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# Symmetry Results For Solutions of a Semilinear Nonhomogeneous Problem

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**Abstract**

In this paper we show the axial symmetry of a particular type of solutions of the following semilinear nonhomogeneous problem

$$\begin{cases} -\Delta u = u_+^p + \lambda u - s\varphi_1(x) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

where $B$ is the unit ball of $\mathbb{R}^N$, $N \geq 3$, $p \in \left(1, \frac{N+2}{N-2}\right)$, $s > 0$, $\lambda < 0$, $u_+ = u$ if $u \geq 0$, $u_+ = 0$ if $u < 0$ and $\varphi_1(x) > 0$ is the eigenfunction of $-\Delta$ corresponding to the first eigenvalue $\lambda_1$, with Dirichlet boundary conditions. The solutions considered exhibit one or two peaks and concentrate, as the parameter $s$ goes to $+\infty$.

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# 1 Introduction

The purpose of this paper is to study symmetry properties of solutions of the problem

$$\begin{cases} -\Delta u = u_+^p + \lambda u - s\varphi_1(x) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.1}$$

where $B$ is the unit ball of $\mathbb{R}^N$, $N \geq 3$, $p \in \left(1, \frac{N+2}{N-2}\right)$, $s > 0$, $\lambda < 0$, $u_+ = u$ if $u \geq 0$, $u_+ = 0$ if $u < 0$ and $\varphi_1(x) > 0$ is the eigenfunction of $-\Delta$ corresponding to the first eigenvalue $\lambda_1$, with Dirichlet boundary conditions.

Problem (1.1) belongs to the class of semilinear elliptic problems of the type

$$\begin{cases} -\Delta u = g(u) - s\varphi_1(x) + \xi(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, $\xi(x)$ is a given function and

$$\lim_{t \to +\infty} \frac{g(t)}{t} = \mu > \lambda_1, \qquad \lim_{t \to -\infty} \frac{g(t)}{t} = \nu < \lambda_1.$$

These problems are known in the literature as Ambrosetti-Prodi type problems, since they were studied in the pioneering paper [1] where a first multiplicity result was obtained independently on the interaction between the nonlinearity and the spectrum of the laplacian. Since then, these kinds of problems have attracted the attention of many mathematicians who have studied mainly the multiplicity or sometimes the exact multiplicity of the solutions under various assumptions on the nonlinearity when the parameter $s$ is large. We refer the interested reader to the introduction of [2] for quite an exhaustive description and motivations of existing results.

In connection with (1.2), a conjecture was formulated by Lazer and McKenna, stating that if $\mu = +\infty$ then, as $s \to +\infty$, the number of solutions of (1.2) should become unbounded. A first step in proving this conjecture was made in [2] where, with an interesting computer assisted proof, it was proved, for a particular "big" value of the parameter $s$, that equation (1.2) has at least four solutions if $\Omega$ is the unit square in $\mathbb{R}^2$. Moreover these solutions can be numerically computed; so one can visualize their shape and their symmetry properties.

Apart from the interest of the result in itself, the paper [2] had the merit to attract again the attention of other mathematicians on the above mentioned conjecture which was finally proved by Dancer and Yan in the recent papers [4] and [5]. In particular, in [5] the problem (1.1) is studied in general bounded domains and, for any positive integer $k$ a family of solutions with $k$ peaks is found, as $s \to +\infty$. We will describe more precisely these solutions later. Moreover in [4] and [5], the asymptotic behaviuor, as $s \to +\infty$, of the mountain pass solution is studied, proving in particular that the single peak of this solution approaches the boundary of the domain. A result of this type had already been obtained in [7] in the ball, proving so that the mountain pass solution is not radial. The paper [7] was also motivated by [2] because the numerical approximation of the solutions helped the attentive reader to guess the geometrical properties of the solutions and their Morse index.

As far as we know these are the only available results about a qualitative study of solutions of problems of the type (1.2). Indeed these are the questions that motivate our results which show that the axial symmetry of the solutions with one or two peaks found in [5], which are namely solutions concentrating, as $s \to +\infty$ around the maximum point of the eigenfunction $\varphi_1(x)$, that, in the case of a ball, is the center. The precise results will be stated in the next section, since we need further notations. Let us remark explicitly that the 1-peak solution we study is not the mountain-pass solution which, as mentioned before, has its peak near the boundary. However, the axial symmetry and the monotonicity with respect to the angular coordinate (foliated Schwartz symmetry) of the mountain pass solution in the case of the nonlinearities considered in [4] and [5] are a consequence of a general result of [10] about solutions of Morse index less than or equal to 1. Note that the case considered in [5], which is problem (1.1) is not really included in [10] because the nonlinearity is not strictly convex everywhere, but can be easily derived by the results of [10] as it is shown in Section 2. The outline of the paper is the following: in Section 2 we state our symmetry results and give some preliminaries. Section 3 and Section 4 are devoted to the proofs.

## 2 Statement of the results

In this paper we will consider solutions of (1.1) of the form

$$u = -\frac{s}{\lambda_1 - \lambda}\varphi_1 + v \tag{2.1}$$

where $-\frac{s}{\lambda_1-\lambda}\varphi_1$ is the minimal negative solution of (1.1) and $v$ satisfies

$$\begin{cases} -\Delta v = \left(v - \dfrac{s}{\lambda_1 - \lambda}\varphi_1\right)_+^p + \lambda v, & \text{in } B \\ v = 0 \ \text{ on } \partial B. \end{cases} \tag{2.2}$$

Solutions of type (2.1) have been recently found by Dancer and Yan in [5]. Let $\varepsilon^2 = \left(\frac{s}{\lambda_1-\lambda}\right)^{1-p}$. Then $\varepsilon \to 0$ as $|s| \to +\infty$. For any solution $v$ of (2.2), the function $u = \frac{\lambda_1-\lambda}{s}v$ is a solution of

$$\begin{cases} -\varepsilon^2\Delta u - \varepsilon^2\lambda u = (u - \varphi_1)_+^p \ \text{ in } B \\ u = 0 \ \text{ on } \partial B. \end{cases} \tag{2.3}$$

Dancer and Yan in [5] use a reduction method to construct, for small values of $\varepsilon$, solutions $u_\varepsilon$ of (2.3) which have $k$ sharp peaks near the maximum points of $\varphi_1(x)$, for any positive integer $k$. Without loss of generality we assume that $\max_{x\in B}\varphi_1(x)=1$. In our case, since $B$ is a ball, $\varphi_1(x)$ has exactly one global maximum point at the center, hence the solutions $u_\varepsilon$ of (2.3) have all peaks near this point. In order to describe more precisely what Dancer and Yan obtained, let us consider, the following problem, for $N\geq 3$:

$$\begin{cases} -\Delta U=(U-1)_+^p,\ U>0 \quad \text{in } \mathbb{R}^N, \\ U(0)=\max\limits_{y\in\mathbb{R}^N} U(y) \\ U(y)\to 0 \quad \text{as } |y|\to+\infty. \end{cases} \tag{2.4}$$

It is known that problem (2.4) has a unique solution $U$ in $D^{1,2}(\mathbb{R}^N)=\{\varphi\in L^{2^*}(\mathbb{R}^N):|\nabla\varphi|\in \mathrm{L}^2(\mathbb{R}^N)\}$ which is radially symmetric. For any $x\in\mathbb{R}^N$, let $U_{\varepsilon,x}(y)=U(\frac{y-x}{\varepsilon})$ and $\mathcal{P}_{\varepsilon,B}U_{\varepsilon,x}$ be the solution of

$$\begin{cases} -\varepsilon^2\Delta v=(U_{\varepsilon,x}-1)_+^p \ \text{ in } B \\ v=0 \ \text{ in } \partial B. \end{cases}$$

We have (see Lemma 3.1 in [5])

**Proposition 2.1** *For any $x\in B$, let $\psi_{\varepsilon,x}:=U_{\varepsilon,x}-\mathcal{P}_{\varepsilon,B}U_{\varepsilon,x}$. Then*

$$\psi_{\varepsilon,x}=(c_0+o(1))\varepsilon^{N-2}H(y,x) \tag{2.5}$$

*where $c_0>0$ is a constant, $H(y,x)$ is the regular part of the Green's function, and $o(1)\to 0$ as $\varepsilon\to 0$.*

**Proposition 2.2** *Let $U$ be the solution of (2.4). Then $U$ is nondegenerate, that is, the kernel of the operator $-\Delta v-p(U-1)_+^{p-1}v$ in $D^{1,2}(\mathbb{R}^N)$ is spanned by $\{\frac{\partial U}{\partial x_1},\dots,\frac{\partial U}{\partial x_N}\}$.*

For the proof we refer to [8].

The main existence result proved by Dancer and Yan in [5] is the following:

**Theorem 2.1** *Suppose that $N\geq 3$ and let $k>0$ be an integer. Then there is an $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0]$, (2.3) has a solution of the form*

$$u_\varepsilon=\sum_{j=1}^{k}\mathcal{P}_{\varepsilon,B}U_{\varepsilon,x_{\varepsilon,j}}+\omega_\varepsilon \tag{2.6}$$

*where $\omega_\varepsilon\in H_0^1(B)$ satisfies*

$$\int_B\varepsilon^2|D\omega_\varepsilon|^2=o(\varepsilon^N)\quad \text{as } \varepsilon\to 0 \tag{2.7}$$

*$\frac{|x_{\varepsilon,i}-x_{\varepsilon,j}|}{\varepsilon}\to+\infty$, for $i\neq j$, $x_{\varepsilon,j}\to x_j\in B$ with $\varphi_1(x_j)=\max_{z\in B}\varphi_1(z)=\varphi_1(0)$.*

Moreover, Dancer and Yan in [5] proved that the mountain-pass solution $u_s$ of the problem (1.1) is not radially symmetric. Let us observe that, since the mountain-pass solution $u_s$ has Morse index equal to one, we can apply the symmetry result proved in [10] and obtain the result described below. Let $P$ be a maximum point of $u_s$ which lies in the interior of $B$ and near the boundary (see [5]) and denote by $r_P$ the axis $\vec{OP}$ passing through the origin and $P$, by $T$ any $(N-1)$-dimensional hyperplane passing through the origin and by $\nu_T$ the normal to $T$, directed towards the half-space containing $P$, in case $T$ does not pass through the axis $r_P$. We have

**Theorem 2.2** [10] *(i) $u_s$ is axially symmetric with respect to the axis $r_P$;*
*(ii) $u_s$ is never symmetric with respect to any hyperpalne $T$ not passing through $r_P$;*
*(iii) all the critical points of $u_s$ belong to the symmetry axis $r_P$; in particular all the maximum points lie on the semiaxis to which $P$ belongs and*

$$\frac{\partial u_s}{\partial \nu_T}(x) > 0 \ \forall x \in B \cap T$$

*for every hyperplane $T$ not passing through $r_P$.*

What about solutions with $k$ peaks? Let us observe that if $u_s$ has more than one peak then arguing as in [6] it is possible to prove that its Morse index must be greater than one.

In this paper we consider the case of solutions of the form (2.6) with one or two peaks.

**Theorem 2.3** *Let $u_\varepsilon$ be a family of solutions of (2.3) with one peak $P_\varepsilon \in B$. Then:*
*(i) for $\varepsilon$ small, $u_\varepsilon$ is symmetric with respect to any hyperplane passing through the axis $r_{P_\varepsilon}$ connecting the origin with the point $P_\varepsilon$*
*(ii) if $P_\varepsilon$ is the origin, then $u_\varepsilon$ is radially symmetric.*
*Moreover,*
*(iii) if*

$$\frac{|P_\varepsilon|}{\varepsilon} \to l \in \mathbb{R}^+ \ \text{ or } \frac{|P_\varepsilon|}{\varepsilon} \to +\infty \ \text{ as } \varepsilon \to 0 \tag{2.8}$$

*all the critical points of $u_\varepsilon$ belong to the symmetry axis $r_{P_\varepsilon}$ and*

$$\frac{\partial u_\varepsilon}{\partial \nu_T}(x) > 0 \ \forall x \in B \cap T \tag{2.9}$$

*where $T$ is any hyperplane passing through the origin but not containing $P_\varepsilon$ and $\nu_T$ is the normal to $T$, directed towards the half-space containing $P_\varepsilon$.*

**Remark** In the case (iii) when

$$\frac{|P_\varepsilon|}{\varepsilon} \to 0 \text{ as } \varepsilon \to 0$$

we expect that $P_\varepsilon$ coincides with the origin for $\varepsilon$ small and hence $u_\varepsilon$ is radially symmetric by (iii). However we have not been able to prove this, so far.

For solutions with two peaks we have the following result

**Theorem 2.4** *Let $u_\varepsilon$ be a family of solutions of (2.3) with two peaks $P_\varepsilon^1$, $P_\varepsilon^2$, belonging to $B$. Then, for $\varepsilon$ small, the points $P_\varepsilon^i$ lay on the same line passing through the origin and $u_\varepsilon$ is axially symmetric with respect to this line.*

The proofs of the above theorems are based on the procedure developped in [3] which relies also on some results of [10]. Note that, since $\varphi_1$ is radially symmetric, Theorem 2.3 and Theorem 2.4 imply that the solutions of (1.1) written as in (2.1) have the same symmetry as $u_\varepsilon$. Before ending this section we prove a proposition which will be used in the following section. Let $T_\nu$ be the hyperplane passing through the origin defined by $T_\nu = \{x \in \mathbb{R}^N,\ x \cdot \nu = 0\}$, $\nu$ being a direction in $\mathbb{R}^N$. We denote by $B_\nu^-$ and $B_\nu^+$ the caps in $B$ determined by $T_\nu$: $B_\nu^- \equiv \{x \in B:\ x \cdot \nu < 0\}$ and $B_\nu^+ \equiv \{x \in B:\ x \cdot \nu > 0\}$.

In $B$ we consider problem (2.3) and denote by $L_\varepsilon$ the linearized operator at a solution $u_\varepsilon$ of (2.3):

$$L_\varepsilon = -\varepsilon^2\Delta - \varepsilon^2\lambda - p(u_\varepsilon - \varphi_1)_+^{p-1}. \tag{2.10}$$

Let $\lambda_1(L_\varepsilon, D)$ be the first eigenvalue of $L_\varepsilon$ in a subdomain $D \subset B$ with zero Dirichlet boundary conditions. By Proposition 1.1 of [10] we have the following:

**Proposition 2.3** *If $\lambda_1(L_\varepsilon, B_\nu^-)$ and $\lambda_1(L_\varepsilon, B_\nu^+)$ are both non-negative, then $u_\varepsilon$ is symmetric with respect to the hyperplane $T_\nu$.*

A slight variation of the previous result is the following:

**Proposition 2.4** *If either $\lambda_1(L_\varepsilon, B_\nu^-)$ or $\lambda_1(L_\varepsilon, B_\nu^+)$ is non-negative and $u_\varepsilon$ has a critical point on $T_\nu \cap B$, then $u_\varepsilon$ is symmetric with respect to the hyperplane $T_\nu$.*

The proofs of the previous propositions are given in [10] and in [3] respectively for solutions of semilinear elliptic problem of the type

$$\begin{cases} -\Delta u = f(x,u) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

when $B$ is a ball or an annulus and $f(x,t)$ is strictly convex in $t$.

Actually, Proposition 2.3 and 2.4 hold even if $f(x,t)$ is convex in $t$ and there exists a real number $c$ such that $f(x,t)$ is strictly convex for $t \in (c, +\infty)$. The proofs are the same of the one given in [10] and [3] using the following proposition.

Let us denote by $v_\varepsilon^-$ and $v_\varepsilon^+$ the reflected functions of $u_\varepsilon$ in the domains $B_\nu^-$ and $B_\nu^+$ with respect to the hyperplane $T_\nu$. Hence the following Proposition holds.

**Proposition 2.5** *If $\lambda_1(L_\varepsilon, B_\nu^-)$ (resp. $\lambda_1(L_\varepsilon, B_\nu^+)$) is non-negative, then $v_\varepsilon^- \geq u_\varepsilon$ (resp. $v_\varepsilon^+ \geq u_\varepsilon$)*

*Proof.* We consider the function $w_\varepsilon = v_\varepsilon^- - u_\varepsilon$ in $B_\nu^-$. By the convexity of the function $(u_\varepsilon - \varphi_1)_+^{p-1}$ we have

$$L_\varepsilon(w_\varepsilon) \geq 0 \text{ in } B_\nu^- \tag{2.11}$$

and the strict inequality holds whenever $v_\varepsilon^- \neq u_\varepsilon$ and whenever $v_\varepsilon^-$ or $u_\varepsilon$ is bigger than $\varphi_1$. Moreover

$$w_\varepsilon \equiv 0 \text{ on } \partial B_\nu^-. \tag{2.12}$$

We have to prove that $w_\varepsilon \geq 0$ in $B_\nu^-$.

Arguing by contradiction, we assume that $w_\varepsilon$ is negative somewhere in $B_\nu^-$. Let us consider the connected component $D$ in $B_\nu^-$ of the set where $w_\varepsilon < 0$. If $D \subset B_\nu^-$, multiplying (2.11) by $w_\varepsilon$, integrating and using (2.12) and the convexity of $(u_\varepsilon - \varphi_1)_+^p$, we get

$$\varepsilon^2 \int_D |\nabla w_\varepsilon|^2 - \lambda\varepsilon^2 \int_D |w_\varepsilon|^2 - \int_D p(u_\varepsilon - \varphi_1)_+^{p-1} w_\varepsilon^2 \leq 0$$

which implies that $\lambda_1(L_\varepsilon, D) \leq 0$; hence $\lambda_1(L_\varepsilon, B_\nu^-) < 0$, contrary to the hypothesis.

If $D = B_\nu^-$, we know that there exists a point $x \in D$ such that $u_\varepsilon(x) > \varphi_1(x)$, by the definition of $u_\varepsilon$ and (2.1). So there exists a neighborhood $I$ of $x$ where $u_\varepsilon > \varphi_1$ and hence the function $(u_\varepsilon - \varphi_1)_+^p$ is strictly convex; so

$$\varepsilon^2 \int_{B_\nu^-} |\nabla w_\varepsilon|^2 - \lambda\varepsilon^2 \int_{B_\nu^-} |w_\varepsilon|^2 - \int_{B_\nu^-} p(u_\varepsilon - \varphi_1)_+^{p-1} w_\varepsilon^2 < 0$$

which implies that $\lambda_1(L_\varepsilon, B_\nu^-) < 0$, contrary to the hypothesis. □

# 3 Proof of Theorem 2.3

By Theorem 2.1 we know that there is an $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, (2.3) has a solution of the form

$$u_\varepsilon = \mathcal{P}_{\varepsilon,B} U_{\varepsilon,P_\varepsilon} + \omega_\varepsilon \tag{3.1}$$

where $\omega_\varepsilon \in H_0^1(B)$ satisfies (2.7).

We start by proving statement (i). Since the solutions can be rotated, without loss of generality, we can assume that the concentration points $P_\varepsilon$ belong to the $x_N$ axis, so that $P_\varepsilon = (0, 0, \ldots, 0, t_\varepsilon)$ with $t_\varepsilon > 0$ (the case $t_\varepsilon = 0$ is considered in the statement (ii)). Let us fix a hyperplane $T$ passing through the $x_N$-axis and denote by $B^-$ and $B^+$ the two open caps determined by $T$. Assume that $T = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 = 0\}$. In order to prove the symmetry of $u_\varepsilon$ with respect to $T$, let us introduce the function

$$w_\varepsilon = v_\varepsilon - u_\varepsilon \text{ in } B^- \tag{3.2}$$

where $v_\varepsilon$ is defined as the reflection of $u_\varepsilon$ with respect to $T$, *i.e.*

$$v(x_1, \ldots, x_N) = u(-x_1, \ldots, x_N).$$

Then $w_\varepsilon$ satisfies the following problem

$$\begin{cases} \varepsilon^2 \Delta w_\varepsilon + \varepsilon^2 \lambda w_\varepsilon + c_\varepsilon w_\varepsilon = 0 \text{ in } B^- \\ w_\varepsilon = 0 \ \text{ on } \partial B^- \end{cases} \tag{3.3}$$

where we define

$$c_\varepsilon = \frac{(v_\varepsilon(x) - \varphi_1(x))_+^p - (u_\varepsilon(x) - \varphi_1(x))_+^p}{v_\varepsilon(x) - u_\varepsilon(x),}. \tag{3.4}$$

We want to show that

$$w_\varepsilon \equiv 0. \tag{3.5}$$

Arguing by contradiction, we suppose that there exists a sequence $\varepsilon_n \to 0$ as $n \to +\infty$ such that $w_n := w_{\varepsilon_n} \neq 0$. Let us define

$$N_n := N_{\varepsilon_n} = \max_{\overline{B^-}} |w_n(x)| = |w_n(m_{\varepsilon_n})| > 0.$$

First of all, in order to prove the result, we claim that

$$\exists R_0 > 0 : \ |m_n - P_n| \leq R_0 \varepsilon, \tag{3.6}$$

where $P_n := P_{\varepsilon_n}$ and $m_n := m_{\varepsilon_n}$. For the sake of simplicity, we define $u_n := u_{\varepsilon_n}$, $v_n := v_{\varepsilon_n}$ and by contradiction, assume that

$$\frac{|m_n - P_n|}{\varepsilon_n} \to +\infty \text{ as } n \to \infty.$$

Then

$$U_{\varepsilon_n, P_n}(m_n) = U\left(\frac{|m_n - P_n|}{\varepsilon_n}\right) \to 0 \text{ as } n \to \infty$$

and by Proposition 2.1

$$P_{\varepsilon_n, B} U_{\varepsilon_n, P_n}(m_n) \to 0 \text{ as } n \to \infty.$$

Since

$$u_n = P_{\varepsilon_n, B} U_{\varepsilon_n, P_n} + \omega_{\varepsilon_n}$$

we find that $u_n(m_n) \to 0$. Furthermore, since the reflection points $\tilde{m}_n$ of $m_n$ with respect to $T$ are such that $|\tilde{m}_n - P_n| = |m_n - P_n|$, also $v_n(m_n) \to 0$. Moreover, applying the Lagrange theorem in (3.4), with $0 \le \theta_n \le 1$ we have

$$c_n(m_n) = c_{\varepsilon_n}(m_n) = p\Big(v_n(m_n) + \theta_n(u_n(m_n) - v_n(m_n)) - \varphi_1(m_n)\Big)_+^{p-1} \to 0.$$

Hence, if $N_n = \max_{\overline{B^-}} w_n(x) = w_n(m_n)$, we obtain

$$0 \le -\varepsilon_n^2 \Delta w_n(m_n) = c_n(m_n) w_n(m_n) + \varepsilon_n^2 \lambda w_n(m_n) < 0,$$

and if $N_n = -\min_{\overline{B^-}} w_n(x) = -w_n(m_n)$, we obtain that

$$0 \ge -\varepsilon_n^2 \Delta w_n(m_n) = c_n(m_n) w_n(m_n) + \varepsilon_n^2 \lambda w_n(m_n) > 0,$$

a contradiction in both cases, which proves (3.6).

Let us consider the rescaled function around $P_n$:

$$\tilde{w}_n(x) = \frac{1}{N_n} w_n(P_n + \varepsilon_n x),\ x \in B_n^- = \frac{B^- - P_n}{\varepsilon_n}. \tag{3.7}$$

Let us observe that the function $\tilde{w}_n$ satisfies

$$\begin{cases} \Delta \tilde{w}_n + \varepsilon_n^2 \lambda \tilde{w}_n + \tilde{c}_n \tilde{w}_n = 0 \text{ in } B_n^- \\ \tilde{w}_n = 0 \ \text{ on } \partial B_n^- \end{cases} \tag{3.8}$$

where we define $\tilde{c}_{\varepsilon_n} = c_{\varepsilon_n}(\varepsilon_n x + P_n)$. Since $|\tilde{w}_n(x)| \le 1$ in $B_n^-$ and $\tilde{c}_n(x)$ is uniformly bounded, we see that $\tilde{w}_n \in C^2_{\text{loc}}$ by the elliptic regularity theorem. Therefore, $\{w_n\}$ contains a subsequence convergent to a function $\tilde{w}$ in $C^2_{\text{loc}}(\mathbb{R}^N_-)$ satisfying

$$\begin{cases} -\Delta \tilde{w} = p(U-1)_+^{p-1} \tilde{w} \ \text{ in } \mathbb{R}^N_- = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : \ x_1 < 0\} \\ \tilde{w} = 0 \ \text{ on } \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : \ x_1 = 0\} \end{cases} \tag{3.9}$$

where $U$ is the solution of (2.4). Then by Proposition 2.2, $\tilde{w} = k \frac{\partial U}{\partial x_1}$. Moreover, using (3.6),

$$\tilde{w}_n \left( \frac{m_n - P_n}{\varepsilon_n} \right) = \frac{w_n(m_n)}{N_n} = \pm 1$$

so $k \neq 0$. Since the points $P_n$ are on the reflection hyperplane $T$ and $\nabla u_n(P_n) = 0$, $\frac{\partial w_n}{\partial x_1}(0) = 0$, so that

$$\frac{\partial \tilde{w}}{\partial x_1}(0) = 0.$$

On the other hand, $\frac{\partial^2 U}{\partial x_1^2}(0) < 0$. Thus, $k = 0$, a contradiction. Consequently, $w_n(x) \equiv 0$ on $B^-$, *i.e.* $v_\varepsilon(x) = u_\varepsilon(x)$ and the proof is complete.

(ii) If $|P_\varepsilon| = 0$, we can repeat the proof of (i), for any hyperplane $T$ passing through the origin, getting that $u_\varepsilon$ is a radial function, for $\varepsilon$ sufficiently small.

(iii) Let us denote by $T_\theta$ the hyperplane $T_\theta = \{x \in \mathbb{R}^N : x_1 \sin\theta + x_N \cos\theta = 0\}$ with $\theta \in \left[0, \frac{\pi}{2}\right]$. For $\theta = \frac{\pi}{2}$ this hyperplane coincides with $T$.

As before, we set $B_\theta^- = \{x \in \mathbb{R}^N : x_1 \sin\theta + x_N \cos\theta < 0\}$ and $B_\theta^+ = \{x \in \mathbb{R}^N : x_1 \sin\theta + x_N \cos\theta > 0\}$. In order to prove statement (iii), let us introduce the function

$$w_{\varepsilon,\theta} = v_{\varepsilon,\theta} - u_\varepsilon \text{ in } B_\theta^- \tag{3.10}$$

where $v_{\varepsilon,\theta}$ is defined as the reflection of $u_\varepsilon$ with respect to $T_\theta$.

Hence (iii) is merely a consequence of Hopf's lemma if we show that for any $\theta \in [0, \frac{\pi}{2})$ the function $w_{\varepsilon,\theta}$ is positive in $B_\theta^-$, since $P_\varepsilon \notin B_\theta^-$. Thus, applying Hopf's lemma to $w_{\varepsilon,\theta}$ (which solves a linear ellitpic equation) at any point on $T_\theta \cap B$ we get (2.9). It remains to prove that $w_{\varepsilon,\theta}$ is positive in $B_\theta^-$. We will do it in two steps.

**Step 1**

Here we prove that for $\theta = 0$ and $\varepsilon$ sufficiently small, $w_{\varepsilon,0} > 0$ for $x \in B_0^- = \{x \in B : x_N < 0\}$.

Let us assume that the limit $l$ that appears in (2.8) is finite, that is $0 < l < \infty$. Let us observe that, in this case, the ball $B(P_\varepsilon, \varepsilon R)$ , $R > 0$ could intersect the cap $B_0^-$ or not. Let us define $E_{\varepsilon,0}^- = B_0^- \setminus \overline{B(P_\varepsilon, \varepsilon R)}$ and $F_{\varepsilon,0}^- := B_0^- \cap \overline{B(P_\varepsilon, \varepsilon R)}$. We claim that for $\varepsilon$ small

$$u_\varepsilon(x) \leq v_{\varepsilon,0}(x)\ x \in F_{\varepsilon,0}^-. \tag{3.11}$$

In fact if (3.11) was not true we could construct a sequence $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence of points $x_n \in F_{\varepsilon_n,0}^-$ such that

$$u_{\varepsilon_n}(x_n) > v_{\varepsilon_n,0}(x_n).$$

Then there would exist a sequence of points $\xi_n \in F_{\varepsilon_n,0}^-$ such that

$$\frac{\partial u_{\varepsilon_n}}{\partial x_N}(\xi_n) < 0. \tag{3.12}$$

Thus, by rescaling $u_{\varepsilon_n}$ around $P_{\varepsilon_n}$, that is $u_{\varepsilon_n}(P_{\varepsilon_n} + \varepsilon_n x)$, and using (2.8), we would get a point $\xi \in F_0^l\,\{x = (x_1, \cdots, x_N) : x_N < -l < 0\}$ such that $\frac{\partial U}{\partial x_N}(\xi) \leq 0$ while $\frac{\partial U}{\partial x_N} > 0$ in $F_0^l$. Hence (3.11) holds. Now we claim that

$$\lambda_1(L_\varepsilon, E_{\varepsilon,0}^-) > 0. \tag{3.13}$$

Indeed, if $x \in E_{\varepsilon,0}^-$ then $|P_\varepsilon - x| > \varepsilon R$ and

$$U_{\varepsilon,P_\varepsilon}(x) = U\left(\frac{|P_\varepsilon - x|}{\varepsilon}\right) \to 0, \text{ as } \varepsilon \to 0\ \forall x \in E_{\varepsilon,0}^-.$$

By (2.5) and by (3.1), it results that $u_\varepsilon \to 0$ uniformly. Hence, the term $p(u_\varepsilon - \varphi_1)_+^{p-1}$ in the expression (2.10) of the linearized operator $L_\varepsilon$ can be made as small as we like as $\varepsilon \to 0$. In particular, for $\varepsilon$ sufficiently small, we have that $p(u_\varepsilon - \varphi_1)_+^{p-1} < \lambda_1(-\Delta, E^-_{\varepsilon,0})$, where $\lambda_1(-\Delta, E^-_{\varepsilon,0})$ is the first eigenvalue of the Laplace operator in $E^-_{\varepsilon,0}$ with zero boundary conditions. Therefore, $\lambda_1(L_\varepsilon, E^-_{\varepsilon,0}) > 0$. It follows that the maximum principle holds in $E^-_{\varepsilon,0}$ and, since $L_\varepsilon(w_{\varepsilon,0}) \geq 0$ in $E^-_{\varepsilon,0}$ (by the convexity of $p(u_\varepsilon - \varphi_1)_+^{p-1}$) and $w_{\varepsilon,0} \geq 0$ on $\partial E^-_{\varepsilon,0}$, we have that

$$w_{\varepsilon,0} \geq 0 \text{ in } E^-_{\varepsilon,0};$$

hence, using (3.11) and by means of the strong maximum principle, it results that

$$w_{\varepsilon,0} > 0 \text{ in } B_0^-.$$

So Step 1 is proved if $0 < l < +\infty$.

Let us now assume that

$$\frac{|P_\varepsilon|}{\varepsilon} \to \infty. \tag{3.14}$$

Obviously, in this situation, the ball $B(P_\varepsilon, \varepsilon R)$ does not intersect the cap $B_0^-$. Hence (3.13) holds with $E^-_{\varepsilon,0} = B_0^-$ and again by means of the strong maximum principle, it results that

$$w_{\varepsilon,0} > 0 \text{ in } B_0^-.$$

So, Step 1 is proved also in this case.

**Step 2**

Now we define

$$\theta_0 = \sup\left\{\tilde\theta \in \left[0, \frac{\pi}{2}\right] : \; w_{\varepsilon,\theta}(x) \geq 0 \qquad x \in B_\theta^- \text{ and } 0 \leq \theta \leq \tilde\theta\right\}.$$

We would like to prove that $\theta_0 = \frac{\pi}{2}$.

If $\theta_0 < \frac{\pi}{2}$, since $P_\varepsilon \notin B^-_{\theta_0}$, denoting by $P'_\varepsilon$ the point in $B^-_{\theta_0}$ which is given by the reflection of $P_\varepsilon$ with respect to $T_{\theta_0}$, we have that

$$w_{\varepsilon,\theta_0}(x) > \eta > 0 \text{ for } x \in \overline{B(P'_\varepsilon, \delta)} \subset B^-_{\theta_0} \tag{3.15}$$

where $B(P'_\varepsilon, \delta)$ is the ball with center in $P'_\varepsilon$ and radius $\delta > 0$ suitably choosen. Thus

$$w_{\varepsilon,\theta_0+\sigma}(x) > \frac{\eta}{2} > 0 \text{ for } x \in \overline{B(P''_\varepsilon, \delta)} \subset B^-_{\theta_0+\sigma} \tag{3.16}$$

for $\sigma > 0$ sufficiently small, where $P''_\varepsilon$ is the reflection of $P_\varepsilon$ with respect to $T_{\theta_0+\sigma}$.
On the other hand, by the monotonicity of the eigenvalues with respect to the domain, we have that $\lambda_1(L_\varepsilon, B^-_{\theta_0} \setminus \overline{B(P'_\varepsilon, \delta)}) > 0$ and hence $\lambda_1(L_\varepsilon, B^-_{\theta_0+\sigma} \setminus \overline{B(P''_\varepsilon, \delta)}) > 0$ for $\sigma$ sufficiently small. This implies, by the maximum principle and (3.16), that

$$w_{\varepsilon,\theta_0+\sigma}(x) > 0 \text{ for } x \in B^-_{\theta_0+\sigma}. \tag{3.17}$$

Since $L_\varepsilon(w_{\varepsilon,\theta_0+\sigma}) \geq 0$ in $B^-_{\theta_0+\sigma}$, the inequality (3.17) implies that $\lambda_1(L_\varepsilon, D) > 0$ in any subdomain $D$ of $B^-_{\theta_0+\sigma}$, and so $\lambda_1(L_\varepsilon, B^-_{\theta_0+\sigma}) \geq 0$ for $\sigma$ positive and sufficiently small. Using Proposition 2.5, we have that $w_{\varepsilon,\theta_0+\sigma} \geq 0$, which contradicts the definition of $\theta_0$ and proves that $\theta_0 = \frac{\pi}{2}$, as we wanted to show. Then $w_{\varepsilon,\frac{\pi}{2}} \equiv 0$; otherwise, by the strong maximum principle we would have $w_{\varepsilon,\frac{\pi}{2}} > 0$ in $B^- = B^-_{\frac{\pi}{2}}$ which, by the Hopf Lemma would imply $\frac{\partial u_\varepsilon}{\partial x_1} \neq 0$ against the fact that $P^1_\varepsilon$ is the maximum point of $u_\varepsilon$. We have thus established the positivity of $w_\varepsilon$ in $B^-_\theta$ for any $\theta \in [0, \frac{\pi}{2})$. □

# 4 Proof of Theorem 2.4

In this section we consider solutions of (2.3) with two peaks, $P^1_\varepsilon$, $P^2_\varepsilon$. By Theorem 2.1 we know that there is an $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, (2.3) has a solution of the form

$$u_\varepsilon = \mathcal{P}_{\varepsilon,B} U_{\varepsilon,P^1_\varepsilon} + \mathcal{P}_{\varepsilon,B} U_{\varepsilon,P^2_\varepsilon} + \omega_\varepsilon \tag{4.1}$$

where $\omega_\varepsilon \in H^1_0(B)$ satisfies (2.7) as $\varepsilon \to 0$, $\frac{|P^1_\varepsilon - P^2_\varepsilon|}{\varepsilon} \to +\infty$, $P^i_\varepsilon \to 0 \in B$ $i = 1, 2$ as $\varepsilon \to 0$. We have:

**Lemma 4.1** *Let $u_\varepsilon$ be a family of solution of (2.3) of the form (4.1). Then, for $\varepsilon$ small, both points $P^i_\varepsilon$, $i = 1, 2$, lay on the same line passing through the origin.*

The proof of this lemma is rather long and will be given later.

### Proof of Theorem 2.4

The first part of the statement is exactly as in Lemma 4.1. Hence we have to prove that $u_\varepsilon$ is symmetric with respect to any hyperplane passing through the axis containing $P^1_\varepsilon$ and $P^2_\varepsilon$. Assume that this axis is the $x_N$-axis. Let us fix a hyperplane $T$ passing through the $x_N$-axis and, for simplicity, assume that $T = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : \ x_1 = 0\}$, so that $B^- = \{x \in B, \ x_1 < 0\}$ and $B^+ = \{x \in B, \ x_1 > 0\}$.

As in the proof of statement (i) of Theorem 2.3, let us consider in $B^-$ the function

$$w_\varepsilon(x) = v_\varepsilon(x) - u_\varepsilon(x), \ x \in B^-$$

where $v_\varepsilon$ is the reflection of $u_\varepsilon$, . *i.e.* $v(x_1, \ldots, x_N) = u(-x_1, \ldots, x_N)$.
Then $w_\varepsilon$ satisfies (3.3). We would like to prove that $w_\varepsilon \equiv 0$ in $B^-$ for $\varepsilon$ small.

Assume, by contradiction, that for a sequence $\varepsilon_n \to 0$, $w_{\varepsilon_n} = w_n \neq 0$. Let $N_n = \max_{\overline{B^-}} |w_n(x)|$ and $m_n \in \overline{B^-}$ be a point such that

$$N_n = |w_n(m_n)| > 0.$$

Firstly we claim:

$$\limsup_{n\to\infty} \min\left(\frac{|m_n - P_n^1|}{\varepsilon_n}, \frac{|m_n - P_n^2|}{\varepsilon_n}\right) < +\infty \tag{4.2}$$

where $P_n^i = P_{\varepsilon_n}^i$, $i = 1, 2$. Let us assume, by contradiction, that

$$\limsup_{n\to\infty} \min\left(\frac{|m_n - P_n^1|}{\varepsilon_n}, \frac{|m_n - P_n^2|}{\varepsilon_n}\right) = +\infty.$$

Then

$$U_{\varepsilon_n, P_n^i}(m_n) = U\left(\frac{|m_n - P_n^i|}{\varepsilon_n}\right) \to 0 \text{ as } n \to \infty,$$

hence, by (2.5)

$$P_{\varepsilon_n, B} U_{\varepsilon_n, P_n^i}(m_n) \to 0$$

as $n \to \infty$. So, by the form of $u_{\varepsilon_n}$:

$$u_{\varepsilon_n} = P_{\varepsilon_n, B} U_{\varepsilon_n, P_n^1} + P_{\varepsilon_n, B} U_{\varepsilon_n, P_n^2} + B_{\varepsilon_n}$$

we find that $u_{\varepsilon_n}(m_n) \to 0$. Furthermore, since the reflection points $\tilde{m}_n$ of $m_n$ with respect to $T$ are such that $|\tilde{m}_n - P_n| = |m_n - P_n|$, we have that $v_{\varepsilon_n}(m_n) \to 0$. Moreover, applying the Lagrange theorem in (3.4), with $0 \le \theta_n \le 1$, we get

$$c_n(m_n) = c_{\varepsilon_n}(m_n) = p\Big(v_{\varepsilon_n}(m_n) + \theta_n(u_{\varepsilon_n}(m_n) - v_{\varepsilon_n}(m_n)) - \varphi_1(m_n)\Big)_+^{p-1} \to 0.$$

Hence, if $N_n = \max_{\overline{B^-}} w_n(x) = w_n(m_n)$, we obtain

$$0 \le -\varepsilon_n^2 \Delta w_n(m_n) = c_n(m_n) w_n(m_n) + \varepsilon_n^2 \lambda w_n(m_n) < 0,$$

and if $N_n = -\min_{\overline{B^-}} w_n(x) = -w_n(m_n)$, we obtain that

$$0 \ge -\varepsilon_n^2 \Delta w_n(m_n) = c_n(m_n) w_n(m_n) + \varepsilon_n^2 \lambda w_n(m_n) > 0,$$

a contradiction in both cases. Therefore we conclude that

$$\frac{|m_n - P_n^1|}{\varepsilon_n} < \infty \text{ or } \frac{|m_n - P_n^2|}{\varepsilon_n} < \infty.$$

Without loss of generality we may assume that

$$\exists R_0 > 0 : \ |m_n - P_n^1| \le R_0 \varepsilon_n. \tag{4.3}$$

Let us consider the rescaled function around $P_n^1$:

$$\tilde{w}_n(x) = \frac{1}{N_n} w_n(P_n^1 + \varepsilon_n x), \ x \in B_{1,n}^- = \frac{B^- - P_n^1}{\varepsilon_n}. \tag{4.4}$$

which is the analogue of (3.7). From now on the proof is exactly the same as that of statement (i) of Theorem 2.3. Hence we get the assertion. □

Finally we prove Lemma 4.1 in several steps

**Proof of Lemma 4.1**

If either $P_\varepsilon^1$ or $P_\varepsilon^2$ is the origin, then there is nothing to prove. Otherwise, since we know by Theorem 2.1 that

$$\frac{|P_\varepsilon^1 - P_\varepsilon^2|}{\varepsilon} \to +\infty \text{ as } \varepsilon \to 0$$

we can assume that

$$\frac{|P_\varepsilon^1|}{\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 \tag{4.5}$$

and that the line connecting $P_\varepsilon^1$ with the origin is the $x_N-$axis. We would like to show that also the point $P_\varepsilon^2$ belongs to the same axis. So we assume by contradiction that for a sequence $\varepsilon_n \to 0$, the points $P_{\varepsilon_n}^2 = P_n^2$ are given by $P_n^2 = (\alpha_n, x_2^n, \ldots, x_N^n)$, $\alpha_n > 0$, where the first coordinate $\alpha_n$ represents the distance of $P_n^2$ from the $x_N$-axis.

**Claim 1** *It is not possible that*

$$\frac{\alpha_n}{\varepsilon_n} \to +\infty. \tag{4.6}$$

Assume that (4.6) holds and consider the hyperplane $T = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 = 0\}$, which obviously passes through the $x_N$-axis and does not contain the point $P_n^2$. We claim that, for $n$ sufficiently large,

$$\lambda_1(L_n, B^-) \geq 0 \tag{4.7}$$

where, as before, $L_n = L_{\varepsilon_n}$ denotes the linearized operator and $B^- = \{x = (x_1, \ldots, x_N) \in B : x_1 < 0\}$. To prove (4.7) let us take the two balls $B(P_n^i, \varepsilon_n R)$ centered at the two points $P_n^i$ and with radii $R\varepsilon_n$, $R > 1$ to be fixed later. By (4.6) we have that $B(P_n^2, \varepsilon_n R)$ does not intersect $B^-$ for large $n$. Moreover, if we take $\theta_0 \in \left[0, \frac{\pi}{2}\right]$ and we consider the hyperplane $T_{\theta_0}$, by (4.6) and (4.5) we can choose $\theta_{0,n} < \frac{\pi}{2}$ and close to $\frac{\pi}{2}$ such that neither one of the balls $B(P_n^i, \varepsilon_n R)$ intersects the cap $B_{n,\theta_0}^- = \{x \in \mathbb{R}^N : x_1 \sin\theta_{0,n} + x_N \cos\theta_{0,n} < 0\}$ for $n$ large enough. Then, arguing as in [9], it is easy to see that it is possible to choose $R$ such that $\lambda_1(L_n, A_{\theta_{0,n}}^-) > 0$ for $n$ large, because $B(P_n^i, \varepsilon_n R) \cap B_{n,\theta_0}^- = \emptyset$, $i = 1, 2$ and $u_n$ concentrates only at $P_n^i$.

Let us fix $n$ sufficiently large and let us set

$$\tilde\theta_n \equiv \sup\left\{\theta \in \left[\theta_{0,n}, \frac{\pi}{2}\right] : \lambda_1(L_n, B_\theta^-) \geq 0\right\}.$$

Repeating the same procedure as in the proof of Theorem 2.3 (iii) Step 2, we get $\tilde\theta_n = \frac{\pi}{2}$, and hence (4.7) holds. So by Proposition 2.4, since $P_n^1 \in T$, we get that $u_n$ is symmetric with respect to the hyperplane $T$, which is not possible, since $P_n^2$ does not belong to $T$. Hence (4.6) cannot hold.

**Claim 2** *It is not possible that*

$$\frac{\alpha_n}{\varepsilon_n} \to l > 0. \tag{4.8}$$

We would like to prove, as in Claim 1, that

$$\lambda_1(L_n, B^-) \geq 0. \tag{4.9}$$

If the points $P_n^1$ and $P_n^2$ have the $N-$th coordinates of the same sign, *i.e.* they lay on the same side with respect to the hyperplane $\{x_N = 0\}$, then it is obvious that we can argue exactly as for the first claim and choose $\theta_0 \in \left[0, \frac{\pi}{2}\right]$ such that neither of the balls $B(P_n^i, \varepsilon_n R)$, $R$ as before, intersects the cap $B_{\theta_0}^-$. Then the proof is the same as before. Hence we assume that the points $P_n^1$ and $P_n^2$ lay on the different sides with respect to the hyperplane $\{x_N = 0\}$. Let us then consider $\theta_n \in \left[0, \frac{\pi}{2}\right]$ such that the points $P_n^1$ and $P_n^2$ have the same distance $d_n > 0$ from the hyperplane

$$T_{\theta_n} = \left\{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 \sin\theta + x_N \cos\theta_n = 0\right\}.$$

Of course, because of (4.8), we have

$$\frac{d_n}{\varepsilon_n} \to l_1 > 0. \tag{4.10}$$

Then, choosing $R > 0$ such that $\lambda_1(L_n, D_n^R) > 0$, for $n$ large, $D_n^R = B \setminus [B(P_n^1, \varepsilon_n R) \cup B(P_n^2, \varepsilon_n R)]$ (see [9]), either neither of the balls $B(P_n^i, \varepsilon_n R)$ intersects the cap $B_{\theta_n}^-$, for $n$ large enough, or, recalling (4.5), only the ball $B(P_n^2, \varepsilon_n R)$ does so. In the first case we argue as in the first claim. In the second case, arguing as in the proof of (3.11) in Theorem 2.3, we observe that in the set $E_{\theta_n}^n = B_{\theta_n}^- \cap B(P_n^2, \varepsilon_n R)$ we have, for $n$ large

$$u_n(x) \leq v_n^{\theta_n}(x)\ x \in E_{\theta_n}^n, \tag{4.11}$$

where $v_n^{\theta_n}(x) = u_n(x^{\theta_n})$, $x^{\theta_n}$ being the reflection of $x$ with respect to $T_{\theta_n}^n$.

Now, arguing again as in [9] and [6], in the set $F_{\theta_n}^n = B_{\theta_n}^- \cap [B(P_n^1, \varepsilon_n R) \cup B(P_n^2, \varepsilon_n R)]$ we have that $\lambda_1(L_n, (F_{\theta_n}^n)^-) \geq 0$. Hence, by (4.11), applying the maximum principle, we have that $w_{n,\theta_n}(x) \geq 0$ in $(F_{\theta_n}^n)^-$, and, again by (4.11) and the strong maximum principle

$$w_{n,\theta_n}(x) > 0 \text{ in } B_{\theta_n}^-. \tag{4.12}$$

As in the proof of Theorem 2.3 (iii) Step 2, this implies that $\lambda_1(L_n, B_{\theta_n}^-) \geq 0$. Then, arguing as in the first claim we get (4.9), which gives the same kind of contradiction because

$P_n^2$ does not belong to $T$.

**Claim 3** *It is not possible that*

$$\frac{\alpha_n}{\varepsilon_n} \to 0. \tag{4.13}$$

Let us argue by contradiction and assume that (4.13) holds. Since the points $P_n^2$ are in the domain $B_n^+$, we have that the function $w_n(x) = v_n(x) - u_n(x),\ x \in B_n^+$ where $v_n$ is the reflection of $u_n$, *i.e.* $v_n(x_1, \dots, x_N) = u_n(-x_1, x_2 \dots, x_N)$ is not identically zero. Then, as in the proof of Theorem 2.4, rescaling the function $w_n$ around $P_n^1$ and $P_n^2$, we have that the functions

$$\tilde{w}_n^i(y) \equiv \frac{1}{N_n} w_n(P_n^i + \varepsilon_n y),\ i = 1, 2, \tag{4.14}$$

defined in the rescaled domain $B_{i,n}^+ = \frac{B^+ - P_n^i}{\varepsilon_n}$, converge both, by (4.13) and standard elliptic estimates, in $C^2_{\text{loc}}$ to a function $w_i$ satisfying (3.9) but in the half space $\mathbb{R}^N_+ = \{x = (x_1, \dots, x_N) \in \mathbb{R} :\ x_1 > 0\}$. Again by Proposition 2.2 it result that $w_i = k_i \frac{\partial U}{\partial x_1}$, where $U$ is the solution of (2.4) and $k_i \in \mathbb{R}$. By (4.2) we have that

$$\frac{|m_n - P_n^1|}{\varepsilon_n} < \infty \qquad \text{or} \qquad \frac{|m_n - P_n^2|}{\varepsilon_n} < +\infty.$$

Hence we can exclude the case that both sequences $\tilde{w}_n^i$ converge to zero in $C^2_{\text{loc}}$. So, for at least one of the two sequences $\tilde{w}_n^i$ we have that the limit $w_i = k_i \frac{\partial U}{\partial x_1}$ with $k_i \neq 0$. If this happens for $\tilde{w}_n^1$, then, since the points $P_n^1$ are on the reflection hyperplane $T$, arguing exactly as in the proof of Theorem 2.4, we get a contradiction.

So we are left with the case when $\tilde{w}_n^1 \to k_1 \frac{\partial U}{\partial x_1}$, $k_1 = 0$ and $\tilde{w}_n^2 \to k_2 \frac{\partial U}{\partial x_1}$, $k_2 \neq 0$ in $C^2_{\text{loc}}$. At the points $P_n^2$, obviously we have that $\frac{\partial u_n}{\partial x_1}(P_n^2) = 0$. Let us denote by $\tilde{P}_n^2$ the reflection of $P_n^2$ with respect to $T$. Hence, for the function $\tilde{w}_n^2$ we have, applying the mean value theorem,

$$\frac{\partial \tilde{w}_n^2}{\partial x_1}(0) = \frac{1}{N_n} \left( \frac{\partial \tilde{u}_n}{\partial x_1}(0) - \frac{\partial \tilde{u}_n}{\partial x_1}\Big(\frac{\tilde{P}_n^2 - P_n^2}{\varepsilon_n}\Big) \right) =$$

$$= \frac{1}{N_n} \frac{\partial^2 \tilde{u}_n}{\partial x_1^2}(\xi) \frac{2\alpha_n}{\varepsilon_n}$$

where $\tilde{u}_n(y) = \varepsilon_n u_n(P_n^2 + \varepsilon_n y)$ and $\xi_n$ belongs to the segment joining the origin with the point $\frac{\tilde{P}_n^2 - P_n^2}{\varepsilon_n}$ in the rescaled domain $B_{2,n}^+$.

Since $\frac{\partial \tilde{w}_n^2}{\partial x_1}(0) \to k_2 \frac{\partial^2 U}{\partial x_1^2}(0)$ and $\frac{\partial^2 \tilde{u}_n}{\partial x_1^2}(\xi_n) \to \frac{\partial^2 U}{\partial x_1^2}(0)$, with $k_2 \neq 0$ and $\frac{\partial^2 U}{\partial x_1^2}(0) < 0$, we get

$$\frac{\alpha_n}{N_n \varepsilon_n} \to \gamma \neq 0. \tag{4.15}$$

Our aim is now to prove that (4.15) implies that $k_1 \neq 0$, which will give a contradiction.

Let us observe that if the function $w_n$ does not change sign near $P_n^1$, then, since $w_n \not\equiv 0$, we would get a contradiction, applying Hopf's lemma to $w_n$ (which solves a linear elliptic equation) at the point $P_n^1$, because $\nabla u_n(P_n^1) = 0$.
Then in any ball $B(P_n^1, \alpha_n)$, $\alpha_n$ as in (4.13), there are points $Q_1^n$ such that $\frac{\partial u_n}{\partial x_1}(Q_n^1) = 0$ and $Q_n^1 \notin T$. Indeed, since $w_n$ changes sign near $P_n^1$, in any set $B(P_n^1, \alpha_n) \cap B^+$ there are points where $w_n$ is zero, *i.e.* $u_n$ coincides with the reflection $v_n$. This implies that there exist points $Q_n^1$ in $B(P_n^1, \alpha_n)$ where $\frac{\partial u_n}{\partial x_1}(Q_n^1) = 0$ and by Hopf's lemma applied to the points of the hyperplane $T$ we have that $Q_n^1 \notin T$. Let us denote by $\tilde{Q}_n^1$ the reflection of $Q_n^1$ with respect to $T$. Assume that $Q_n^1 \in B^-$ (the argument is the same if $B^+$). Then, as before, we have

$$\frac{\partial \tilde{w}_n^1}{\partial x_1}\left(\frac{Q_n^1 - P_n^1}{\varepsilon_n}\right) = -\frac{1}{\varepsilon_n}\left(\frac{\partial \tilde{u}_n}{\partial x_1}\Big(\frac{\tilde{Q}_n^1 - P_n^1}{\varepsilon_n}\Big) - \frac{\partial \tilde{u}_n}{\partial x_1}\Big(\frac{Q_n^1 - P_n^1}{\varepsilon_n}\Big)\right) =$$

$$= -\frac{1}{N_n}\frac{\partial^2 \tilde{u}_n}{\partial x_1^2}(\xi_n)\frac{2\alpha_n}{\varepsilon_n}$$

where $\xi_n$ belongs to the segment joining $\frac{\tilde{Q}_n^1 - P_n^1}{\varepsilon_n}$ and $\frac{Q_n^1 - P_n^1}{\varepsilon_n}$ in the rescaled domain $B_{1,n}^+$. Since $\frac{\partial \tilde{w}_n^1}{\partial x_1}\left(\frac{Q_n^1 - P_n^1}{\varepsilon_n}\right) \to k_1 \frac{\partial^2 U}{\partial x_1^2}(0)$, $\frac{\partial^2 \tilde{u}_n}{\partial x_1^2}(\xi_n) \to \frac{\partial^2 U}{\partial x_1^2}(0) < 0$ and using (4.15), we get $k_1 \neq 0$ and hence a contradiciton. So the third claim is also true and the proof of Lemma 4.1 is complete. □

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