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On Stable and Meta-stable Solutions of the Shadow System For the Geirer-Meinhardt Equation

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Abstract

We show how the theory of finite Morse index solutions for small diffusion problems can be adapted to the shadow system for the Geirer-Meinhardt equation. In particular, we prove that the only possible stable solution of this shadow system is the one peak solution discussed by Wei and Ni, Takagi and Yanagida.

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1 Introduction

In this short paper, we prove that the only positive solution of the shadow system (with a Neumann boundary condition for u)

$$\dot{u} = \varepsilon^2 \Delta u - u^r h^{-q} \qquad \text{in } \Omega$$
$$\tau \dot{h} = -h + |\Omega|^{-1} \int_\Omega u^r h^{-s}$$

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which is meta-stable for small ε is the well known one peak solutions constructed in [8] and [10]. There solutions are rather well understood and are proved to be meta-stable in [11] (under some conditions). By [9] and [11] we have a good understanding of when peak solutions are stable and meta-stable. Here Ω is a bounded smooth domain in $\mathbb{R}^2$ and we assume $r = 2$ or $r = p + 1$. We also prove that positive solutions of bounded Morse index are solutions with finitely many sharp peaks. Solutions of this type are known to exist by [4], [5] and [12]. Note that our restrictions on p and r include the case originally studied in [7].

Our system arises in the theory of shells in biology except our version is a limit equation (the shadow system) where the diffusion in the second equation tends to infinity.

We prove our results by combining our recent work on finite Morse index solutions of scalar equations [1] or [2] with some very simple results for the spectrum of one-dimensional linear perturbations of self-adjoint problems. These have been extensively studied in the literature by Aronszajn, Foguel, Lifsic, Wolf and many others. For references, see Kato [6]. However, they do not seem to include exactly the results we need.

It seems likely that our techniques can be used for other problems. Note that the alternative strategy of directly blowing up our linearized equation seems to run into difficulties because it is not clear if one term has a limit for a large enough part of the domain of the self-adjoint part.

We prove the linear operator results in Section 2 and prove our main results in Section 2. Note that a number of our results were announced in our survey article [2]. It seems that our methods can also be used for a number of other systems.

2 One-dimensional perturbations of self-adjoints

We assume L is a self-adjoint closed linear operator on a Hilbert space H with dense domain $\mathcal{D}(L)$ such that $\sigma(L) \cap (-\infty, 0]$ consists of a finite number of points of finite multiplicity $\{\lambda_i\}_{i=1}^k$ where $\lambda_1 < \lambda_2 < \ldots < \lambda_k < 0$. By standard theory [6], $L - \lambda_i I$ has closed range. Let M_- be the sum of the multiplicities of the λ_i, $i = 1, \ldots, k$. We assume f is a continuous linear functional on H, $h \in H$ and $g : (-\infty, 0] \to \mathbb{R}$ is continuous and positive. We want to study the negative real spectrum of $L_1(\lambda) = L + g(\lambda) f(\,.\,)h$. In other words, we study the negative real λ for which $L_1(\lambda) - \lambda I$ has a non-trivial kernel. For simplicity of notation, we usually write L_1 rather than $L_1(\lambda)$. Note that L_1 may have complex spectrum (provided g is defined for complex λ) but we do not study this. Note that by standard perturbation theorems [6], L_1 has only finitely many negative points in its spectrum, each of finite algebraic multiplicity, if g is real analytic.

Proposition 2.1 (i) *If $f(x) = \beta\langle x, h\rangle$ where β is real (equivalently L_1 is self-adjoint for real λ), then L_1 has at least $M_- - 1$ negative real eigenvalues counting geometric multiplicity.*

(ii) If $f(\tilde{\phi}_i)\langle h, \tilde{\phi}_i\rangle$ has fixed sign (possibly zero) for every eigenfunction $\tilde{\phi}_i$ of L corresponding to a negative simple eigenvalue of L, then L_1 has at least $\frac{1}{2}(M_- - 1)$ negative real eigenvalues counting geometric multiplicity (and at least 1 if $M_- \geq 2$).

Remark 2.2 We will give an example later in the section showing that to have any result similar to (ii), we must have some restrictions on f and h. Note that (ii) includes the case where L_1 is self-adjoint.

Proof. (i) This is rather easy. Let $\tilde{M}$ be the subspace of H spanned by the eigenfunctions of L corresponding to negative eigenvalues. Thus $\dim \tilde{M} = M_-$. Since f is a continuous linear function on $\tilde{M}$, there is a subspace M_1 of $\tilde{M}$ of codimension 1 such that f vanishes on M_1. Note that $\langle Lx, x\rangle < 0$ on $\tilde{M}$. Hence we see that $\langle (L_1 - \lambda)x, x\rangle < 0$ on M_1 if λ is small and M_1 has dimension $M_- - 1$.

On the other hand it is easy to see that $L_1 - \lambda I$ is positive definite if λ is large negative. Hence the result follows by a simple counting argument on the Morse index of $L - \lambda I$ if we note that the Morse index of $L_1 - \lambda I$ on H (where $\lambda < 0$) can only change at $\tilde{\lambda}$ by at most $\dim N(L_1 - \tilde{\lambda} I)$, as is easily proved. (Remember that $L_1 - \lambda$ is Fredholm if $\lambda < 0$.)
(ii) Here we first need to consider the possibility that λ_i is also an eigenvalue of L. There are two subcases to consider:

(a) f vanishes at some nonzero point of N_i, the eigenspaces of L corresponding to the eigenvalue λ_i, and

(b) N_i is one-dimensional, spanned by h_i where $f(h_i) \neq 0$.

Note that these cover all cases, because if $\dim N_i > 1$, f must vanish on a subspace $\tilde{N}_i$ of N_i with $\dim \tilde{N}_i = \dim N_i - 1$ (since f is a linear functional).

Now in case (a), it is easy to see that $\tilde{N}_i$ consists of eigenvectors of L_1 corresponding to the eigenvalue λ_i and hence λ_i is an eigenvalue of L_1 of geometric multiplicity at least $\dim \tilde{N}_i = \dim N_i - 1$. (If N_i is one-dimensional and f vanishes on N_i, we set $\tilde{N}_i = N_i$ and see λ_i is an eigenvalue of L_1 of geometric multiplicity at least 1.)

We now consider case (b). In this case, we see that λ_i can only be an eigenvalue if $h \in R(L - \lambda_i I)$ (since otherwise $L_1 x = \lambda_i x$ implies that $f(x) = 0$ and $Lx = \lambda_i x$). In this case, choose u such that $Lu - \lambda_i u = h$ and the only possible solutions of $L_1 x = \lambda_i x$ are $su + qn$ where n spans N_i. Substituting this in the equation $Lx - \lambda_i x = -g(\lambda_i)f(x)h$, we find that our equation reduces to $-g(\lambda_i)f(su + qn) = s$ and we can solve this for q since $f(n) \neq 0$ and $g(\lambda_i) \neq 0$.

Summarizing, we have proved that λ_i is an eigenvalue of L_1 of geometric multiplicity at least $\max\{\frac{1}{2}\dim N_i, 1\}$ unless $\dim N_i = 1$, f does not vanish on N_i and $h \notin R(L - \lambda_i I)$, that is, $\langle h, \tilde{\phi}_i\rangle \neq 0$ where $\tilde{\phi}_i$ spans N_i.

We next consider the case where λ_i, λ_{i+1} are both simple eigenvalues of L such that $f(\tilde{\phi}_i) \neq 0$, $f(\tilde{\phi}_{i+1}) \neq 0$, $\langle h, \tilde{\phi}_i\rangle \neq 0$ and $\langle h, \tilde{\phi}_{i+1}\rangle \neq 0$. In this case, we prove that L_1 has a real eigenvalue in $(\lambda_i, \lambda_{i+1})$.

First note that if $\lambda < 0$ and $\lambda \neq \lambda_i$ for all i, the equation $L_1 x = \lambda x$ can be rewritten as $(L - \lambda I)x = -g(\lambda)f(x)h$ and hence $x = -g(\lambda)f(x)R_\lambda h$, where $R_\lambda = (L - \lambda I)^{-1}$. Hence $x = R_\lambda h$ (up to scalar multiples). Hence we have a non-trivial solution if and only if $g(\lambda)f(R_\lambda h) = -1$. Now $f(R_\lambda h)$ is continuous on $(\lambda_i, \lambda_{i+1})$ and hence it suffices to prove that $f(R_{\lambda_i+\delta}h)$ and $f(R_{\lambda_{i+1}-\delta}h)$ are both large and have opposite signs if δ is small and positive. Now by eigenfunction expansions of $R(\lambda)$ we see that

$$f(R_{\lambda_i+\delta}h) = -\delta^{-1} f(\tilde{\phi}_i)\langle \tilde{\phi}_i, h\rangle + \text{ bounded terms (in } \delta)$$

if δ is small and positive. Similarly,

$$f(R_{\lambda_{i+1}-\delta}h) = \delta^{-1} f(\tilde{\phi}_{i+1})\langle\tilde{\phi}_{i+1}, h\rangle + \text{ bounded terms (in } \delta)$$

if δ is small and positive. Hence by our assumptions $f(R_{\lambda_i+\delta}h)$ and $f(R_{\lambda_{i+1}-\delta}h)$ have opposite signs and our claim follows. The final results now follow easily by a simple counting argument.

Remark 2.3 There are many variants. Our main interest is in proving that, if L has many negative real eigenvalues counting multiplicity, then so does L_1. Clearly, we could still do this if we could bound the number of negative simple eigenvalues of L for which $\langle f, \tilde{\phi}_i\rangle\langle h, \tilde{\phi}_i\rangle < 0$ or the number where it is strictly positive (or indeed the number of adjacent pairs of such eigenvalues). We have also not tried to obtain optimal estimates for the number of eigenvalues.

Lastly, for this section, we have a simple counterexample which shows that some restrictions on f and h are necessary not to be able to move all the real eigenvalues onto the positive real axis if all the negative eigenvalues of L are simple. It suffices to choose f, h such that $\langle f, \tilde{\phi}_i\rangle \neq 0$, $\langle h, \tilde{\phi}_i\rangle \neq 0$ for $1 \leq i \leq k$ and

$$f(R_\lambda h) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{-1} \tag{2.1}$$

because then $f(R_\lambda, h) > 0$ if $\lambda < \lambda_1$ and $|f(R_\lambda, h)|$ has a positive lower bound on $[\lambda_1, 0]$. Thus if c is large positive the equation $cg(\lambda)f(R_\lambda h) = -1$ has no negative solution. (Our earlier arguments imply that λ_i is not an eigenvalue for each i.) By a partial fraction decomposition, we see that

$$f(R_\lambda h) = \sum_{i=1}^{k} c_i(\lambda_i - \lambda)^{-1}. \tag{2.2}$$

By comparing the asymptotics of (2.1) and (2.2) near λ_i, we see that $c_i \neq 0$. Thus our proof reduces to choosing f, h such that $\langle f, \tilde{\phi}_i\rangle \neq 0$, $\langle h, \tilde{\phi}_i\rangle \neq 0$ for all i and (2.2) holds. We choose $h \in span\{\tilde{\phi}_i\}_{i=1}^{k}$. Hence by eigenfunction expansions,

$$R_\lambda h = \sum_{i=1}^{k} (\lambda_i - \lambda)^{-1} \langle h, \tilde{\phi}_i\rangle \tilde{\phi}_i$$

(where $\tilde{\phi}_i$ are normalized) and hence

$$f(R_\lambda h) = \sum_{i=1}^{k} (\lambda_i - \lambda)^{-1} \langle h, \tilde{\phi}_i\rangle \langle f, \tilde{\phi}_i\rangle.$$

Hence we only have to choose f, h such that $\langle h, \tilde{\phi}_i\rangle\langle f, \tilde{\phi}_i\rangle = c_i$ for $1 \leq i \leq k$. This is easy to do. This completes the construction. Note that there are no restrictions on L except for the simplicity of its negative eigenvalues.

If g is constant, we can with care modify our construction to produce examples where there are no complex eigenvalues in the left half plane. The idea is to construct f, h so that

$$f(R_\lambda h) = cP(\lambda)\prod_{i=1}^{k}(\lambda_i - \lambda)^{-1}$$

where P is a polynomial of degree $k-1$ with all its roots in the right half plane and suitable sign on the negative real axis and choose c large positive. Note that this construction is valid for any L with simple negative eigenvalues. With somewhat more care, one can give similar examples for $g(\lambda) = (a - b\lambda)^{-1}$ (which is the g of Section 3).

Finally, note that if L has discrete spectrum and if the extra condition in Proposition 2.1(ii) holds for a complete set of orthogonal eigenfunctions, it is not difficult to prove as in [9] that $f(R_\lambda h)$ is monotone on each interval $(\lambda_i, \lambda_{i+1})$ and we can keep better control on the number of negative real eigenvalues.

3 The main results

In this section, we discuss "nearly stable" solutions of the shadow system for the Geirer-Meinhardt system. More precisely, we assume that Ω is a smooth bounded domain in $\mathbb{R}^2$ and consider the system

$$\begin{aligned} \dot{u} &= \varepsilon^2 \Delta u - u^r h^{-q} \qquad \text{in } \Omega \\ \tau \dot{h} &= -h + |\Omega|^{-1}\int_\Omega u^r h^{-s} \end{aligned} \tag{3.1}$$

where $u = u(x,t)$, $h = h(t)$, $\epsilon, \tau, q, r > 0$, $s \geq 0$, $p > 1$, $\cdot$ denotes time derivatives and u satisfies a Neumann boundary condition on $\partial\Omega$. Let $\gamma_0 = \frac{qr}{(p-1)(s+1)}$. We are interested in solutions with u positive on Ω. As in [8], it is easy to see stationary solutions of the system are given by $u = tw$ where t is constant and positive and

$$-\varepsilon^2 \Delta w = -w + w^p \qquad \text{in } \Omega \tag{3.2}$$

with Neumann boundary conditions. By [8] there are simple formulae for t and h in terms of integrals of powers of w. Moreover, as in [11] it is easy but tedious to prove that the spectra of the linearization of (3.1) at a solution of w of (3.2) with real part non-positive (and with non-small absolute value), are given by the eigenvalues of

$$Lz + (s+1-\tau\lambda)^{-1} qr \,\frac{\int_\Omega w^{r-1} z}{\int_\Omega w^r}\, w^p = \lambda z \tag{3.3}$$

on Ω with Neumann boundary conditions, where $Lz = -\varepsilon^2 \Delta z + z - p w^{p-1} z$. Moreover, geometric multiplicities of eigenvalues are preserved.

Solutions (u_i, h_i) of (3.1) for $\varepsilon = \varepsilon_i \to 0$ as $i \to \infty$ are said to be meta-stable if for each $\mu > 0$, (3.3) for $\varepsilon = \varepsilon_i$, $w = w_i$ has no eigenvalue λ with $\operatorname{Re}\lambda \leq -\mu$ for large i. Here w_i is the solution of (3.2) corresponding to u_i. It is said to be linearized stable if we can take $\mu = 0$. Note that, for small ε, a meta-stable solution may be unstable but the instability is only likely to be apparent under very long time intervals (particularly if eigenvalues with non-positive small real part also have small imaginary part).

Similarly (u_i, h_i) are said to have meta-finite Morse index if there is a $\tilde{K} > 0$ such that for each $\mu > 0$, (3.3) for $\varepsilon = \varepsilon_i$, $w = w_i$ has at most $\tilde{K}$ eigenvalues counting multiplicity with Re $\lambda \leq -\mu$ for large i. If we can take $\mu = 0$, we say that it has finite Morse index. Thus finite Morse index solutions are not too unstable.

Theorem 3.1 *Assume that $r = p + 1$ or $r = 2$.*

(i) If (u_i, h_i) are meta-stable solutions of (3.1) for $\varepsilon = \varepsilon_i \to 0$ as $i \to \infty$, then for large i, u_i has its maximum on $\overline{\Omega}$ at a single point and is exponentially small away from the peak. (This is called a one peak solution.)

(ii) If (u_i, h_i) are meta-finite Morse index solutions of (3.1) for $\varepsilon = \varepsilon_i \to 0$ as $i \to \infty$, then for large i, u_i has $\tilde{s}_i \leq m$ sharp peaks in Ω (where m is determined by $\tilde{K}$) and u_i is exponentially small away from the peaks. (These are so-called $\tilde{s}_i$ peak solutions.) These are unstable if $\tilde{s}_i > 1$.

Remark 3.2 The $\tilde{s}$ peak solutions are reasonably (but not completely) understood (by [4], [5], [8], [11], [12], where many further references can be found) and so is their stability and hence we have a good understanding of the meta-finite Morse index solutions (which are in fact always finite Morse index solutions). In particular, the one peak solutions are rather well understood. It follows from (i) that the only possible meta-stable solutions are found in Wei [11] (for certain parameter values). Wei's theory frequently shows that the only eigenvalues of real part less than or equal to zero have small absolute values. If the peak is on the boundary, it can be shown much as in [9] or [11] that the one peak solutions are sometimes stable. In these last two results whether $\gamma_0 < 1$ or $\gamma_0 > 1$ plays an important role as does the size of τ.

Proof. We first prove (ii) and then indicate how it must be modified to prove (i). If $u_i = t_i w_i$ where t_i is a positive constant and w_i is a solution of (3.2), it suffices to prove that w_i has meta-finite Morse index as a solution of the linearizeation of (3.2) because we can apply the theory in [1] and [2] to deduce that w_i is of the required peak form, completing the proof. To do this we use the theory in Section 2, with L as above (for $w = w_i$). We immediately see that if L has many real eigenvalues less than or equal to μ then so does (3.3) for $w = w_i$. There are two comments here. We apply Proposition 2.1 with λ replaced by $\lambda - \mu$. Secondly we need to check the technical condition in the statement of Proposition 2.1 (which we do below). Hence we see that, for our meta-finite Morse index solution, L has only finitely many eigenvalues less than or equal to μ for large i (with the bound independent of i). Hence w_i are solutions of (3.2) of meta-finite Morse index.

It remains to check the technical conditions in Proposition 2.1. If $r = p + 1$, $f(z) = \int_\Omega w^{r-1} z$, and $h = w^p$, it is easy to see that the map $z \to f(z)h$ is self-adjoint. Hence there is no problem. If $r = 2$,

$$-\varepsilon^2 \Delta w_i = w_i^p - w_i$$

in Ω and

$$-\varepsilon^2 \Delta s_i = p w_i^{p-1} s_i - s_i + \gamma_i s_i$$

(where $\gamma_i < 0$) in Ω with w_i and s_i both satisfying Neumann boundary conditions on $\partial\Omega$, then by a simple integration by parts

$$(p-1) \int_\Omega w^p s_i + \gamma_i \int_\Omega w_i s_i = 0. \tag{3.4}$$

Hence

$$\begin{aligned} f(s_i)\langle h, s_i \rangle &= \int_\Omega s_i w_i \int_\Omega w^p s_i \\ &= -(p-1)^{-1} \gamma_i \Big(\int_\Omega s_i w_i \Big)^2 \\ &\geq 0 \end{aligned}$$

since $\gamma_i \leq 0$. Hence the technical condition is satisfied and we have proved our claim.

The proof of (i) is almost the same as (ii) if we note that the meta-stability of u_i implies that L has meta Morse index at most 1 (by Proposition 2.1) and hence u_i must be a one peak solution by [2].

Remark 3.3 Once we know w_i decay exponentially away from the peaks we can blow up as in [3] and [11] to obtain limiting equations of (3.3) on $\mathbb{R}^n$ or a half space. These ideas could also be used to deduce (i) from (ii) in Theorem 3.1.

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