Advanced Nonlinear Studies **6** (2006), 563 – 590

Non-Degeneracy and Periodic Solutions of Semilinear Differential Equations with Deviation

Gang Meng, Ping Yan

Department of Mathematical Sciences
Tsinghua University
Beijing 100084, China

Xiaoyan Lin

Department of Mathematics, Huaihua Teachers College
Huaihua 418008, Hunan, China

Meirong Zhang *†

Department of Mathematical Sciences and
Zhou Pei-Yuan Center for Applied Mathematics
Tsinghua University
Beijing 100084, China

Received in revised form 29 May 2006
Communicated by Rafael Ortega

Abstract

In this paper, we will use some Sobolev constants to construct certain classes of non-degenerate potentials for linear differential equations with deviation. As an application of these potentials, we will give some existence results of periodic solutions of semilinear equations with deviation.

2000 Mathematics Subject Classification. Primary: 34K13; Secondary: 34K06

Key words. Non-degenerate potential, Sobolev constant, periodic solution, semilinear equation, deviation

*Supported by the National Natural Science Foundation of China (Grant no. 10325102 and no. 10531010) and TRAPOYT-M.O.E. of China (2001).

†Correspondence author, E-mail: mzhang@math.tsinghua.edu.cn.

1 Introduction

We recall first some known results on existence conditions of periodic solutions of semilinear ordinary differential equations (ODE). Consider the second order ODE

$$x'' + g(t,x) = 0, \tag{1.1}$$

where $g : \mathbb{R}/T\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and grows semilinearly in the following sense: There exist $a(t)$ and $b(t)$ in $L^1(\mathbb{S}_T)$, $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$, such that

$$a(t) \le \liminf_{|x|\to\infty} \frac{g(t,x)}{x} \le \limsup_{|x|\to\infty} \frac{g(t,x)}{x} \le b(t) \tag{1.2}$$

uniformly in t. In 1989, Fonda and Mawhin [2] obtained the existence of T-periodic solutions of (1.1) under the following condition: For any $q(t) \in L^1(\mathbb{S}_T)$ with

$$a(t) \le q(t) \le b(t) \quad \text{for a.e. } t, \tag{1.3}$$

the linear equation

$$x'' + q(t)x = 0 \tag{1.4}$$

has only the trivial T-periodic solution. Such an existence condition is very sharp for non-autonomous differential equations. It can be extended to various boundary value problems of ODEs or PDEs. See, for example, [1, 4] for systems of ODEs, [14, 12] for the scalar p-Laplacian, and [5] for semilinear elliptic PDEs. A similar existence result has been established even for equations of Emarkov-Pinney type which has a singularity [16].

In this paper, we will give an extension of such an existence result to semilinear differential equations with deviation

$$x''(t) + g(t, x(\varphi(t))) = 0, \tag{1.5}$$

where $g(t,x)$ is as above and the deviation function φ is in $C(\mathbb{S}_T, \mathbb{S}_T)$. Typical examples for the deviation are $\varphi(t) = t - r$, $\varphi(t) = mt - r$, where $m \in \mathbb{Z}\backslash\{0\}$ and $r \in \mathbb{R}$, or $\varphi(t) = mt - r + a\sin(2\pi t/T)$, $m \in \mathbb{Z}\backslash\{0\}$, $r,\ a \in \mathbb{R}$.

In order to study T-periodic solutions of (1.5), we will reduce to the study of linear equation with deviation

$$x''(t) + q(t)x(\varphi(t)) = 0, \tag{1.6}$$

where $q(t) \in L^1(\mathbb{S}_T)$ satisfies (1.3). Equation (1.6) is an extension of (1.4) and can be referred to as the Hill's equation with deviation. Define the order-interval in $L^1(\mathbb{S}_T)$ by

$$\langle a, b\rangle := \left\{q(t) \in L^1(\mathbb{S}_T) : q(t) \text{ staisfies (1.3)}\right\}.$$

Definition 1.1 Let $\varphi \in C(\mathbb{S}_T, \mathbb{S}_T)$ be fixed. We say that $q(t) \in L^1(\mathbb{S}_T)$, or equation (1.6), is *non-degenerate* with respect to φ, if (1.6) has only the trivial T-periodic solution. We say that the order-interval $\langle a, b\rangle$ is *non-degenerate* with respect to φ if for any $q(t) \in \langle a, b\rangle$, equation (1.6) is non-degenerate.

One of the main results of this paper is the following existence principle for equation (1.5). If $\langle a, b\rangle$ is non-degenerate, then (1.5) has at least one T-periodic solution. See Theorem 2.1. The proof is based on degree argument with the aid of an estimate result from [13] on families of positively homogeneous operators.

Since the interval $\langle a, b\rangle$ is 'large', it is not an easy work to verify the non-degeneracy of $\langle a, b\rangle$ for general deviation $\varphi(t)$. However, for the ODE case, i.e., $\varphi(t) \equiv t$, it is easy to use eigenvalues to express the non-degeneracy of $\langle a, b\rangle$ and the remaining work is how to estimate eigenvalues. Let $q(t) \in L^1(\mathbb{S}_T)$. We use

$$\lambda_0(q) < \lambda_1(q) \le \lambda_2(q) < \cdots < \lambda_{2n-1}(q) \le \lambda_{2n}(q) < \cdots$$

to denote all eigenvalues of

$$x''(t) + (\lambda + q(t))x(t) = 0$$

with the T-periodic boundary condition: $u(0) = u(T)$ and $u'(0) = u'(T)$. Using the comparison results for eigenvalues [8], the non-degeneracy of $\langle a, b\rangle$ is now equivalent to

$$\lambda_{k-1}(a) < 0 \quad \text{and} \quad \lambda_k(b) > 0 \tag{1.7}$$

for some $k \ge 0$. (Here $\lambda_{-1}(a)$ is understood as $-\infty$.) It should be noticed that some optimal estimates on conditions (1.7) have been worked out recently by Zhang [15]. For example, in order that the first non-trivial existence condition

$$\lambda_0(a) < 0 \quad \text{and} \quad \lambda_1(b) > 0 \tag{1.8}$$

holds, it suffices that $a,\ b$ satisfy, for some exponent $\alpha \in [1, \infty]$,

$$\bar{a} > 0 \quad \text{and} \quad \|b_+\|_\alpha < 4K(2\alpha^*, T) \quad (\alpha^* = \alpha/(\alpha - 1)). \tag{1.9}$$

Here $b_+(t) = \max(b(t), 0)$ and $\|\cdot\|_\alpha = \|\cdot\|_{L^\alpha(\mathbb{S}_T)}$ is the L^α norm in the space $L^\alpha(\mathbb{S}_T)$. For a function $a(t) \in L^1(\mathbb{S}_T)$, $\bar{a} = (1/T)\int_{\mathbb{S}_T} a(t)$ denotes the mean value. The constant $K(\gamma, T)$ is the best Sobolev constant in the following inequality

$$C\|x\|_\gamma^2 \le \|x'\|_2^2 \quad \text{for all} \quad x \in H_0^1(0, T). \tag{1.10}$$

Moreover, the constants $4K(2\alpha^*, T)$ in (1.9) are optimal to guarantee (1.8). See [15].

In the literature, conditions (1.8) and (1.9) are referred to as the non-resonance condition between the zeroth and the first eigenvalues of the periodic problem. Since we have no complete eigenvalue theory for the Hill's equations with deviation, we do not have the generalization of condition (1.8) to the deviation case. Another important contribution of this paper is that we will provide a partial generalization of the non-degeneracy condition (1.9) to the deviation case. That is, given $\alpha \in [1, \infty]$ and $T > 0$, we will give some explicitly expressed constants $Q(\alpha, T) > 0$ such that if

$$a \succ 0 \quad \text{and} \quad \|b\|_\alpha < Q(\alpha, T), \tag{1.11}$$

then $\langle a, b\rangle$ is non-degenerate with respect to $\varphi(t) = t - r$, or $\varphi(t) = mt - r$. Here $a \succ 0$ means that $a(t) \ge 0$ for a.e. t and $\bar{a} > 0$. For general $\varphi(t)$, some generalization of (1.9) with $\alpha = 1$ will be given also. See Theorem 3.4. Again, the constants $Q(\alpha, T)$ are related with Sobolev constants $K(\gamma, T)$. Note that conditions (1.9) have not only applications in semilinear equations, but also in some superlinear equations [6, 9]. In a separate paper, we will develop also some interesting applications of (1.11) to superlinear delay differential equations (DDEs).

In Section 4, special attention is paid to the non-degeneracy of linear DDEs ($\varphi(t) = t - r$). Some reasonable estimates on non-degenerate potentials will be given in Theorem 4.4, Corollary 4.6 and Theorem 4.7, where some new Sobolev-type inequalities related with delay equations are important in our estimates.

2 An existence principle

In this section, we give an existence principle of periodic solutions of equation (1.5), where $\varphi \in C(\mathbb{S}_T, \mathbb{S}_T)$ and $g = g(t, x) : \mathbb{S}_T \times \mathbb{R} \to \mathbb{R}$ is semilinear in the sense of (1.2).

Theorem 2.1 *Suppose that $g(t, x)$ satisfies* (1.2) *for some $a(t)$ and $b(t)$. If $\langle a, b\rangle$ is non-degenerate with respect to φ, then equation* (1.5) *has at least one T-periodic solution.*

It is standard to use the Leray-Schauder degree to prove this result. In obtaining *a priori* bounds for homotopy equations, we will adopt here the estimating principle developed by Zhang [13, Theorem 3.1]. Due to this principle, all estimates necessary in the application of degree theory will be reduced to elementary inequalities. This estimating principle has many successful applications to different problems. See [10, 12, 14].

The estimating principle is as follows. Let X be a normed space and M be a sequentially compact space. A mapping $F : X \to X$ is said to be positively homogeneous if $F(kx) = kF(x)$ for all $k \geq 0$ and all $x \in X$. We say that a mapping $F : M \times X \to X$ is uniformly completely continuous if (i) F is continuous on $M \times X$; and (ii) for any sequence $\{(q_n, x_n)\}$ in $M \times X$ such that $\{x_n\}$ is bounded in X, the sequence $\{F(q_n, x_n)\}$ has a convergent subsequence in X.

Theorem 2.2 [13, Theorem 3.1] *Let X be a normed space and M be a sequentially compact space. Let $F : M \times X \to X$ be a uniformly completely continuous mapping. Assume that, for each $q \in M$, $F(q, \cdot)$ is positively homogeneous and the equation*

$$u - F(q, u) = 0 \tag{2.1}$$

has only the trivial solution $u = 0$ in X. Then there exists a constant $c_0 > 0$ such that

$$\|u - F(q, u)\| \geq c_0 \|u\| \quad \text{for all } q \in M \text{ and } u \in X.$$

Recall that for a given $h(t) \in L^1(\mathbb{S}_T)$, T-periodic solutions $x(t)$ of the equation

$$x''(t) + h(t) = 0$$

are equivalent to $x(t) \in C(\mathbb{S}_T)$ and $x(t)$ satisfies the fixed point equation

$$x = Hx,$$

where $H : C(\mathbb{S}_T) \to C(\mathbb{S}_T)$ is defined by

$$(Hx)(t) = x(0) + \int_0^T G(t,s)h(s)ds, \quad t \in [0,T],$$

and the Green function is

$$G(t,s) = \begin{cases} \frac{2s(T-t)+(T/2-t)^2}{2T} & \text{if } 0 \le s \le t \le T, \\ \frac{2t(T-s)+(T/2-t)^2}{2T} & \text{if } 0 \le t \le s \le T. \end{cases}$$

Note that $G(0,s) = G(T,s)$ and $G(t,0) = G(t,T)$. Thus G can be extended continuously to 2-torus $\mathbb{S}_T \times \mathbb{S}_T$, and H can be rewritten as

$$(Hx)(t) = x(0) + \int_{\mathbb{S}_T} G(t,s)h(s)ds, \qquad t \in \mathbb{S}_T.$$

We take $X = C(\mathbb{S}_T)$ with the supremum norm $\|\cdot\|$. Let $M = \langle a,b\rangle \subset L^1(\mathbb{S}_T)$ with the topology of weak convergence in $L^1(\mathbb{S}_T)$. Then M is sequentially compact. Define $F : M \times X \to X$ by

$$F(q,x)(t) = x(0) + \int_{\mathbb{S}_T} G(t,s)q(s)x(\varphi(s))ds, \quad t \in \mathbb{S}_T. \tag{2.2}$$

Since $\varphi \in C(\mathbb{S}_T, \mathbb{S}_T)$, it is clear that $F(q,x) \in X$. It is obvious that $F(q,x)$ is linear in $x \in X$.

Proposition 2.3 *Given φ and a, b as above, the operator $F : M \times X \to X$ is uniformly completely continuous.*

Proof. Compactness. Let $\{(q_n, x_n)\}$ be a sequence in $M \times X$ such that $\|x_n\| \le C$ for some $C > 0$. Then $\|q_n\|_1 \le \|a\|_1 + \|b\|_1$. By (2.2), one has

$$\|F(q_n,x_n)\| \le \|x_n\| + \|G\|_\infty \|q_n\|_1 \|x_n\| \le C\left(1 + \|G\|_\infty(\|a\|_1 + \|b\|_1)\right) \quad \text{for all } n.$$

For the equi-continuity of $\{y_n = F(q_n,x_n)\}$, letting $t_1,\ t_2 \in \mathbb{S}_T$, we have

$$\begin{aligned} |y_n(t_2) - y_n(t_1)| &= \left| \int_{\mathbb{S}_T} (G(t_2,s) - G(t_1,s))q_n(s)x_n(\varphi(s))ds \right| \\ &\le \max_s |G(t_2,s) - G(t_1,s)| \cdot (\|a\|_1 + \|b\|_1)C. \end{aligned}$$

It follows from the uniform continuity of G that $\{y_n\}$ is equi-continuous. Hence $\{F(q_n,x_n)\}$ has a subsequence converging to some $y_0(t)$ in X by the Arzela-Ascoli theorem.

Continuity. Now we are ready to check the continuity of F on $M \times X$. Assume that $\{q_n\} \subset M$ is a sequence such that $q_n \rightharpoonup q_0$ and $\{x_n\} \subset X$ is a sequence converging to x_0 in X. The continuity is proved by proving the following assertion: Any subsequence of $\{F(q_n,x_n)\}$ has a sub-subsequence converging to $F(q_0,x_0)$ in X. In order to simplify notations, we only prove that $\{F(q_n, x_n)\}$ has a subsequence converging to $F(q_0, x_0)$ in X.

First, we prove the pointwise convergence, which means that for any given t, one has

$$F(q_n, x_n)(t) \to F(q_0, x_0)(t) \tag{2.3}$$

as $n \to \infty$. Note that

$$\begin{aligned}
&|F(q_n, x_n)(t) - F(q_0, x_0)(t)| \\
&= \left|(x_n(0) - x_0(0)) + \int_{\mathbb{S}_T} G(t,s)q_n(s)x_n(\varphi(s))ds - \int_{\mathbb{S}_T} G(t,s)q_0(s)x_0(\varphi(s))ds\right| \\
&\le \|x_n - x_0\| + \left|\int_{\mathbb{S}_T} G(t,s)q_n(s)(x_n(\varphi(s)) - x_0(\varphi(s)))ds\right| \\
&\quad + \left|\int_{\mathbb{S}_T} G(t,s)(q_n(s) - q_0(s))x_0(\varphi(s))ds\right| \\
&\le \|x_n - x_0\| + \|x_n - x_0\| \cdot \|G\|_\infty(\|a\|_1 + \|b\|_1) \\
&\quad + \left|\int_{\mathbb{S}_T} G(t,s)x_0(\varphi(s))(q_n(s) - q_0(s))ds\right|.
\end{aligned}$$

Since $x_n \to x_0$ in X, the first two terms go to zero as $n \to \infty$. For the third term, one notices that for given t, $G(t,\cdot)x_0(\varphi(\cdot)) \in X \subset L^\infty(\mathbb{S}_T) = \left(L^1(\mathbb{S}_T)\right)^*$ and $q_n - q_0 \rightharpoonup 0$ in $L^1(\mathbb{S}_T)$. Thus, by definition of weak convergence, the third term also goes to zero as $n \to \infty$. We have proved (2.3) for each t.

Next, since the convergence of $\{x_n\}$ in X implies that $\{x_n\}$ is bounded, from the proof of compactness, we know that there is a subsequence $\{F(q_{n'}, x_{n'})\}$ of $\{F(q_n, x_n)\}$ and some $y_0 \in X$ such that

$$F(q_{n'}, x_{n'}) \to y_0 \qquad \text{in } X. \tag{2.4}$$

Form (2.3) and (2.4), it is necessary that $F(q_0, x_0)(t) \equiv y_0(t)$ for all t. Now we conclude from (2.4) that

$$F(q_{n'}, x_{n'}) \to F(q_0, x_0) \quad \text{in } X.$$

The assertion is proved and the continuity is obtained.

From the compactness and the continuity, one knows that $F : M \times X \to X$ is uniformly completely continuous. □

Now we give the proof of Theorem 2.1.

Step 1. *Estimates for linear problems.* Note that the non-degeneracy of $M = \langle a, b\rangle$ is equivalent to the corresponding operator equation (2.1) has only the trivial solution for each $q \in M$. Thus the operator F defined by (2.2) fulfills all requirements of Theorem 2.2. Hence there exists $c_0 > 0$ such that

$$\|u - F(q, u)\| \ge c_0\|u\| \quad \text{for all } (q, u) \in M \times X. \tag{2.5}$$

Step 2. *The homotopy equation and estimates for nonlinear equations.* Choose any $q_0(t) \in M$. Consider the following homotopy equation

$$x''(t) + g_\tau(t, x(\varphi(t))) = 0, \quad (\tau \in [0, 1]) \tag{2.6}$$

where $g_\tau(t,x) = (1-\tau)q_0(t)x + \tau g(t,x)$. Then T-periodic solutions of (2.6) are fixed points of

$$x = N_\tau(x), \quad x \in X, \tag{2.7}$$

where

$$N_\tau(x)(t) = x(0) + \int_{\mathbb{S}_T} G(t,s) g_\tau(s, x(\varphi(s)))ds, \quad t \in \mathbb{S}_T.$$

It can be proved that $N(\tau,x) = N_\tau(x) : [0,1] \times X \to X$ is completely continuous. See the proof of Proposition 2.3.

Next, the semilinearity assumption (1.2) implies that for any $\varepsilon > 0$, one has the following decomposition for $g(t,x)$: there exist Carathéodory functions $m_\varepsilon(t,x),\ l_\varepsilon(t,x)$ such that

$$\begin{gathered} g(t,x) \equiv m_\varepsilon(t,x)x + l_\varepsilon(t,x), \\ a(t) \le m_\varepsilon(t,x) \le b(t), \qquad |l_\varepsilon(t,x)| \le \varepsilon|x| + h_\varepsilon(t), \end{gathered} \tag{2.8}$$

for all $x \in \mathbb{R}$ and a.e. $t \in \mathbb{S}_T$, where $h_\varepsilon \in L^1(\mathbb{S}_T, \mathbb{R}^+)$. See the constructions in [13, Example 2.2]. Hence $g_\tau(t,x) = m_{\varepsilon,\tau}(t,x)x + l_{\varepsilon,\tau}(t,x)$, where $m_{\varepsilon,\tau}(t,x) = (1-\tau)q_0(t) + \tau m_\varepsilon(t,x)$ and $l_{\varepsilon,\tau}(t,x) = \tau l_\varepsilon(t,x)$. Now N_τ can be decomposed as follows:

$$N_\tau(x) = N_\tau^0(x) + N_\tau^1(x),$$

where

$$N_\tau^0(x)(t) = x(0) + \int_{\mathbb{S}_T} G(t,s) m_{\varepsilon,\tau}(s, x(\varphi(s)))x(\varphi(s))ds, \quad t \in \mathbb{S}_T,$$

$$N_\tau^1(x)(t) = \int_{\mathbb{S}_T} G(t,s) l_{\varepsilon,\tau}(s, x(\varphi(s)))ds, \quad t \in \mathbb{S}_T.$$

Suppose now that $x \in X$ is a fixed point of N_τ. Then

$$x - N_\tau^0(x) = N_\tau^1(x). \tag{2.9}$$

For the left-hand side, let $q(t) = m_{\varepsilon,\tau}(t, x(\varphi(t)))$. Then $q(t) \in M$. Thus, by (2.5), we have

$$\|x - N_\tau^0(x)\| \ge c_0\|x\|. \tag{2.10}$$

For the right-hand side of (2.9), using estimate (2.8),

$$\begin{aligned} \|N_\tau^1(x)\| &= \max_t \left| \int_{\mathbb{S}_T} G(t,s) l_{\varepsilon,\tau}(s, x(\varphi(s)))ds \right| \\ &\le \max_t \int_{\mathbb{S}_T} G(t,s)(\varepsilon|x(\varphi(s))| + h_\varepsilon(s))ds \\ &\le A\varepsilon\|x\| + B_\varepsilon, \end{aligned} \tag{2.11}$$

where the constants are

$$A = T \cdot \max_{(t,s)} G(t,s), \quad B_\varepsilon = \max_{(t,s)} G(t,s) \cdot \|h_\varepsilon\|_1.$$

If we choose $0 < \varepsilon < c_0/A$, from (2.9), (2.10) and (2.11), all solutions of (2.7) for all $\tau \in [0,1]$ satisfy

$$\|x\| \le B_\varepsilon/(c_0 - A\varepsilon) =: R_0.$$

Step 3. *Existence of T-periodic solutions.* Let $B(2R_0) = \{x \in X : \|x\| < 2R_0\}$. From Step 2 and the homotopy invariance of the Leray-Schauder degree, we have

$$\deg(I - N_1, B(2R_0), 0) = \deg(I - N_0, B(2R_0), 0).$$

Since N_0 is a linear operator, one has $\deg(I - N_0, B(2R_0), 0) = \pm 1$. Thus

$$\deg(I - N_1, B(2R_0), 0) \ne 0$$

and hence one has at least $x \in B(2R_0)$ such that $x = N_1 x$. Equivalently, x is a T-periodic solution of (1.5). □

3 Non-degeneracy for general deviations

3.1 Asymptotically linear case

Consider the DDE

$$x''(t) + g(x(t-r)) = h(t), \tag{3.1}$$

where $h \in L^1(\mathbb{S}_T)$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous and is asymptotically linear: There exists $m \in \mathbb{R}$ such that

$$\lim_{|x|\to\infty} \frac{g(x)}{x} = m.$$

For the ODE case ($r = 0$), it is well-known that (3.1) has at least one T-periodic solution for each $h \in L^1(\mathbb{S}_T)$ if $m \ne (2k\pi/T)^2$ for all $k \in \mathbb{Z}^+ := \{0, 1, 2, \cdots\}$. For the DDE (3.1), by Theorem 2.1, we need to study when the following linear equation

$$x''(t) + mx(t-r) = 0 \tag{3.2}$$

has only the trivial T-periodic solution. We use the Fourier series to address this problem. Suppose

$$x(t) = \sum_{k\in\mathbb{Z}} c_k e^{i2k\pi t/T},$$

where $c_k \in \mathbb{C}$ and $\overline{c_k} = c_{-k}$. Substituting into (3.2), we have

$$\sum_{k\in\mathbb{Z}} c_k \left(-(2k\pi/T)^2 + me^{-i2k\pi r/T}\right) e^{i2k\pi t/T} = 0.$$

Then (3.2) has non-trivial T-periodic solutions if and only if m satisfies for some k

$$m = (2k\pi/T)^2 e^{i2k\pi r/T}.$$

From Theorem 2.1, we have

Theorem 3.1 *Suppose that in* (3.1) g *is asymptotically linear with some* $m \in \mathbb{R}$. *If* m *satisfies*

$$m \neq (2k\pi/T)^2 e^{i2k\pi r/T} \quad \text{for all } k \in \mathbb{Z}, \tag{3.3}$$

then (3.1) *has at least one* T*-periodic solution.*

Remark 3.2 Condition (3.3) depends upon r/T in a delicate way. If r/T is irrational, condition (3.3) is simply

$$m \neq 0.$$

Assume now that $r/T = q/p$ is rational, where $q,\ p \in \mathbb{Z}$, $p \geq 1$, and $q,\ p$ are co-prime. If p is odd, then (3.3) is

$$m \neq (2kp\pi/T)^2 \quad \text{for all } k \in \mathbb{Z}^+.$$

If p is even, then (3.3) becomes

$$m \neq (-1)^k (2kp\pi/T)^2 \quad \text{for all } k \in \mathbb{Z}^+.$$

Given $r \in \mathbb{R}$, we say that m *is an eigenvalue of (3.2)* if (3.2) has non-zero T-periodic solutions. Then $m = 0$ is always an eigenvalue of (3.2). However, if $r \neq lT$, $l \in \mathbb{Z}$, the first positive eigenvalue is always $> (2\pi/T)^2$ or is $+\infty$. In fact the first positive eigenvalue is $\geq (4\pi/T)^2$, and $(4\pi/T)^2$ is realized for $r = lT + T/2$. Thus if $r \neq lT$ and $0 < m < (4\pi/T)^2$, equation (3.1) has at least one T-periodic solution. This shows that the appearance of the delay r has positive impact on existence of periodic solutions of (3.1). See also Example 4.10.

This also shows that for the existence of periodic solutions of semilinear differential equations with deviation, some restriction on the linear growth of nonlinearities is necessary.

3.2 Semilinear case

In this subsection, we will construct some class of non-degenerate potentials for (1.6) using Sobolev inequalities.

For an exponent $1 \leq \gamma \leq \infty$, the best Sobolev constant $K(\gamma, T)$ in (1.10) is

$$K(\gamma, T) = \inf_{x \in H_0^1(0,T),\ x \neq 0} \frac{\|x'\|_2^2}{\|x\|_\gamma^2}.$$

Denote $K(\gamma) := K(\gamma, 1)$. Using a scaling, one has

$$K(\gamma, T) = K(\gamma)/T^{1+2/\gamma}.$$

The constants $K(\gamma, T)$ can be computed explicitly. See, for example, [11, 14]. In fact,

$$K(\gamma) = \begin{cases} \frac{2\pi}{\gamma}\left(\frac{2}{2+\gamma}\right)^{1-2/\gamma}\left(\frac{\Gamma(1/\gamma)}{\Gamma(1/2+1/\gamma)}\right)^2 & \text{if} \quad 1 \leq \gamma < \infty, \\ 4 & \text{if} \quad \gamma = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler. When γ goes from 1 to ∞, $K(\gamma)$ runs from 12 to 4.

We need also the following Sobolev constants. Let $\tilde{H}^1_T = \left\{x \in H^1(\mathbb{S}_T) : \bar{x} = 0\right\}$. For an exponent $1 \le \gamma \le \infty$, we use $L(\gamma, T)$ to denote the best Sobolev constant in the following inequality

$$C\|x\|_\gamma^2 \le \|x'\|_2^2 \quad \text{for all} \quad x \in \tilde{H}^1_T.$$

That is,

$$L(\gamma, T) = \inf_{x \in \tilde{H}^1_T,\ x \ne 0} \frac{\|x'\|_2^2}{\|x\|_\gamma^2}. \tag{3.4}$$

It is not difficult to prove that the infimum of (3.4) can be attained by some $x \in \tilde{H}^1_T$ which are called the minimizers of (3.4). See, for example, the proof of Proposition 4.1(i).

It is well-known that $L(\infty, T) = 12/T$ and $L(2, T) = (2\pi/T)^2 = 4K(2, T)$. We will not give the explicit formula for $L(\gamma, T)$, although it is possible to use the variational method to address this problem as in [14, Theorem 4.1] for the Dirichlet problems. Some properties of $L(\gamma, T)$ to be used later are listed here.

Proposition 3.3 (i) $L(\gamma, T) = L(\gamma, 1)/T^{1+2/\gamma}$.

(ii) *The minimizers of* (3.4) *have the minimal period* T.

Proof. The equality in (i) is trivial by a scaling technique. Let us prove (ii). Suppose that $x \in \tilde{H}^1_T$ is a minimizer of (3.4), i.e., $\|x'\|^2_{L^2[0,T]} = L(\gamma, T)\|x\|^2_{L^\gamma[0,T]}$. If $x(t)$ has the minimal period $0 < S < T$, then $S = T/N$ for some $N \in \mathbb{N}$, $N \ge 2$. Now $\|x'\|^2_{L^2[0,T]} = N\|x'\|^2_{L^2[0,S]}$ and $\|x\|^2_{L^\gamma[0,T]} = N^{2/\gamma}\|x\|^2_{L^\gamma[0,S]}$. Thus $x \in \tilde{H}^1(\mathbb{S}_S)$ and satisfies $\|x'\|^2_{L^2[0,S]} = N^{2/\gamma-1}L(\gamma, T)\|x\|^2_{L^\gamma[0,S]}$. Hence, $L(\gamma, S) \le N^{2/\gamma-1}L(\gamma, T)$. However, by (i), $L(\gamma, S) = N^{1+2/\gamma}L(\gamma, T)$. We have a contradiction. □

Theorem 3.4 *Suppose that* $q \in L^\alpha(\mathbb{S}_T)$, $1 \le \alpha \le \infty$.

(i) *Let* $\varphi(t) = mt - r$, *where* $m \in \mathbb{Z}$, $m \ne 0$. *If* $q \succ 0$ *and*

$$\|q\|_\alpha < \frac{1}{1/L(2\alpha^*, T) + 1/\sqrt{L(\infty, T)L(\alpha^*, T)}} =: Q(\alpha, T), \tag{3.5}$$

then equation (1.6) *is non-degenerate.*

(ii) *If* $q \succ 0$ *and* $\|q\|_1 < Q(1, T) = 6/T$, *then for each deviation function* $\varphi \in C(\mathbb{S}_T, \mathbb{S}_T)$, *equation* (1.6) *is non-degenerate.*

Proof. (i) Suppose that (1.6) has a non-zero T-periodic solution x. Write $x = \tilde{x} + \bar{x}$. By (1.6),

$$\tilde{x}''(t) + q(t)\tilde{x}(\varphi(t)) + \bar{x}q(t) = 0. \tag{3.6}$$

Integration over $\mathbb{S}_T$,

$$\bar{x} = -\frac{\int_{\mathbb{S}_T} q(t)\tilde{x}(\varphi(t))}{\int_{\mathbb{S}_T} q(t)}. \tag{3.7}$$

Multiplying (3.6) by $\tilde{x}(t)$ and integrating over $\mathbb{S}_T$,

$$\begin{aligned}\|\tilde{x}'\|_2^2 &= -\int_{\mathbb{S}_T} \tilde{x}''(t)\tilde{x}(t) = \int_{\mathbb{S}_T} \left(q(t)\tilde{x}(\varphi(t)) + \bar{x}q(t)\right)\tilde{x}(t) \\ &= \int_{\mathbb{S}_T} q(t)\tilde{x}(t)\tilde{x}(\varphi(t)) + \bar{x}\int_{\mathbb{S}_T} q(t)\tilde{x}(t).\end{aligned}$$

Since $q \succ 0$ and $q \in L^\alpha(\mathbb{S}_T)$,

$$\begin{aligned}\left|\int_{\mathbb{S}_T} q(t)\tilde{x}(t)\tilde{x}(\varphi(t))\right| &\le \|q\|_\alpha \|\tilde{x}(\cdot)\tilde{x}(\varphi(\cdot))\|_{\alpha^*} \\ &\le \|q\|_\alpha \|\tilde{x}\|_{2\alpha^*}\|\tilde{x}(\varphi(\cdot))\|_{2\alpha^*} = \|q\|_\alpha \|\tilde{x}\|_{2\alpha^*}^2,\end{aligned}$$

where the last equality follows from the following equality

$$\begin{aligned}\|\tilde{x}(\varphi(\cdot))\|_\gamma &= \left(\int_{\mathbb{S}_T} |\tilde{x}(mt-r)|^\gamma dt\right)^{1/\gamma} \\ &= \left(\frac{1}{m}\int_{-r}^{mT-r} |\tilde{x}(s)|^\gamma ds\right)^{1/\gamma} = \|\tilde{x}\|_\gamma.\end{aligned}$$

By (3.7), $|\bar{x}| \le \|\tilde{x}\|_\infty$. Moreover, $\left|\int_{\mathbb{S}_T} q(t)\tilde{x}(t)\right| \le \|q\|_\alpha \|\tilde{x}\|_{\alpha^*}$. Thus

$$\|\tilde{x}'\|_2^2 \le \|q\|_\alpha \left(\|\tilde{x}\|_{2\alpha^*}^2 + \|\tilde{x}\|_\infty \|\tilde{x}\|_{\alpha^*}\right). \tag{3.8}$$

Using the definition of $L(\gamma, T)$, we get $\|\tilde{x}\|_\gamma \le \|\tilde{x}'\|_2/\sqrt{L(\gamma, T)}$. Now (3.8) implies that $\tilde{x}$ satisfies

$$\|\tilde{x}'\|_2^2 \le \|q\|_\alpha \left(\frac{1}{L(2\alpha^*, T)} + \frac{1}{\sqrt{L(\infty, T)L(\alpha^*, T)}}\right)\|\tilde{x}'\|_2^2.$$

If (3.5) is satisfied, one has $\|\tilde{x}'\|_2 = 0$ and $\tilde{x} = 0$, and $\bar{x} = 0$ by (3.7). Hence $x = \tilde{x} + \bar{x} = 0$, which is a contradiction.

(ii) Let $\alpha = 1$. Then $\alpha^* = \infty$. The conclusion can be obtained using the same argument as above by noticing that $\|\tilde{x}(\varphi(\cdot))\|_\infty \le \|\tilde{x}\|_\infty$ for any deviation function φ. □

Combining Theorems 2.1 and 3.4, we have the following existence results for equation (1.5).

Corollary 3.5 *Suppose that $g(t, x)$ satisfies (1.2) for some $a(t)$, $b(t) \in L^\alpha(\mathbb{S}_T)$.*

(i) *Let $\varphi(t) = mt - r$, where $m \in \mathbb{Z}$, $m \ne 0$. If*

$$a \succ 0 \quad \textit{and} \quad \|b\|_\alpha < Q(\alpha, T), \quad (\alpha \in [1, \infty]) \tag{3.9}$$

then equation (1.5) has at least one T-periodic solution.

(ii) *If $a \succ 0$ and $\|b\|_1 < 6/T$, then, for each deviation function $\varphi \in C(\mathbb{S}_T, \mathbb{S}_T)$, equation (1.5) has at least one T-periodic solution.*

Remark 3.6 Corollary 3.5 is remarkable because a single condition like (3.9) will not only guarantee that the ODE (1.1) has a T-periodic solution, but also guarantee the existence of families of T-periodic solutions $x = \psi_{m,r},\ m \in \mathbb{Z}\backslash\{0\},\ r \in \mathbb{R}$, of the deviation equations (1.5).

Note that our result in Theorem 3.4(i) for non-degenerate potentials is independent of m and r. For the L^1 estimate in Theorem 3.4(ii), it is even independent of the form of the deviation functions $\varphi(t)$. In general, the non-degeneracy condition on potentials is related with the deviation function φ. It is then an interesting problem to find the optimal bound on $\|q\|_\alpha$ for which (1.6) is non-degenerate. In the next section, we will find some interesting estimates on the optimal bounds for the delay case.

4 Non-degenerate potentials for the delay equations

Let us take $\varphi(t) = t - r$ and consider the linear DDE

$$x''(t) + q(t)x(t-r) = 0. \tag{4.1}$$

Given $\alpha \in [1, \infty]$ and $r \in \mathbb{R}$, define

$$R(\alpha, T, r) = \sup\left\{R > 0 :\ q \succ 0 \text{ and } \|q\|_\alpha < R \text{ imply that (4.1) is non-degenerate}\right\}.$$

For the ODE case, i.e., $r = 0$, the optimal bound is known

$$R(\alpha, T, 0) = 4K(2\alpha^*, T).$$

See [15]. To achieve these optimal bounds, the most important fact used is that for Hill's equations, the zeroes of eigenfunctions are very clear. However, for the linear DDE (4.1), we have no results on the distribution of zeros of non-zero solutions of (4.1). For example, we do not know if the zeros of the non-zero solutions of (4.1) are non-degenerate.

In the following we aim at improving the non-degeneracy condition (3.5) for the DDEs. That is, we will provide some lower bounds for $R(\alpha, T, r)$. Before going to the non-degeneracy problem, we discuss two minimal and maximal problems related with DDEs.

Let $\gamma \in [1, \infty]$, $T > 0$ and $r \in \mathbb{R}$. For any $x \in \tilde{H}^1_T$, $x(t_0) = 0$ for some $t_0 \in [0, T)$. We have rough estimates as follows. For any $t \in [t_0, t_0 + T)$,

$$|x(t)| = \left|\int_{t_0}^{t} x'(s)ds\right| \le \int_{t_0}^{t_0+T} |x'(s)|ds \le T^{1/2}\|x'\|_2,$$

i.e., $\|x\|_\infty \le T^{1/2}\|x'\|_2$. In particular,

$$T^{-(1+1/\gamma)}\|(x(\cdot)x(\cdot - r))_+\|_\gamma \le \|x'\|_2^2. \tag{4.2}$$

The first minimal problem is

$$\tilde{L}(\gamma, T, r) = \inf_{x \in \tilde{H}^1_T,\ x \ne 0} \frac{\|x'\|_2^2}{\|(x(\cdot)x(\cdot - r))_+\|_\gamma}\ (< \infty). \tag{4.3}$$

That is, $\tilde{L}(\gamma, T, r)$ is the best constant C in the following inequality

$$C\|(x(\cdot)x(\cdot - r))_+\|_\gamma \leq \|x'\|_2^2 \qquad \text{for all } x \in \tilde{H}_T^1.$$

By (4.2), $\tilde{L}(\gamma, T, r) \geq T^{-(1+1/\gamma)} > 0$. The constants $\tilde{L}(\gamma, T, r)$ are an extension of the Sobolev constants $L(2\gamma, T)$ defined by (3.4) to the delay case. One has $\tilde{L}(\gamma, T, 0) = L(2\gamma, T)$ and $\tilde{L}(\gamma, T, r)$ is T-periodic in $r \in \mathbb{R}$.

The following function will be useful in this section.

$$A(r) = \sum_{n=1}^{\infty} \frac{\cos nr}{n^2}, \quad r \in \mathbb{R}. \tag{4.4}$$

Then $A(r)$ is even and is 2π-periodic in r. This function can be rewritten as

$$A(r) = \frac{\pi^2}{6} - \frac{1}{4}|r|(2\pi - |r|), \quad r \in [-2\pi, 2\pi]. \tag{4.5}$$

Some properties of the constants $\tilde{L}(\gamma, T, r)$ are presented in the following proposition.

Proposition 4.1 (i) *As a function of $r \in \mathbb{R}$, $\tilde{L}(\gamma, T, r)$ is even in r and is T-periodic in r. Moreover, $\tilde{L}(\gamma, T, r)$ is globally $\frac{1}{2}$-Hölder continuous in r.*

(ii) $\tilde{L}(\gamma, kT, kr) = \tilde{L}(\gamma, T, r)/k^{1+1/\gamma}$ *for all $k > 0$.*

(iii) $\tilde{L}(\gamma, T, r) > L(2\gamma, T)$ *for all $r \neq lT$, $l \in \mathbb{Z}$.*

(iv) $\tilde{L}(\infty, T, r) = \frac{4\pi^2}{T(A(0)+A(2\pi r/T))}$, *where $A(r)$ is given by (4.4) and (4.5).*

(v) $\tilde{L}(1, T, T/2) = 4(2\pi/T)^2 = 4L(2, T)$.

Proof. (i) For any $x \in \tilde{H}_T^1$, $\|(x(\cdot)x(\cdot - r))_+\|_\gamma = \|(x(\cdot + r/2)x(\cdot - r/2))_+\|_\gamma$. By (4.3), it is easy to see that $\tilde{L}(\gamma, T, -r) = \tilde{L}(\gamma, T, r)$.

Note that the space $\tilde{H}_T^1$ with the norm $\|x'\|_2$ is a Hilbert space. Given $r \in \mathbb{R}$. Define a functional $J_r : \tilde{H}_T^1 \to \mathbb{R}$ by

$$J_r(x) = \|(x(\cdot)x(\cdot - r))_+\|_\gamma, \qquad x \in \tilde{H}_T^1.$$

Note that J_r is homogeneous: $J_r(kx) = k^2 J_r(x)$ for all $x \in \tilde{H}_T^1$ and $k \in \mathbb{R}$. Let

$$\Omega = \left\{x \in \tilde{H}_T^1 : \|x'\|_2 \leq 1\right\}.$$

By (4.2), J_r is bounded from above on Ω. Let $x_n \in \Omega$ be such that $J_r(x_n) \to \sup_\Omega J_r$. Then x_n has a sub-sequence x_{n_i} converging weakly to some x in $\tilde{H}_T^1$ as $i \to \infty$. Thus,

$$\|x'\|_2 \leq \liminf_{i\to\infty} \|x'_{n_i}\|_2 \leq 1.$$

Hence $x \in \Omega$. Moreover, x_{n_i} converges uniformly to x. Since $J_r(x)$ is continuous in x with respect to the uniform norm, we have $J_r(x) = \sup_\Omega J_r$. By the homogeneity of J_r, it is easy to see that $\|x'\|_2 = 1$. Thus the supremum $\sup_\Omega J_r$ is attainable and

$$\hat{L}(r) := \frac{1}{\tilde{L}(\gamma, T, r)} = \sup_{x\in\Omega} J_r(x) = \max_{x\in\partial\Omega} J_r(x) = J_r(x_r) \tag{4.6}$$

for some $x_r \in \partial\Omega := \{x \in \Omega : \|x'\|_2 = 1\}$.

Note that $\Omega \subset \tilde{H}_T^1$ is embedded into $C(\mathbb{S}_T)$ (with the uniform norm) compactly. In particular, there exists $C_0 > 0$ such that

$$\|x\|_\infty \le C_0 \qquad \text{for all } x \in \partial\Omega.$$

Let $r,\ r' \in \mathbb{R}$. Then for any $x \in \partial\Omega$, one has

$$|x(t)x(t-r') - x(t)x(t-r)| = \left| x(t) \int_{t-r}^{t-r'} x'(u)du \right| \le C_0 |r'-r|^{1/2}.$$

Using the inequality $|u_+ - v_+| \le |u-v|$ for all $u,\ v \in \mathbb{R}$, one has

$$|(x(t)x(t-r'))_+ - (x(t)x(t-r))_+| \le C_0|r'-r|^{1/2}. \tag{4.7}$$

In case $\gamma = \infty$, one can deduce from (4.7) that

$$|J_{r'}(x) - J_r(x)| \le C_0 |r'-r|^{1/2} \tag{4.8}$$

because $|\max_t f(t) - \max_t g(t)| \le \|f-g\|_\infty$. Let $x = x_r$ in (4.8). We get

$$\hat{L}(r') \ge J_{r'}(x_r) \ge \hat{L}(r) - C_0|r'-r|^{1/2}.$$

That is,

$$\hat{L}(r') - \hat{L}(r) \ge -C_0|r'-r|^{1/2}.$$

Changing the positions of r and r', we have also

$$\hat{L}(r) - \hat{L}(r') \ge -C_0|r'-r|^{1/2}.$$

Thus

$$|\hat{L}(r') - \hat{L}(r)| \le C_0|r'-r|^{1/2},$$

which proves the $\frac{1}{2}$-Hölder continuity of $\hat{L}(\cdot)$.

In case $\gamma \in [1,\infty)$, we let $u = (x(t)x(t-r'))_+ \ge 0$ and $v = (x(t)x(t-r))_+ \ge 0$. Then $u,\ v \le C_0^2$. By (4.7), we have

$$|u^\gamma - v^\gamma| = |\gamma\xi^{\gamma-1}(u-v)| \le \gamma(\max(u,v))^{\gamma-1}|u-v| \le \gamma C_0^{2\gamma-1}|r'-r|^{1/2}.$$

Thus

$$\begin{aligned} |(J_{r'}(x))^\gamma - (J_r(x))^\gamma| &= \left| \int_{\mathbb{S}_T} (x(t)x(t-r'))_+^\gamma dt - \int_{\mathbb{S}_T} (x(t)x(t-r))_+^\gamma \right| \\ &\le T\gamma C_0^{2\gamma-1}|r'-r|^{1/2}. \end{aligned}$$

Arguing as before, one has

$$\left| \left(\hat{L}(r')\right)^\gamma - \left(\hat{L}(r)\right)^\gamma \right| \le T\gamma C_0^{2\gamma-1}|r'-r|^{1/2}. \tag{4.9}$$

From this, the continuity of $(\hat{L}(\cdot))^\gamma$ can be obtained. Thus $\hat{L}(\cdot)$ is also continuous. Furthermore, as $\hat{L}(r) > 0$ and $\hat{L}(r)$ is T-periodic, we know that there exists some $K \geq 1$ such that for all r,

$$1/K \leq \hat{L}(r) \leq K.$$

Now it follows from (4.9) that

$$\begin{aligned}
|\hat{L}(r') - \hat{L}(r)| &= \left|\Big((\hat{L}(r'))^\gamma\Big)^{1/\gamma} - \Big((\hat{L}(r))^\gamma\Big)^{1/\gamma}\right| \\
&= \frac{\Big((\hat{L}(\xi))^\gamma\Big)^{1/\gamma-1}}{\gamma}\left|(\hat{L}(r'))^\gamma - (\hat{L}(r))^\gamma\right| \\
&\leq \frac{K^{\gamma-1}}{\gamma}\cdot T\gamma C_0^{2\gamma-1}|r'-r|^{1/2} \\
&= K^{\gamma-1}TC_0^{2\gamma-1}|r'-r|^{1/2}.
\end{aligned}$$

Thus $\hat{L}(r)$ is also $\frac{1}{2}$-Hölder continuous in r in this case.

By equality (4.6), it is now easy to obtain the $\frac{1}{2}$-Hölder continuity of $\tilde{L}(\gamma, T, r) = 1/\hat{L}(r)$.

(ii) The equality is trivial by a scaling technique.

(iii) We need only to prove the inequality for the case $\gamma \in [1, \infty)$, because for the case $\gamma = \infty$, the inequality follows simply from the formula of $\tilde{L}(\infty, T, r)$ in (iv).

Let $x \in \tilde{H}_T^1$ and $x \neq 0$. Using the Hölder inequality, we have

$$\|(x(\cdot)x(\cdot - r))_+\|_\gamma \leq \|x(\cdot)x(\cdot - r)\|_\gamma \tag{4.10}$$

$$= \left[\int_{\mathbb{S}_T} |x(t)|^\gamma |x(t-r)|^\gamma dt\right]^{1/\gamma}$$

$$\leq \left[\left(\int_{\mathbb{S}_T} |x(t)|^{2\gamma} dt\right)^{1/2}\left(\int_{\mathbb{S}_T} |x(t-r)|^{2\gamma} dt\right)^{1/2}\right]^{1/\gamma} \tag{4.11}$$

$$= \|x\|_{2\gamma}^2. \tag{4.12}$$

Now the inequality $\tilde{L}(\gamma, T, r) \geq L(2\gamma, T)$ follows immediately from (3.4) and (4.3).

In order to obtain the strict inequality when $r \neq lT$, $l \in \mathbb{Z}$, we need only to consider the case $0 < r \leq T/2$. From the proof of (i), $\hat{L}(r) = J_r(x_r)$ for some $x_r \in \partial\Omega$. We need to prove that

$$\hat{L}(r) = J_r(x_r) < \hat{L}(0) = 1/L(2\gamma, T) = \max_{x \in \partial\Omega} \|x\|_{2\gamma}^2. \tag{4.13}$$

Note that we have proved that $\hat{L}(r) \leq \hat{L}(0)$. Assume that (4.13) is false. It is then necessary that $\hat{L}(r) = \hat{L}(0)$ and now (4.13) becomes

$$J_r(x_r) = \hat{L}(0) = \max_{x \in \partial\Omega} \|x\|_{2\gamma}^2. \tag{4.14}$$

In particular,

$$J_r(x_r) = \|(x_r(\cdot)x_r(\cdot - r))_+\|_\gamma \geq \|x_r\|_{2\gamma}^2.$$

By noticing inequality (4.12), it is necessary that

$$J_r(x_r) = \|x_r\|_{2\gamma}^2. \tag{4.15}$$

Equality (4.15) has the following two implications. The first one is that (4.14) becomes

$$\hat{L}(0) = \max_{x \in \partial\Omega} \|x\|_{2\gamma}^2 = \|x_r\|_{2\gamma}^2. \tag{4.16}$$

That is, x_r is also a maximizer of $J_0(x) = \|x\|_{2\gamma}^2$, $x \in \partial\Omega$.

Another implication is obtained from (4.10)–(4.12) and (4.15). One has both (4.10) and (4.11) are necessarily equalities for $x = x_r$. That is,

$$\|(x_r(\cdot)x_r(\cdot - r))_+\|_\gamma = \|x_r(\cdot)x_r(\cdot - r)\|_\gamma,$$

and

$$\int_{\mathbb{S}_T} |x_r(t)|^\gamma |x_r(t-r)|^\gamma dt = \left(\int_{\mathbb{S}_T} |x_r(t)|^{2\gamma} dt\right)^{1/2} \left(\int_{\mathbb{S}_T} |x_r(t-r)|^{2\gamma} dt\right)^{1/2}.$$

From the first equality, one has $x_r(t)x_r(t-r) \geq 0$ for all t. The second equality means that there holds an equality in the Hölder inequality. Thus there exists some $k > 0$ such that $|x_r(t)| \equiv k|x_r(t-r)|$. Since $x_r(t)$ and $x_r(t-r)$ have the same sign, one deduces $x_r(t) \equiv kx_r(t-r)$. As $x_r(t)$ is T-periodic and $\|x_r'\|_2 = 1$, one sees $k = 1$ and $x_r(t) \equiv x_r(t-r)$. That is, x_r is also r-periodic.

However, as x_r is also a maximizer of problem (4.16), we know from Proposition 3.3(ii) that x_r shall have the minimal period T. We arrive at a contradiction.

(iv) We first use the Fourier expansions to compute $1/\tilde{L}(\infty, 2\pi, r)$. Let

$$x(t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \in \tilde{H}_{2\pi}^1. \tag{4.17}$$

The condition $\|x'\|_2 = 1$ is

$$\pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) = 1. \tag{4.18}$$

We need to maximize $\|(x(\cdot)x(\cdot - r))_+\|_\infty = (x(t_0)x(t_0 - r))_+$, where t_0 depends on x. After a translation of t, we may assume that $t_0 = 0$ and consider the functional

$$F(x) = x_1 x_2,$$

where

$$x_1 = x(0) = \sum_{n=1}^{\infty} a_n, \quad x_2 = x(-r) = \sum_{n=1}^{\infty} (a_n \cos nr - b_n \sin nr).$$

To find the maximum of F under (4.18), by the Lagrangian multiplier method, there exists λ such that

$$x_1 \cos nr + x_2 = \lambda n^2 a_n, \quad -x_1 \sin nr = \lambda n^2 b_n, \quad n \in \mathbb{N}.$$

It is reasonable to assume that $\lambda \neq 0$. Thus

$$a_n = \frac{x_1 \cos nr + x_2}{\lambda n^2}, \quad b_n = \frac{-x_1 \sin nr}{\lambda n^2}. \tag{4.19}$$

Now

$$x_1 = \sum_{n=1}^{\infty} \frac{x_1 \cos nr + x_2}{\lambda n^2} = \frac{x_1 A(r) + x_2 A(0)}{\lambda},$$

$$x_2 = \sum_{n=1}^{\infty} \frac{(x_1 \cos nr + x_2)\cos nr + x_1 \sin^2 nr}{\lambda n^2} = \frac{x_1 A(0) + x_2 A(r)}{\lambda}.$$

That is,

$$(A(r) - \lambda)x_1 + A(0)x_2 = 0, \quad A(0)x_1 + (A(r) - \lambda)x_2 = 0. \tag{4.20}$$

Since we are considering the maximum of F, it is reasonable to assume $x_1 x_2 > 0$. Hence we have $(A(r) - \lambda)^2 = (A(0))^2$, i.e., $\lambda = A(r) \pm A(0)$. Note that the case $-$ will result in $x_2 = -x_1$. We have $\lambda = A(0) + A(r) > 0$. Now we get from (4.19) and (4.20)

$$x_2 = x_1, \quad a_n = x_1(\cos nr + 1)/(\lambda n^2), \quad b_n = -x_1 \sin nr/(\lambda n^2).$$

By (4.18), one has

$$\frac{1}{\pi} = \frac{x_1^2}{\lambda^2} \sum_{n=1}^{\infty} n^2 \frac{(\cos nr + 1)^2 + \sin^2 nr}{n^4} = \frac{2x_1^2}{\lambda^2}(A(0) + A(r)) = \frac{2x_1^2}{\lambda}.$$

Thus the maximum of F is

$$F = x_1^2 = \frac{\lambda}{2\pi} = \frac{A(0) + A(r)}{2\pi} = \frac{1}{\tilde{L}(\infty, 2\pi, r)}.$$

Now we can apply Property (ii) to $k = 2\pi/T$ and $\gamma = \infty$ to obtain the equality required.

(v) We consider the case $T = 2\pi$. By (4.6), we need to prove that

$$\frac{1}{\tilde{L}(1, 2\pi, \pi)} = \max_{x \in \tilde{H}^1_{2\pi},\ \|x'\|_2 = 1} J_\pi(x) = \frac{1}{4}. \tag{4.21}$$

On the one hand, let $x(t) = a\cos 2t + b \sin 2t \in \tilde{H}^1_{2\pi}$, where $a^2 + b^2 = 1/(4\pi)$. It is easy to see that $\|x'\|_2 = 1$ and $J_\pi(x) = \int_0^{2\pi} x^2(t)dt = 1/4$. Thus $\max J_\pi \geq 1/4$.

On the other hand, let $x(t)$ be as in (4.17) and (4.18). It is not easy to maximize $J_\pi(x)$ directly. We will work out a upper bound for $J_\pi(x)$ which will yield an ideal result. Let

$$x_1(t) = a_1 \cos t + b_1 \sin t, \quad x_2(t) = \sum_{n=2}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Then $x(t) = x_1(t) + x_2(t)$. Note that $x_1(t - \pi) \equiv -x_1(t)$. Thus $x(t - \pi) = -x_1(t) + x_2(t - \pi)$ and

$$x(t)x(t - \pi) = -x_1^2(t) + x_1(t)(x_2(t - \pi) - x_2(t)) + x_2(t)x_2(t - \pi).$$

Taking the positive part, we have the following trivial inequality

$$(x(t)x(t-\pi))_+ \le \big(x_1(t)(x_2(t-\pi)-x_2(t))\big)_+ + |x_2(t)x_2(t-\pi)|.$$

Thus

$$J_\pi(x) \le \int_0^{2\pi} \big(x_1(t)(x_2(t-\pi)-x_2(t))\big)_+ dt + \int_0^{2\pi} |x_2(t)x_2(t-\pi)|dt. \tag{4.22}$$

For the first term, we notice that

$$\int_0^{2\pi} x_1(t)(x_2(t-\pi)-x_2(t))dt = 0$$

because

$$x_2(t-\pi)-x_2(t) = -2\sum_{n=2}^{\infty} \left(a_{2n-1}\cos(2n-1)t + b_{2n-1}\sin(2n-1)t\right).$$

Thus

$$\begin{aligned}
&\int_0^{2\pi} \big(x_1(t)(x_2(t-\pi)-x_2(t))\big)_+ dt \\
&= \frac{1}{2}\int_0^{2\pi} |x_1(t)(x_2(t-\pi)-x_2(t))|dt \\
&\le \frac{1}{2}\|x_1\|_2\|x_2(\cdot-\pi)-x_2(\cdot)\|_2 \\
&= \pi(a_1^2+b_1^2)^{1/2}\left(\sum_{n=2}^{\infty}(a_{2n-1}^2+b_{2n-1}^2)\right)^{1/2}.
\end{aligned}$$

For the second term in (4.22), one has

$$\begin{aligned}
&\int_0^{2\pi} |x_2(t)x_2(t-\pi)|dt \le \|x_2\|_2\|x_2(\cdot-\pi)\|_2 \\
&= \|x_2\|_2^2 = \pi\sum_{n=2}^{\infty}(a_n^2+b_n^2).
\end{aligned}$$

Now (4.22) shows that

$$J_\pi(x) \le \hat{J}(x) := \pi\sum_{n=2}^{\infty}(a_n^2+b_n^2) + \pi(a_1^2+b_1^2)^{1/2}\left(\sum_{n=2}^{\infty}(a_{2n-1}^2+b_{2n-1}^2)\right)^{1/2}. \tag{4.23}$$

In the following, we will show that the maximum of $\hat{J}(x)$ under (4.18) is also $1/4$. This implies that $\max J_\pi \le 1/4$ and now (4.21) is clear. Let $\alpha_n = \pi^{1/2}a_n$ and $\beta_n = \pi^{1/2}b_n$, $n \in \mathbb{N}$. The constraint (4.18) reads as

$$\sum_{n=1}^{\infty} n^2(\alpha_n^2+\beta_n^2) = 1. \tag{4.24}$$

The functional $\hat{J}$ in (4.23) can be rewritten as

$$\hat{J}(x) = \hat{J}(\alpha_n, \beta_n) := \sum_{n=2}^{\infty}(\alpha_n^2 + \beta_n^2) + (\alpha_1^2 + \beta_1^2)^{1/2} \left(\sum_{n=2}^{\infty}(\alpha_{2n-1}^2 + \beta_{2n-1}^2) \right)^{1/2}. \quad (4.25)$$

In order to discuss (4.25), we have four cases. Note that $c_1 := (\alpha_1^2 + \beta_1^2)^{1/2} \in [0, 1]$.
Case 1: $c_1 = 0$. One has

$$\hat{J}(\alpha_n, \beta_n) = \sum_{n=2}^{\infty}(\alpha_n^2 + \beta_n^2) \leq \frac{1}{4} \sum_{n=2}^{\infty} n^2(\alpha_n^2 + \beta_n^2) \leq \frac{1}{4}.$$

Case 2: $c_1 = 1$. From (4.24), $\alpha_n = \beta_n = 0$ for all $n \geq 2$. Thus $\hat{J}(\alpha_n, \beta_n) = 0$.
Case 3: The coefficients (α_n, β_n) satisfy

$$c_1 \in (0, 1) \quad \text{and} \quad C := \left(\sum_{n=2}^{\infty}(\alpha_{2n-1}^2 + \beta_{2n-1}^2) \right)^{1/2} = 0. \quad (4.26)$$

In this case,

$$\hat{J}(\alpha_n, \beta_n) = \sum_{n=1}^{\infty}(\alpha_{2n}^2 + \beta_{2n}^2) \leq \frac{1}{4} \sum_{n=2}^{\infty} n^2(\alpha_n^2 + \beta_n^2) = \frac{1}{4}\left(1 - c_1^2\right) < \frac{1}{4}.$$

Case 4: The coefficients (α_n, β_n) satisfy

$$c_1 \in (0, 1) \quad \text{and} \quad C > 0. \quad (4.27)$$

Note that, in this region, the function $\hat{J}(\alpha_n, \beta_n)$ and the constraint in (4.24) are differentiable in (α_n, β_n). In the following we will find all critical values of $\hat{J}$ using the Lagrangian multiplier method. Let

$$F = \hat{J}(\alpha_n, \beta_n) - \lambda \left(\sum_{n=1}^{\infty} n^2(\alpha_n^2 + \beta_n^2) - 1 \right).$$

From

$$\frac{\partial F}{\partial \alpha_n} = 0, \quad \frac{\partial F}{\partial \beta_n} = 0, \quad (n \in \mathbb{N}),$$

we have

$$\frac{C}{c_1}\alpha_1 = 2\lambda\alpha_1, \quad (4.28)$$

$$\frac{C}{c_1}\beta_1 = 2\lambda\beta_1, \quad (4.29)$$

$$2\alpha_{2n-1} + \frac{c_1}{C}\alpha_{2n-1} = 2\lambda(2n-1)^2\alpha_{2n-1}, \quad n = 2, 3, \cdots \quad (4.30)$$

$$2\beta_{2n-1} + \frac{c_1}{C}\beta_{2n-1} = 2\lambda(2n-1)^2\beta_{2n-1}, \quad n = 2, 3, \cdots \quad (4.31)$$

$$\alpha_{2n} = \lambda(2n)^2\alpha_{2n}, \quad n = 1, 2, \cdots \quad (4.32)$$

$$\beta_{2n} = \lambda(2n)^2\beta_{2n}, \quad n = 1, 2, \cdots. \quad (4.33)$$

Since $\alpha_1^2+\beta_1^2>0$, we get from (4.28) and (4.29) that $\lambda=C/(2c_1)>0$. Since $C>0$, some coefficients of $(\alpha_{2n-1},\beta_{2n-1})$, $n\geq 2$, are non-zero. See (4.26) and (4.27). By (4.30) and (4.31), there exists some $n_0\geq 2$ such that λ satisfies

$$2\lambda(2n_0-1)^2=2+c_1/C=2+1/(2\lambda).$$

Hence

$$\lambda=\lambda(n_0):=\frac{1+\sqrt{(2n_0-1)^2+1}}{2(2n_0-1)^2}.$$

From (4.30) and (4.31) again, it is easy to see that $\alpha_{2n-1}=\beta_{2n-1}=0$ for all $n\geq 2$, $n\neq n_0$. Thus $C=(\alpha_{2n_0-1}^2+\beta_{2n_0-1}^2)^{1/2}$ and $c_1=C/(2\lambda(n_0))$.

We assert that $\alpha_{2n}=\beta_{2n}=0$ for all $n\in\mathbb{N}$. Otherwise, by (4.32) and (4.33),

$$\lambda(n_0)=1/(2m_0)^2$$

for some $m_0\in\mathbb{N}$. However, for any $n_0\geq 2$, it is standard to verify that $\lambda(n_0)$ is an irrational number. Thus we have a contradiction.

In conclusion, in this case, all critical points of $\hat{J}(\alpha_n,\beta_n)$ under (4.24) are those (α_n,β_n) such that $(\alpha_1^2+\beta_1^2)^{1/2}=C/(2\lambda(n_0))$, $(\alpha_{2n_0-1}^2+\beta_{2n_0-1}^2)^{1/2}=C$ and the other (α_n,β_n) are zero, where $n_0\geq 2$ and $C=C(n_0)>0$ is determined by (4.24), i.e.,

$$C^2\left(1/(4\lambda(n_0)^2)+(2n_0-1)^2\right)=1.$$

Now the critical values of $\hat{J}$ are

$$\begin{aligned}\hat{J}(\alpha_n,\beta_n)&=(\alpha_{2n_0-1}^2+\beta_{2n_0-1}^2)+(\alpha_1^2+\beta_1^2)^{1/2}(\alpha_{2n_0-1}^2+\beta_{2n_0-1}^2)^{1/2}\\&=(1/(2\lambda(n_0))+1)\,C^2=\frac{1/(2\lambda(n_0))+1}{1/(4\lambda(n_0)^2)+(2n_0-1)^2}\\&=\lambda(n_0),\quad n_0=2,3,\cdots\end{aligned}$$

As a function of $n_0\geq 2$, it is easy to verify that $\lambda(n_0)$ is decreasing in n_0. Thus

$$\lambda(n_0)\leq\lambda(2)=\frac{1+\sqrt{10}}{18}<\frac{1}{4}\quad\text{for all } n_0\geq 2.$$

We conclude that the maximum value of $\hat{J}(\alpha_n,\beta_n)$ under constraint (4.24) is attained in Case 1 and $\max\hat{J}=1/4$. Thus the maximum in (4.21) is also $1/4$ and the maximizers are $x(t)=a\cos 2t+b\sin 2t$, $a^2+b^2=1/(4\pi)$. □

Another maximal problem is as follows. Let $\gamma\in[1,\infty]$, $T>0$ and $r\in\mathbb{R}$. For any $x\in\tilde{H}_T^1$, it is necessary that

$$\|x(\cdot)-x(\cdot-r)\|_\gamma\leq C\|x'\|_2$$

for some constant $C>0$. Let us use $M(\gamma,T,r)$ to denote the optimal constant C in the inequality above. That is,

$$M(\gamma,T,r)=\sup_{x\in\tilde{H}_T^1,\ x\neq 0}\frac{\|x(\cdot)-x(\cdot-r)\|_\gamma}{\|x'\|_2}\ (<\infty). \tag{4.34}$$

Obviously, $M(\gamma, T, 0) = 0$.

Some properties of $M(\gamma, T, r)$ are as follows.

Proposition 4.2 (i) *As a function of* $r \in \mathbb{R}$, $M(\gamma, T, r)$ *is even in* r *and is* T*-periodic in* r. *Moreover,* $M(\gamma, T, r)$ *is globally* $\frac{1}{2}$*-Hölder continuous in* r.

(ii) $M(\gamma, kT, kr) = k^{1/2+1/\gamma} M(\gamma, T, r)$ *for all* $k > 0$.

(iii) $M(\gamma, T, r) > 0$ *for all* $r \neq lT$, $l \in \mathbb{Z}$.

(iv) $M(\infty, T, r) = \frac{T^{1/2}}{\pi} \left(A(0) - A(2\pi r/T)\right)^{1/2}$.

(v) $M(2, T, r) = \frac{T}{\pi} |\sin(\pi r/T)|$.

Proof. We need only to prove (iv) and (v) for the case $T = 2\pi$. Let $x(t)$ be as in (4.17) and (4.18). Then $y(t) = x(t) - x(t - r)$ has the Fourier expansion

$$y(t) = \sum_{n=1}^{\infty} (a_n(r) \cos nt + b_n(r) \sin nt), \tag{4.35}$$

where

$$a_n(r) = a_n(1 - \cos nr) + b_n \sin nr, \quad b_n(r) = -a_n \sin nr + b_n(1 - \cos nr). \tag{4.36}$$

(iv) In order to maximize $\|y\|_\infty$ under condition (4.18), we notice that for any given $x \in \tilde{H}^1_{2\pi}$ with (4.18) satisfied, $\|y\|_\infty = |y(t_0)|$ for some t_0. After translating $x(t)$, we may assume, without loss of generality, that $t_0 = 0$. Thus we need to maximize $|y(0)|$ under (4.18). Note that

$$y(0) = \sum_{n=1}^{\infty} a_n(r) = \sum_{n=1}^{\infty} (a_n(1 - \cos nr) + b_n \sin nr).$$

By the Hölder inequality and (4.18), we have

$$\begin{aligned} \|y\|_\infty &= |y(0)| = \left| \sum_{n=1}^{\infty} \left(n a_n \frac{1 - \cos nr}{n} + n b_n \frac{\sin nr}{n} \right) \right| \\ &\leq \left(\sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{(1 - \cos nr)^2 + \sin^2 nr}{n^2} \right)^{1/2} \\ &= \left(2(A(0) - A(r))/\pi\right)^{1/2}. \end{aligned}$$

Hence $M(\infty, 2\pi, r) \leq \left(2(A(0) - A(r))/\pi\right)^{1/2}$. Next let us take $x(t)$ as

$$a_n = \beta \frac{1 - \cos nr}{n^2}, \quad b_n = \beta \frac{\sin nr}{n^2},$$

where $\beta > 0$ is so that $\|x'\|_2 = 1$, i.e., $2\pi\beta^2(A(0) - A(r)) = 1$. (Here we assume that $r \neq 2l\pi$, $l \in \mathbb{Z}$.) Then one has

$$\|y\|_\infty \geq y(0) = 2\beta(A(0) - A(r)) = \left(2(A(0) - A(r))/\pi\right)^{1/2}.$$

This proves the equality required.

(v) First we notice that $|\sin(nx)|/n \le |\sin x|$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Thus

$$\sup_{n\in\mathbb{N}} \frac{|\sin(nr/2)|}{n} = |\sin(r/2)|.$$

Now it follows from (4.35) and (4.36) that

$$\begin{aligned}\|y\|_2 &= \left(\pi\sum_{n=1}^{\infty}(a_n^2(r)+b_n^2(r))\right)^{1/2}\\ &= \left(\pi\sum_{n=1}^{\infty}n^2(a_n^2+b_n^2)\cdot 4\left(\frac{|\sin(nr/2)|}{n}\right)^2\right)^{1/2}\\ &\le 2\sup_{n\in\mathbb{N}}|\sin(nr/2)|/n\cdot\left(\pi\sum_{n=1}^{\infty}n^2(a_n^2+b_n^2)\right)^{1/2}\\ &= 2|\sin(r/2)|.\end{aligned}$$

This proves that $M(2,2\pi,r) \le 2|\sin(r/2)|$. The converse inequality can be proved by taking $x(t) = (\sin t)/\sqrt{\pi}$. □

Remark 4.3 By the Hölder inequality, $M(\gamma,T,r)$, $2<\gamma<\infty$, can be estimated using $M(\infty,T,r)$, and $M(\gamma,T,r)$, $1\le\gamma<2$, can be estimated using $M(2,T,r)$. For example, $M(1,T,r) \le (T^{3/2}/\pi)|\sin(\pi r/T)|$.

In the following theorem, we will prove that the constants $\tilde{L}(\gamma,T,r)$ and $M(\gamma,T,r)$ play an important role in the non-degeneracy of (4.1).

Theorem 4.4 *Let $\alpha\in[1,\infty]$, $T>0$, $r\in\mathbb{R}$, and $q\in L^\alpha(\mathbb{S}_T)$, $q\succ 0$. If $q(t)$ satisfies*

$$\|q\|_\alpha < \frac{1}{1/\tilde{L}(\alpha^*,T,r)+M(\infty,T,r)M(\alpha^*,T,r)/4} =: P(\alpha,T,r), \tag{4.37}$$

then (4.1) is non-degenerate.

Proof. Suppose that (4.1) has a non-zero T-periodic solution x. Write $x=\tilde{x}+\bar{x}$. By (4.1),

$$-\tilde{x}''(t) = q(t)\tilde{x}(t-r)+\bar{x}q(t).$$

Multiplying the equation by $\tilde{x}(t)-\bar{x}$ and then integrating over $\mathbb{S}_T$, we have

$$\begin{aligned}\|\tilde{x}'\|_2^2 &= -\int_{\mathbb{S}_T}\tilde{x}''(t)\tilde{x}(t) = \int_{\mathbb{S}_T}\left(q(t)\tilde{x}(t-r)+\bar{x}q(t)\right)\left(\tilde{x}(t)-\bar{x}\right)\\ &= \int_{\mathbb{S}_T}q(t)\tilde{x}(t)\tilde{x}(t-r)+\left(\bar{x}\int_{\mathbb{S}_T}q(t)(\tilde{x}(t)-\tilde{x}(t-r))-\bar{x}^2\int_{\mathbb{S}_T}q(t)\right)\\ &=: \mathrm{I}+\mathrm{II}.\end{aligned} \tag{4.38}$$

Notice that $q \succ 0$. The term I can be estimated as follows:

$$\begin{aligned} \mathrm{I} &= \int_{\mathbb{S}_T} q(t)(\tilde{x}(t)\tilde{x}(t-r))_+ - \int_{\mathbb{S}_T} q(t)(\tilde{x}(t)\tilde{x}(t-r))_- \\ &\leq \int_{\mathbb{S}_T} q(t)(\tilde{x}(t)\tilde{x}(t-r))_+ \leq \|q\|_\alpha \|(\tilde{x}(\cdot)\tilde{x}(\cdot - r))_+\|_{\alpha^*}. \end{aligned} \tag{4.39}$$

For the term II, we use the elementary inequality $bx - ax^2 \leq b^2/(4a)$ to deduce

$$\mathrm{II} \leq B^2/(4\|q\|_1), \quad B = \int_{\mathbb{S}_T} q(t)(\tilde{x}(t-r) - \tilde{x}(t)).$$

Write $y(t) = \tilde{x}(t) - \tilde{x}(t-r)$. Then

$$|B| = \left| \int_{\mathbb{S}_T} q(t)y(t) \right| \leq \|q\|_1 \|y\|_\infty, \quad |B| \leq \|q\|_\alpha \|y\|_{\alpha^*}.$$

Thus

$$\mathrm{II} \leq \|q\|_\alpha \|y\|_\infty \|y\|_{\alpha^*}/4. \tag{4.40}$$

From (4.38)–(4.40), we have

$$\|\tilde{x}'\|_2^2 \leq \|q\|_\alpha \left(\|(\tilde{x}(\cdot)\tilde{x}(\cdot - r))_+\|_{\alpha^*} + \|y\|_\infty \|y\|_{\alpha^*}/4\right). \tag{4.41}$$

Now we use the constants $\tilde{L}(\gamma, T, r)$ and $M(\gamma, T, r)$ to obtain the estimate for (4.41)

$$\|\tilde{x}'\|_2^2 \leq \|q\|_\alpha \left(1/\tilde{L}(\alpha^*, T, r) + M(\infty, T, r)M(\alpha^*, T, r)/4\right) \|\tilde{x}'\|_2.$$

If (4.37) is satisfied, we have $\|\tilde{x}'\|_2 = 0$ and $x \equiv \bar{x}$. From (4.1), $\bar{x}$ satisfies $q(t)\bar{x} = 0$ for a.e. t. Hence $\bar{x} = 0$ and $x = 0$, which is a contradiction. □

Remark 4.5 Let us make a comparison between conditions (4.37) and (3.5). By (4.12), $\|(\tilde{x}(\cdot)\tilde{x}(\cdot - r))_+\|_{\alpha^*} \leq \|\tilde{x}\|_{2\alpha^*}^2$. One has also the trivial inequalities $\|y\|_\infty = \|\tilde{x}(\cdot) - \tilde{x}(\cdot - r)\|_\infty \leq 2\|\tilde{x}\|_\infty$ and $\|y\|_{\alpha^*} \leq 2\|\tilde{x}\|_{\alpha^*}$. Thus $\|y\|_\infty \|y\|_{\alpha^*}/4 \leq \|\tilde{x}\|_\infty \|\tilde{x}\|_{\alpha^*}$. Hence estimate (4.41) is always better than estimate (3.8) in the delay case. Consequently, condition (4.37) in Theorem 4.4 is better than condition (3.5) in Theorem 3.4.

Let us consider $P(\alpha, T, r)$, where $\alpha = 1$. By Propositions 4.1(iv) and 4.2(iv),

$$1/\tilde{L}(\infty, T, r) = \frac{T}{4\pi^2}(A(0) + A(2\pi r/T)),$$

$$(M(\infty, T, r))^2/4 = \frac{T}{4\pi^2}(A(0) - A(2\pi r/T)).$$

Thus

$$P(1, T, r) \equiv \frac{2\pi^2}{A(0)T} = \frac{12}{T} = L(\infty, T),$$

and Theorem 4.4 asserts the non-degeneracy of (4.1) for all r if $\|q\|_1 < 12/T$. Notice that if $r = lT$, $l \in \mathbb{Z}$, we arrive at ODEs and the corresponding optimal condition is $\|q\|_1 \le 16/T$. If $r \neq lT$, one can notice from the proof of Propositions 4.1(iv) and 4.2(iv) that the minimizers for $\tilde{L}(\infty, 2\pi, r)$ and the maximizers of $M(\infty, 2\pi, r)$ are different functions. Thus for any $\tilde{x} \in \tilde{H}^1_{2\pi}$, $\tilde{x} \neq 0$, one has the strict inequality

$$\|(\tilde{x}(\cdot)\tilde{x}(\cdot - r))_+\|_\infty + \|y\|_\infty^2/4 < (T/12)\|\tilde{x}'\|_2^2,$$

where $y(t) = \tilde{x}(t) - \tilde{x}(t-r)$. Hence, if $\|q\|_1 \le 12/T$ and $\tilde{x} \neq 0$, one has from (4.41) that

$$\|\tilde{x}'\|_2^2 < \|q\|_1(T/12)\|\tilde{x}'\|_2^2 \le \|\tilde{x}'\|_2^2.$$

We have a contradiction. That is to say, $\tilde{x} = 0$ if $\|q\|_1 \le 12/T$.

Corollary 4.6 *There holds the equality $P(1, T, r) = 12/T$ for all r. Hence, for semilinear delay differential equations, the existence condition in Corollary 3.5(ii) can be weakened as $\|b\|_1 \le 12/T$.*

The next result shows that when $q(t)$ has the period less than T, Theorem 4.4 can be improved in some cases. This result will yield an existence result for DDEs which illustrates the difference between ODEs and DDEs. See Example 4.10 below.

Theorem 4.7 *Let $\alpha \in [1, \infty]$, $T > 0$, $r \in (0, T)$, and $q \in L^\alpha(\mathbb{S}_T)$, $q \succ 0$. If $q(t)$ is also r-periodic and*

$$\|q\|_\alpha < \tilde{L}(\alpha^*, T, r), \tag{4.42}$$

then (4.1) is non-degenerate.

Proof. If $q(t)$ is also r-periodic, we have

$$\int_{\mathbb{S}_T} q(t)(\tilde{x}(t) - \tilde{x}(t-r)) = \int_{\mathbb{S}_T} q(t)\tilde{x}(t) - \int_{\mathbb{S}_T} q(t+r)\tilde{x}(t) = 0.$$

Thus the term II in (4.38) is $\mathrm{II} = -\bar{x}^2 \int_{\mathbb{S}_T} q(t) \le 0$. It follows from (4.38) and (4.39) that

$$\|\tilde{x}'\|_2^2 \le \|q\|_\alpha \|\tilde{x}'\|_2^2/\tilde{L}(\alpha^*, T, r).$$

Now we can argue as in the proof of Theorem 4.4 to obtain the non-degeneracy condition (4.42). □

We will present an application of Theorem 4.7 to periodic solutions of semilinear DDEs. In the theory of periodic solutions of DDEs, it is of particular interest to study the case where the period T is a multiple of the delay r. For example, when $r = T/4$, the T-periodic solutions of autonomous systems of DDEs

$$x'(t) = g(x(t - T/4))$$

can be studied using the variational method [3, 7].

In the following we will give a result for non-autonomous second order DDE

$$x''(t) + g(t, x(t - T/2)) = h(t), \tag{4.43}$$

where the delay r is $T/2$. The result obtained will reveal some difference between ODEs and DDEs.

In order to state our result for DDEs, we recall briefly the proof for the optimality of $4K(2\alpha^*, T)$ for ODEs in [17, 15]. Let $\alpha \in (1, \infty]$. On the one hand, it is proved in [15] that if $q \in L^\alpha(\mathbb{S}_T)$ satisfies $q \succ 0$ and $\|q\|_{\alpha,[0,T]} := \|q\|_{L^\alpha[0,T]} < 4K(2\alpha^*, T)$, then

$$x'' + q(t)x = 0, \quad t \in \mathbb{R}, \tag{4.44}$$

is non-degenerate with respect to the T-periodic boundary condition

$$x(0) = x(T), \;\; x'(0) = x'(T). \tag{4.45}$$

On the other hand, for $1 \le \gamma < \infty$ and $S > 0$, we define

$$E_\gamma(v) = \int_0^v \frac{dv}{(1 - v^\gamma)^{1/2}}, \quad v \in [0, 1].$$

Let $u_{\gamma,S} : [0, S] \to \mathbb{R}$ be such that $u_{\gamma,S}(S - t) \equiv u_{\gamma,S}(t)$ and

$$u_{\gamma,S}(t) = E_\gamma^{-1}\big(2E_\gamma(1)t/S\big) \quad \text{for} \quad t \in [0, S/2].$$

Define then $x_{\alpha,S},\ q_{\alpha,S} : [0, S] \to \mathbb{R}$, $\alpha \in (1, \infty]$, by

$$x_{\alpha,S}(t) \equiv u_{2\alpha^*,S}(t), \quad q_{\alpha,S}(t) = S^{-1/\alpha} K(2\alpha^*, S)(x_{\alpha,S}(t))^{2/(\alpha-1)},$$

where for the case $\alpha = \infty$, $q_{\infty,S}(t) \equiv K(2, S) = (\pi/S)^2$. Then $x(t) = x_{\alpha,S}(t)$ and $q_{\alpha,S}(t)$ satisfy

$$x''(t) + q_{\alpha,S}(t)x(t) = 0, \quad t \in [0, S],$$
$$x(0) = x(S) = 0,$$
$$\|q_{\alpha,S}\|_{\alpha,[0,S]} = K(2\alpha^*, S).$$

See [17].

Now let $q(t)$ in (4.44) be the function $\hat{q}_{\alpha,T}(t)$ which is the $T/2$-periodic extension of $q_{\alpha,T/2}(t)$. Then $\hat{q}_{\alpha,T} \in L^\alpha(\mathbb{S}_T)$ and $\hat{q}_{\alpha,T} \succ 0$, $\|\hat{q}_{\alpha,T}\|_{\alpha,[0,T]} = 4K(2\alpha^*, T)$. Equation (4.44) has at least one non-zero T-periodic solution $x(t) = \hat{x}_{\alpha,T}(t)$ defined by

$$\hat{x}_{\alpha,T}(t) = \begin{cases} x_{\alpha,T/2}(t), & t \in [0, T/2], \\ -x_{\alpha,T/2}(t - T/2), & t \in [T/2, T]. \end{cases}$$

This proves the optimality of $4K(2\alpha^*, T)$ for the T-periodic problem of ODEs.

It is a remarkable fact that the potential $\hat{q}_{\alpha,T}(t)$ is actually $T/2$-periodic. In conclusion, for $q \in L^\alpha(\mathbb{S}_{T/2})$, $q \succ 0$, if we view $q(t)$ as a T-periodic potential and study the non-degeneracy of (4.44) with respect to (4.45), then the optimal bound on $\|q\|_{\alpha,[0,T]}$ is also $4K(2\alpha^*, T)$.

This observation, together with Theorem 2.1, can yield the following result. Consider the ODE

$$x''(t) + g(t, x(t)) = h(t), \tag{4.46}$$

where the nonlinearity $g(t, x)$ is $T/2$-periodic in t and satisfies the semilinearity condition (1.2) for some $a,\ b \in L^\alpha(\mathbb{S}_{T/2})$, $1 < \alpha \le \infty$. However, the forcing $h(t)$ is assumed to be T-periodic. We will seek T-periodic solutions of (4.46).

Corollary 4.8 *Assume that $g(t, x)$ is $T/2$-periodic in t and is semilinear in x with $a,\ b \in L^\alpha(\mathbb{S}_{T/2})$ for some $\alpha \in (1, \infty]$. If $a \succ 0$ and*

$$\|b\|_{\alpha,[0,T]} < 4K(2\alpha^*, T), \tag{4.47}$$

then, for any $h \in L^1(\mathbb{S}_T)$, equation (4.46) has at least one T-periodic solution. Moreover, condition (4.47) is optimal to guarantee the existence of T-periodic solutions of (4.46) for any $h \in L^1(\mathbb{S}_T)$.

Proof. We need only to prove the optimality of (4.47) for T-periodic solutions of nonlinear ODEs. To this end, let $g(t, x) = \hat{q}_{\alpha,T}(t)x$ and consider an inhomogeneous linear equation

$$x'' + \hat{q}_{\alpha,T}(t)x = h(t). \tag{4.48}$$

Then $a = b = \hat{q}_{\alpha,T} \in L^\alpha(\mathbb{S}_{T/2})$ and $\|b\|_{\alpha,[0,T]} = 4K(2\alpha^*, T)$. The Fredholm principle shows that (4.48) will have T-periodic solutions only if $h(t)$ satisfies

$$\int_{\mathbb{S}_T} h(t)\hat{x}_{\alpha,T}(t)dt = 0.$$

This proves the optimality. □

Now we consider the DDE (4.43), where $g(t, x)$, α, a, b are as in Corollary 4.8. By Theorems 2.1 and 4.7, we have

Corollary 4.9 *If $a \succ 0$ and*

$$\|b\|_{\alpha,[0,T]} < \tilde{L}(\alpha^*, T, T/2), \tag{4.49}$$

then, for any $h \in L^1(\mathbb{S}_T)$, equation (4.43) has at least one T-periodic solution.

Example 4.10 Let $\alpha = \infty$. The optimal condition (4.47) for ODE (4.46) is

$$\|b\|_\infty < (2\pi/T)^2.$$

By Proposition 4.1(v), $\tilde{L}(1, T, T/2) = 4(2\pi/T)^2$. Now condition (4.49) for DDE (4.43) is

$$\|b\|_\infty < 4(2\pi/T)^2. \tag{4.50}$$

By the discussion in Remark 3.2, if $b(t) = b_0 := 4(2\pi/T)^2$, equation

$$x''(t) + b_0 x(t - T/2) = 0$$

has a non-zero T-periodic solution denoted by $\psi(t)$. If $h \in L^1(\mathbb{S}_T)$ is such that the equation

$$x''(t) + b_0 x(t - T/2) = h(t) \tag{4.51}$$

has a T-periodic solution $x(t)$, then one has

$$\begin{aligned}\int_{\mathbb{S}_T} h(t)\psi(t)dt &= \int_{\mathbb{S}_T} x''(t)\psi(t)dt + \int_{\mathbb{S}_T} b_0 x(t-T/2)\psi(t)dt \\ &= \int_{\mathbb{S}_T} x(t)\psi''(t)dt + \int_{\mathbb{S}_T} b_0 x(t)\psi(t+T/2)dt \\ &= \int_{\mathbb{S}_T} x(t)(\psi''(t) + b_0\psi(t-T/2))dt = 0.\end{aligned}$$

This shows that it is impossible for (4.51) to have T-periodic solutions for all $h \in L^1(\mathbb{S}_T)$. Thus condition (4.50) for DDEs is also optimal.

Acknowledgment

The authors would like to thank Professor Rafael Ortega for his careful reading of the manuscript and valuable suggestions.

References

[1] A. Fonda and P. Habets, Periodic solutions of asymptotically positively differential equations, *J. Differential Equations* **81** (1989), 68–97.

[2] A. Fonda and J. Mawhin, Quadratic forms, weighted eigenfunctions and boundary value problems for non-linear second order differential equations, *Proc. Royal Soc. Edinburgh Sect. A* **112** (1989), 145–153.

[3] Z. Guo and J. Yu, Multiplicity results for periodic solutions of delay differential equations via critical point theory, *J. Differential Equations* **218** (2005), 15–35.

[4] P. Habets and G. Metzen, Existence of periodic solutions of Duffing equations, *J. Differential Equations* **78** (1989), 1–32.

[5] M. Jiang, Solutions of a resonant elliptic equation, *Preprint*, 2005.

[6] M. Kunze and R. Ortega, On the number of solutions to semilinear boundary value problems, *Advanced Nonlinear Stud.* **4** (2004), 237–249.

[7] J. Li and X. He, Proof and generalization of Kaplan-Yorke's conjecture on periodic solutions of differential delay equations, *Sci. China Ser. A* **42** (1999), 957–964.

[8] W. Magnus and S. Winkler, *Hill's Equations*, corrected reprint of 1966 edition, Dover, New York, 1979.

[9] R. Ortega and M. Zhang, Some optimal bounds for bifurcation values of a superlinear periodic problem, *Proc. Royal Soc. Edinburgh Sect. A* **135** (2005), 119–132.

[10] D. B. Qian and X. Y. Li, Periodic solutions for ordinary differential equations with sublinear impulsive effects, *J. Math. Anal. Appl.* **303** (2005), 288–303.

[11] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl. (4)* **110** (1976), 353–372.

[12] P. Yan, Nonresonance of one-dimensional p-Laplacian with regular restoring, *J. Math. Anal. Appl.* **285** (2003), 141–154.

[13] M. Zhang, Nonresonance conditions for asymptotically positively homogeneous differential systems: The Fucik spectrum and its generalization, *J. Differential Equations* **145** (1998), 332–366.

[14] M. Zhang, Nonuniform nonresonance of semilinear differential equations, *J. Differential Equations* **166** (2000), 33–50.

[15] M. Zhang, Certain classes of potentials for p-Laplacian to be non-degenerate, *Math. Nachr.* **278** (2005), 1823–1836.

[16] M. Zhang, Periodic solutions of equations of Emarkov-Pinney type, *Advanced Nonlinear Stud.* **6** (2006), 57–67.

[17] M. Zhang and W. Li, A Lyapunov-type stability criterion using L^α norms, *Proc. Amer. Math. Soc.* **130** (2002), 3325–3333.