w_1 + aP\delta + b\frac{\partial P\delta}{\partial \lambda} + \sum_{i=1}^{n} c_i \frac{\partial P\delta}{\partial x_i} \tag{4.67}$$

with $w_1 \in E_{\lambda,x}$, $a, b, c_i \in \mathbb{R}$. Furthermore if we take the scalar product in $\mathcal{H}(\Omega)$ of (4.67) with respectively $P\delta$, $\frac{\partial P\delta}{\partial \lambda}$, $\frac{\partial P\delta}{\partial x_j}$, $j = 1, \ldots, n$ we get a quasi-diagonal linear system in

a, b, c_i, whose coefficients are given by (3.50) and the left side by

$$\begin{aligned}
&\langle \frac{\partial \overline{v}}{\partial \lambda}, P\delta\rangle = -\langle \overline{v}, \frac{P\delta}{\partial \lambda}\rangle = 0\\
&\langle \frac{\partial \overline{v}}{\partial \lambda}, \frac{\partial P\delta}{\partial \lambda}\rangle = -\langle \overline{v}, \frac{\partial^2 P\delta}{\partial \lambda^2}\rangle = O\big(\frac{\|\overline{v}\|}{\lambda^2}\big)\\
&\langle \frac{\partial \overline{v}}{\partial \lambda}, \frac{\partial P\delta}{\partial x_j}\rangle = -\langle \overline{v}, \frac{\partial^2 P\delta}{\partial x_j \partial \lambda}\rangle = O\big(\|\overline{v}\|\big),
\end{aligned}$$

where we have used the facts that $\left\|\frac{\partial^2 P\delta}{\partial \lambda^2}\right\| = O(\frac{1}{\lambda^2})$ and $\left\|\frac{\partial^2 P\delta}{\partial \lambda \partial x_j}\right\| = O(1)$. The solution of this system then yields

$$a = O\big(\frac{\|\overline{v}\|}{\lambda^{n-2}}\big), \quad b = O\big(\|\overline{v}\|\big), \quad c = O\big(\frac{\|\overline{v}\|}{\lambda^2}\big). \tag{4.68}$$

Let us estimate $\|w_1\|$. From (2.8) we have

$$\frac{\partial K}{\partial v} = \overline{v} - (\Delta^2)^{-1}\big(|\tilde{u}|^{p-1}\tilde{u}\big) = AP\delta + B\frac{\partial P\delta}{\partial \lambda} + \sum_{i=1}^{n} C_i \frac{\partial P\delta}{\partial x_i} \tag{4.69}$$

where $\tilde{u} = \alpha P\delta + \overline{v} + \varepsilon \tilde{f}$ and $(\Delta^2)^{-1}(g)$ denotes the solution of

$$\Delta^2 \psi = g \ \text{ in } \Omega; \quad \psi = \Delta\psi = 0 \ \text{ on } \partial\Omega.$$

Differentiating (4.69) with respect to λ and taking the scalar product with w_1 in $\mathcal{H}(\Omega)$, we obtain

$$\begin{aligned}
&\int_\Omega |\Delta w_1|^2 - p\int_\Omega |\alpha P\delta + \overline{v} + \varepsilon\tilde{f}|^{p-1}\big(\alpha\frac{\partial P\delta}{\partial \lambda} + \frac{\partial \overline{v}}{\partial \lambda}\big)w_1\\
&\quad = B\int_\Omega \Delta\big(\frac{\partial^2 P\delta}{\partial \lambda^2}\big)\Delta w_1 + \sum_{i=1}^{n} C_i \int_\Omega \Delta\big(\frac{\partial^2 P\delta}{\partial x_i \partial \lambda}\big)\Delta w_1.
\end{aligned} \tag{4.70}$$

We recall that at the point $(\alpha_x, \lambda_x) = (c_0 - \rho_x, \lambda_x)$, we have

$$\frac{1}{\lambda_x^{(n-4)/2}} = \frac{\varepsilon\tilde{f}(x)}{\alpha\varphi(x)}(1+\xi_x), \quad \rho_x = O(\varepsilon^2), \quad \xi_x = O(\varepsilon^{4/(n-4)}). \tag{4.71}$$

Then, a computation using (4.70), (2.11), (2.14) and (3.52) shows that

$$\|w_1\| = O\big(\varepsilon^{2/(n-4)}\|\overline{v}\|\big). \tag{4.72}$$

In the same way, we prove that

$$\left\|\frac{\partial \overline{v}}{\partial \alpha}\right\| = O\big(\|\overline{v}\|\big). \tag{4.73}$$

Then using (4.67), (4.68), (4.71), (4.72) and (4.73) a computation shows that

$$\begin{aligned}\langle\frac{\partial^2 K}{\partial\alpha\partial v},\frac{\partial\overline{v}}{\partial\lambda}\rangle &= \int_\Omega \Delta\frac{\partial\overline{v}}{\partial\alpha}\Delta\frac{\partial\overline{v}}{\partial\lambda} - p\int_\Omega |\alpha P\delta+\overline{v}+\varepsilon\tilde{f}|^{p-1}\big(P\delta+\frac{\partial\overline{v}}{\partial\alpha}\big)\frac{\partial\overline{v}}{\partial\lambda}\\ &= O\big(\varepsilon^{2/(n-4)}\|\overline{v}\|^2\big)\end{aligned} \tag{4.74}$$

and

$$\begin{aligned}\langle\frac{\partial^2 K}{\partial\lambda\partial v},\frac{\partial\overline{v}}{\partial\lambda}\rangle &= \int_\Omega \big|\Delta\frac{\partial\overline{v}}{\partial\lambda}\big|^2 - p\int_\Omega |\alpha P\delta+\overline{v}+\varepsilon\tilde{f}|^{p-1}\big(\alpha\frac{\partial P\delta}{\partial\lambda}+\frac{\partial\overline{v}}{\partial\lambda}\big)\frac{\partial\overline{v}}{\partial\lambda}\\ &= O\big(\varepsilon^{4/(n-4)}\|\overline{v}\|^2\big).\end{aligned} \tag{4.75}$$

Lastly, we write

$$\frac{\partial^2\overline{v}}{\partial\lambda^2} = w_2 + a'P\delta + b'\frac{\partial P\delta}{\partial\lambda} + \sum_{i=1}^{n} c_i'\frac{\partial P\delta}{\partial x_i} \tag{4.76}$$

with $w_2\in E_{\lambda,x},\ a',b',c_i'\in\mathbb{R}$. We take the scalar product of (4.76) with respectively $P\delta,\frac{\partial P\delta}{\partial\lambda},\frac{\partial P\delta}{\partial x_j},\ j=1,\dots,n$. On the left side we have

$$\langle\frac{\partial^2\overline{v}}{\partial\lambda^2},P\delta\rangle = -2\langle\frac{\partial\overline{v}}{\partial\lambda},\frac{\partial P\delta}{\partial\lambda}\rangle - \langle\overline{v},\frac{\partial^2 P\delta}{\partial\lambda^2}\rangle = O(\varepsilon^{4/(n-4)}\|\overline{v}\|) \tag{4.77}$$

since $\overline{v}\in E_{\lambda,x},\ \|\frac{\partial^2 P\delta}{\partial\lambda^2}\| = O\big(\frac{1}{\lambda^2}\big)$ and using (4.67), (4.68) and (3.50). In the same way, using also (4.72), $\|\frac{\partial^2 P\delta}{\partial\lambda\partial x_i}\| = O(1),\ \|\frac{\partial^3 P\delta}{\partial\lambda^3}\| = O\big(\frac{1}{\lambda^3}\big),$ and $\|\frac{\partial^3 P\delta}{\partial\lambda^2\partial x_i}\| = O\big(\frac{1}{\lambda}\big)$, we get

$$\langle\frac{\partial^2\overline{v}}{\partial\lambda^2},\frac{\partial P\delta}{\partial\lambda}\rangle = -2\langle\frac{\partial\overline{v}}{\partial\lambda},\frac{\partial^2 P\delta}{\partial\lambda^2}\rangle - \langle\overline{v},\frac{\partial^3 P\delta}{\partial\lambda^3}\rangle = O(\varepsilon^{6/(n-4)}\|\overline{v}\|), \tag{4.78}$$

$$\langle\frac{\partial^2\overline{v}}{\partial\lambda^2},\frac{\partial P\delta}{\partial x_j}\rangle = -2\langle\frac{\partial\overline{v}}{\partial\lambda},\frac{\partial^2 P\delta}{\partial x_j\partial\lambda}\rangle - \langle\overline{v},\frac{\partial^3 P\delta}{\partial\lambda^2\partial x_j}\rangle = O(\varepsilon^{2/(n-4)}\|\overline{v}\|). \tag{4.79}$$

On the right side we use again (3.50), and from the linear system that we obtain, we derive

$$a' = O(\varepsilon^{4/(n-4)}\|\overline{v}\|),\ \ b' = O(\varepsilon^{2/(n-4)}\|\overline{v}\|),\ \ c' = O(\varepsilon^{6/(n-4)}\|\overline{v}\|). \tag{4.80}$$

From (4.69), (4.76), (4.80), (3.50) and (3.52) we find

$$\langle\frac{\partial K}{\partial\overline{v}},\frac{\partial^2\overline{v}}{\partial\lambda^2}\rangle = O\Big(\big(|\rho|^2\varepsilon^{4/(n-4)} + \varepsilon^{(2n-4)/(n-4)}\big)\|\overline{v}\|\Big). \tag{4.81}$$

Coming back to (4.65), It only remains to estimate the second derivatives of K with respect to α and λ. Using (4.71), a computation shows

$$\frac{\partial^2 K}{\partial\alpha^2} = C_1' + O(|\rho|+\varepsilon^2) \tag{4.82}$$

$$\frac{\partial^2 K}{\partial\alpha\partial\lambda} = O(\varepsilon^{(2n-6)/(n-4)}) \tag{4.83}$$

$$\frac{\partial^2 K}{\partial \lambda^2} = C_2' \frac{\big(\tilde{f}(x)\big)^{(2n-4)/(n-4)}}{\big(\varphi(x)\big)^{n/(n-4)}} \varepsilon^{(2n-4)/(n-4)} + O(\varepsilon^{2n/(n-4)}) \tag{4.84}$$

where C_1' and C_2' are positive constants which depend on n only. Then using (4.65), (4.66), (4.74), (4.75), (4.81),..., (4.84) together with (3.49) we can compute the determinant of the matrix (4.64), and we prove that it is equal to

$$C_1' C_2' \frac{\big(\tilde{f}(x)\big)^{(2n-4)/(n-4)}}{\big(\varphi(x)\big)^{n/(n-4)}} \varepsilon^{(2n-4)/(n-4)} + O(\varepsilon^{2n/(n-4)}).$$

This ends the proof of proposition 4.1.

Step 3. *Optimization with respect to x.* We consider the following C^1 map $\mathcal{K}$ defined on $\Omega_d := \{x \in \Omega \,|\, d(x, \partial\Omega) > d\}$ by

$$\mathcal{K}(x) = K(\alpha_x, \lambda_x, x, \overline{v}_x) = J(\alpha_x P\delta_{\lambda_x, x} + \overline{v}_x).$$

As $H(x,x) \simeq c/d^{n-4}$ as $d = d(x, \partial\Omega) \to 0$, it follows from proposition 2.3 that, for d and ε small enough, there exists c such that the level set

$$\mathcal{K}_c := \{x \in \Omega_d \,/ \mathcal{K}(x) \le c\}$$

satisfies

$$\Omega_{2d} \subset \mathcal{K}_c \subset \Omega_d.$$

Now we will apply the Ljusternik-Schnirelman theory to the function $\mathcal{K}$ defined on $\mathcal{K}_c$. For this we need to prove that the gradient of $-\mathcal{K}$ is pointing inward on the boundary of $\mathcal{K}_c$. Using proposition4.1 we have

$$\frac{\partial \mathcal{K}}{\partial x_i}(\alpha_x, \lambda_x, x, \overline{v}_x) = \frac{\partial K}{\partial x_i}(\alpha_x, \lambda_x, x, \overline{v}_x) + \Big\langle \frac{\partial K}{\partial v}(\alpha_x, \lambda_x, x, \overline{v}_x), \frac{\partial \overline{v}_x}{\partial x_i} \Big\rangle, \text{ for } i = 1, \dots, n.$$

The expansion of $\frac{\partial K}{\partial x}(\alpha_x, \lambda_x, x, \overline{v}_x)$ is given by (2.26), and it is easy to obtain

$$\begin{aligned}\Big\langle \frac{\partial K}{\partial v}, \frac{\partial \overline{v}}{\partial x_i} \Big\rangle &= -B \Big\langle \frac{\partial^2 P\delta}{\partial \lambda \partial x_i}, \overline{v} \Big\rangle - \sum_{j=1}^{n} C_j \Big\langle \frac{\partial^2 P\delta}{\partial x_i \partial x_j}, \overline{v} \Big\rangle \\ &= O\Big(\big(|B| + \lambda^2 \sum_{j=1}^{n} |C_j|\big) \|\overline{v}\| \Big)\end{aligned}$$

since $\big\| \frac{\partial^2 P\delta}{\partial \lambda \partial x_i} \big\| = O(1)$ and $\big\| \frac{\partial^2 P\delta}{\partial x_i \partial x_j} \big\| = O(\lambda^2)$. Therefore (2.26), (3.52), (4.71) and proposition 2.1 show that

$$\mathcal{K}'(x) = C_3 \Big(\frac{\alpha^2 \varphi'(x)}{2\lambda^{n-4}} - \frac{\alpha \varepsilon \tilde{f}'(x)}{\lambda^{(n-4)/2}} \Big) + \mathcal{V}_x(\alpha_x, \lambda_x, x) \tag{4.85}$$

where $\mathcal{V}_x$ is a continuous function which satisfies the same estimate as V_x, given by (2.29). It follows

$$\frac{\partial \mathcal{K}}{\partial n_x}(x) = \nabla \mathcal{K}(x) n_x = \frac{C_3 \alpha}{\lambda^{(n-4)/2}} \Big(\frac{\alpha}{2\lambda^{(n-4)/2}} \frac{\partial \varphi(x)}{\partial n_x} - \varepsilon \frac{\partial \tilde{f}(x)}{\partial n_x} \Big) + \mathcal{V}_x n_x.$$

Here n_x is the outward normal to $\partial\Omega_d$ at the point x.

Now, since $\partial\varphi/\partial n_x \simeq c/d^{n-3}$ as $d = d(x,\partial\Omega) \longrightarrow 0$, one has $\partial\varphi/\partial n_x > 0$ for d small enough. We have also by Hopf Lemma $\partial\tilde{f}/\partial n_x < 0$. Therefore we deduce that the gradient of $-\mathcal{K}$ is pointing inward on the boundary of $\mathcal{K}_c$. Then applying the Ljusternik-Shnirelman theory to the function $\mathcal{K}$ defined on $\mathcal{K}_c$, we conclude that $\mathcal{K}$ has at least as many critical points in Ω_d as the category of $\mathcal{K}_c$. Moreover, since Ω is smooth, we have for d small enough

$$Cat(\mathcal{K}_c) = Cat(\mathcal{K}_c, \mathcal{K}_c) \geq Cat(\Omega_{2d}, \Omega) = Cat(\Omega,\Omega) = Cat(\Omega).$$

Thus, there exist at least $p = Cat(\Omega)$ distinct points $x_1, \ldots, x_p$ of Ω such that $\mathcal{K}'(x_i) = 0$. Consequently, the functions $\alpha_i P\delta_{x_i,\lambda_{x_i}} + \overline{v}_{x_i},\ 1 \leq i \leq p$ are critical points of J. It follows that $u_i = \alpha_i P\delta_{x_i,\lambda_{x_i}} + \overline{v}_{x_i} + \varepsilon\tilde{f},\ 1 \leq i \leq p$ are solutions of (Q_ε). Proceeding as in Section 3 these solutions are positive. By construction, each of them concentrates at a point of Ω as ε goes to zero. Using (4.85) and Proposition 4.1, it is easy to see that for each $i = 1, ..., p$, $\overline{x_i} = \lim_{\varepsilon\to 0} x_i$ is a critical point of $\tilde{f}/\sqrt{\varphi}$. Thereby, Theorem 1.2 is proved.

Remark 4.1 We believe the results of theorems 1.1, 1.2 and 1.3 to be true for $n = 5$. More careful estimate of the function v_ε, defined in proposition 2.1, should include the case $n = 5$.

5 Proof of Theorem 1.4

Making the change of variable in (R_ε)

$$u = \tilde{u} + \varepsilon\tilde{f},$$

we are led to consider the equivalent problem

$$(\tilde{R}_\varepsilon) \qquad \begin{cases} \Delta^2\tilde{u} = |\tilde{u} + \varepsilon\tilde{f}|^{p-1}(\tilde{u} + \varepsilon\tilde{f}) + \varepsilon(\tilde{u} + \varepsilon\tilde{f}) & \text{in } \Omega \\ \Delta\tilde{u} = \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

We will use the same framework and notations as in section 2 and 3. We look for solutions of the form

$$\tilde{u} = \alpha P\delta_{x,\lambda} + v$$

where $\alpha \to c_0$ (c_0 is defined by (1.2)), $x \to x_0 \in \Omega$, $\lambda \to +\infty$, $v \in E_{\lambda,x}$ with $\|v\| \to 0$, as $\varepsilon \to 0$.

For $\varepsilon > 0$, we define on $\mathcal{H}(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ the functional

$$J(\tilde{u}) = \frac{1}{2}\int_\Omega |\Delta\tilde{u}|^2 - \frac{1}{p+1}\int_\Omega |\tilde{u} + \varepsilon\tilde{f}|^{p+1} - \frac{\varepsilon}{2}\int_\Omega (\tilde{u} + \varepsilon\tilde{f})^2$$

whose critical points are solutions of $(\tilde{R}_\varepsilon)$. We define also the functional

$$\begin{aligned} K : M \quad &\longrightarrow \quad \mathbb{R} \\ (\alpha, \lambda, x, v) \quad &\longmapsto \quad J(\alpha P\delta_{x,\lambda} + v). \end{aligned}$$

Then $(\alpha, \lambda, x, v) \in M$ is a critical point of K if and only if $\tilde{u} = \alpha P\delta_{x,\lambda} + v$ is a critical point of J. That means there exists $(A, B, C_1 \dots, C_n) \in \mathbb{R}^{n+2}$ such that the system (2.8) is satisfied. We have

Proposition 5.1 *There exist $\varepsilon_1 > 0$, $T_1 > 0$ and a C^1 map which to any $(\varepsilon, \alpha, \lambda, x)$ in $(0, \varepsilon_1) \times \mathbb{R} \times \mathbb{R}_+^* \times \Omega$ with $\lambda d > T_1$, associates $\overline{v} = v_{\varepsilon,\alpha,\lambda,x} \in E_{\lambda,x}$ such that (E_v) of (2.8) is satisfied for some $(A, B, C_1, \dots, C_n) \in \mathbb{R}^{n+2}$. Such a $\overline{v}$ is unique and minimizes $K(\alpha, \lambda, x, v)$ with respect to $v \in E_{\lambda,x}$. Moreover*

$$\|\overline{v}\| \leq C \begin{cases} \frac{1}{(\lambda d)^{n-4}} + \frac{\varepsilon}{\lambda^{(n-4)/2}} + \varepsilon^2 & \text{if } n < 12 \\ \frac{(\log(\lambda d))^{2/3}}{(\lambda d)^8} + \frac{\varepsilon(\log \lambda)^{2/3}}{\lambda^4} + \varepsilon^p & \text{if } n = 12 \\ \frac{1}{(\lambda d)^{(n+4)/2}} + \frac{\varepsilon}{\lambda^4} + \varepsilon^p & \text{if } n > 12 \end{cases}$$

where C is a positive constant independent of ε and $d = d(x, \partial\Omega)$.

The proof is the same as the one of proposition 2.1 and will be omitted. Now combining the estimates that we obtain in proposition 2.2 with other in [15], we obtain

$$\frac{\partial K}{\partial \alpha}(\alpha, \lambda, x, \overline{v}) = C_\alpha \rho - \frac{c\alpha\varepsilon}{\lambda^4} + V_\alpha^1(\alpha,\, \lambda,\, x) \tag{5.86}$$

$$\frac{\partial K}{\partial \lambda}(\alpha, \lambda, x, \overline{v}) = C_\lambda \Big(\frac{\alpha\varepsilon \tilde{f}(x)}{\lambda^{(n-2)/2}} + \frac{2cc_0}{C_\lambda}\frac{\alpha\varepsilon}{\lambda^5} - \frac{\alpha^2 \varphi(x)}{\lambda^{n-3}} \Big) + V_\lambda^1(\alpha, \lambda, x) \tag{5.87}$$

$$\frac{\partial K}{\partial x}(\alpha, \lambda, x, \overline{v}) = C_x \Big(\frac{\alpha^2 \varphi'(x)}{2\lambda^{n-4}} - \frac{\alpha\varepsilon \tilde{f}'(x)}{\lambda^{(n-4)/2}} \Big) + V_x^1(\alpha,\, \lambda,\, x) \tag{5.88}$$

where C_α, $C_\lambda = \frac{n-4}{2} c_0^{8/(n-4)} \int_{\mathbb{R}^n} \delta_{0,1}^{(n+4)/(n-4)}$ and C_x are positive constants which depend on n only, c_0 is defined by (1.2), $c = \int_{\mathbb{R}^n} \delta_{0,1}^2$ and V_α^1, V_λ^1, V_x^1 are smooth functions which verify

$$V_\alpha^1(\alpha,\, \lambda,\, x) = O\Big(\rho^2 + \frac{1}{(\lambda d)^{n-4}} + \frac{\varepsilon}{\lambda^{\frac{n-4}{2}}} + \frac{\varepsilon^2}{\lambda^4} + \varepsilon^{p+1} \Big) \tag{5.89}$$

$$\begin{aligned} V_\lambda^1(\alpha,\, \lambda,\, x) = O\Big(&\frac{\varepsilon|\rho|}{\lambda^5} + \frac{1}{\lambda^{n-1} d^{n-2}} + \frac{|\rho|}{\lambda^{n-3} d^{n-4}} + \frac{\varepsilon|\rho|}{\lambda^{\frac{n-2}{2}}} \\ &+ \frac{\varepsilon}{\lambda^{\frac{n+2}{2}} d^2} + \frac{\varepsilon^{p+1}}{\lambda} + \frac{\varepsilon^2}{\lambda^5} + \frac{\varepsilon^2}{\lambda^{\frac{n-2}{2}}} + \frac{\varepsilon}{\lambda^{n-3} d^{n-4}} \Big) \end{aligned} \tag{5.90}$$

$$\begin{aligned} V_x^1(\alpha,\, \lambda,\, x) = O\Big(&\frac{1}{\lambda^{n-2} d^{n-1}} + \frac{|\rho|}{\lambda^{n-4} d^{n-3}} + \frac{\varepsilon|\rho|}{\lambda^{\frac{n-4}{2}}} + \frac{\varepsilon}{\lambda^{\frac{n-2}{2}}} + \lambda\varepsilon^{p+1} \\ &+ \frac{\varepsilon \log \lambda d}{\lambda^{\frac{n+2}{2}} d^4} + \frac{\varepsilon^2}{\lambda^3} + \frac{\varepsilon}{\lambda^{n-4} d^{n-3}} + \frac{\varepsilon^2}{\lambda^{\frac{n-6}{2}}} \Big). \end{aligned} \tag{5.91}$$

If $(n-2)/2 < 5$, i.e. $n < 12$, we have

$$\frac{c\alpha\varepsilon}{\lambda^4} = o\Big(\frac{\varepsilon}{\lambda^{(n-4)/2}} \Big) \text{ and } \frac{\varepsilon}{\lambda^5} = o\Big(\frac{\alpha\varepsilon \tilde{f}(x)}{\lambda^{(n-2)/2}} \Big).$$

Therefore (5.86),(5.87) and (5.88) become similar to (2.24), (2.25) and (2.26). Note that other terms appear in $V_\alpha^1(\alpha,\lambda,x), V_\lambda^1(\alpha,\lambda,x)$ and $V_x^1(\alpha,\lambda,x)$. Those terms will modify the estimates of (2.27), (2.28) and (2.29) but, we obtain other estimates which do not change the argument used in Section 3. Thus (i) follows.

Regarding claim (ii), since $x_0 = 0$ is a critical point of $\tilde{f}$ with $\tilde{f}(0) = 0$, we have

$$\tilde{f}(x) = \tilde{f}(0) + O(|x|^2) = O(|x|^2) \quad ; \quad \tilde{f}'(x) = Hess\tilde{f}(0)x + O(|x|^2).$$

Now making the change of variable

$$\frac{1}{\lambda^{n-8}} = \varepsilon \frac{cc_0}{C_\lambda \alpha\varphi(x)}(1+\xi), \quad |\xi| < \frac{1}{2}, \tag{5.92}$$

and using the fact that $n = 10$, 11, equations (5.86),(5.87) and (5.88) become

$$\begin{aligned}
\frac{\partial K}{\partial \alpha}(\alpha,\lambda,x,\overline{v}) &= C_\alpha \rho + O\Big(\rho^2 + \varepsilon^{5/2}(\text{ if } n=10) + \varepsilon^{13/6}(\text{ if } n=11)\Big)\\
\frac{\partial K}{\partial \lambda}(\alpha,\lambda,x,\overline{v}) &= \frac{\alpha\varepsilon}{\lambda^5}\Big(-c\xi + O\big(\varepsilon^{-\gamma_1}(|x|^2+|\rho|) + \varepsilon^{\gamma_2}\big)\Big)\\
\frac{\partial K}{\partial x}(\alpha,\lambda,x,\overline{v}) &= \frac{-\alpha\varepsilon}{\lambda^{(n-4)/2}}\Big(cHess\tilde{f}(0)x + O\big(|x|^2+|\rho|+\varepsilon^{\gamma_3}\big)\Big).
\end{aligned}$$

where $\gamma_1 = 1/2$, $\gamma_2 = 1/3$, $\gamma_3 = 1/3$ if $n = 10$ and $\gamma_1 = 1/6$, $\gamma_2 = 1/2$, $\gamma_3 = 1/6$ if $n = 11$.

Using the following change of variables $x = \varepsilon^{\gamma_1/2}x_1$, $\rho = \varepsilon^{\gamma_1}\rho_1$ and arguing as in Section 3, (ii) follows.

If $(n-2)/2 = 5$, i.e. $n = 12$, a simple computation shows that $\frac{2cc_0}{C_\lambda} = \frac{1}{2}$, then the expansion (5.87) becomes

$$\frac{\partial K}{\partial \lambda}(\alpha,\lambda,x,\overline{v}) = C_\lambda\Big(\frac{\alpha\varepsilon(\tilde{f}(x)+1/2)}{\lambda^5} - \frac{\alpha^2\varphi(x)}{\lambda^9}\Big) + V_\lambda^1(\alpha,\lambda,x).$$

Hence arguing as in Section 3, (iii) follows.

If $(n-2)/2 > 5$, i.e. $n > 12$, we have $\frac{\alpha\varepsilon\tilde{f}(x)}{\lambda^{(n-2)/2}} = o\big(\frac{\varepsilon}{\lambda^5}\big)$. Then (5.87) becomes

$$\frac{\partial K}{\partial \lambda}(\alpha,\lambda,x,\overline{v}) = C_\lambda\Big(\frac{cc_0}{C_\lambda}\frac{\alpha\varepsilon}{\lambda^5} - \frac{\alpha^2\varphi(x)}{\lambda^{n-3}}\Big) + O\Big(\frac{\varepsilon}{\lambda^{(n-2)/2}}\Big) + V_\lambda^1(\alpha,\lambda,x).$$

Assume that x_0 is a nondegenerate critical point of φ. Without loss of generality we will assume that $x_0 = 0$. Since $\varphi'(x) \simeq cd(x,\partial\Omega)^{3-n}$ for x close to the boundary of Ω, one can see that there exists $d_0 > 0$ such that $d(x,\partial\Omega) > d_0$ for x near 0. We have for x in a neighborhood of 0 in Ω

$$\varphi(x) = \varphi(0) + O(|x|^2) \quad ; \quad \varphi'(x) = Hess\varphi(0)x + O(|x|^2).$$

Then taking the change of variable defined by (5.92), we derive

$$\begin{aligned}
\frac{\partial K}{\partial \alpha}(\alpha,\lambda,x,\overline{v}) =& C_\alpha \rho + O\Big(\rho^2 + \frac{1}{\lambda^{n-4}} + \frac{\varepsilon}{\lambda^{\frac{n-4}{2}}} + \frac{\varepsilon^2}{\lambda^4} + \varepsilon^{p+1}\Big)\\
\frac{\partial K}{\partial \lambda}(\alpha,\lambda,x,\overline{v}) =& -\frac{cc_0}{C_\lambda}\frac{\alpha\varepsilon}{\lambda^5}\xi\\
&+ O\Big(\frac{|x|^2}{\lambda^{n-3}} + \frac{\varepsilon}{\lambda^{\frac{n-2}{2}}} + \frac{\varepsilon|\rho|}{\lambda^5} + \frac{1}{\lambda^{n-1}} + \frac{|\rho|}{\lambda^{n-3}} + \frac{\varepsilon^{p+1}}{\lambda} + \frac{\varepsilon^2}{\lambda^5}\Big)\\
\frac{\partial K}{\partial x}(\alpha,\lambda,x,\overline{v}) =& \frac{C_x\alpha^2}{2\lambda^{n-4}} Hess\varphi(0)x\\
&+ O\Big(\frac{|x|^2}{\lambda^{n-4}} + \frac{\varepsilon}{\lambda^{\frac{n-4}{2}}} + \frac{1}{\lambda^{n-2}} + \frac{|\rho|}{\lambda^{n-4}} + \lambda\varepsilon^{p+1} + \frac{\varepsilon^2}{\lambda^3}\Big).
\end{aligned}$$

Now arguing as in Section 3, (iv) follows.

To complete the proof of the theorem, it only remains to show that if $f \geq 0$, the solutions constructed in each previous case are positive functions. For this purpose we proceed in the same way as in section 3 and the proof of theorem 1.4 is thereby complete. □

6 Appendix

In this appendix, we collect some estimates used in the paper. These estimates were originally introduced by Bahri [2] for the case of the Laplace operator. In this appendix, we suppose that $\lambda d(x,\partial\Omega)$ is large enough. We have

Proposition 6.1 [9] *Let $n \geq 5$. For $x \in \Omega$, $\lambda > 0$ and $\varphi_{x,\lambda} = \delta_{x,\lambda} - P\delta_{x,\lambda}$, we have the following estimates :*

$$(a)\quad 0 \leq \varphi_{x,\lambda} \leq \delta_{x,\lambda}, \qquad (b)\quad \varphi_{x,\lambda} = c_0\frac{H(x,.)}{\lambda^{\frac{n-4}{2}}} + f_{x,\lambda},$$

where c_0 is defined by (1.2) and $f_{x,\lambda}$ satisfies

$$f_{x,\lambda} = O\left(\frac{1}{\lambda^{\frac{n}{2}}d^{n-2}}\right), \quad \lambda\frac{\partial f_{x,\lambda}}{\partial\lambda} = O\left(\frac{1}{\lambda^{\frac{n}{2}}d^{n-2}}\right), \quad \frac{1}{\lambda}\frac{\partial f_{x,\lambda}}{\partial x} = O\left(\frac{1}{\lambda^{\frac{n+2}{2}}d^{n-1}}\right),$$

where d is the distance $d(x,\partial\Omega)$.

$$(c)\qquad |\varphi_{x,\lambda}|_{L^{\frac{2n}{n-4}}} = O\big(\frac{1}{(\lambda d)^{\frac{n-4}{2}}}\big), \quad |\lambda\frac{\partial\varphi_{x,\lambda}}{\partial\lambda}|_{L^{\frac{2n}{n-4}}} = O\big(\frac{1}{(\lambda d)^{\frac{n-4}{2}}}\big),$$

$$\|\varphi_{x,\lambda}\| = O\big(\frac{1}{(\lambda d)^{\frac{n-4}{2}}}\big), \quad |\frac{1}{\lambda}\frac{\partial\varphi_{x,\lambda}}{\partial x}|_{L^{\frac{2n}{n-4}}} = O\big(\frac{1}{(\lambda d)^{\frac{n-2}{2}}}\big).$$

Using proposition 6.1, a computation similar to the one performed in [2] and [20] allows us to state the following

Proposition 6.2 *We have*

$$\int_\Omega \Delta P\delta \Delta \frac{\partial P\delta}{\partial \lambda} = \frac{n-4}{2} c_0^{p-1} J_n \frac{H(x,x)}{\lambda^{n-3}} + O\left(\frac{1}{\lambda^{n-1} d^{n-2}}\right)$$
$$\int_\Omega (\alpha P\delta)^p \alpha \frac{\partial P\delta}{\partial \lambda} = (n-4)\alpha^{p+1} J_n \frac{n-4}{2} \frac{H(x,x)}{\lambda^{n-3}} + O\left(\frac{1}{\lambda^{n-1} d^{n-2}}\right)$$
$$\int_\Omega |\Delta P\delta|^2 = c_0^{p-1} S_n - \frac{c_0^{p-1} J_n}{\lambda^{n-4}} H(x,x) + O\left(\frac{1}{(\lambda d)^{n-2}}\right)$$
$$\int_\Omega (P\delta)^{p+1} = S_n - \frac{2n}{n-4} \frac{J_n}{\lambda^{n-4}} H(x,x) + O\left(\frac{1}{(\lambda d)^{n-2}}\right),$$

where $J_n = \int_{\mathbb{R}^n} \delta_{0,1}^{(n+4)/(n-4)}$, *and* c_0 *is defined by* (1.2).

Proposition 6.3

$$p\int_\Omega (P\delta)^{p-1} \frac{\partial P\delta}{\partial \lambda} \varepsilon \tilde{f} = -\frac{n-4}{2} \varepsilon J_n \frac{\tilde{f}(x)}{\lambda^{(n-2)/2}} + O\left(\frac{\varepsilon}{\lambda^{(n+2)/2} d^2}\right). \tag{6.93}$$

$$\int_\Omega P\delta^p \tilde{f} = \frac{J_n}{\lambda^{(n-4)/2}} \tilde{f}(x) + O\left(\frac{\log(\lambda d)}{\lambda^{(n+4)/2} d^4}\right) + O\left(\frac{1}{\lambda^{(n-2)/2}}\right). \tag{6.94}$$

Proof. To prove Claim (6.94), since $|\tilde{f}|_\infty$ is bounded, it is easy to see that

$$\int_{\Omega \setminus B_d} (P\delta)^p \tilde{f} = O\left(\frac{1}{\lambda^{(n+4)/2} d^4}\right) \tag{6.95}$$

where B_d denotes the ball of center x and radius d.

In the ball B_d, expanding $\tilde{f}$ around x and using Proposition 6.1, we find

$$\int_{B_d} P\delta^p \tilde{f} = \frac{J_n}{\lambda^{(n-4)/2}} \tilde{f}(x) + O\left(\frac{\log(\lambda d)}{\lambda^{(n+4)/2} d^4}\right) + O\left(\frac{1}{\lambda^{(n-2)/2}}\right). \tag{6.96}$$

Hence, (6.95) and (6.96) imply Claim (6.94).

The proof of Claim (6.93) is similar to the proof of Claim (6.94). So we omit it. □

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