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On the Prescribed Paneitz Curvature Problem on the Standard Spheres

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Abstract

In this paper, we study some fourth order conformal invariants on the standard n-spheres, $n \geq 5$. Using topological arguments, we give a variety of classes of functions that can be realized as Paneitz curvature .

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1 Introduction

Given (M, g) a smooth 4-dimensional Riemannian manifold, let S, Rc denote respectively, the scalar curvature and the Ricci curvature of g.

The Paneitz operator, discovered in [35], is the fourth order operator defined by

$$P^4 u = \Delta^2 u - div(\frac{2}{3} S g - 2Rc) du,$$

where $\Delta u = div \nabla u$ is the Laplacien of u with respect to g. This operator enjoys many interesting propertie (in particular, it is conformally invariant) and can be seen as a natural extension of the well known second order conformal operator Δ on 2-manifolds.

In [12], Branson generalized the Paneitz operator P^4 to higher dimension. More precisely, let (M^n, g) be a compact Riemannian n-manifold, $n \geq 5$. The Paneitz-Branson operator is defined by

$$P^n u = \Delta^2 u - div(a_n Sg + b_n Rc)du + \frac{n-4}{2} Q^n u,$$

where

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \qquad b_n = \frac{-4}{n-2}$$

and,

$$Q^n = \frac{-1}{2(n-1)}\Delta S + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} S^2 - \frac{2}{(n-2)^2}|Rc|^2.$$

Such a Q^n is a fourth order invariant called Q-curvature or Paneitz curvature. As for P^4, the operator P^n, $n \geq 5$, is conformally invariant; if $\tilde{g} = \varphi^{\frac{4}{n-4}} g$ is a conformal metric to g, then for all $\psi \in C^\infty(M)$ we have,

$$P^n(\psi\varphi) = \varphi^{\frac{n+4}{n-4}}\ P^n(\psi).$$

In particular, taking $\psi = 1$, we have

$$P^n(\varphi) = \frac{n-4}{2}\ Q^n\ \varphi^{\frac{n+4}{n-4}}. \tag{1}$$

Many interesting results on the Paneitz operator and related topics have been recently obtained by several authors. Among them, let us mention Branson [13], Branson-Chang-Yang [14], Chang-Yang [21], Chang-Gursky-Yang [17], Chang-Qing-Yang [19] and [20], Gursky [29], Hebey-Robert [31] and Qing-Raske [36]. See also the surveys S. A. Chang [16] and Chang-Yang [22], for more results on basic property of the Paneitz operator.

In view of relation (1) it is natural to study the problem of prescribing the Paneitz curvature, that is: given a smooth function

$$K:\ M^n \longrightarrow \mathbb{R},$$

does there exist a metric $\tilde{g}$ conformally equivalent to g such that, $Q^n = K$? This problem is equivalent to finding a smooth positive solution of the following equation,

$$\begin{cases} P^n(u) &= \dfrac{n-4}{2}\ K\ u^{\frac{n+4}{n-4}}, \\ \quad u &> 0 \qquad\qquad\qquad \text{in } M. \end{cases} \tag{2}$$

In this paper, we study the case of the standard sphere (S^n, g), $n \geq 5$, endowed with it's standard metric g; the operator $P^n := \mathcal{P}$ is coercive on the Sobolev space $H_2^2(S^n)$, and has the expression

$$\mathcal{P}u = \Delta^2 u - c_n \Delta u + d_n u,$$

where

$$c_n = \frac{1}{2}(n^2-2n-4), \qquad d_n = \frac{n-4}{16} n(n^2-4).$$

Thus, we are have reduced it to looking for positive solution u of the following problem:

$$(P)\begin{cases} \Delta^2 u - c_n\Delta u + d_n u & = \dfrac{n-4}{2}\, K\, u^{\frac{n+4}{n-4}} \\ u & > 0 \quad \text{on } S^n. \end{cases}$$

Problem (P) is the analogue for Paneitz operator, of the so-called Scalar curvature problem, to which many works are devoted with various methods. Among them, let us cite Bahri and Coron see [5] (topological method), Schoen and Zhang [37] (compactness and topology), Chang, Gursky and Yang [18] (perturbation method) and Y. Y. Li [40] (subcritical approximations and blow-up analysis).

It is easy to see that a necessary condition for solving (P) is that K has to be positive somewhere. Moreover, due to topological obstructions, very close to Kazdan-Warner obstructions for the Scalar curvature problem (see [24] and [39]), the problem of realizing a function as a Q curvature of some metric conformal to a given one does not always have a solution.

The special nature of problem (P) appears when we consider it from a variational viewpoint. Indeed, the Euler-Lagrange functional associated to (P) does not satisfy the Palais-Smale condition, that is there exist non compact sequences along which the functional is bounded and it's gradient goes to zero. This fact is due to the presence of the critical exponent. Furthermore, as is already known for problems related to the scalar curvature, there is a new phenomenon in dimension $n \geq 7$, due to the fact that the self interaction of the functions failing the Palais-Smale condition dominates the interaction of two of those functions. In the five dimensional case, the reverse happens. In the six dimensional case, we have a balance phenomenon, that is the self interaction and the interaction are of the same size (see [11]).

Our aim in the present work is to give sufficient conditions on K such that problem (P) possesses a solution. Our approach follows closely the ideas developed in Aubin-Bahri [1] and Bahri [3] where the problem of prescribing the scalar curvature on closed manifold was studied using some topological and dynamical tools of the theory of critical points at infinity (see [2]). The main idea is to use the difference of topology of the critical points at infinity (see definition below) between the level sets of the associated Euler-Lagrange functional and the main issue is under our conditions on K, there remains some difference of topology which is not due to the critical points at infinity and therefore the existence of solution of (P) in our statement.

In order to state our results, we need to fix some notations and assumptions that we are using.

In this paper, we assume that K has only non degenerate critical points $y_0, y_1, \ldots, y_h$ such that

$$K(y_0) \geq K(y_1) \geq \cdots \geq K(y_h)$$

and

$$\Delta K(y_i) \neq 0 \text{ for each } i = 0, \ldots, h.$$

We denote by F_+ the set

$$F_+ = \{y_i / \ \nabla K(y_i) = 0 \ \text{ and } \Delta K(y_i) < 0\}.$$

Let Z be a pseudo-gradient of K of Morse-Smale type (that is the intersections of the stable and the unstable manifolds of the critical points of K are transverse).

$(\mathbf{H_0})$ Assume that $W_s(y_i) \cap W_u(y_j) = \emptyset$ for each critical points y_i, and y_j such that $y_i \in F_+$ and $y_j \notin F_+$, where $W_s(y_i)$ is the stable manifold of y_i and $W_u(y_j)$ denotes the unstable manifold of y_j for the pseudo-gradient Z.

Remark 1.1 Recall that $W_s(y_i)$ is the set of all points $x \in S^n$ attracted by y_i through the decreasing flow $\eta(.,x)$ of $Z : \{x \in S^n, \ \text{s. t} \ \eta(s,x) \to y_i \text{ when } s \to +\infty\}$ and $W_u(y_i) = \{x \in S^n \ \text{s. t} \ \eta(s,x) \to y_i \text{ when } s \to -\infty\}$.

For each index $i, \ 0 \leq i \leq h$, we let

$$X_i = \bigcup_{\substack{0 \leq s \leq i \\ y_s \in F_+}} \overline{W}_s(y_s).$$

Our first main result is the following,

Theorem 1.1 *Let $n = 5$. Suppose that under the assumption $(\mathbf{H_0})$, there exists an index $i, \ 0 \leq i \leq h$ satisfying the two following conditions:*
$(\mathbf{H_1})$ *X_i is not contractible, we denote by k the dimension of the first nontrivial reduced homological group.*
$(\mathbf{H_2})$ *For each $y_j \in F_+$ such that $j \in \{i+1, \dots, h\}$, we have*

$$\frac{1}{K(y_j)^{\frac{1}{4}}} > \frac{1}{K(y_0)^{\frac{1}{4}}} + \frac{1}{K(y_i)^{\frac{1}{4}}}.$$

Then (P) has a solution of Morse index $\geq k$.

This result allows us to prove the following:

Corollary 1.1 *Let $n = 5$. Under the assumption $(\mathbf{H_0})$, if*

$$\sum_{y_j \in F_+} (-1)^{5 - ind(y_j)} \neq 1$$

then (P) has a solution. Here, $ind(y_j)$ is the Morse index of K at y_j.

Remark 1.2 This kind of result given by Corollary 1.1 was first proved by Bahri-Coron [5], concerning the scalar curvature problem on the standard three dimensional sphere. Their approach consisted of studying the critical points at infinity of the associated variational problem, computing their total Morse index and comparing this index to the Euler-poincaré characteristic of the space of variation. For the four dimensional case, the result was proved by the second author and Ben

Ayed-Chen-Hammami [8]. On the other hand, this kind of result has been proved by Djadli-Malchiodi-Ould Ahmedou [25] for the Paneitz curvature problem on five and six dimensional spheres. Their approach involved a fine blow up analysis of some subcritical approximations and the use of the topological degree tools.

In Remark 1.3, we exhibit a situation in which Theorem 1.1 applies but not Djadli-Malchiodi-Ould Ahmedou's [25]

To state our next result, we need to introduce the following assumptions:

$(\mathbf{H_3})$ Assume that there exist $G_+ \subset F_+$ such that

$$X = \bigcup_{y_i \in G_+} \overline{W_s}(y_i)$$

is a stratified set in dimension $k \geq 1$ without boundary (in the topological sense, i.e. $X \in S_k(S^n)$, the group of chains of dimension k and $\partial X = 0$).

$(\mathbf{H_4})$ Assume that for all $y \in F_+ \setminus G_+$ we have,

$$ind(y) \neq \{n-k, n-(k+1)\}.$$

Theorem 1.2 *Let $n = 5$. Under the assumptions $(\mathbf{H_0}), (\mathbf{H_3})$ and $(\mathbf{H_4})$, (P) admits a solution.*

Remark 1.3 Here, we give some situation where the result of Corollary 1.1., (see also Theorem 1.7 of [25]), does not give solution to problem (P). But, by Theorem 1.1. or Theorem 1.2. we derive that problem (P) admits a solution.

For this, let $K : S^5 \longrightarrow \mathbb{R}$, be a function such that $F_+ = \{y_0, y_1, y_2\}$ with, $K(y_0) \geq K(y_1) \geq K(y_2)$, $ind(y_0) = 5$, $ind(y_1) \in \{1, 3\}$, $ind(y_2) \in \{2, 4\}$ and $K(y) < K(y_1)$ for any critical point y of K which is not in F_+. It is easy to see that,

$$\sum_{y_j \in F_+} (-1)^{5-ind(y_j)} = 1.$$

Let

$$X_1 = \overline{W}_s(y_1) = W_s(y_1) \cup \{y_0\}.$$

X_1 is a stratified set in dimension ≥ 1, without boundary. Thus, X_1 is not contractible. We distinguish two cases:

First Case: If $K(y_2)^{-1/4} > K(y_0)^{-1/4} + K(y_1)^{-1/4}$, we deduce from Theorem 1.1 that problem (P) admits a solution.

Second Case: If $\Big(ind(y_1), ind(y_2)\Big) = (1, 2)$ or $(3, 4)$, by Theorem 1.2. we derive that (P) has a solution (in this case we choose $G_+ = \{y_1\}$).

Remark 1.4 For $n=6$, we set the following hypothesis:

$(\mathbf{H_5})$ $\dfrac{\Delta K(y_i)\Delta K(y_j)}{K(y_i)K(y_j)} < \Big(15G(y_i,y_j)\Big)^2$ for each $y_i \neq y_j$ in F_+, where G denote the green function associated to the operator $\mathcal{P}$. Under the assumption $(\mathbf{H_5})$, the results of Theorem 1.1 and Theorem 1.2 are true in dimension six.

In the above results, we have assumed that the hypotheses $(\mathbf{H_0})$ holds; in the second part of this paper, we propose to handle problem (P) without the assumption $(\mathbf{H_0})$. To this aim we give a contribution in the same direction as in the paper of Aubin-Bahri [1] concerning the scalar curvature problem.

Let $\ell' \leq \ell \leq h$, ℓ be chosen so that for each $s \geq \ell+1$, $y_s \notin F_+$. We let

$$X' = \bigcup_{\substack{0\leq i\leq \ell' \\ y_i\in F_+}} W_s(y_i) \quad \text{and} \quad X = \bigcup_{\substack{0\leq i\leq \ell \\ y_i\in F_+}} \overline{W}_s(y_i).$$

Let

$$g : X' \longrightarrow X \text{ the natural embedding.}$$

We assume

$(\mathbf{A_1})$ X' is contractible in $T = \bigcup_{0\leq i\leq \ell} \overline{W}_s(y_i)$. Now we define the set

$$I_- = \Big\{y_p,\ 0\leq p\leq \ell, \text{ such that } y_p \notin F_+ \text{ and there exists } y_j,\ 0\leq j\leq \ell,$$
$$\text{satisfying } y_j \in F_+ \text{ and } W_s(y_j)\cap W_u(y_p)\neq \emptyset\Big\}.$$

Let

$$\Gamma = \bigcup_{y_p\in I_-} W_s(y_p).$$

We then have

Theorem 1.3 *Let $n=5$. Under the assumption $(\mathbf{A_1})$ if we have:*
$(\mathbf{A_2})$ g_* is nontrivial in dimension m and the critical points of Γ of index $5-m$ has an intersection number equal to zero (modulo 2) with the critical points of index $5-(m+1)$ in X. Then (P) has a solution.

The last result can be extended to higher dimensional spheres.

Theorem 1.4 *Let $n\geq 6$. Adding to the hypothesis $(\mathbf{A_1})$ and $(\mathbf{A_2})$ the condition* $(\mathbf{A_3})$

$$\frac{K(y_0)}{K(y_\ell)} \leq 2^{\frac{4}{n-4}},$$

then K is a Paneitz curvature for a conformal metric to the standard metric of S^n.

Remark 1.5 To see how to construct an example of a function K satisfying the assumptions of Theorem 1.3 and Theorem 1.4, we refer the interested reader to [1].

The remainder of the present paper is organized as follows. In section 2, we set up the general framework of the problem (P), and we recall the characterization of the critical points at infinity of the associated variational problem. Section 3 will be devoted to the proof of our results.

2 Variational structure and preliminaries

In this section, we recall the functional setting and the variational problem and it's main features. The functional is

$$J(u)=\frac{\displaystyle\int_{S^n}\mathcal{P}u.udv}{\Big(\displaystyle\int_{S^n}Ku^{\frac{2n}{n-4}}dv\Big)^{\frac{n-4}{n}}}\,,\qquad u\in H_2^2(S^n)$$

where dv is the volume element of (S^n,g). The space $H_2^2(S^n)$ is equipped with the norm:

$$\|u\|^2=\int_{S^n}\mathcal{P}u.udv=\int_{S^n}|\Delta u|^2+c_n\int_{S^n}|\nabla u|^2+d_n\int_{S^n}|u|^2.$$

Problem (P) is equivalent to finding critical points of J subject to the constraint $u\in\Sigma^+$ where,

$$\Sigma^+=\Big\{u\in H_2^2(S^n)/\,u>0,\ \text{ and }\|u\|=1\Big\}.$$

The functional J is known not to satisfy the Palais-Smale condition which leads to the failure of the classical existence mechanisms. The failure of the Palais-Smale condition can be described using arguments similar to those of of [15], [32] and [38], see also [30]. In order to characterize the sequence failing the Palais-Smale condition, we need to introduce some notations.

For $a\in S^n$ and $\lambda>0$, let

$$\tilde{\delta}_{(a,\lambda)}(x)=c_n\frac{\lambda^{\frac{n-4}{2}}}{(\lambda^2+1+(1-\lambda^2)\cos d(a,x))^{\frac{n-4}{2}}}\,,$$

where d is the geodesic distance on (S^n,g) and,

$$c_n=\Big((n-4)(n-2)n(n+2)\Big)^{\frac{n-4}{8}}.$$

After performing a stereographic projection π with the point $-a$ as pole, the function $\tilde{\delta}_{(a,\lambda)}$ is transformed in to

$$\delta_{(0,\lambda)}=c_n\frac{\lambda^{\frac{n-4}{2}}}{(1+\lambda^2|y|^2)^{\frac{n-4}{2}}}\,,$$

which is solution of the following problem,

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \ u > 0 \text{ on } \mathbb{R}^n \quad \text{(see [33])}.$$

For $p \in \mathbb{N}$ and $\varepsilon > 0$, we define the set $V(p, \varepsilon)$ of the function $u \in \Sigma^+$ satisfying: there exist $\alpha_1, \dots, \alpha_p > 0, \ a_1, \dots, a_p \in S^n, \text{ and } \lambda_1, \dots, \lambda_p > \varepsilon^{-1}$ with

$$\Big\| u - \sum_{i=1}^{p} \alpha_i \tilde{\delta}_{(a_i, \lambda_i)} \Big\| < \varepsilon, \ \Big| 1 - J(u)^{\frac{n}{n-4}} \alpha_i^{\frac{8}{n-4}} K(a_i) \Big| < \varepsilon, \text{ and } \quad \varepsilon_{ij} < \varepsilon \text{ for each } i \neq j$$

where,

$$\varepsilon_{ij}^{-1} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{\lambda_i \lambda_j}{2} (1 - cos\ d(a_i, a_j)) \Big)^{\frac{n-4}{2}}.$$

The failure of the Palais-Smale condition can be described as follows,

Proposition 2.1 ([30])*Assume that J has no critical point in Σ^+. Let $(u_j) \in \Sigma^+$ be a sequence such that $\partial J(u_j)$ tends to zero and $J(u_j)$ is bounded. Then, there exist an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_j > 0, (\varepsilon_j$ tends to zero), and an extract of u_j's, again denoted by u_j such that $u_j \in V(p, \varepsilon_j)$.*

The following result defines a parameterization of the set $V(p, \varepsilon)$.

Proposition 2.2 *There exists $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $u \in V(p, \varepsilon)$, then the problem*

$$\min \Big\{ \| u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \|, \alpha_i > 0, \ \lambda_i > 0, \ a_i \in S^n \Big\}$$

has a unique solution $(\bar{\alpha}, \bar{\lambda}, \bar{a})$ (up to permutation). In particular, we can write u as follows: $u = \sum_{i=1}^{p} \bar{\alpha}_i \delta_{(\bar{a}_i, \bar{\lambda}_i)} + v$, where v belongs to $H_2^2(S^n)$ and satisfies

$$(V_0) \begin{cases} \Big(v, \delta_{(a_i, \lambda_i)} \Big) & = 0 \\ \Big(v, \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial a_i} \Big) & = 0 \qquad \qquad \textit{for each } i = 1, \dots, p \\ \Big(v, \frac{\partial \delta_{(a_i, \lambda_i)}}{\partial \lambda_i} \Big) & = 0. \end{cases}$$

The proof of Proposition 2.2 is similar up to minor modifications to the corresponding statements in [3] and [5] (see also Lemma 2.2 of [26] for the fourth order elliptic equation).

Definition 2.1 ([2]) *The critical points at infinity* of the associated variational problem are the noncompact orbits of the gradient flow associated to the function $(-J)$. It follows from Proposition 2.1 that under the assumption that J has no critical points, these noncompact orbits remain in $V(p, \varepsilon(s))$, where $\varepsilon(s)$ is given function such that $\varepsilon(s)$ goes to zero when s goes to $+\infty$.

The following proposition which is proved in [11] (see Proposition 4.3 of [11]), characterizes the critical points at infinity of the associated variational problem. Their argument involves the construction of a pseudo-gradient for the associated variational problem, the Palais-Smale condition is satisfied along the decreasing flow lines as long as these flow lines do not enter a neighborhood of the critical points at infinity.

Proposition 2.3 Let $n = 5$. *Assume that J has no critical point in Σ^+. Then the only critical points at infinity are $\delta_{(y,\infty)}$ such that $y \in F_+$, ($\delta_{(y,\infty)}$ correspond to the limit of the noncompact orbit, notation of [3] p335). Such a critical point at infinity has a Morse index equal to $5 - ind(y)$. Its level is $S_5^{4/5}/K(y)^{1/5}$, where $S_5 = \int_{\mathbb{R}^5} \delta_{(0,1)}^{10} dx$.*

Remark 2.1 Proposition 2.3 can follow from section 3.2 of [25], where the authors studied problem (P) for $n = 5$ and proving in this section that blow ups of solutions (of subcritical approximations) occur at one point.

We end this section by the following lemma the proof of which is very similar to the proof of Corollary B.3 of [6], see also [3].

Lemma 2.1 *Let a_1, a_2 in S^n, $\alpha_1, \alpha_2 > 0$, and λ large enough. For $u = \alpha_1 \delta_{(a_1,\lambda)} + \alpha_2 \delta_{(a_2,\lambda)}$, we have*

$$J(u) \leq \Big(S_n \sum_{i=1}^{2} \frac{1}{K(a_i)^{\frac{n-4}{4}}} \Big)^{\frac{4}{n}} (1 + o(1)).$$

3 Proof of Theorems

Proof of Theorem 1.1. We argue by contradiction. We assume that J has no critical point in Σ^+. Using Proposition 2.3, the only critical points at infinity of J correspond to $\delta_{(y,\infty)}$ such that $y \in F_+$. The level of such critical point at infinity is $C_\infty(y) = S_5^{4/5}(\frac{1}{K(y)^{1/5}})$. We then have the following Morse Lemma at infinity for J in $V(1, \varepsilon)$ (we consider the case of a single masse); it follows from the corresponding statement in [10] (see Theorem 1.2 of [10], see also [3]).

For $\varepsilon > 0$ small enough and $\alpha \tilde{\delta}_{(a,\lambda)} + v \in V(1, \varepsilon)$, we can find a change of variable

$$(a, \lambda, v) \longrightarrow (\tilde{a}, \tilde{\lambda}, V)$$

where V belongs to a neighborhood of zero in a fixed Hilbert space, such that

$$J\Big(\alpha \tilde{\delta}_{(a,\lambda)} + v\Big) = J\Big(\alpha \tilde{\delta}_{(\tilde{a},\tilde{\lambda})}\Big) + \|V\|^2. \tag{3.1}$$

Moreover, (3.1) can be improved where the concentration point is near a critical point y of K with $y \in F_+$, leading to the following normal form:

$$J\Big(\alpha \tilde{\delta}_{(\tilde{a},\tilde{\lambda})}\Big) = \frac{S_5^{4/5}}{K(\tilde{a})^{1/5}} \Big(1 - \frac{4c(1-\eta)\Delta K(y)}{5 S_5 \tilde{\lambda}^2 K(y)}\Big)$$

where $c = \frac{1}{10}\int_{\mathbb{R}^5} |x|^2 \delta_{(0,1)}^{10}$ and η is a small positive constant.

Thus, the unstable manifolds at infinity for the vector field $(-\partial J)$ of such critical points at infinity $W_u(y)_\infty$ can be described as the product of $W_s(y)$ (for a decreasing pseudo-gradient of K) by $[A, \infty[$ domain of the variable λ, for some positive number A large enough. Let

$$C_\infty(y_0, y_i) = S_5^{4/5}\Big(\frac{1}{K(y_0)^{1/4}} + \frac{1}{K(y_i)^{1/4}}\Big)^{1/5}.$$

It follows from assumption $(\mathbf{H_2})$ of Theorem 1.1 that the only critical points at infinity of J under the level $c_1 = C_\infty(y_0, y_i) + \varepsilon$, for ε small enough, are $\delta_{(y_j,\infty)}$ where $y_j \in F_+$ and $j \in \{0, \dots, i\}$. Since J has no critical points, it follows from Proposition 7.24 and Theorem 8.2 of [7], that $J_{c_1} = \{u \in \Sigma^+ / J(u) \leq c_1\}$ retracts by deformation onto

$$(X_i)_\infty = \bigcup_{\substack{0\leq j\leq i \\ y_j \in F_+}} \overline{W}_u(y_j)_\infty,$$

which can be parameterized by $X_i \times [A, \infty[$, where

$$X_i = \bigcup_{\substack{0\leq j\leq i \\ y_j \in F_+}} \overline{W}_s(y_j).$$

Observe that by assumption $(\mathbf{H_1})$ of Theorem 1.1, X_i is not a contractible set.

Now, we claim that $(X_i)_\infty$ is contractible in J_{c_1}. Indeed, let

$$\begin{aligned} f : [0,1] \times (X_i)_\infty &\longrightarrow \Sigma^+ \\ (t, x, \lambda) &\longmapsto \frac{t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)}}{\Big\|t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)}\Big\|}. \end{aligned}$$

f is continuous and satisfies for $t = 0$, $f(0, x, \lambda) = \frac{1}{S}\delta_{(x,\lambda)} \in (X_i)_\infty$, and $f(1, x, \lambda) = \frac{1}{S}\delta_{(y_0,\lambda)}$.

For $a_1, a_2 \in S^5$, $\alpha_1, \alpha_2 > 0$ and λ large enough, let $u = \alpha_1\delta_{(a_1,\lambda)} + \alpha_2\delta_{(a_2,\lambda)}$. By Lemma 2.1 we have the following estimate:

$$J\Big(\frac{u}{\|u\|}\Big) \leq \Big(S(\frac{1}{K(a_1)^{\frac{1}{4}}} + \frac{1}{K(a_2)^{\frac{1}{4}}})\Big)^{\frac{4}{5}}(1 + o(1)).$$

Hence,

$$J\Big(f(t, x, \lambda)\Big) \leq \Big(S(\frac{1}{K(y_0)^{\frac{1}{4}}} + \frac{1}{K(x)^{\frac{1}{4}}})\Big)^{\frac{4}{5}}(1 + o(1)).$$

Since $K(x) \geq K(y_i)$ for any $x \in X_i$, we derive

$$J\Big(f(t, x, \lambda)\Big) < c_1 \text{ for any } (t, x, \lambda) \in [0,1] \times (X_i)_\infty.$$

Thus, the contraction f is performed under the level c_1. We derive that $(X_i)_\infty$ is contractible in J_{c_1}, which retractss by deformation on $(X_i)_\infty$. Therefore $(X_i)_\infty$ is contractible leading to the contractibility of X_i which is a contradiction. Hence, J has a critical point. Arguing as in the scalar curvature case (see [8], see also the proof of Theorem 1.1 of [11]), we derive that such critical point is positive. Therefore (P) has a solution. Now we are going to show that such critical point has a Morse index $\geq k$.

Arguing by contradiction, we may assume that the Morse index of the solution provided in Theorem 1.1 is $\leq k-1$. Perturbing J if necessary, we may assume that all the critical points of J are non degenerate and have their Morse index $\leq k-1$. Such critical points do not change the homological group in dimension k of the level of J.

As $(X_i)_\infty$ defines a homological class in dimension k which is non trivial in $J_{C_\infty(y_i)+\varepsilon}$ for ε small enough but $(X_i)_\infty$ defines a homological class in dimension k, which is trivial in J_{c_1}, our result follows and the proof of Theorem 1.1 is thereby complete.

Proof of Corollary 1.1. Assume that P has no solution. Observe that for $i = h$, we have

$$X_h = \bigcup_{y_j \in F_+} \overline{W}_s(y_j).$$

Let $\chi(X_h)$ the Euler-poincaré characteristic of X_h. We claim that

$$\chi(X_h) = \sum_{y_j \in F_+} (-1)^{5-ind(y_j)}.$$

Indeed, by the same arguments used in [5] (see the proof of Theorem 1 of [5]), we have $\chi(\Sigma^+) = \sum_{y_j \in F_+} (-1)^{5-ind(y_j)}$. Also, Σ^+ retractss by deformation on $\bigcup_{y_j \in F_+} \overline{W}_u(y_j)_\infty :=$ $(X_h)_\infty$ (see sections 7 and 8 of [7]), therefore $\chi(\Sigma^+) = \chi((X_h)_\infty)$. Since $(X_h)_\infty$ can be described as $(X_h) \times [A, +\infty[$, for some positive number A large enough, our claim follows. Under the assumption of Corollary 1.1, we derive that X_h is not contractible. Thus, the result of Corollary 1.1 follows from Theorem 1.1.

Proof of Theorem 1.2. Let

$$X = \bigcup_{y_i \in G_+} \overline{W_s}(y_i).$$

By the assumption $(\mathbf{H_3})$ of Theorem 1.2, X is a stratified set in dimension $k \geq 1$ without boundary. For λ large enough, we define the following set

$$C_\lambda(y_0, X) = \Big\{\alpha\tilde{\delta}_{(y_0,\lambda)} + (1-\lambda)\alpha\tilde{\delta}_{(x,\lambda)},\ x \in X \ \text{ and } \alpha \in [0,1]\Big\}.$$

$C_\lambda(y_0, X)$ is a contractible manifold in dimension $k+1$, that is its singularities arise in dimension $k-1$ and lower.

Let

$$X_\infty = \bigcup_{y_i \in G_+} \overline{W_u}(y_i)_\infty$$

where $W_u(y)_\infty$ denote the unstable manifold at infinity of the critical points at infinity $\tilde{\delta}_{(y,\infty)}$. Using the same arguments as those used in Theorem 1.1, we derive that

$$X_\infty = X \times [A, +\infty[.$$

We argue by contradiction. Suppose that J has no critical points in Σ^+. It follows from Proposition 7.24 and Theorem 8.2 of [7], that $C_\lambda(y_0, X)$ retracts by deformation on $\bigcup_{y \in H} \overline{W}_u(y)_\infty$, where

$$H = \Big\{ y \in F_+ / \quad C_\lambda(y_0, X) \cap W_s(y)_\infty \neq \emptyset \Big\}.$$

$W_s(y)_\infty$ denote the stable manifold at infinity of the critical point at infinity $\delta_{(y,\infty)}$ (the definitions of the stable manifold and the unstable manifold are very clear for usual critical points (see Remark 1.1). For critical points at infinity which in our problem are combinations of classical critical points with a 1-dimensional asymptote correspond to λ (see [3]).

Since $C_\lambda(y_0, X)$ is a manifold in dimension $k+1$, this manifold can be assumed to avoid the unstable manifold of every critical point at infinity $\tilde{\delta}_{(y,\infty)}$ of Morse index $> k+1$ (i.e., $ind(y) < 5 - (k+1)$). Thus,

$$H \subset \{ y \in F_+ / \, ind(y) \geq 5 - k \}.$$

More precisely, $C_\lambda(y_0, X)$ retracts by deformation on $X_\infty \cup D_\infty$, where $D_\infty = \bigcup_{y \in D} W_u(y)_\infty$ and $D = \{ y \in H \setminus G_+ \}$.

Using the assumption $(\mathbf{H_4})$ of Theorem 1.2, we derive that

$$ind(y) > 5 - k \text{ for each } y \in D.$$

Thus, the Morse index at infinity of the critical point at infinity $\tilde{\delta}_{(y,\infty)}$, $y \in D$ is $\leq k-1$, and therefore D_∞ is a stratified set of dimension at most $k-1$. Since $C_\lambda(y_0, X)$ is a contractible set, then $H_k(X_\infty \cup D_\infty) = 0$ for all $* \in \mathbb{N}^*$. Using the exact homology sequence of $\Big(X_\infty \cup D_\infty, X_\infty\Big)$, we have

$$\ldots \longrightarrow H_{k+1}(X_\infty \cup D_\infty) \longrightarrow H_{k+1}(X_\infty \cup D_\infty, X_\infty) \longrightarrow H_k(X_\infty) \longrightarrow H_k(X_\infty \cup D_\infty) \longrightarrow \ldots$$

Since $H_*\Big(X_\infty \cup D_\infty\Big) = 0$ for all $* \in \mathbb{N}^*$, then

$$H_k\Big(X_\infty\Big) = H_{k+1}\Big(X_\infty \cup D_\infty, X_\infty\Big).$$

In addition, $\Big(X_\infty \cup D_\infty, X_\infty\Big)$ is a stratified set of dimension at most k, so $H_{k+1}\Big(X_\infty \cup D_\infty, X_\infty\Big) = 0$. Thus, $H_k(X_\infty) = 0$ and therefore $H_k(X) = 0$ which is in contradiction to the assumption $(\mathbf{H_4})$ of Theorem. Hence our result follows.

Proof of Remark 1.4. Observe that under the assumption $(\mathbf{H_5})$, the least eigenvalue of the $k \times k$ symmetric matrix (defined in [25] p 5) is negative (when $k \geq 2$ and under the assumption that the matrix is non degenerate). Using now Proposition 3.4 of [25], we derive that the only critical points at infinity of J in Σ^+ correspond to $\tilde{\delta}_{(y,\infty)}$, $y \in F_+$. Thus our Remark follows from the above arguments.

Proof of Theorem 1.3. Arguing by contradiction and assuming that J has no critical point in Σ^+, it follows from Proposition 2.3 that the only critical points at infinity of the associated variational problem are in one to one correspondence with the critical points y_j of K such that $y_j \in F_+$.

For $k \in \mathbb{N}$, we denote by A_k the set

$$A_k = \{y \in S^5,\ s.t.\ \nabla K(y) = 0, \quad \text{and } ind(y) = 5 - k\}.$$

Let ∂ the boundary operator in the sense of Floer [28], ∂ acting on critical points of K. For any w critical point of index ℓ, we have

$$\partial(w) = \partial(W_s(w)) = \sum_{w' \ of index\ \ell+1} i(w, w').W_s(w'),$$

where $W_s(w)$ is the stable manifold of w for Z (pseudo-gradient for K), viewed as a simplex of codimension ℓ of S^5, and $i(w, w')$ is the number of flow-lines (modulo 2) in $W_u(w') \cap W_s(w)$. The operator ∂ can be found in Milnor [34].

Under the assumption $(\mathbf{A_2})$ of Theorem 1.3, let $\mathcal{O}_{X'}$ the $\mathbb{Z}_2$ homology orientation class of X' in dimension m which survive in X. $\mathcal{O}_{X'}$ is defined as a sum with $\mathbb{Z}_2$ coefficients of stable manifolds of critical points of K in X' of index $5 - m$.

$$\mathcal{O}_{X'} = \sum_{x_m \in X'_m} W_s(x_m),$$

where X'_m is a subset of $X' \cap A_m$. The boundary of $\mathcal{O}_{X'}$ is given by

$$\partial(\mathcal{O}_{X'}) = \sum_{x_{m-1} \in A_{m-1}} \Big(\sum_{x_m \in X'_m} i(x_m, x_{m-1}) \Big).W_s(x_{m-1}).$$

Since $\mathcal{O}_{X'}$ is a cycle, we derive that $\partial(\mathcal{O}_{X'}) = 0$ and therefore,

$$\sum_{x_m \in X'_m} i(x_m, x_{m-1}) = 0 \quad \text{for any} \quad x_{m-1} \in A_{m-1}. \tag{3.2}$$

On the other hand, given a decreasing pseudo-gradient V for J in Σ^+, let

$$(\mathcal{O}_{X'})_\infty = \sum_{x_m \in X'_m} W_u^\infty(x_m^\infty) \tag{3.3}$$

where $W_u^\infty(x_m^\infty)$ is the unstable manifolds for V of the critical point at infinity x_m^∞ ($x_m^\infty := \delta_{(x_m,+\infty)}$ notation of [3]). $(\mathcal{O}_{X'})_\infty$ is the equivalent of $\mathcal{O}_{X'}$ in the space of variation Σ^+.

Following again A. Floer [28] (see also J. Milnor [34] and C. C. Conley [23]), for w^∞ a critical point at infinity of index k, We define
$\partial_\infty(w^\infty)$ to be

$$\partial_\infty\Big(W_u^\infty(w^\infty)\Big) = \sum_{w'^\infty \text{ of index } k-1} i(w^\infty, w'^\infty).W_u(w'^\infty)$$

where $i(w^\infty, w'^\infty)$ is the intersection number of the critical points at infinity w^∞ and w'^∞ in an intermediate level surface of $W_u^\infty(w^\infty)$ with $W_s(w'^\infty)$ (modulo 2).

We claim that $\partial_\infty(\mathcal{O}'_X)_\infty = 0$. Indeed, from the description of the critical points at infinity given by Proposition 2.3, we have

$$\begin{aligned}\partial_\infty(\mathcal{O}'_X)_\infty &= \sum_{x_m\in X'_m} \partial_\infty\Big(W_u^\infty(x_m^\infty)\Big)\\ &= \sum_{x_{m-1}\in A_{m-1}\cap F_+} \Big(\sum_{x_m\in X'_m} i(x_m^\infty, x_{m-1}^\infty)\Big).W_u^\infty(x_{m-1}^\infty).\end{aligned}$$

The intersection number between the critical points at infinity X_m^∞ and X_{m-1}^∞ can be computed as follows:

$$i(x_m^\infty, x_{m-1}^\infty) = i(x_m, x_{m-1}) + \sum_{y_{m-1}\in A_{m-1}\cap\Gamma} i(x_m, y_{m-1}).\, i(y_{m-1}^\infty, x_{m-1}^\infty).$$

y_{m-1}^∞ is a (false) critical point at infinity, since $y_{m-1}\notin F_+$ and $i(y_{m-1}^\infty, x_{m-1}^\infty)$ is the intersection number (modulo 2) in an intermediate level surface of $W_u^\infty(y_{m-1}^\infty)$ with $W_s^\infty(x_{m-1}^\infty)$.

$W_u^\infty(y_{m-1}^\infty)$ is not really an unstable manifold at infinity, but is the set of flow lines emanating from a neighborhood of the point y_{m-1}^∞, (see Lemma 1.6 of [4]). Therefore,

$$\begin{aligned}\sum_{x_m\in X'_m} i(x_m^\infty, x_{m-1}^\infty) &= \sum_{x_m\in X'_m} i(x_m, x_{m-1}) \quad +\\ &\sum_{y_{m-1}\in A_{m-1}\cap\Gamma}\Big(\sum_{x_m\in X'_m} i(x_m, y_{m-1})\Big).\, i(y_{m-1}^\infty, x_{m-1}^\infty).\end{aligned}$$

Using (3.2), we derive that

$$\sum_{x_m\in X'_m} i(x_m^\infty, x_{m-1}^\infty) = 0,$$

and thus $\partial_\infty(\mathcal{O}'_X)_\infty = 0$. Hence our claim follows.

For λ large enough, we define

$$\begin{aligned} f_\lambda : S^5 &\longrightarrow \Sigma^+\\ a &\longmapsto \frac{\delta_{(a,\lambda)}}{\|\delta_{(a,\lambda)}\|}.\end{aligned}$$

We have $(\mathcal{O}_{X'})_\infty = f_\lambda(\mathcal{O}_{X'})$ in $\mathrm{S}_m(\Sigma^+)$, the group of chains of dimension m. Indeed, $\mathcal{O}_{X'}$ is a cycle; so the intersection number with any critical point in A_{m-1} (so in $A_{m-1}\cap\Gamma$) is equal to zero. Thus $(\mathcal{O}_{X'})_\infty$ is made only by direct connection (see definition 2.3 of [9], see also [4]). In addition, $W_u^\infty(x_m^\infty)$, by the argument used in the proof of Theorem 1.1, can be viewed as combination of $W_s(x_m)$ (for K) with a 1-dimensional asymptote (the variable is λ). Since this exists only as a limit, we will use $f_\lambda(W_s(x_m))$, λ very large, for $W_u^\infty(x_m^\infty)$. Thus, the identity follows.

By the assumption $(\mathbf{A_1})$ of Theorem 1.3, it is easy to see that $f_\lambda(X')$ is contractible in $f_\lambda(T)$. Since J has no critical points in Σ^+, it follows from Proposition 7.24 and Theorem 8.2 of [7] that $f_\lambda(T)$ retracts by deformation on

$$T_\infty := \bigcup_{\substack{0\le j\le \ell \\ y_j\in F_+}} W_u^\infty(y_j^\infty).$$

Therefore $f_\lambda(X')$ is contractible in T_∞.

Observe that $(\mathcal{O}_{X'})_\infty = f_\lambda(\mathcal{O}'_X)$ is a cycle of dimension m in $f_\lambda(X')$. Then we can contract $(\mathcal{O}'_X)_\infty$ in T_∞. Thus, let W be a chain of dimension $m+1$ in T_∞ such that the boundary of W is precisely $(\mathcal{O}'_X)_\infty$. W is defined as a sum with $\mathbb{Z}_2$ coefficient of unstable manifolds at infinity of critical points at infinity (y_j^∞) such that $y_j \in A_{m+1}\cap F_+$. We then have

$$(\mathcal{O}_{X'})_\infty = \partial\Big(\sum_{x_{m+1}\in \widetilde{A}_{m+1}} W_u^\infty(x_{m+1}^\infty)\Big)$$

where $\widetilde{A}_{m+1}$ is a subset of $A_{m+1}\cap F_+$.

$$(\mathcal{O}_{X'})_\infty = \sum_{z_m\in A_m\cap F_+}\Big(\sum_{x_{m+1}\in \widetilde{A}_{m+1}} i(x_{m+1}^\infty, z_m^\infty)\Big).\, W_u^\infty(z_m^\infty).$$

Using the above argument, we have

$$i(x_{m+1}^\infty, z_m^\infty) = i(x_{m+1}, z_m) + \sum_{\gamma_m\in A_m\cap\Gamma} i(x_{m+1},\gamma_m).\, i(\gamma_m^\infty, z_m^\infty),$$

by the assumption $(\mathbf{A_2})$ we have $i(x_{m+1}\gamma_m)=0$. Thus, we derive that,

$$i(x_{m+1}^\infty, z_m^\infty) = i(x_{m+1}, z_m).$$

We then have

$$(\mathcal{O}_{X'})_\infty = \sum_{z_m\in A_m\cap F_+}\Big(\sum_{x_{m+1}\in \widetilde{A}_{m+1}} i(x_{m+1}, z_m)\Big).\, W_u^\infty(z_m^\infty).$$

From (3.3), we have

$$\sum_{x_m\in X'_m} W_u^\infty(x_m^\infty) = \sum_{z_m\in A_m\cap F_+}\Big(\sum_{x_{m+1}\in \tilde{A}_{m+1}} i(x_{m+1}, z_m)\Big).W_u^\infty(z_m^\infty).$$

So, let $z_m \in A_m\cap F_+$. If $z_m\in X'_m$ we then have

$$\sum_{x_{m+1}\in\tilde{A}_{m+1}} i(x_{m+1}, z_m) = 1 \qquad (3.4)$$

if $z_m \notin X'_m$ we have

$$\sum_{x_{m+1}\in\tilde{A}_{m+1}} i(x_{m+1}, z_m) = 0 \qquad (3.5)$$

and if $z_m \in \Gamma$, by the assumption $(\mathbf{A_2})$ we derive that,

$$i(x_{m+1}, z_m) = 0. \qquad (3.6)$$

On the other hand, we have $\mathcal{O}_{X'} = \sum_{x_m\in X'_m} W_s(x_m)$. Using (3.4), (3.5) and (3.6), we derive that

$$\mathcal{O}_{X'} = \sum_{z_m\in A_m}\Big(\sum_{x_{m+1}\in\widetilde{A}_{m+1}} i(x_{m+1}, z_m)\Big).\, W_s(z_m)$$

where the second sum is expanded on all critical points in A_m. Using the fact that

$$\partial(W_s(x_{m+1})) = \sum_{z_m\in A_m} i(x_{m+1}, z_m).\, W_s(z_m),$$

we derive that,

$$\mathcal{O}_{X'} = \partial\Big(\sum_{x_{m+1}\in\widetilde{A}_{m+1}} W_s(x_{m+1})\Big).$$

Thus, $\mathcal{O}_{X'}$ is the boundary of chain of dimension $m+1$ of X, which is a contradiction with the assumption $(\mathbf{A_2})$ of Theorem 1.3. This concludes the proof of our Theorem.

Proof of Theorem 1.4. Under the assumption $(\mathbf{A_3})$ of Theorem 1.4, we can assume that λ is so large that all the critical points at infinity of J of two masses or more are above $f_\lambda(T)$. This is a simple consequence of the expansion of J.

Thus, the sequel of the proof of our Theorem is exactly the same as in the proof of Theorem 1.3; so our result follows.

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References

[1] T. Aubin and A. Bahri, *Une hypothese topologique pour le probleme de la courbure scalaire prescritr,* J. Math. Pure Appl. **76** (1997), 843-850.

[2] A. Bahri, Critical Point at Infinity in Some Variational Problems, Pitman Res. Notes Math, Ser **182**, Longman Sci. Tech. Harlow 1989.

[3] A. Bahri, *An invariant for yamabe-type flows with applications to scalar curvature problems in high dimensions,* A celebration of J. F. Nash Jr., Duke Math. J. **81** (1996), 323-466.

[4] A. Bahri, *The scalar curvature problem on sphere of dimension* $n \geq 7$*,* Preprint.

[5] A. Bahri and J. M. Coron, *The scalar curvature problem on the standard three dimensional spheres,* J. Funct. Anal. **95** (1991), 106-172.

[6] A. Bahri and J. M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of topology of the domain,* Comm. Pure Appli. Math. **41** (1988), 255-294.

[7] A. Bahri and P. Rabinowitz, *Periodic orbits of hamiltonian systems of three body type,* Ann. Inst. H. Poincar Anal. Non Linaire **8** (1991), 561-649.

[8] M. Ben Ayed, Y. Chen, H. Chtioui and M. Hammami, *On the prescribed scalar curvature problem on 4-manifolds,* Duke Math. J. **84** (1996), 633-677.

[9] M. Ben Ayed, H. Chtioui and M. Hammami, *The scalar curvature problem on higher dimensional spheres,* Duke Math. J. **93** (1998), 377-424.

[10] M. Ben Ayed, H. Chtioui and M. Hammami, *Morse lemma at infinity for Yamabe-type problems on domains of* $\mathbb{R}^n$*,* Ann. Inst. H. Poincar Anal. Non Lineaire, (2003), 543-577.

[11] M. Ben Ayed and K. El Mehdi, *The Paneitz curvature problem on lower dimensional spheres,* Preprint ICTP, Trieste, Italy, IC/2003/48 (to appear in Ann. of Global Analysis and Geometry).

[12] T. P. Branson, *Differential operators canonically associated to a conformal structure,* Math. Scand. **57** (1985), 293-345.

[13] T. P. Branson, *Group representations arising from Lorentz conformal geometry,* J. Funct. Anal. **74** (1987), 199-291.

[14] T. P. Branson, S. A. Chang and P. C. Yang, *Estimates and extremal problems for the log-determinant on 4-manifolds,* Comm. Math. Phys. **149** (1992), 241-262.

[15] H. Brezis and J. M. Coron, *Convergence of solutions of H-systems or how to blow bubbles,* Arch. Rational Mech. Anal. **89**, (1985), 21-56.

[16] S. A. Chang, *On a fourth order PED in conformal geometry,* Preprint, 1997.

[17] S. A. Chang, M. J. Gursky and P. C. Yang, *Regularity of a fourth order non linear PDE with critical exponent,* Amer. J. Math. **121** (1999), 215-257.

[18] S. A. Chang, M. Gursky and P. Yang, *The scalar curvature equation on 2- and 3-spheres,* Calc. Var. Partial Differential Equations **1** (1993), no. 2, 205-229.

[19] S. A. Chang, J. Qing and P. C. Yang, *On the chern-Gauss-Bonnet integral for conformal metrics on* $\mathbb{R}^4$*,* Duke. Math. J. **103** (2000), 523-544.

[20] S. A. Chang, J. Qing and P. C. Yang, *Compactification for a class of conformally flat 4-manifolds,* Invent. Math. **142** (2000), 65-93.

[21] S. A. Chang and P. C. Yang, *Extremal metrics of zeta fonctional determinants on 4-manifolds,* Annals of Mathematics, **142** (1995), 171-212.

[22] S. A. Chang and P. C. Yang, *On a fourth order curvature invariant,* Spectral problems in Geometry and Arithmetic, Contemporary Math. **237** (1999), 9-28.

[23] C. C. Conley, *Isolated invariant sets and the Morse index,* CBMS Reg. conf-Series in Math, **38**, (1978), AMS.

[24] Z. Djadli, E. Hebey and M. Ledoux, *Paneitz-type operators and applications,* Duke Math. J. **104** (2000), 129-169.

[25] Z. Djadli, A. Malchiodi and M. Ould Ahmedou, *Prescribing a fourth order conformal invariant on the standard sphere,* **Part II:** *blow up analysis and applications,* Annali della Scuola Normale Sup. di Pisa. **5** (2002), 387-434.

[26] F. Ebobisse and M. Ould Ahmedou, *On a nonlinear fourth order elliptic equation involving the critical Sobolev exponent,* Nonlinear analysis **52** (2003), 1535-1552.

[27] V. Felli, *Existence of conformal metrics on S^n with prescribed fourth-order invariant,* Adv. Differential Equations. **7** (2002), 47-76.

[28] A. Floer, *Cuplength Estimates on Lagrangian intersections comm,* On Pure and Applied Math, vol XLII No 4, (1989).

[29] M. J. Gursky, *the weyl functional, de Rham cohomology and Khaler-Einstein metrics,* Ann. of Math. **148** (1998), 315-337.

[30] E. Hebey and F. Robert, *Cercivity and Struwe's compactness for Paneitz type operators with constant coefficients,* Calc. Var. (2001).

[31] E. Hebey and F. Robert, *Compactness and global estimates for the geometric Paneitz equation in high dimensions,* Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 135-141.

[32] P. L. Lions, *The concentration compactness principle in the calculus of variations. The limit case,* Rev. Mat. Iberoamericana **1** (1985), I: 165-201, II: 45-121.

[33] C. S. Lin, *A classification of solution of a conformally invariant fourth order equation in $\mathbb{R}^n$,* Commentari Mathematici Helvetici **73**(1998), 206-231.

[34] J. Milnor, *Lectures on the h-cobordism theorem,* Princeton University Press, (1965).

[35] S. Paneitz, *A quadratic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds,* Preprint.

[36] J. Qing and D. Raske, *Compactness for conformal metrics with Constant Q curvature on locally conformally flat manifolds,* to appear in Calculus of Variations.

[37] R. Schoen and D. Zhang, *Prescribed scalar curvature on the n-sphere* , Calc. Var. Partial Differential Equations **4** (1996), no. 1, 1-25.

[38] M. Struwe, *A global compactness result for elliptic boundary value problems involving nonlinearities,* Math. Z. **187** (1984), 511-517.

[39] J. Wei and X. Xu, *On conformal deformations of metrics on S^n,* J. Funct. Anal. **157** (1998), 292-325.

[40] Y. Y. Li, *Prescribing scalar curvture on S^n and related problems. II. Existence and compactness,* Comm. Pure Appl. Math. **49** (1996), no. 6, 541-597.