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Existence of Multi-Bump Solutions For a Class of Quasilinear Problems

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Abstract

Using variational methods we establish existence of multi-bump solutions for the following class of quasilinear problems

$$-\Delta_p u + (\lambda V(x) + Z(x))u^{p-1} = f(u), \;\; u > 0 \;\text{ in }\; I\!R^N$$

where $\Delta_p u$ is the p-Laplacian operator, $2 \leq p < N$, $\lambda \in (0, \infty)$, f is a continuous function with subcritical growth and $V, Z : I\!R^N \to I\!R$ are continuous functions verifying some hypothesis.

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1 Introduction

We are concerned with the existence of positive solutions for the following class of quasilinear elliptic problems

$$\begin{cases} -\Delta_p u + (\lambda V(x) + Z(x))u^{p-1} = f(u), & I\!R^N \\ u > 0 \ \text{ in } \ I\!R^N \\ u \in W^{1,p}(I\!R^N) \end{cases} \qquad (P)_\lambda$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u), 2 \leq p < N, f$ is a continuous function with subcritical growth, $\lambda \in (0, \infty)$ and $V, Z : I\!R^N \to I\!R$ are continuous functions with $V(x) \geq 0$ $\forall x \in I\!R^N$. The general hypotheses considered in this work are the following:

(H1) The potential well $\Omega := intV^{-1}(0)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$ and $V^{-1}(0) = \overline{\Omega}$.

(H2) There exist two positive constants M_o and M_1 verifying

$$0 < M_o \leq \lambda V(x) + Z(x) \ \ \forall x \in I\!R^N \ \text{ and } \ \forall \lambda \geq 1$$

and

$$|Z(x)| \leq M_1 \ \ \forall x \in I\!R^N.$$

$(\mathbf{f_1})$ There exists $p < q < p^* = \frac{Np}{N-p}$ such that

$$\limsup_{|t|\to\infty} \frac{f(t)}{|t|^{q-1}} = m < +\infty.$$

$(\mathbf{f_2})$ $\quad f(t) = o(|t|^{p-1}) \quad$ as $\quad t \to 0.$

$(\mathbf{f_3})$ There exist $\theta \in (p, p^*)$ such that

$$0 < \theta F(t) \leq t f(t) \ \ \forall\, t > 0$$

where $F(t) = \int_0^t f(\tau)d\tau$.

$(\mathbf{f_4})$ The function $f(t)/t^{p-1}$ is increasing for $t \in [0, +\infty)$.

For the case $p = 2$, there exist a lot of papers concerning the existence and multiplicity of positive solutions, where the behavior of functions V and Z as well as their geometry are main points to make a careful study of solutions. We cite the papers of Bartsch & Wang [3, 4], Clapp & Ding [7], Bartsch, Pankov & Wang [5], de Figueiredo & Ding [10], Gui [12], Séré [14] and references therein. For the case $p \geq 2$, in [1], Alves & Ding considered the existence, multiplicity and concentration of solution for a class of quasilinear problem in $I\!R^N$ involving the p-Laplacian operator and a nonlinearity with critical growth.

In [9], Ding & Tanaka considered $(P)_\lambda$ assuming that Ω has k connected components and $p = 2$. The authors showed that $(P)_\lambda$ has at least $2^k - 1$ solutions for large λ and established the existence of multi-bump solutions with the following characteristics :

If Ω has k components Ω_j, for each non-empty subset $\Gamma \subset \{1,2,...,k\}$ there exists a positive solution u_λ of $(P)_\lambda$ for large λ verifying:

$$\Big| \int_{\Omega_j} |\nabla u_\lambda|^2 + (\lambda V(x) + Z(x))u_\lambda^2 - \Big(\frac{1}{2} - \frac{1}{q+1}\Big)^{-1} c_j \Big| \to 0 \ \ as \ \lambda \to \infty \ \forall j \in \Gamma$$

and

$$\int_{\mathbb{R}^N \setminus \Omega_\Gamma} |\nabla u_\lambda|^2 + |u_\lambda|^2 \to 0 \ \ as \ \lambda \to \infty$$

where $\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j$ and c_j is the minimax level of energy functional related to the problem

$$\begin{cases} -\Delta u + Z(x)u = u^q, \ \ \Omega_j \\ u > 0, \ in \ \Omega_j \ \ and \ u = 0 \ on \ \partial\Omega_j. \end{cases} \qquad (P_j)$$

Motivated by papers [1] and [9], we show the existence of multi-bump solutions to $(P)_\lambda$ for general case $p \geq 2$ with the same above characteristics. However, here we use different approach in some estimates, because the p-Laplacian is not linear, and some properties that occur for 2-Laplacian (Laplacian operator), in our opinion, are not standard that they hold for general case, $p \geq 2$, therefore, a careful analysis it is necessary. Moreover, our nonlinearity is nonhomogeneous and some arguments developed in [9] can not be applied. So, we modify the sets that appear in the minimax arguments explored in [9]. The arguments developed in this paper are variational, and our main result completes the study made in [9], in the sense that, we are working with p-Laplacian operator and a general class of nonlinearity.

Before enunciating our main results, it is necessary to fix some notations. Since we intend to find positive solutions, throughout this paper, we assume that

$$f(t) = 0 \ \ \forall t \in (-\infty, 0].$$

Moreover, let us assume that the set Ω consists of k connected components denoted by $\Omega_j, \ j \in \{1,..,k\}$ satisfying $d(\Omega_i, \Omega_j) > 0$ for $i \neq j$, that is

$$\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_k.$$

Our main result is the following

Theorem 1.1 *Assume that $(H1)-(H2), (f_1)-(f_4)$ occur. Then, for any non-empty subset Γ of $\{1,2,..,k\}$, there exists λ^* such that, for $\lambda \geq \lambda^*, (P)_\lambda$ has a family $\{u_\lambda\}$ of positive solution verifying: For any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that $u_{\lambda_{n_i}}$ converges strongly in $W^{1,p}(\mathbb{R}^N)$ to a function u which satisfies $u(x) = 0$ for $x \notin \Omega_\Gamma$, and the restriction $u|_{\Omega_j}$ is a least energy solution of*

$$-\Delta_p u + Z(x)u^{p-1} = f(u), \ u > 0 \ in \ \Omega_j, \ \ u|_{\partial\Omega_j} = 0 \ \ for \ j \in \Gamma$$

where $\Omega_\Gamma = \cup_{j\in\Gamma}\Omega_j$.

Corollary 1.1 *Under the assumptions of Theorem 1.1, there exists $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$, $(P)_\lambda$ has at least $2^k - 1$ positive solutions.*

2 Preliminary remarks

In this section, we fix some notations and recall the definition of some functionals that will be used in this work.

Hereafter, when h is a mensurable function, we denote by $\int_{\mathbb{R}^N} h$ the following integral $\int_{\mathbb{R}^N} h dx$. Moreover, we will use the symbols $\|u\|, |u|_r$ $(r > 1)$ and $\|u\|_\infty$ to denote the usual norms in the spaces $W^{1,p}(\mathbb{R}^N), L^r(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ respectively. For an open set $\Theta \subset \mathbb{R}^N$, the symbols $\|u\|_\Theta, |u|_{r,\Theta}(r > 1)$ and $|u|_{\infty,\Theta}$ denote the usual norms in the spaces $W^{1,p}(\Theta), L^r(\Theta)$ and $L^\infty(\Theta)$.

The nonnegative weak solutions of $(P)_\lambda$ are the critical points of functional $J : E_\lambda \to \mathbb{R}$ given by

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + (\lambda V(x) + Z(x))|u|^p - \int_{\mathbb{R}^N} F(u)$$

where E_λ is the space of functions defined by

$$E_\lambda = \Big\{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p < \infty\Big\}$$

endowed with the norm

$$\|u\|_\lambda = \Big(\int_{\mathbb{R}^N} |\nabla u|^p + (\lambda V(x) + Z(x))|u|^p\Big)^{\frac{1}{p}}.$$

It is easy to see that $(E_\lambda, \|.\|_\lambda)$ is a Banach space and $E_\lambda \subset W^{1,p}(\mathbb{R}^N)$ for $\lambda \geq 1$.

We also write for an open set $\Theta \subset \mathbb{R}^N$

$$E(\Theta) = \Big\{u \in W^{1,p}(\Theta); \int_\Theta V(x)|u|^p < \infty\Big\}$$

and

$$\|u\|_{\lambda,\Theta} = \Big(\int_\Theta |\nabla u|^p + (\lambda V(x) + Z(x))|u|^p\Big)^{\frac{1}{p}}.$$

In view of $(H2)$,

$$M_o|u|^p_{p,\Theta} \leq \int_\Theta |\nabla u|^p + (\lambda V(x) + Z(x))|u|^p \text{ for all } u \in E(\Theta) \text{ and } \lambda \geq 1$$

or equivalently

$$\|u\|^p_{\lambda,\Theta} \geq M_o|u|^p_{p,\Theta} \text{ for all } u \in E(\Theta) \text{ and } \lambda \geq 1.$$

The next result is an immediate consequence of the above considerations

Lemma 2.1 *There exist $\delta_0, \nu_0 > 0$ such that for all open sets $\Theta \subset \mathbb{R}^N$*

$$\delta_0\|u\|^p_{\lambda,\Theta} \leq \|u\|^p_{\lambda,\Theta} - \nu_0|u|^p_{p,\Theta} \text{ for all } u \in E(\Theta) \text{ and } \lambda \geq 1.$$

To finish this section, in what follows, for each $j \in \{1, 2, ..., k\}$, we fix a bounded open subset $\Omega_j^{'}$ with smooth boundary such that

$$(i)\ \overline{\Omega_j} \subset \Omega_j^{'}$$
$$(ii)\ \overline{\Omega_j^{'}} \cap \overline{\Omega_l^{'}} = \emptyset \text{ for all } j \neq l.$$

3 An auxiliary problem

In this section, we adapt for our case some arguments developed by Ding & Tanaka [9], del Pino & Felmer [8] and Alves & Figueiredo [2].

Let $\nu_0 > 0$ be the constant given in Lemma 2.1, $a > 0$ verifying $f(a)/a^{p-1} = \nu_0$ and $\tilde{f}, \tilde{F} : I\!R \to I\!R$ the functions

$$\tilde{f}(s) = \begin{cases} f(s), & \text{if } s \leq a \\ \nu_0 s^{p-1}, & \text{if } s > a \end{cases}$$

and

$$\tilde{F}(s) = \int_0^s \tilde{f}(\tau) d\tau.$$

From now on we fix a non-empty subset $\Gamma \subset \{1, 2, ..., k\}$ and

$$\Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j,\ \Omega_\Gamma^{'} = \bigcup_{j \in \Gamma} \Omega_j^{'},\ \chi_\Gamma(x) = \begin{cases} 1 & \text{for } x \in \Omega_\Gamma^{'} \\ 0 & \text{for } x \notin \Omega_\Gamma^{'} \end{cases}$$

and the functions

$$g(x, s) = \chi_\Gamma(x) f(s) + (1 - \chi_\Gamma(x)) \tilde{f}(s) \qquad (g_1)$$

and

$$G(x, s) = \int_0^s g(x, t) dt = \chi_\Gamma(x) F(s) + (1 - \chi_\Gamma(x)) \tilde{F}(s). \qquad (g_2)$$

Moreover, $\Phi_\lambda : E_\lambda \to I\!R$ denotes the functional given by

$$\Phi_\lambda(u) = \frac{1}{p} \int_{I\!R^N} (|\nabla u|^p + (\lambda V(x) + Z(x))|u|^p - \int_{I\!R^N} G(x, u).$$

Under the conditions $(H1), (H2), (f_1)$ and (f_2), $\Phi_\lambda \in C^1(E_\lambda, I\!R)$ and its critical points are non-negative weak solutions of

$$-\Delta_p u + (\lambda V(x) + Z(x))|u|^{p-2} u = g(x, u) \text{ in } I\!R^N. \qquad (A)_\lambda$$

We would like to detach that if u_λ is a positive solution of $(A)_\lambda$ verifying $u(x) \leq a$ in $I\!R^N \setminus \Omega_\Gamma^{'}$, then it is a positive solution to $(P)_\lambda$.

3.1 The Palais-Smale condition and its consequences

We start this subsection studying the boundedness of $(PS)_c$ sequences, that is, a sequence $\{u_n\} \subset E_\lambda$ verifying

$$\Phi_\lambda(u_n) \to c \text{ and } \Phi_\lambda^{'}(u_n) \to 0 \tag{3.1}$$

for some $c \in I\!R$.

Lemma 3.1 *Suppose that a sequence $\{u_n\} \subset E_\lambda$ satisfies (3.1). Then there exists a positive constant K which is independent of $\lambda \geq 1$ such that*

$$\|u_n\|_\lambda^p \leq K \ \ \forall n \in I\!N.$$

Proof. It follows from the definition of Palais-Sequence that

$$\Phi_\lambda(u_n) - \frac{1}{\theta}\Phi_\lambda^{'}(u_n)u_n = c + o_n(1) + \epsilon_n\|u_n\|_\lambda,$$

where $\epsilon_n \to 0$. From (g_1) and (g_2)

$$\Big(\frac{1}{p} - \frac{1}{\theta}\Big)\|u_n\|_\lambda^p - \int_{I\!R^N\setminus\Omega_\Gamma'}(\tilde{F}(u_n) - \frac{1}{\theta}\tilde{f}(u_n)u_n) \leq c + o_n(1) + \epsilon_n\|u_n\|_\lambda. \tag{3.2}$$

Since $\tilde{f}$ and $\tilde{F}$ verify

$$\tilde{F}(s) - \frac{1}{\theta}\tilde{f}(s)s \leq (\frac{1}{p} - \frac{1}{\theta})\nu_0|s|^p \text{ for all } s \in I\!R,$$

we obtain,

$$\Big(\frac{1}{p} - \frac{1}{\theta}\Big)(\|u_n\|_\lambda^p - \nu_0|u_n|_p^p) \leq c + o_n(1) + \epsilon_n\|u_n\|_\lambda,$$

and by Lemma 2.1

$$\Big(\frac{1}{p} - \frac{1}{\theta}\Big)\delta_0\|u_n\|_\lambda^p \leq c + o_n(1) + \epsilon_n\|u_n\|_\lambda.$$

Therefore,

$$\limsup_{n\to\infty}\|u_n\|_\lambda^p \leq \Big(\frac{1}{p} - \frac{1}{\theta}\Big)^{-1}\delta_0^{-1}c,$$

from which it follows that there exists $K > 0$ such that

$$\|u_n\|_\lambda^p \leq K \ \ \forall n \in I\!N.$$

■

Proposition 3.2 *For $\lambda \geq 1, \Phi_\lambda$ satisfies $(PS)_c$ condition for all $c \in I\!R$, that is, any $(PS)_c$ sequence $\{u_n\} \subset E_\lambda$ has a strongly convergent subsequence in E_λ.*

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(PS)_c$ sequence. By Lemma 3.1, $\{u_n\}$ is bounded in E_λ and we may assume

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } E_\lambda \text{ and } W^{1,p}(I\!R^N)\\ &u_n \to u \text{ in } L^{q+1}_{loc}(I\!R^N) \text{ and } L^p_{loc}(I\!R^N). \end{aligned}$$

First, we observe that the limit function u is a critical point of Φ_λ. In fact, because for any bounded sequence $\{\varphi_n\} \subset E_\lambda$, we can easily see that $\Phi'_\lambda(u_n)\varphi_n \to 0$. Using the sequence

$$\varphi_n(x) = \eta(x)u_n(x)$$

where $\eta \in C^\infty(I\!R^N)$ is given by

$$\eta(x) = 1 \ \forall x \in B^c_R(0), \eta(x) = 0 \ \forall x \in B_{\frac{R}{2}}(0),\ \eta(x) \in [0,1] \text{ with } \Omega'_\Gamma \subset B_{\frac{R}{2}}(0)$$

and adapting the arguments used in [8, Lemma 1.1](see also [2]), for each $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{\{x\in I\!R^N:|x|\geq R\}} |\nabla u_n|^p + (\lambda V(x) + Z(x))|u_n|^p \leq \epsilon \text{ for } n \in I\!N.$$

The last inequality together with subcritical growth of g imply

$$\lim_{n\to\infty} \int_{I\!R^N} P^1_n + (\lambda V(x) + Z(x))P^2_n = 0$$

where

$$P^1_n = \langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u_o|^{p-2}\nabla u_o, \nabla u_n - \nabla u_o\rangle$$

and

$$P^2_n = (|u_n|^{p-2}u_n - |u_o|^{p-2}u_o)(u_n - u_o)).$$

Using the same type of arguments found in Jianfu [13, Lemma 4.2] (see Tolksdorff [15]), it follows that

$$u_n \to u \text{ in } E_\lambda.$$

■

Remark 3.3

• Using well-known arguments, we can assume that all $(PS)_c$ sequences are nonnegative functions

• Since Φ_λ verifies the mountain pass geometry, the above results imply the existence of a non-trivial critical point to Φ_λ.

Our next step it is to study the behavior of a $(PS)_\infty$ sequence, that is, a sequence $\{u_n\} \subset W^{1,p}(I\!R^N)$ satisfying:

$$\begin{aligned} &u_n \in E_{\lambda_n} \text{ and } \lambda_n \to \infty,\\ &\Phi_{\lambda_n}(u_n) \to c,\\ &\|\Phi'_{\lambda_n}(u_n)\| \to 0. \end{aligned} \qquad (PS)_\infty$$

Proposition 3.4 *Let $\{u_n\}$ be a $(PS)_\infty$ sequence. Then, for some subsequence, still denoted by $\{u_n\}$, there exists $u \in W^{1,p}(\mathbb{R}^N)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\mathbb{R}^N).$$

Moreover,

(i) $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega_\Gamma$ and u is a non-negative solution of

$$\begin{cases} -\Delta_p u + Z(x)|u|^{p-2}u = f(u) & \text{in } \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j \end{cases} \qquad (P)_j$$

for each $j \in \Gamma$.

(ii) u_n converges to u in a stronger sense;

$$\|u_n - u\|_{\lambda_n} \to 0.$$

Hence,

$$u_n \to u \text{ strongly in } W^{1,p}(\mathbb{R}^N).$$

(iii) u_n also satisfies

$$\begin{aligned} &\lambda_n \int_{\mathbb{R}^N} V(x)|u_n|^p \to 0 \\ &\|u_n\|^p_{\lambda_n,\mathbb{R}^N\setminus\Omega_\Gamma} \to 0 \\ &\|u_n\|^p_{\lambda_n,\Omega'_j} \to \int_{\Omega_j} |\nabla u|^p + Z(x)|u|^p \text{ for all } j \in \Gamma. \end{aligned}$$

Proof. As in the proof of Lemma 3.1, it is easy to check that there exists $K > 0$ such that

$$\|u_n\|^p_{\lambda_n} \le K \quad \forall n \in \mathbb{N}.$$

Thus $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and we may assume that for some $u \in W^{1,p}(\mathbb{R}^N)$

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\mathbb{R}^N)$$

and

$$u_n(x) \to u(x) \text{ a.e in } \mathbb{R}^N.$$

To show (i), we fix the set $C_m = \{x \in \mathbb{R}^N : V(x) \ge \frac{1}{m}\}$. Then,

$$\int_{C_m} |u_n|^p \le \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x)|u_n|^p$$

that is

$$\int_{C_m} |u_n|^p \le \frac{m}{\lambda_n} \|u_n\|^p_{\lambda_n}.$$

The last inequality together with Fatou's Lemma imply

$$\int_{C_m} |u|^p = 0 \;\; \forall m \in I\!N.$$

Thus $u(x) = 0$ on $\bigcup_{m=1}^{+\infty} C_m = I\!R^N \setminus \overline{\Omega}$, from which we can assert that $u|_{\Omega_j} \in W_o^{1,p}(\Omega_j)$ $\forall j \in \{1, ..., k\}$.

Using again arguments explored in the above section, we get

$$\lim_{n\to\infty} \int_{I\!R^N} P_n^1 + (\lambda_n V(x) + Z(x))P_n^2 = 0$$

where

$$P_n^1 = \langle |\nabla u_{\lambda_n}|^{p-2}\nabla u_{\lambda_n} - |\nabla u_o|^{p-2}\nabla u_o, \nabla u_{\lambda_n} - \nabla u_o\rangle$$

and

$$P_n^2 = (|u_{\lambda_n}|^{p-2}u_{\lambda_n} - |u_o|^{p-2}u_o)(u_{\lambda_n} - u_o)).$$

Thus,

$$u_n \to u \;\text{ in }\; W^{1,p}(I\!R^N).$$

From the last limit, for each $\varphi \in C_o^\infty(\Omega_j)$

$$\int_{\Omega_j} |\nabla u|^{p-2}\nabla u \nabla\varphi + Z(x)|u|^{p-2}u\varphi - \int_{\Omega_j} g(x,u)\varphi = 0,$$

showing that u is a solution to (P_j) for each $j \in \Gamma$.

For each $j \in \{1, 2, ..., k\} \setminus \Gamma$, setting $\varphi = u$, we have

$$\int_{\Omega_j} |\nabla u|^p + Z(x)|u|^p - \int_{\Omega_j} \tilde{f}(u)u = 0$$

that is,

$$\|u\|^p_{\lambda,\Omega_j'} - \int_{\Omega_j'} \tilde{f}(u)u = 0.$$

By Lemma 2.1 and the fact that $\tilde{f}(s)s \le \nu_0 s^p$ for all $s \in I\!R$, we have

$$\delta_0|u|^p_{p,\Omega_j'} \le \|u\|^p_{\lambda,\Omega_j'} - \nu_0|u|^p_{p,\Omega_j'} \le \|u\|^p_{\lambda,\Omega_j'} - \int_{\Omega_j'} \tilde{f}(u)u = 0.$$

Thus, $u = 0$ in Ω_j for $j \in \{1, 2, ..., k\} \setminus \Gamma$ showing that (i) holds.

For (ii), we use again [13, Lemma 4.2] to get the following inequality:

$$\begin{aligned}&\|u_n - u\|^p_{\lambda_n} - \textstyle\int_{I\!R^N\setminus\Omega_\Gamma'}(\tilde{f}(u_n) - \tilde{f}(u))(u_n - u) - \int_{\Omega_\Gamma'}(f(u_n) - f(u))(u_n - u)\\ &\le \Phi'_{\lambda_n}(u_n)(u_n - u) - \Phi'_{\lambda_n}(u)(u_n - u).\end{aligned}$$

Using the limits below

$$\int_{\Omega'_\Gamma} (f(u_n) - f(u))(u_n - u) = o_n(1),$$

$$\Phi'_{\lambda_n}(u)(u_n - u) = \int_{\Omega_\Gamma} |\nabla u|^{p-2}\nabla u \nabla(u_n - u) + Z(x)u^{p-1}(u_n - u) - \\ - \int_{\Omega_\Gamma} f(u)(u_n - u) = o_n(1)$$

and

$$|\Phi'_{\lambda_n}(u_n)(u_n - u)| \leq \|\Phi'_{\lambda_n}(u_n)\|(\|u_n\|_{\lambda_n} + \|u\|_{\lambda_n}) = o_n(1)$$

it follows that

$$\|u_n - u\|^p_{\lambda_n} - \int_{I\!R^N \setminus \Omega'_\Gamma} (\tilde{f}(u_n) - \tilde{f}(u))(u_n - u) \leq o_n(1).$$

Now, using Lemma 2.1, the fact that $u \equiv 0$ in $I\!R^N \setminus \Omega_\Gamma$ and the last estimate, we obtain

$$\|u_n - u\|^p_{\lambda_n} \to 0 \text{ as } n \to \infty.$$

To prove (iii), notice that from (H_2)

$$\int_{I\!R^N} \lambda_n V(x)|u_n|^p \leq C\|u_n - u\|^p_{\lambda_n}$$

so,

$$\int_{I\!R^N} \lambda_n V(x)|u_n|^p \to 0 \text{ as } n \to \infty.$$

■

Proposition 3.5 *Let $\{u_\lambda\}$ be a family of positive solutions of (A_λ) with $u_\lambda \to 0$ in $W^{1,p}(I\!R^N \setminus \Omega_\Gamma)$ as $\lambda \to +\infty$. Then, there exists $\lambda^* > 0$ such that*

$$|u_\lambda|_{\infty, I\!R^N \setminus \Omega'_\Gamma} \leq a \;\; \forall \lambda \geq \lambda^*.$$

Proof. In this proof we adapt some arguments developed in Gongbao [11, Theorem 1.11] (see also [2]). In the sequel, u denotes u_λ.

For each $\eta \in C^\infty(I\!R^N)$, we define the functions

$$v = \eta^p u u_L^{p(\beta-1)} \text{ and } W_L = \eta u u_L^{\beta-1}$$

where $u_L = \min\{u, L\}$. By direct computation, using the fact that ν_o given in Lemma 2.1 can be chosen sufficiently small, the growth of function g and the definition of weak solution, we have

$$|W_L|^p_{p^*} \leq C \int_{I\!R^N} |\nabla W_L|^p \leq C\beta^p \Big(\int_{I\!R^N} |\nabla \eta|^p u^p u_L^{p(\beta-1)} \Big).$$

Fixing $\Omega_j^{'} \subset \widetilde{\Omega}_j$ and η verifying

$$\begin{array}{l} 0 \leq \eta(x) \leq 1 \ \forall x \in I\!R^N \\ \eta(x) = 0 \ \forall x \in \Omega_j^{'} \\ \eta(x) = 1 \ \forall x \in I\!R^N \setminus \bigcup_{j\in\Gamma} \widetilde{\Omega}_j, \end{array}$$

the above estimate implies

$$|W_L|_{p^*,\mathcal{B}}^p \leq C_1\beta^p\Big(\int_\Upsilon u^p u_L^{p(\beta-1)}\Big)$$

where $\Upsilon = \bigcup_{j\in\Gamma} \widetilde{\Omega}_j \setminus \Omega_j^{'}$ and $\mathcal{B} = I\!R^N \setminus \bigcup_{j\in\Gamma} \widetilde{\Omega}_j$.

Fixing $\beta = \frac{p^*}{p}$, the last inequality implies that

$$u \in L^{\frac{p^{*2}}{p}}(\mathcal{B}).$$

Now, if $\beta = \frac{p^*(t-1)}{pt}$ with $t = \frac{p^{*2}}{(p^*-p)p}$, then $\beta > 1$ and $\frac{pt}{(t-1)} \in (p, p^*)$. Thus, using Holder's inequality,

$$|W_L|_{p^*,\mathcal{B}}^p \leq C_2\beta^p\Big(\int_\Upsilon u^{\frac{p\beta t}{(t-1)}}\Big)^{\frac{t-1}{t}}.$$

Hence, Letting $L \to \infty$, we obtain

$$|u|_{\beta p^*,\mathcal{B}}^{p\beta} \leq C_2\beta^p|u|_{\frac{p\beta t}{(t-1)},\Upsilon}^{p\beta}.$$

Defining $\xi = \frac{p^*(t-1)}{pt}$, $s = \frac{pt}{(t-1)}$ and $\beta = \xi^m(m = 1, 2, 3, ...)$, it is possible to show that there exists $C_3 > 0$ such that

$$|u|_{\xi^{m+1}s,\mathcal{B}} \leq C_3|u|_{\xi s,\Upsilon} \ \ \forall m \in \{1, 2, 3, ...\}.$$

Letting $m \to +\infty$, we get

$$|u|_{\infty,\mathcal{B}} \leq C_3|u|_{p^*,\Upsilon}.$$

Using the fact that

$$u_\lambda \to 0 \ \text{ in } \ W^{1,p}(I\!R^N \setminus \Omega_\Gamma) \ \text{ as } \ \lambda \to \infty$$

from the last inequality, we can conclude that there exists $\lambda^* > 0$ such that

$$|u_\lambda|_{\infty, I\!R^N\setminus\Omega_\Gamma'} \leq a \ \ \forall\lambda \geq \lambda^*.$$

■

The last proposition implies the following result

Corollary 3.6 *Under the hypothesis of Proposition 3.5, there exists $\lambda^* > 0$ such that for $\lambda \geq \lambda^*$, u_λ is a positive solution of $(P)_\lambda$.*

4 The existence of multi-bump positive solutions

In this section, we denote by $I_j : W_o^{1,p}(\Omega_j) \to I\!R$ and $\Phi_{\lambda,j} : W^{1,p}(\Omega_j^{'}) \to I\!R$ the functionals given by

$$I_j(u) = \frac{1}{p}\int_{\Omega_j} |\nabla u|^p + Z(x)|u|^p - \int_{\Omega_j} F(u) \tag{4.1}$$

and

$$\Phi_{\lambda,j}(u) = \frac{1}{p}\int_{\Omega_j^{'}} |\nabla u|^p + (\lambda V(x) + Z(x))|u|^p - \int_{\Omega_j^{'}} F(u). \tag{4.2}$$

We know that the critical points of the above functionals are related with the weak solutions to the problems

$$\begin{cases} -\Delta_p u + Z(x)|u|^{p-2}u = f(u), \text{ in } \Omega_j \\ u = 0, \text{ on } \partial\Omega_j \end{cases} \tag{4.3}$$

and

$$\begin{cases} -\Delta_p u + (\lambda V(x) + Z(x))|u|^{p-2}u = f(u), \text{ in } \Omega_j^{'} \\ \frac{\partial u}{\partial \eta} = 0, \text{ on } \partial\Omega_j^{'}. \end{cases} \tag{4.4}$$

It is easy to check that functionals I_j and $\Phi_{\lambda,j}$ satisfy the mountain pass geometry. In what follows, we denote by c_j and $c_{\lambda,j}$ the minimax level related to the above functions defined by

$$c_j = \inf_{\gamma\in\Gamma_j} \max_{t\in[0,1]} I_j(\gamma(t))$$

and

$$c_{\lambda,j} = \inf_{\gamma\in\Gamma_{\lambda,j}} \max_{t\in[0,1]} \Phi_{\lambda,j}(\gamma(t))$$

where

$$\Gamma_j = \{\gamma \in C([0,1], W_o^{1,p}(\Omega_j)); \gamma(0) = 0, I_j(\gamma(1)) < 0\}$$

and

$$\Gamma_{\lambda,j} = \{\gamma \in C([0,1], W^{1,p}(\Omega_j^{'})); \gamma(0) = 0, \Phi_{\lambda,j}(\gamma(1)) < 0\}.$$

Moreover, the Palais - Smale conditions implies that there exist two nonnegative functions $w_j \in W_o^{1,p}(\Omega_j)$ and $w_{\lambda,j} \in W^{1,p}(\Omega_j^{'})$ verifying

$$I_j(w_j) = c_j \text{ and } I_j^{'}(w_j) = 0$$

and

$$\Phi_{\lambda,j}(w_{\lambda,j}) = c_{\lambda,j} \text{ and } \Phi_{\lambda,j}^{'}(w_{\lambda,j}) = 0.$$

4.1 A special critical value to Φ_λ

In what follows, let us fix $R > 0$ such that

$$\Big|I_j\Big(\frac{1}{R}w_j\Big)\Big| < \frac{c_j}{2} \quad \forall j \in \Gamma$$

and

$$|I_j(Rw_j) - c_j| \geq 1 \quad \forall j \in \Gamma.$$

From the definition of c_j, the equality below is standard.

$$\max_{s\in[1/R^2,1]} I_j(sRw_j) = c_j \ \ \forall j \in \Gamma.$$

Hereafter, $\Gamma = \{1, ..., l\}(l \leq k)$,

$$\gamma_o(s_1, s_2, ..., s_l)(x) = \sum_{j=1}^{l} s_j Rw_j(x) \ \ \forall(s_1, ..., s_l) \in [1/R^2, 1]^l, \tag{4.5}$$

$$\Gamma_* = \{\gamma \in C([1/R^2, 1]^l, E_\lambda \setminus \{0\}); \gamma = \gamma_o \ \text{ on } \ \partial([1/R^2, 1]^l)\}$$

and

$$b_{\lambda,\Gamma} = \inf_{\gamma\in\Gamma_*} \max_{(s_1,...,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\gamma(s_1, ..., s_l)).$$

We remark that $\gamma_o \in \Gamma_*$, so $\Gamma_* \neq \emptyset$ and $b_{\lambda,\Gamma}$ is well-defined.

Lemma 4.1 *For any $\gamma \in \Gamma_*$ there exists $(t_1, ..., t_l) \in [1/R^2, 1]^l$ such that*

$$\Phi'_{\lambda,j}(\gamma(t_1, ..., t_l))(\gamma(t_1, ..., t_l)) = 0 \ \textit{ for } \ j \in \{1, ..., l\}.$$

Proof. For a given $\gamma \in \Gamma_*$, let us consider the map $\tilde{\gamma} : [1/R^2, 1]^l \to \mathbb{R}^l$ defined by

$$\tilde{\gamma}(s_1, ..., s_l) = \big(\Phi'_{\lambda,1}(\gamma)(\gamma), ..., \Phi'_{\lambda,l}(\gamma)(\gamma)\big)$$

where

$$\Phi'_{\lambda,j}(\gamma)(\gamma) = \Phi'_{\lambda,j}(\gamma(s_1, ..., s_l))(\gamma(s_1, ..., s_l)) \ \ \forall j \in \Gamma.$$

For $(s_1, ..., s_l) \in \partial([1/R^2, 1]^l)$, it follows that

$$\gamma(s_1, ..., s_l) = \gamma_0(s_1, ..., s_l)$$

and

$$\Phi'_{\lambda,j}(\gamma_0(s_1, ..., s_l))(\gamma_0(s_1, ..., s_l)) = 0 \Rightarrow s_j \notin \{1/R^2, 1\} \ \ \forall \ \ j \in \Gamma.$$

Thus $(0, ..., 0) \notin \tilde{\gamma}(\partial([1/R^2, 1]^l))$. Using this fact, it follows from the topological degree

$$deg(\tilde{\gamma}, (1/R^2, 1)^l, (0, ..., 0)) = (-1)^l \neq 0$$

and hence, there exists $(t_1, ..., t_l) \in (1/R^2, 1)^l$ satisfying

$$\Phi'_{\lambda,j}(\gamma(t_1, ..., t_l))(\gamma(t_1, ..., t_l)) = 0 \ \text{ for } \ j \in \{1, ..., l\}.$$

■

In the sequel, we denote by c_Γ the following number

$$c_\Gamma = \sum_{j=1}^{l} c_j.$$

Proposition 4.2

a) $\displaystyle\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,\Gamma} \leq c_\Gamma$ *for all* $\lambda \geq 1$.

b) $\Phi_\lambda(\gamma(s_1, ..., s_l)) < c_\Gamma$ *for all* $\lambda \geq 1, \gamma \in \Gamma_*$ *and* $(s_1, ..., s_l) \in \partial([1/R^2, 1]^l)$.

Proof.

a) Since γ_o defined in (4.5) belongs to Γ_*, we have

$$\begin{aligned} b_{\lambda,\Gamma} \leq & \max_{(s_1,....,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\gamma_o(s_1, ..., s_l)) \\ &= \max_{(s_1,...,s_l)\in[1/R^2,1]^l} \sum_{j=1}^{l} I_j(sRw_j) \\ &= \sum_{j=1}^{l} c_j = c_\Gamma. \end{aligned}$$

Fixing $(t_1, ...t_l) \in [1/R^2, 1]$ given in Lemma 4.1 and recalling that $c_{\lambda,j}$ can be characterizated by

$$c_{\lambda,j} = \inf\Big\{\Phi_{\lambda,j}(u)\,;\, u \in E_\lambda \setminus \{0\} \ \text{ and } \ \Phi'_{\lambda,j}(u)(u) = 0\Big\},$$

it follows that

$$\Phi_{\lambda,j}(\gamma(t_1, ..., t_l)) \geq c_{\lambda,j} \ \ \forall j \in \Gamma.$$

On the other hand, recalling that $\Phi_{\lambda,I\!R^N\setminus\Omega'_\Gamma}(u) \geq 0$ for all $u \in W^{1,p}(I\!R^N \setminus \Omega'_\Gamma)$, we have

$$\Phi_\lambda(\gamma(s_1, ..., s_l)) \geq \sum_{j=1}^{l} \Phi_{\lambda,j}(\gamma(s_1, ..., s_l)).$$

Thus,

$$\max_{(s_1,....,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\gamma(s_1, ..., s_l)) \geq \Phi_\lambda(\gamma(t_1, ..., t_l)) \geq \sum_{j=1}^{l} c_{\lambda,j}.$$

From definition of $b_{\lambda,\Gamma}$, we can conclude

$$b_{\lambda,\Gamma} \geq \sum_{j=1}^{l} c_{\lambda,j}$$

finishing the proof of a).

b) Since $\gamma(s_1, ..., s_l) = \gamma_o(s_1, ..., s_l)$ on $\partial([1/R^2, 1]^l)$, for $(s_1, ..., s_l) \in \partial([1/R^2, 1]^l)$ we have

$$\Phi_\lambda(\gamma_o(s_1, ..., s_l)) = \sum_{j=1}^{l} I_j(s_j R w_j).$$

Moreover, $I_j(s_j R w_j) \leq c_j$ for all $j \in \Gamma$ and for some $j_o \in \Gamma, s_{jo} \in \{1/R^2, 1\}$ and $I_{j_o}(s_{j_o} R w_{j_o}) \leq \frac{c_{j_o}}{2}$. Therefore,

$$\Phi_\lambda(\gamma_o(s_1, ..., s_l)) \leq c_\Gamma - \epsilon$$

for some $\epsilon > 0$. ■

Corollary 4.3
a) $b_{\lambda,\Gamma} \to c_\Gamma$ *as* $\lambda \to \infty$
b) $b_{\lambda,j}$ *is a critical value of* Φ_λ *for large* λ.

Proof.
a) Since $c_{\lambda,j} \to c_j$ for each j (see Ding & Tanaka [9]), it follows from Proposition 4.2 that $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$.
b) Using the fact that Φ_λ verifies the Palais-Smale condition, we can use well known arguments involving deformation lemma to conclude that $b_{\lambda,\Gamma}$ is a critical level to Φ_λ for $\lambda \geq 1$.

4.2 Proof of the main theorem

To prove Theorem 1.1, we need to find a positive solution u_λ for a large λ which approaches a least energy solution in each Ω_j $(j \in \Gamma)$ and to 0 elsewhere as $\lambda \to \infty$. To this end, we will show two propositions which imply together with the estimates made in the previous section that Theorem 1.1 holds.

Hereafter,

$$M = 1 + \sum_{j=1}^{k} \sqrt{\left(\frac{1}{p} - \frac{1}{\theta}\right)^{-1} c_j} \ ,$$

$$\bar{B}_{M+1}(0) = \{u \in E_\lambda;\ \|u\|_\lambda \leq M + 1\}$$

and for small $\mu > 0$, we define

$$A_\mu^\lambda = \Big\{u \in \bar{B}_{M+1}(0); \|u\|_{\lambda, I\!R^N \setminus \Omega'_\Gamma} \leq \mu,\ \ \Big|\Phi_{\lambda,j}(u) - c_j\Big| \leq \mu\ \ \forall j \in \Gamma\Big\}.$$

We also use the notation:

$$\Phi_\lambda^{c_\Gamma} = \{u \in E_\lambda; \Phi_\lambda(u) \leq c_\Gamma\}$$

and remark that $w = \sum_{j=1}^{l} w_j \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$, showing that $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma} \neq \emptyset$. Fixing

$$0 < \mu < \frac{1}{3} \min\{c_j ; j \in \Gamma\} \tag{4.6}$$

we have the following uniform estimate of $\|\Phi_\lambda^{'}(u)\|$ on the annulus $(A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma}$.

Proposition 4.4 *Let $\mu > 0$ satisfy (4.6). Then there exist $\sigma_o > 0$ and $\Lambda_* \geq 1$ independent of λ such that*

$$\|\Phi_\lambda^{'}(u)\|_\lambda \geq \sigma_o \ \textit{for} \ \lambda \geq \Lambda_* \ \textit{and for all} \ u \in (A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma}. \tag{4.7}$$

Proof. Arguing by contradiction, we assume that there exist $\lambda_n \to \infty$ and

$$u_n \in (A_\mu^{\lambda_n} \cap A_\mu^{\lambda_n}) \cap \Phi_{\lambda_n}^{c_\Gamma}$$

such that $\|\Phi_{\lambda_n}^{'}(u_n)\| \to 0$.

Since $u_n \in A_{2\mu}^{\lambda_n}$ and $\{\|u_n\|_{\lambda_n}\}$ is a bounded sequence, it follows that $\{\Phi_{\lambda_n}(u_n)\}$ is also bounded. Thus we may assume

$$\Phi_{\lambda_n}(u_n) \to c \in (-\infty, c_\Gamma]$$

after extracting a subsequence if necessary.

Applying Proposition 3.4, we can extract a subsequence $u_n \to u$ in $W^{1,p}(I\!R^N)$ where $u \in W_o^{1,p}(\Omega_\Gamma)$ is a non-negative solution of (4.3) with

$$u_n \to u \ \text{ in } \ W^{1,p}(I\!R^N), \tag{4.8}$$

$$\lambda_n \int_{I\!R^N} V(x)|u_n|^p \to 0 \tag{4.9}$$

and

$$\|u_n\|_{\lambda_n, I\!R^N \setminus \Omega_\Gamma} \to 0. \tag{4.10}$$

Since c_j is the least energy level for I_j, we have two possibilities:
(i) $I_j(u|_{\Omega_j}) = c_j$ for all $j \in \Gamma$.
(ii) $I_{j_o}(u|_{\Omega_{j_o}}) = 0$, that is, $u|_{\Omega_{j_o}} \equiv 0$ for some $j_o \in \Gamma$.

If (i) occurs, it follows from (4.8)-(4.10) that $u_n \in A_\mu^{\lambda_n}$ for large n, which is in contradiction to the assumption $u_n \in A_{2\mu}^{\lambda_n} \setminus A_\mu^{\lambda_n}$.

If (ii) occurs, from (4.8)-(4.9) it follows that

$$\left|\Phi_{\lambda_n, j_o}(u_n) - c_{j_o}\right| \to c_{j_o} \geq 3\mu$$

which is a contradiction with the fact that $u_n \in A_{2\mu}^{\lambda_n} \setminus A_\mu^{\lambda_n}$. Thus neither (i) nor (ii) can hold, and we have completed the proof of Proposition 4.4. ■

Proposition 4.5 *Let μ satisfy (4.6) and $\Lambda_* \geq 1$ be a constant given in Proposition 4.4. Then for $\lambda \geq \Lambda_*$ there exists a positive solution u_λ of $(P)_\lambda$ satisfying $u_\lambda \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$.*

Proof. Assuming by contradiction that there are no critical points in $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$, since the Palais-Smale condition holds for Φ_λ (see Proposition 3.2), there exists a constant $d_\lambda > 0$ such that

$$\|\Phi_\lambda^{'}(u)\| \geq d_\lambda \ \text{ for all } \ u \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}.$$

From hypothesis, we also have

$$\|\Phi_\lambda^{'}(u)\| \geq \sigma_o \ \text{ for all } \ u \in (A_{2\mu}^\lambda \setminus A_\mu^\lambda) \cap \Phi_\lambda^{c_\Gamma}$$

where $\sigma_o > 0$ is independent of λ. In what follows, $\Psi : E_\lambda \to I\!R$ and $H : \Phi_\lambda^{c_\Gamma} \to I\!R$ are continuous functions that verify

$$\begin{array}{l} \Psi(u) = 1 \ \text{ for } \ u \in A_{3\mu/2}^\lambda \\ \Psi(u) = 0 \ \text{ for } \text{ for } \ u \notin A_{2\mu}^\lambda, \\ 0 \leq \Psi(u) \leq 1 \ \text{ for } \ u \in E_\lambda \end{array}$$

and

$$H(u) = \begin{cases} -\Psi(u)\|Y(u)\|^{-1}\|Y(u)\|, & u \in A_{2\mu}^\lambda \\ 0, & u \notin A_{2\mu}^\lambda \end{cases}$$

where Y is a pseudo-gradient vector field for Φ_λ on $\mathcal{M} = \{u \in E_\lambda : \Phi_\lambda' \neq 0\}$. Hence, using the properties involving Y and Φ_λ, we have the inequality

$$\|H(u)\| \leq 1 \ \ \forall \lambda \geq \Lambda_* \ \text{ and } \ u \in \Phi_\lambda^{c_\Gamma}.$$

Considering the deformation flow $\eta : [0, \infty) \times \Phi_\lambda^{c_\Gamma} \to \Phi_\lambda^{c_\Gamma}$ defined by

$$\frac{d\eta}{dt} = H(\eta), \ \ \eta(0, u) = u \in \Phi_\lambda^{c_\Gamma}$$

and observing that there exits $K_* > 0$ such that

$$|\Phi_{\lambda,j}(u) - \Phi_{\lambda,j}(v)| \leq K_*\|u - v\|_{\lambda,\Omega_j'} \ \ \forall u, v \in \bar{B}_{M+1}(0) \ \text{ and } \ \forall j \in \Gamma$$

we obtain, using similar arguments explored by Ding & Tanaka [9], two numbers $T = T(\lambda) > 0$ and $\epsilon_* > 0$ independent of $\lambda \geq \Lambda_*$ satisfying

$$\gamma^*(s_1, ..., s_l) = \eta(T, \gamma_o(s_1, ..., s_l)) \in \Gamma_*$$

and

$$\max_{(s_1,\ldots,s_l)\in[1/R^2,1]^l} \Phi_\lambda(\gamma^*(s_1, ..., s_l)) \leq c_\Gamma - \epsilon_*.$$

Combining the definition of $b_{\lambda,\Gamma}$ and the above informations, we get the inequality

$$b_{\lambda,\Gamma} \leq c_\Gamma - \epsilon_*, \quad \forall \lambda \geq \Lambda_*$$

which contradicts Corollary 4.3. ■

4.3 Final conclusion

From Proposition 4.5 there exists a family $\{u_\lambda\}$ of positive solutions to (A_λ) verifying the following properties:
I) For fixed $\mu > 0$ there exists λ^* such that

$$\|u_\lambda\|_{\lambda, I\!R^N \setminus \Omega'_\Gamma} \leq \mu \ \forall \lambda \geq \lambda^*.$$

Thus, from proof of Proposition 3.5, μ fixed sufficiently small, we can conclude that

$$|u_\lambda|_{\infty, I\!R^N \setminus \Omega'_\Gamma} \leq a \ \forall \lambda \geq \lambda^*$$

showing that u_λ is a positive solution to $(P)_\lambda$.

II) Fixing $\lambda_n \to \infty$ and $\mu_n \to 0$, the sequence $\{u_{\lambda_n}\}$ verifies:

- $\Phi'_{\lambda_n}(u_{\lambda_n}) = 0 \ \forall n \in I\!N$
- $\|u_{\lambda_n}\|_{\lambda_n, I\!R^N \setminus \Omega'_\Gamma} \to 0$
- $\Phi'_{\lambda_n, j}(u_{\lambda_n}) \to c_j \quad \forall j \in \Gamma.$

Thus from Proposition 3.4, we have that

- $u_{\lambda_n} \to u$ in $W^{1,p}(I\!R^N)$ with $u \in W_o^{1,p}(\Omega_\Gamma)$

from which follows the proof of Theorem 1.1. ■

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