Advanced Nonlinear Studies **6** (2006), 461–481

Uniqueness and Exact Multiplicity of Solutions For Non-autonomous Dirichlet Problems

Philip Korman*

Department of Mathematical Sciences,
University of of Cincinnati, Cincinnati Ohio 45221-0025
e-mail: kormanp@math.uc.edu

Received in revised form 9 March 2006
Communicated by Ireneo Peral

Abstract

We consider positive solutions for a class of non-autonomous equations on a unit ball in R^n

$$\Delta u + f(|x|, u) = 0 \text{ for } |x| < 1, \quad u = 0 \text{ when } |x| = 1.$$

We assume that $f_r(r, u) \leq 0$, so that in view of [8] all positive solutions are radially symmetric. If $f(|x|, u) = a(|x|)g(u)$, we obtain several exact multiplicity results for a class of convex-concave $g(u)$, with $g(0) = 0$. In another direction, we obtain uniqueness and non-degeneracy of positive solutions for a class of equations, modeled on $f(|x|, u) = -a(|x|)u + b(|x|)u^p$, with $1 < p < \frac{n+2}{n-2}$.

1991 Mathematics Subject Classification. 35J60.
Key words. Semilinear Dirichlet problems, exact number of solutions

1 Introduction

We study uniqueness and exact multiplicity of positive solutions for the problem

$$\Delta u + f(|x|, u) = 0, \quad \text{for } |x| < 1, \quad u = 0 \text{ if } |x| = 1, \tag{1.1}$$

*Supported in part by the Taft Faculty Grant at the University of Cincinnati

with nonlinearity depending explicitly on x. Denoting $r = |x|$, we assume that the function $f(r, u)$ is differentiable, and

$$f_r(r, u) \leq 0, \quad \text{for all } r \in [0, 1], \text{ and } u > 0. \tag{1.2}$$

In view of the classical results of B. Gidas, W.-M. Ni and L. Nirenberg [8] positive solutions of (1.1) are radially symmetric, with $u'(r) < 0$ for all $r \in (0, 1)$, and hence they satisfy

$$u'' + \frac{n-1}{r}u' + f(r, u) = 0, \;\; u'(0) = 0, \; u(1) = 0. \tag{1.3}$$

In case $f = f(u)$, i.e. without explicit r dependence, exact multiplicity of positive solutions have been studied extensively in recent years, starting with P. Korman, Y. Li and T. Ouyang [12], and continued by T. Ouyang and J. Shi [17], [18] (and in a number of other papers by various authors). In [12] a general scheme for proving such results was developed. It involves several steps: proving positivity of solutions to linearized equation, studying the direction of bifurcation, showing uniqueness of the solution curve, etc. Here we follow the same approach, however when one allows the r-dependence, i.e. $f = f(r, u)$, all of the above mentioned steps become much more complicated. For example, it appears impossible to follow the steps in [12] to study the direction of bifurcation. We establish exact multiplicity results for a subclass of (1.3), depending on a positive parameter λ

$$u'' + \frac{n-1}{r}u' + \lambda a(r)g(u) = 0, \;\; u'(0) = 0, \; u(1) = 0, \tag{1.4}$$

with $g(0) = 0$, $g(u) > 0$ for $u > 0$, and $g(u)$ convex-concave, i.e. convex for small $u > 0$, and concave after a certain point. As for $a(r)$, we assume it to be positive, non-increasing, and its derivative is not too large, relative to $a(r)$. We can treat the general problem (1.3) as well, but the conditions become messy.

There are several possibilities for positive convex-concave $g(u)$. The case of $g'(u) > 0$ and $\lim_{u\to\infty} \frac{g(u)}{u} = 0$, e.g. $g(u) = \frac{u^2}{1+u^2}$, is covered by the Theorem 6.1. Assume first that $g'(0) = 0$. We prove existence of a critical $\lambda_0 > 0$, so that for $\lambda < \lambda_0$ the problem (1.4) has no positive solutions, it has exactly one positive solution at $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda > \lambda_0$, see Figure 1. Moreover, we show that all solutions lie on a single smooth solution curve, and study monotonicity of its branches. In case $g'(0) > 0$, the situation is similar, except that the lower branch enters zero at $\frac{\lambda_1}{g'(0)}$, where λ_1 is the principal eigenvalue of $\Delta u + \lambda a(r)u = 0$ on the unit ball. Another possibility is that $g'(u) > 0$ and $0 < \lim_{u\to\infty} \frac{g(u)}{u} < \infty$, e.g. $g(u) = \frac{u^3}{1+u^2}$. Here we prove uniqueness of solutions, see Figure 2. As in the previous case, we show that only turns to the right are possible, but since bifurcation from infinity is supercritical (i.e. toward increasing λ), no turns are possible at all. The remaining case is when $g(c) = 0$ for some $c > 0$. In this case we can give a complete result only in the one-dimensional case, see Figure 3, because of a technical difficulty when proving the direction of bifurcation.

As we mentioned, we follow the general scheme developed in [12]. One of our additional tools (used throughout the proofs) involves certain generalized Wronskians, introduced recently by M. Tang [20], which are some combinations of solutions of (1.3) and of the corresponding linearized problem. While using generalized Wronskians is equivalent to using test functions, as in [12], [17], [18], they do appear to shorten the proofs somewhat. In another direction we use the same tools to extend recent non-degeneracy and uniquess results of A. Aftalion and F. Pacella [2], which they proved for a class of p-Laplace equations with f modelled on $f(r,u) = -a(r)u + b(r)u^q$. In case of $p = 2$ our approach appears to be easier, and we also consider a more general model problem. Previous works in this direction include E. Yanagida [21], Adimurthi and S.L. Yadava [1], and M. Chaves and J. García-Azorero [4].

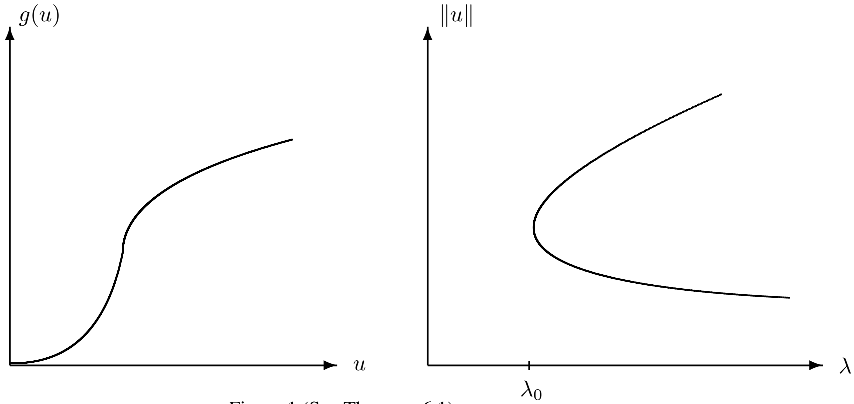

Figure 1 (See Theorem 6.1)

We recall the Crandall-Rabinowitz bifurcation theorem [5], which is one of our main tools.

Theorem 1.1 [5] *Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) = span\ \{x_0\}$ be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span $\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + sx_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

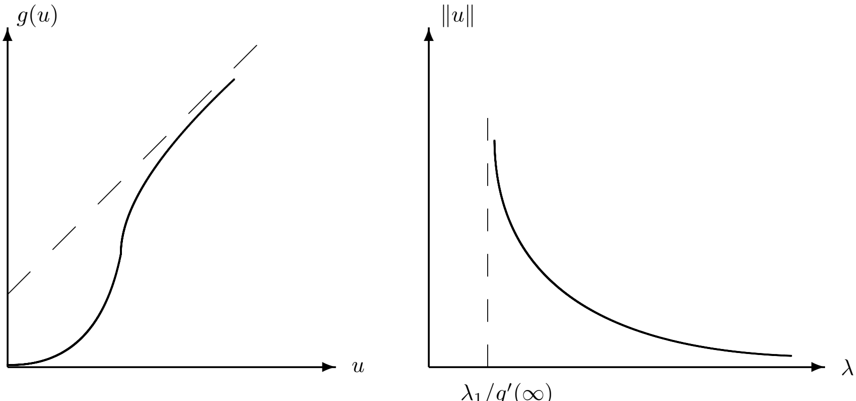

Figure 2 (See Theorem 6.3)

Throughout the paper we consider only the classical solutions.

2 Two preliminary results

We consider positive solutions of the problem

$$u'' + \frac{n-1}{r}u' + f(r,u) = 0, \quad u'(0) = 0, \; u(1) = 0. \tag{2.1}$$

If $u(r)$ is a solution of (2.1), the following functions come up often, see [12], [18], [20], and [2]

$$Q(r) = r^n \left[u'(r)^2 + u(r)f(r,u(r)) \right] + (n-2)r^{n-1}u'(r)u(r), \tag{2.2}$$

$$P(r) = r^n \left[u'(r)^2 + 2F(r,u(r)) \right] + (n-2)r^{n-1}u'(r)u(r), \tag{2.3}$$

where we denote $F(r,u) = \int_0^u f(r,t)\,dt$. Clearly,

$$Q(r) = P(r) + r^n \left[u(r)f(r,u(r)) - 2F(r,u(r)) \right]. \tag{2.4}$$

Observe that $P(0) = 0$, $P(1) = u'(1)^2 > 0$, and

$$\begin{aligned} P'(r) &= r^{n-1} \left[2nF(r,u(r)) - (n-2)u(r)f(r,u(r)) + 2rF_r(r,u(r)) \right] \\ &\equiv r^{n-1}I(r). \end{aligned}$$

The following lemma then follows immediately. It gives three sets of conditions for the positivity of $P(r)$.

Lemma 2.1 **(i)** *Assume there is an $0 < r_0 \le 1$, such that $I(r) > 0$ on the interval $(0, r_0)$. Then $P(r) > 0$ on $(0, r_0]$.*

(ii) *Assume there is an $0 \le r_0 < 1$, such that $I(r) < 0$ on the interval $(r_0, 1)$. Then $P(r) > 0$ on $[r_0, 1)$.*

(iii) *Assume there is an $0 < r_0 < 1$, such that $I(r) > 0$ on the interval $(0, r_0)$ and $I(r) < 0$ on the interval $(r_0, 1)$. Then $P(r) > 0$ on $(0, 1]$.*

We will consider a special form of (2.1)

$$u'' + \frac{n-1}{r}u' + a(r)g(u) = 0, \quad u'(0) = 0, \; u(1) = 0, \tag{2.5}$$

with a positive and non-increasing function $a(r) \in C^1[0,1]$, and $g(u) \in C^2(\bar{R}_+)$, and its corresponding linearized problem

$$L[w] \equiv w'' + \frac{n-1}{r}w' + a(r)g'(u)w = 0, \quad w'(0) = 0, \; w(1) = 0. \tag{2.6}$$

The following technical lemma will be used to study the direction of bifurcation.

Lemma 2.2 *Assume that the problem (2.6) admits a positive solution $w(r) > 0$ on $[0,1)$, and the function $a(r)$ is non-increasing. Assume that $g(u)$ satisfies*

$$g(0) = 0, \quad \textit{and} \;\; g(u) > 0 \;\; \textit{for} \; u \in (0,c), \tag{2.7}$$

$$g''(u) > 0 \;\; \textit{when} \; u \in (0,\alpha), \;\; g''(u) < 0 \;\; \textit{when} \; u \in (\alpha,c), \tag{2.8}$$

for some $0 < c \le \infty$, and $\alpha \in (0,c)$, and assume that $u(0) > \alpha$. Then

$$\int_0^1 a(r) g''(u) {u'}^2 w r^{n-1}\, dr \le 0. \tag{2.9}$$

Proof. Consider a function (which was used previously in [20])

$$\theta(r) = r^{n-1} \left(g(u)' w - g(u) w' \right). \tag{2.10}$$

Here, of course, $g(u)' = g'(u)u'$. Using the equations (2.5) and (2.6), we compute

$$\theta'(r) = r^{n-1} g''(u) {u'}^2 w. \tag{2.11}$$

Integrating (2.11), and using that $g(0) = 0$,

$$\int_0^1 g''(u) {u'}^2 w r^{n-1}\, dr = 0. \tag{2.12}$$

Since $g''(u(r))$ is negative for small r and positive near $r = 1$, with exactly one sign change, while $a(r)$ is non-increasing, the inequality (2.9) follows. (The integrand in (2.12) changes sign exactly once, and $a(r)$ is larger where the integrand is negative.) ♢

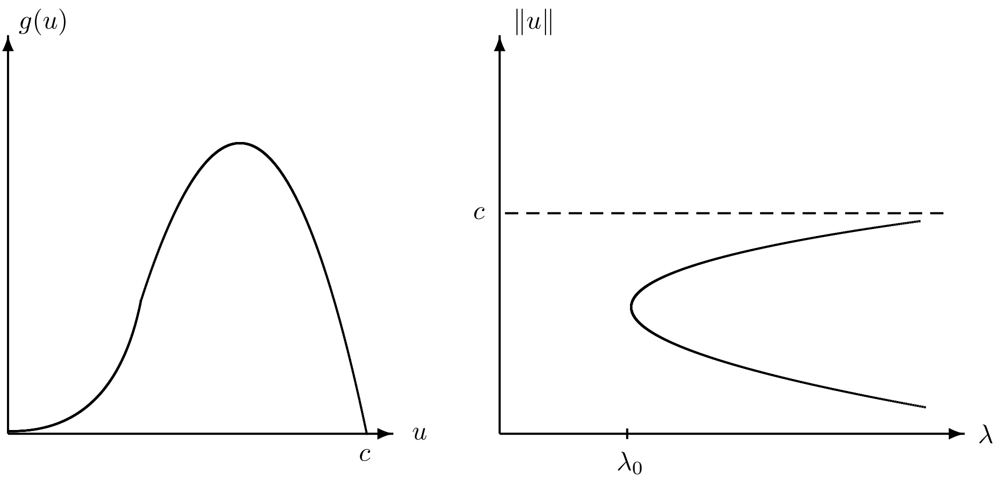

Figure 3 (See Theorem 6.5, $n = 1$)

3 Four generalized Wronskians

It turns out that certain expressions involving $u(r)$ and $w(r)$, the solutions of (2.5) and (2.6) respectively, play an important role for most steps of our analysis. One such generalized Wronskian, the function $\theta(r) = r^{n-1}\left(g(u)'w - g(u)w'\right)$ has already been defined and used above. We shall also need the following functions $\xi(r)$ and $\zeta(r)$, introduced by M. Tang [20] (in the case of constant $a(r)$)

$$\xi(r) = r^{n-1}\left(u'w - uw'\right),$$

and

$$\zeta(r) = r^n\left[u'w' + a(r)g(u)w\right] + (n-2)r^{n-1}u'w.$$

As was observed in M. Tang [20], these functions have the following derivatives

$$\xi'(r) = a(r)\left[ug'(u) - g(u)\right]wr^{n-1}, \tag{3.1}$$

$$\zeta'(r) = \left(2a(r) + ra'(r)\right)g(u)wr^{n-1}. \tag{3.2}$$

The Wronskian $\zeta(r)$ corresponds to the important test function $ru'(r)$, in the sense that $L[-ru'(r)] = \left(2a(r) + ra'(r)\right)g(u)$. Similarly, $\xi(r)$ corresponds to the test function $u(r)$, and the Wronskian $\theta(r)$ to the test function $g(u(r))$. We list another Wronskian $\eta(r) = r^{n-1}w'(r)$, which corresponds to constant test functions. Clearly,

$$\eta'(r) = -a(r)g'(u)wr^{n-1}. \tag{3.3}$$

It seems natural to ask: are there any other generalized Wronskians? (I.e. expressions involving u,u',w and w', whose derivatives do not involve u' and w'.)

We remark that the generalized Wronskians can be traced back to the works of P.H. Rabinowitz [19], W.-M. Ni and J. Serrin [16], and B. Franchi, E. Lanconelli and J. Serrin [7].

4 Positivity for the linearized problem

In this section we give conditions for any non-trivial solution of the linearized problem (2.6) to be of one sign. Following T. Ouyang and J. Shi [18], for the function $g(u)$ satisfying the conditions (2.7) and (2.8), we define $\rho = \alpha - \frac{g(\alpha)}{g'(\alpha)}$. The following lemma is essentially due to [18], however our proof appears to be simpler.

Lemma 4.1 *Assume that $g(u)$ satisfies the conditions (2.7) and (2.8). Then any solution of the linearized problem (2.6), $w(r)$, cannot vanish in the region where $u(r) > \rho$.*

Proof. We begin by observing that

$$h(u) \equiv g(u) - g'(u)(u-\rho) > 0, \quad \text{for all } u \in (\rho, c) \setminus \{\alpha\}. \tag{4.1}$$

Indeed, $h(\alpha) = 0$ and $h'(u) = -g''(u)(u-\rho)$, so that $h(u)$ has a minimum at α (on the interval (ρ, c)), and (4.1) follows. We may assume that $w(0) > 0$, and if the lemma is false, let τ be the first root of $w(r)$ in the region where $u(r) > \rho$ (so that $w(r) > 0$ on $[0, \tau)$). Define $z(r) = \xi(r) + \rho\eta(r)$, where $\xi(r)$ and $\eta(r)$ are the generalized Wronskians, defined above. In view of (3.1) and (3.3),

$$z'(r) = -a(r)w(r)\left(g(u) - g'(u)(u-\rho)\right)r^{n-1} < 0, \quad \text{where } u(r) > \rho.$$

Since $z(0) = 0$, we conclude that $z(\tau) < 0$, but $z(\tau) = -\tau^{n-1}u(\tau)w'(\tau) + \rho\tau^{n-1}w'(\tau) = -\tau^{n-1}w'(\tau)\left(u(\tau) - \rho\right) > 0$, a contradiction. ♢

We now prove positivity for the linearized problem. Our argument is inspired by M. Tang [20]. Given a solution $u(r)$ of (2.5), we denote by $r_1 \in (0,1)$ the point where $u(r_1) = \rho$. The following result has a number of conditions, but we will see later on that these conditions hold for many natural examples. We denote $G(u) = \int_0^u g(t)\,dt$.

Theorem 4.1 *Assume that $g(u)$ satisfies the conditions (2.7) and (2.8). Assume that the function $I(r) = 2na(r)G(u(r)) - (n-2)u(r)a(r)g(u(r)) + 2ra'(r)G(u(r))$ satisfies either the condition* **(i)** *of Lemma 2.1 with $r_0 = 1$, or the condition* **(ii)** *of Lemma 2.1 with $r_0 = r_1$ or the condition* **(iii)** *of Lemma 2.1. Assume that*

$$u(r)g(u(r)) - 2G(u(r)) > 0, \qquad \text{for } r \in (r_1, 1). \tag{4.2}$$

Assume that the function $K(u) \equiv \dfrac{ug'(u)}{g(u)}$ satisfies

$$\begin{gathered} K(u) \text{ is decreasing on } (0,\rho), \quad K(\rho) > 1, \\ K(u) < K(\rho) \text{ for } u \in (\rho, c). \end{gathered} \tag{4.3}$$

Assume that the function $a(r)$ is positive, and the function $\phi(r) \equiv 1 + \dfrac{1}{2}\dfrac{ra'(r)}{a(r)}$ is positive and non-increasing on $[0,1)$. Then any non-trivial solution of the linearized problem (2.6) is of one sign, i.e. we may assume that $w(r) > 0$ on $[0,1)$.

Proof. Using the generalized Wronskians $\xi(r)$ and $\zeta(r)$, we define a function $O(r) = 2\xi(r) - \gamma\zeta(r)$, with a constant $\gamma > 0$ to be selected. In view of (3.1) and (3.2), we compute

$$O'(r) = 2a(r)g(u(r))w(r)r^{n-1}\left[K(u(r)) - 1 - \gamma\phi(r)\right]. \tag{4.4}$$

Let $w(0) > 0$, and assuming the theorem to be false, let $\tau_1 \in (0,1)$ denote the smallest root of $w(r)$. By Lemma 4.1 $u(\tau_1) < \rho$, and then our assumptions on $K(u)$ imply that

$$\begin{gathered} K(u(\tau_1)) > 1, \quad K(u(r)) < K(u(\tau_1)) \text{ for } r \in [0, \tau_1), \\ K(u(r)) > K(u(\tau_1)) \text{ for } r \in (\tau_1, 1). \end{gathered} \tag{4.5}$$

We now fix γ so that the square bracket in (4.4) vanishes at τ_1, i.e.

$$K(u(\tau_1)) - 1 = \gamma\phi(\tau_1).$$

Using monotonicity of $\phi(r)$ and (4.5), we see that the square bracket in (4.4) changes sign exactly once on $(0,1)$, at $r = \tau_1$, and $K(u(r)) - 1 - \gamma\phi(r)$ is negative on $(0, \tau_1)$ and positive on $(\tau_1, 1)$. If we now denote by $\tau_2 \in (\tau_1, 1]$ the second root of $w(r)$, we see from (4.4) that $O(r)$ is a decreasing function on $[0, \tau_2)$. Since $O(0) = 0$, we conclude that

$$O(r) < 0 \quad \text{for all } r \in [0, \tau_2]. \tag{4.6}$$

There are two cases to consider.

Case (i) $\tau_2 = 1$. From (4.6) we have $O(1) < 0$. On the other hand, $O(1) = -\gamma\zeta(1) = -\gamma u'(1)w'(1) > 0$, a contradiction.

Case (ii) $\tau_2 < 1$. Since $\xi(\tau_1) > 0$, while $\xi(\tau_2) < 0$, we can find $t \in (\tau_1, \tau_2)$, such that $\xi(t) = 0$, i.e.

$$\frac{u(t)}{w(t)} = \frac{u'(t)}{w'(t)}. \tag{4.7}$$

Since $u(\tau_1) < \rho$, we have $\tau_1, t \in (r_1, 1)$. (Recall, $u(r_1) = \rho$.) Hence, by Lemma 2.1, (2.4) and (4.2), we conclude that

$$Q(t) > 0. \tag{4.8}$$

In view of (4.6),

$$\zeta(t) = -\frac{1}{\gamma} O(t) > 0.$$

On the other hand, using (4.7) and (4.8),

$$\zeta(t) = \left[t^n \left(u'w'\frac{u}{w} + a(t)g(u)u \right) + (n-2)t^{n-1}u'u \right] \frac{w}{u} = Q(t)\frac{w(t)}{u(t)} < 0,$$

giving us a contradiction. ♢

Remark We say that a function $f(u) \in C^1(u_1, u_2)$ belongs to the class $\mathbf{K}$ on (u_1, u_2) if $K_f(u) \equiv \frac{uf'(u)}{f(u)}$ is non-increasing on (u_1, u_2). In the above theorem both $g(u)$ and $a(r)$ were assumed to be of class $\mathbf{K}$ on some intervals, so we want to say more on this class. Since

$$K_{fg} = K_f + K_g,$$

we see that the class $\mathbf{K}$ is closed under the multiplication. Also

$$K_{f^r} = rKf,$$

for any $r > 0$. This implies, in particular, that we can deform any function of class $\mathbf{K}$ to a constant, while staying in the class. Indeed, $f^{1-\theta}$ for $0 \le \theta \le 1$ gives such a deformation.

5 The direction of bifurcation

We now consider the problem

$$u'' + \frac{n-1}{r}u' + \lambda a(r)g(u) = 0, \quad u'(0) = 0, \; u(1) = 0, \tag{5.1}$$

depending on a positive parameter λ. Recall that a solution $(\lambda, u(x))$ of (5.1) is called *degenerate* if the corresponding linearized problem

$$L[w] \equiv w'' + \frac{n-1}{r} w' + \lambda a(r) g'(u) w = 0, \;\; w'(0) = 0, \; w(1) = 0 \qquad (5.2)$$

has a nontrivial solution.

We shall need the following technical lemma. It is only at this step that we need to assume that $g'(u) > 0$.

Lemma 5.1 *Assume that $g(u) > 0$ and $g'(u) > 0$ for all $u > 0$, and*

$$A(r) \equiv (n-1)a(r) + (r^2 a'(r))' + (n-1) r a'(r) > 0 \qquad (5.3)$$
$$\text{for all } r \in [0, 1).$$

Assume $n > 1$. Then for any solution of (5.1)

$$\phi(r) \equiv (n-1)u'(r) - \lambda r^2 a'(r) g(u(r)) < 0 \quad \text{for all } r \in (0, 1].$$

Proof. Observe that $\phi(0) = 0$, while in view of the equation (5.1), and our assumptions

$$\phi'(r) + \tfrac{1}{n-1} b(r)\phi(r) = \qquad (5.4)$$
$$-\lambda g(u(r)) \left[A(r) + \tfrac{\lambda}{n-1} r^4 a'^2(r) g'(u(r)) \right] < 0,$$

where we have denoted $b(r) = \left[\frac{(n-1)^2}{r} + \lambda r^2 a'(r) g'(u) \right]$. Integrating (5.4), we conclude the lemma. (The function $b(r)$ is singular at $r = 0$. However, the integrating factor for the inequality (5.4) is a continuous function.) ♢

Lemma 5.2 *Assume that $a(r)$ and $g(u)$ satisfy the conditions of the Theorem 4.1, and of Lemma 2.2, and also (5.3). If (λ_0, u_0) is a degenerate solution of (5.1), then the Crandall-Rabinowitz Theorem 1.1 applies, and moreover only turns to the right are possible on the solution curve, i.e.,*

$$\lambda''(0) > 0. \qquad (5.5)$$

Proof. We show that the Crandall-Rabinowitz Theorem 1.1 applies at any degenerate solution (λ_0, u_0) of (5.1). Let B denote the unit ball $|x| < 1$ in R^n. We define the function spaces $X = \{u \in C^{2,\alpha}(\bar{B}) \mid u = 0 \text{ on } \partial B\}$ and $Y = C^\alpha(\bar{B})$. Let $F : R_+ \times X \to Y$ be given by $F(\lambda, u) = \Delta u + \lambda a(r) g(u)$. Positive solutions of the problem (5.1) are solutions of the operator equation $F(\lambda, u) = 0$. The left hand side of (5.2) is then $F_u(\lambda, u)w$, since by [6] any solution of the linearized equation $F_u(\lambda, u)w = 0$ is radially symmetric.

Observe that the null-space of $F_u(\lambda_0, u_0)$ is one-dimensional, since it can be parameterized by $w'(1)$. Since $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero, it follows that $codim R(F_u(\lambda_0, u_0)) = 1$. Finally, if the condition $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$ was violated, we could find some $z \in X$ satisfying $L[z] = a(r) g(u)$. By the Fredholm alternative this will imply that $a(r) g(u)$ is orthogonal to $w(r)$, which is impossible since

$a(r)$, $g(u)$ and $w(r)$ are all positive. Applying the Crandall-Rabinowitz theorem, we conclude that (λ_0, u_0) is a bifurcation point, near which the solutions of (5.1) form a curve $(\lambda(s), u_0 + sw + z(s))$, with $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, and $z(0) = z'(0) = 0$.

It remains to prove (5.5). For this we need the formula

$$\lambda''(0) = -\lambda_0 \frac{\int_0^1 a(r) g''(u_0) w^3 r^{n-1}\, dr}{\int_0^1 a(r) g(u_0) w r^{n-1}\, dr}. \tag{5.6}$$

We omit the proof of (5.6), since it is similar to [12], and also a more general formula is derived in the next lemma. Since $w(r) > 0$, the denominator in (5.6) is positive. Following the crucial trick from [12], we will show that

$$\int_0^1 a(r) g''(u_0) w^3 r^{n-1}\, dr < \int_0^1 a(r) g''(u_0) u_{0_r}^2 w r^{n-1}\, dr. \tag{5.7}$$

Since by Lemma 2.2 the right hand side of (5.7) is non-positive, it will follow that $\lambda''(0) > 0$.

We begin our proof of (5.7) by differentiating (5.1) (we intentionally mix two notations for the derivative, and also we write $u(r)$ in place of $u_0(r)$)

$$u_r'' + \frac{n-1}{r} u_r' + \lambda a(r) g'(u) u_r = \frac{1}{r^2} \left[(n-1) u'(r) - \lambda r^2 a'(r) g(u) \right] < 0, \tag{5.8}$$

in view of Lemma 5.1, in case $n > 1$. (In case $n = 1$ the inequality (5.7) is proved the same way as in the Theorem 6.5 below.) It is only at this point that we need this lemma, with its extra conditions. It follows that $-u_r$ is a subsolution of the linearized equation (2.6). We claim that the functions $-u_r$ and w intersect exactly once on $(0, 1)$. (We consider only the points of intersection where these functions change their order, and ignore any points where they merely "touch".) If, on the contrary, these functions intersect more than once, then there is an interval $(r_1, r_2) \subset (0, 1)$, on which $w(r) < -u_r$, and $w(r_i) = -u_r(r_i)$, $i = 1, 2$. We can then find a constant $\mu > 1$, such that $-u_r \le \mu w$ for all $r \in (r_1, r_2)$, and $-u_r(r_0) = \mu w(r_0)$ for some $r_0 \in (r_1, r_2)$. Since $-u_r$ and μw are respectively a subsolution and solution of the linearized equation (2.6), we obtain a contradiction by the strong maximum principle.

Let now $\bar{r}$ be the point where $g''(u_0(r))$ changes sign (i.e. $u(\bar{r}) = \alpha$). By considering μw with a proper constant μ, we may assume that $-u_r$ and w intersect (and change order) at the same point $\bar{r}$. Then (5.7) follows, since the integrand on the right is pointwise larger than the one on the left. $\diamondsuit$

We shall need a more general version of our problem (5.1)

$$u'' + \frac{n-1}{r} u' + a(r, \mu) g(u) = 0, \quad u'(0) = 0, \ u(1) = 0, \tag{5.9}$$

depending on a positive parameter $0 \le \mu \le 1$.

Lemma 5.3 *Assume that $a(r, \mu)$ and $g(u)$ satisfy the conditions of the Theorem 4.1, and of Lemma 2.2, and also (5.3), for all $0 \le \mu \le 1$. Assume also that $a_\mu(r, \mu) < 0$*

for all $r \in [0,1)$ and $0 \le \mu \le 1$. If (μ_0, u_0) is a degenerate solution of (5.9) then the Crandall-Rabinowitz Theorem 1.1 applies, and moreover only turns to the right are possible on the solution curve, i.e., $\mu''(0) < 0$.

Proof. Similarly to the previous lemma, we show that the Crandall-Rabinowitz Theorem 1.1 applies at any degenerate solution (μ_0, u_0), which implies that the solution set of (5.9) near that point forms a curve of the form $(\mu(s), u_0+sw+z(s))$, with $\mu(0) = \lambda_0$, $\mu'(0) = 0$, and $z(0) = z'(0) = 0$. To complete the proof, we need a formula for $\mu''(0)$, generalizing (5.6):

$$\mu''(0) = -\frac{\int_0^1 a(r,\mu_0) g''(u_0) w^3 r^{n-1}\, dr}{\int_0^1 a_\mu(r,\mu_0) g(u_0) w r^{n-1}\, dr}. \tag{5.10}$$

To prove (5.10), we differentiate the PDE version of (5.9) twice in s, denoting for convenience $f(r,u,\mu) = a(r,\mu)g(u)$,

$$\Delta u_{ss} + f_u u_{ss} + f_{uu} u_s^2 + f_\mu \mu'' + 2 f_{\mu u} \mu' u_s + f_{\mu\mu} {\mu'}^2 = 0.$$

Setting here $s = 0$, and using that $\mu'(0) = 0$ and $u_s|_{s=0} = w(r)$, we obtain

$$\Delta u_{ss} + f_u u_{ss} + f_{uu} w^2 + f_\mu \mu''(0) = 0. \tag{5.11}$$

Multiplying (5.11) by w, the linearized equation (2.6) by u_{ss}, integrating and subtracting, we conclude (5.10). By the previous lemma, and by our assumptions, both integrals in (5.10) are negative. ♢

6 Exact multiplicity results

We can now prove our exact multiplicity results. We define $g'(\infty) = \lim_{u\to\infty} \frac{g(u)}{u}$. We denote by $\lambda_1 > 0$ the principal eigenvalue of

$$u'' + \frac{n-1}{r} u' + \lambda a(r) u = 0, \;\; u'(0) = 0, \; u(1) = 0. \tag{6.1}$$

Theorem 6.1 *Assume that $a(r)$and $g(u)$ satisfy the conditions of the Theorem 4.1, and the function $a_\mu \equiv a(0)^{1-\mu} a(r)^\mu$ satisfies the condition (5.3), for all $0 \le \mu \le 1$. Assume that $g(u)$ satisfies the conditions (2.7) and (2.8), with $c = \infty$, and in addition $g'(u) > 0$ for all $u > 0$, and $g'(\infty) = 0$. Assume first that $g'(0) = 0$. Then there is a critical $\lambda_0 > 0$, so that for $\lambda < \lambda_0$ the problem (5.1) has no positive solutions, it has exactly one positive solution at $\lambda = \lambda_0$, and exactly two positive solutions for $\lambda > \lambda_0$. Moreover, all solutions lie on a single smooth solution curve, which for $\lambda > \lambda_0$ has two ordered branches $0 < u_-(r,\lambda) < u_+(r,\lambda)$, for all $r \in [0,1)$ and $\lambda > \lambda_0$, with $u_+(r,\lambda)$ strictly monotone increasing in λ, and $\lim_{\lambda\to\infty} u_+(r,\lambda) = \infty$ for all $r \in [0,1)$. For the lower branch we have $\lim_{\lambda\to\infty} u_-(r,\lambda) = 0$ for all $r \in [0,1)$. (See Figure 1.) In case when $g'(0) > 0$, the situation is similar, except that the lower branch enters zero at $\frac{\lambda_1}{g'(0)}$ (and hence there is exactly one positive solution for $\lambda > \frac{\lambda_1}{g'(0)}$).*

Proof. In case $a(r)$ is a constant, the theorem is proved by using the arguments of [12] and [18] (and we repeat most of those arguments below.) In particular, in case $a(r)$ is a constant, we have a maximal solution for large λ, which tends to infinity as $\lambda \to \infty$.

We shall prove that the problem (5.1) has a positive solution, if λ is large enough. Define $U = U(r, \lambda) > 0$ to be the maximal solution of

$$U'' + \frac{n-1}{r}U' + \lambda a(0)g(U) = 0, \;\; U'(0) = 0, \; U(1) = 0. \tag{6.2}$$

By above, this maximal solution tends to infinity when $\lambda \to \infty$, providing us with arbitrary large supersolution of (5.1) at any fixed λ. Similarly to (6.2), we consider an autonomous problem

$$\bar{u}'' + \frac{n-1}{r}\bar{u}' + \lambda a(1)g(\bar{u}) = 0, \;\; \bar{u}'(0) = 0, \; \bar{u}(1) = 0. \tag{6.3}$$

Either one of its two solutions for large λ will provide us with a subsolution to the problem (5.1). We conclude that the problem (5.1) has a positive solution for large λ, moreover it has a maximal solution, in view of existence of arbitrary large supersolutions. The maximal solution is increasing in λ, since maximal solution at any $\bar{\lambda}$ is a subsolution at all $\lambda > \bar{\lambda}$.

We now continue the maximal solution for decreasing λ. At a non-degenerate solution this can be done by the implicit function theorem, and the case of a degenerate solution will be discussed below. We claim that a degenerate solution will be reached eventually, and the solution curve will turn back. Indeed, the solution curve has no other place to go for decreasing λ. Since $u(r) < U$, the only possibility is for it to go to zero. Notice that $g(u) = o(u)$ for u small. Multiplying the equation (5.1) by u, integrating by parts and using the Poincare's inequality, we see that the solution curve cannot go to zero.

It follows that a degenerate solution must be eventually reached. By Lemma 5.2 at this solution the Crandall-Rabinowitz Theorem 1.1 applies, and a simple turn to the right occurs. According to the Theorem 1.1 near the turning point (λ_0, u_0) there is a solution asymptotic to $u_0 + sw$, which is increasing in s for small s, and a decreasing solution asymptotic to $u_0 - sw$. Arguing as in [12], we see that the increasing branch stays increasing until a possible next turn occurs. (Here is a brief outline: by above, $u_\lambda > 0$ near the turning point, and by maximum principle we will get a contradiction at the first λ where this inequality is violated.)

We claim that after the turn at (λ_0, u_0), the lower branch of the solution curve continues for all $\lambda > \lambda_0$, without any more turns. Indeed, at the next turn (to the left), we would have $u(0) < \alpha$, since otherwise only turns to the right are possible, in view of Lemma 5.2. After that turn the solutions are decreasing, as we discussed above. Hence the inequality $u(0) < \alpha$ continues to hold. But then $g(u)$ is convex for all values of $u(r)$, and only turns to the left are possible, see (5.6). Hence our branch, which is traveling to the left now, cannot turn, and it has no place to go, a contradiction. As for the upper branch, emerging at (λ_0, u_0), no other turns are possible in view Lemma 5.2, and hence it continues for all $\lambda > \lambda_0$. Arguing as in [11], we see that the upper branch tends to ∞, and the lower one tends to 0 as $\lambda \to \infty$.

It remains to show that the parabola-like curve of solutions, described above, exhausts the set of all positive solutions of (5.1). This is certainly true in case of constant $a(r)$. (For autonomous problems the value of $u(0)$ uniquely identifies the solution $(\lambda, u(r))$, and on our curve all possible values of $u(0)$, from zero to infinity, are "taken".) Assume on the contrary that at some $\lambda_1 > \lambda_0$ there is a third solution of (5.1). Consider a family of problems

$$u'' + \frac{n-1}{r}u' + \lambda a(0)^{1-\mu}a(r)^{\mu}g(u) = 0, \ \ u'(0) = 0, \ u(1) = 0, \tag{6.4}$$

with a parameter $0 \le \mu \le 1$. At $\mu = 0$ the problem is autonomous, and so it has at most two positive solutions. The function $a(x)^{\mu}$ satisfies the conditions of both Theorem 4.1 and Lemma 5.3, so that at any degenerate solution any nontrivial solution of the linearized problem is positive, and since $f_\mu = f \ln \frac{a(r)}{a(0)} < 0$, we have $\mu''(0) < 0$. (Here $f \equiv a(0)^{1-\mu}a(r)^{\mu}g(u)$.) It follows that only turns to the left are possible in the (μ, u) "plane", and hence no spontaneous bifurcation of solutions may occur. So that either the two solutions coalesce at some $\mu_0 \in (0, 1)$, and we have no solutions at $\mu = 1$, or we have exactly two solutions at $\mu = 1$. Both possibilities contradict our assumption of three solutions. ♢

We show that the theorem applies to the problem

$$u'' + \frac{n-1}{r}u' + \lambda a(r)\frac{u^2}{u^2+1} = 0, \ \ u'(0) = 0, \ u(1) = 0. \tag{6.5}$$

To verify that, we need an elementary lemma.

Lemma 6.1 *Denote* $g(u) = \frac{u^2}{u^2+1}$, $G(u) = \int_0^u g(t)\,dt$. *Then* $\frac{d}{du}\left(\frac{ug(u)}{G(u)}\right) < 0$ *for all* $u > 0$.

Proof. Compute

$$\frac{d}{du}\left(\frac{ug(u)}{G(u)}\right) = -\frac{u^2}{(3+u^2)(1+u^2)^2(u-\tan^{-1}u)^2}\left[-\frac{3u}{3+u^2} + \tan^{-1}u\right].$$

We claim that the function $p(u) \equiv -\frac{3u}{3+u^2} + \tan^{-1}u$ is positive for all $u > 0$. Indeed, $p(0) = 0$, and $p'(u) = \frac{4u^4}{(1+u^2)(3+u^2)^2} > 0$, and the lemma follows. ♢

Theorem 6.2 *Assume that $a(r)$ satisfies all of the assumptions of the Theorem 6.1. Then all of the conclusions of Theorem 6.1 hold for the problem (6.5), and its bifurcation diagram is given by Figure 1.*

Proof. We show how to verify that the Lemma 2.1 applies (which was the key ingredient in proving positivity for the linearized problem), with the other conditions of the Theorem 6.1 being straightforward to verify. We write $I(r) = a(r)G(u(r))J(r)$, where

$$J(r) = 2n - (n-2)\frac{u(r)g(u(r))}{G(u(r))} + 2\frac{ra'(r)}{a(r)}.$$

It suffices to show that the function $J(r)$ is decreasing on $(0, 1)$, since then it is either positive or negative on $(0, 1)$, or else it changes sign exactly once, and in the way we want it, i.e. J is positive near $r=0$ and negative near $r=1$. And the sign of $I(r)$ is the same as that of $J(r)$. In view of Lemma 6.1 we have

$$\frac{d}{dr}J(r) = -(n-2)\frac{d}{du}\left[\frac{ug(u)}{G(u)}\right]u'(r) + 2\left(\frac{ra'(r)}{a(r)}\right)' < 0.$$

Turning to other conditions of the Theorem 6.1, we see that $g(u)=\frac{u^2}{u^2+1}$ is a positive and increasing function, which changes concavity only once at $\alpha=\frac{1}{\sqrt{3}}$. We compute $\rho=\frac{1}{3\sqrt{3}}$. Here $K(u)=\frac{2}{1+u^2}$, which is a decreasing function for all $u>0$, and $K(\rho)=\frac{27}{14}>1$. Finally, letting $h(u)=ug(u)-2G(u)$, we compute $h'(u)=\frac{u^2-u^4}{(1+u^2)^2}$, which means that $h(u)$ is an increasing (and hence positive) function when $u\in(0,\rho)$, verifying the condition (4.2). ♢

We turn next to the case when $0<g'(\infty)<\infty$.

Theorem 6.3 *Assume that $a(r)$ and $g(u)$ satisfy the conditions of the Theorem 4.1, and the function $a_\mu \equiv a(0)^{1-\mu}a(r)^\mu$ satisfies the condition (5.3), for all $0\le\mu\le 1$. Assume that $g(u)$ satisfies the conditions (2.7) and (2.8), with $c=\infty$, and in addition $g'(u)>0$ for all $u>0$, and $0<g'(\infty)<\infty$. Assume also that $\frac{g(u)}{u}<g'(\infty)$ for all $u>0$. Assume first that $g'(0)=0$. Then for $0<\lambda<\frac{\lambda_1}{g'(\infty)}$ the problem (5.1) has no positive solutions, and it has a unique positive solution for $\frac{\lambda_1}{g'(\infty)}<\lambda<\infty$. Moreover, all solutions lie on a single smooth solution curve, which tends to infinity when $\lambda\downarrow\frac{\lambda_1}{g'(\infty)}$, and to zero when $\lambda\to\infty$. (See Figure 2.) In case $g'(0)>0$, the situation is similar, except that the solution curve enters zero at $\frac{\lambda_1}{g'(0)}$ (i.e. there no positive solutions for $\lambda>\frac{\lambda_1}{g'(0)}$).*

Proof. By a standard analysis, bifurcation from infinity occurs at $\frac{\lambda_1}{g'(\infty)}$, and it is supercritical, i.e. the solution curve emerges from infinity toward increasing λ, see e.g. Proposition 3.4 in [18]. As we continue the solution to the right, no degenerate solutions are encountered, since by Lemma 5.2 at any degenerate solution a turn to the right would have to occur, which is not possible. As before we show that the solution curve tends to zero for increasing λ, and that the solution curve is unique. ♢

Remark Since $g'(\infty)>0$, it follows that $f(u)\sim au+b$ for large u, with some $a>0$ and $b\in R$. The assumption that $\frac{g(u)}{u}<g'(\infty)$ implies that $b\le 0$. We needed that assumption to show that bifurcation from infinity is toward increasing λ. In case $b>0$ we cannot tell the direction of bifurcation from infinity. If it is to the right, then the same result holds. Assume bifurcation from infinity is to the left, and $g'(0)=0$. Then the solution curve makes exactly one turn to the left, and then continues for all λ, since only turns to the left are possible. In case $g'(0)>0$, the solution curve enters zero at $\frac{\lambda_1}{g'(0)}$, travelling always to the left if $g'(0)>g'(\infty)$, and making exactly one turn to the left in case $g'(0)\le g'(\infty)$. So that in any case we obtain an exact multiplicity result.

We show that the theorem 6.3 applies to the problem

$$u''+\frac{n-1}{r}u'+\lambda a(r)\frac{u^3}{u^2+1}=0,\ \ u'(0)=0,\ u(1)=0. \tag{6.6}$$

Again, we need an elementary lemma.

Lemma 6.2 *Denote* $g(u) = \frac{u^3}{u^2+1}$, $G(u) = \int_0^u g(t)\,dt = \frac{u^2}{2} - \frac{1}{2}\ln(1+u^2)$. *Then* $\frac{d}{du}\left(\frac{ug(u)}{G(u)}\right) < 0$ *for all* $u > 0$.

Proof. Compute

$$\frac{d}{du}\left(\frac{ug(u)}{G(u)}\right) = \frac{4u^3(2+u^2)}{(1+u^2)^2\,(u^2-\ln(1+u^2))^2}\left[\frac{2u^2}{2+u^2} - \ln(1+u^2)\right].$$

We claim that the function $p(u) \equiv \frac{2u^2}{2+u^2} - \ln(1+u^2)$ is negative for all $u > 0$. Indeed, $p(0) = 0$, and $p'(u) = -\frac{2u^5}{(1+u^2)(2+u^2)^2} < 0$, and the lemma follows. ♢

Similarly to the Theorem 6.2, we obtain the following result.

Theorem 6.4 *Assume that* $a(r)$ *satisfies all of the assumptions of the Theorem 6.3. Then all of the conclusions of Theorem 6.3 hold for the problem (6.6), and its bifurcation diagram is given by Figure 2.*

Proof. Thanks to Lemma 6.2, we see as above that the Lemma 2.1 applies here. So let us run through the verification of other conditions. Here $g(u) = \frac{u^3}{u^2+1}$ is a positive and increasing function, which changes concavity only once at $\alpha = \sqrt{3}$. Here $K(u) = \frac{u^2+3}{u^2+1}$, and $K'(u) = -\frac{4u}{(u^2+1)^2}$, which implies that $K(u)$ is a decreasing function, greater than 1, for all $u > 0$. Finally, letting $h(u) = ug(u) - 2G(u)$, we compute $h'(u) = \frac{2u^3}{(1+u^2)^2}$, which means that $h(u)$ is an increasing (and hence positive) for all $u > 0$, verifying the condition (4.2). ♢

In the remaining case, when $g(c) = 0$, we have to assume that $n = 1$, although in this case we can relax our conditions on $a(r)$. In case $n > 1$, we can still say quite a bit about the solution set (since we still have positivity for the linearized equation), but we do not have a complete exact multiplicity result.

Theorem 6.5 *Assume* $n = 1$. *Assume that* $a(r)$ *and* $g(u)$ *satisfy the conditions of the Theorem 4.1. Assume that* $g(u)$ *satisfies the conditions (2.7) and (2.8), with* $c < \infty$, *and* $g(c) = 0$. *Assume first that* $g'(0) = 0$. *Then there is a critical* $\lambda_0 > 0$, *so that for* $\lambda < \lambda_0$ *the problem (5.1) has no positive solutions, it has exactly one positive solution at* $\lambda = \lambda_0$, *and exactly two positive solutions for* $\lambda > \lambda_0$. *Moreover, all solutions lie on a single smooth solution curve, which for* $\lambda > \lambda_0$ *has two ordered branches* $0 < u_-(r,\lambda) < u_+(r,\lambda)$, *with* $u_+(r,\lambda)$ *strictly monotone increasing in* λ, *and* $\lim_{\lambda\to\infty} u_+(r,\lambda) = c$ *for all* $r \in [0,1)$. *For the lower branch we have* $\lim_{\lambda\to\infty} u_-(r,\lambda) = 0$ *for all* $r \in [0,1)$. *(See Figure 3.) In case* $g'(0) > 0$, *the situation is similar, except that the lower branch enters zero at* $\frac{\lambda_1}{g'(0)}$.

Proof. All of the steps of our analysis in the previous cases carry over here, except the step when we were proving that w and $-u_r$ cannot intersect more than once. We can no longer rely on Lemma 5.1, since we do not have the positivity of $g'(u)$. We observe that this time $-u_r$ is a supersolution of the linearized problem (2.6). If the functions w and $-u_r$ intersected more than once, we could find an interval $(r_1, r_2) \subset (0,1)$, on which $w(r) > -u_r$, and $w(r_i) = -u_r(r_i)$, $i = 1,2$. We can then find a constant $\mu < 1$, such that $-u_r \geq \mu w$ for all $r \in (r_1, r_2)$, and $-u_r(r_0) = \mu w(r_0)$ for some $r_0 \in (r_1, r_2)$. Since $-u_r$ and μw are respectively a supersolution and solution of the linearized equation (2.6), we obtain a contradiction by the strong maximum principle. ♢

7 A simple approach to non-degeneracy and uniqueness

We consider positive solutions of the problem

$$\Delta u + f(|x|, u) = 0, \quad \text{for } |x| < 1, \quad u = 0 \ \text{ if } |x| = 1. \tag{7.1}$$

We assume that $f(r,u)$ is a continuously differentiable function (with $r = |x|$), and

$$f(r,0) = 0, \quad \text{and} \quad f_r(r,u) \leq 0, \qquad \text{for all } r \in [0,1], \text{ and } u > 0. \tag{7.2}$$

In view of the classical results of B. Gidas, W.-M. Ni and L. Nirenberg [8] positive solutions of (7.1) are radially symmetric, with $u'(r) < 0$ for all $r \in (0,1)$, and hence they satisfy

$$u'' + \frac{n-1}{r}u' + f(r,u) = 0, \quad u'(0) = 0, \ u(1) = 0. \tag{7.3}$$

By [6] (see also [15]) any solution of the linearized problem for (7.1) is also radial, and hence it satisfies

$$w'' + \frac{n-1}{r}w' + f_u(r,u)w = 0, \quad w'(0) = 0, \ w(1) = 0. \tag{7.4}$$

In addition to (7.2), we assume that $f(r,u)$ satisfies the following conditions (which have appeared before in [2])

$$uf_u(r,u) - f(r,u) > 0 \qquad \text{for all } r \in [0,1], \text{ and } u > 0, \tag{7.5}$$

$$\alpha(r) = \frac{2f(r,u(r)) + rf_r(r,u(r))}{u(r)f_u(r,u(r)) - f(r,u(r))} \tag{7.6}$$

is a non-increasing function of r, for $r \in (0,1)$.

We will show that under these conditions any positive solution of (7.3) is non-degenerate. We start with two technical lemmas.

Lemma 7.1 *Let $u(r)$ be a positive solution of (7.3), and assume that the function $f(r,u)$ satisfies the conditions (7.2) and (7.5). Then the function $f(r,u(r))$ can change sign at most once on $(0,1)$.*

Proof. Let $\xi \in (0,1)$ be such that $f(\xi, u(\xi)) = 0$. We claim that $f(r, u(r)) > 0$ for all $r \in [0, \xi)$. Indeed, from (7.5) we conclude that $f_u(\xi, u(\xi)) > 0$, and in general $f_u(r, u(r)) > 0$, so long as $f(r, u(r)) > 0$, and hence

$$\frac{d}{dr} f(r, u(r)) = f_r(r, u(r)) + f_u(r, u(r))u'(r) < 0,$$

and the claim follows. So that, if the function $f(r, u(r))$ is positive near $r = 1$, it is positive for all $r \in [0, 1)$. If, on the other hand, $f(r, u(r))$ is negative near $r = 1$, it will change sign exactly once on $[0, 1)$ (since it cannot stay negative for all r, by the maximum principle). $\diamondsuit$

The lemma implies that there is a $r_2 \in (0, 1]$, so that $f(r, u(r)) > 0$ on $[0, r_2)$ and $f(r, u(r)) < 0$ on $(r_2, 1)$.

Lemma 7.2 *In the conditions of the preceeding lemma, any solution of the linearized problem (7.4) $w(r)$ cannot vanish in the region where $f(r, u(r)) < 0$ (i.e. on $(r_2, 1)$) .*

Proof. Assume the contrary is true, and let τ denote the largest root of $w(r)$ in the region where $f(r, u(r)) < 0$. We may assume that $w(r) > 0$ on $(\tau, 1)$. Then integrating the formula (7.9) below over $(\tau, 1)$, we have

$$u'(1)w'(1) - \tau^n u'(\tau)w'(\tau) = \int_\tau^1 \left(2f(r, u(r)) + r f_r(r, u(r))\right) w r^{n-1}\, dr.$$

We have a contradiction, since the left hand side is positive, while the integral on the right is negative. $\diamondsuit$

Theorem 7.1 *Assume that the conditions (7.2), (7.5) and (7.6) hold. Assume also that either the condition* **(i)** *of Lemma 2.1 holds with $r_0 = r_2$, or the condition* **(ii)** *of Lemma 2.1 holds with $r_0 = 0$, or the condition* **(iii)** *of that lemma holds. Assume finally that*

$$u(r)f(r, u(r)) - 2F(r, u(r)) > 0 \quad \text{for } r \in (0, r_2). \tag{7.7}$$

Then any positive solution of (7.3) is non-degenerate, i.e. the corresponding linearized problem (7.4) admits only the trivial solution.

Proof. We use again the generalized Wronskians $\xi(r) = r^{n-1}(u'w - uw')$, and $\zeta(r) = r^n [u'w' + f(r, u)w] + (n-2)r^{n-1}u'w$. We have

$$\xi'(r) = [u f_u(r, u) - f(r, u)]\, w r^{n-1}, \tag{7.8}$$

$$\zeta'(r) = 2f(r, u) w r^{n-1} + r^n f_r(r, u) w. \tag{7.9}$$

As before, we introduce the function $O(r) = 2\gamma\xi(r) - \zeta(r)$, which in view of (7.8) and (7.9) satisfies

$$O'(r) = 2\left[u(r)f_u(r, u(r)) - f(r, u(r))\right] w(r) r^{n-1} \left[\gamma - \alpha(r)\right], \tag{7.10}$$

with $\alpha(r)$ as defined by (7.6).

We may assume that $w(0) > 0$. We claim that the function $w(r)$ cannot have any roots inside $(0, 1)$. Assuming otherwise, let τ_1 be the smallest root of $w(r)$, i.e. $w(r) > 0$ on $[0, \tau_1)$. Let $\tau_2 \in (\tau_1, 1]$ denote the second root of $w(r)$. We now fix $\gamma = \alpha(\tau_1)$. By monotonicity of $\alpha(r)$ the function $\gamma - \alpha(r)$ is non-positive on $[0, \tau_1)$ and non-negative on (τ_1, τ_2). Since $O(0) = 0$, we then conclude from (7.10) that

$$O(r) \le 0 \quad \text{for all } r \in [0, \tau_2]. \tag{7.11}$$

We consider two cases.

Case (i) $\tau_2 = 1$. From (7.11) we have $O(1) \le 0$. On the other hand, $O(1) = -\zeta(1) = -u'(1)w'(1) > 0$, a contradiction.

Case (ii) $\tau_2 < 1$. Observe that $f(\tau_2, u(\tau_2)) > 0$. Indeed, assuming otherwise, we conclude by Lemma 7.1 that $f(r, u(r)) \le 0$ on $(\tau_2, 1)$. But this contradicts Lemma 7.2. Applying Lemma 7.1 again, we conclude that $f(r, u(r)) > 0$ over $[0, \tau_2)$.

Since $\xi(\tau_1) > 0$, while $\xi(\tau_2) < 0$, we can find $t \in (\tau_1, \tau_2)$, such that $\xi(t) = 0$, i.e.

$$\frac{u(t)}{w(t)} = \frac{u'(t)}{w'(t)}. \tag{7.12}$$

By Lemma 7.2 $t \in (0, r_2)$, and then by Lemma 2.1, (2.4) and (7.7)

$$Q(t) > 0. \tag{7.13}$$

In view of (7.11)

$$\zeta(t) = -O(t) \ge 0.$$

On the other hand, using (7.12) and (7.13),

$$\zeta(t) = \left[t^n \left(u'w'\frac{u}{w} + f(t,u)u\right) + (n-2)t^{n-1}u'u\right]\frac{w}{u} = Q(t)\frac{w(t)}{u(t)} < 0,$$

giving us a contradiction.

It follows that $w(r)$ cannot have any roots, i.e. we may assume that $w(r) >$ on $[0, 1)$. But that is impossible, as can be seen by integrating (7.8) over $(0, 1)$. Hence $w \equiv 0$. ♢

We consider next the problem (here $r = |x|$)

$$\Delta u - a(r)u + b(r)u^p = 0, \quad r \in (0,1), \tag{7.14}$$
$$u = 0 \quad \text{for } r = 1.$$

We assume that $p < \frac{n+2}{n-2}$, and the functions $a(r), b(r) \in C^1[0, 1]$ satisfy

$$a(r) > 0, \;\; b(r) > 0, \;\; a'(r) > 0, \;\; b'(r) < 0 \quad \text{for } r \in (0,1). \tag{7.15}$$

These assumptions imply, in particular, that any positive solution of (7.14) is radially symmetric, in view of [8]. We define the functions

$$A(r) \equiv 2a(r) + ra'(r),$$

$$B(r) \equiv \left(\frac{n}{p+1} - \frac{n-2}{2}\right) b(r) + \frac{rb'(r)}{p+1}.$$

Observe that $\frac{n}{p+1} - \frac{n-2}{2} > 0$ for subcritical p, i.e. when $p < \frac{n+2}{n-2}$.

Theorem 7.2 *In addition to the conditions (7.15) assume that the function $A(r)$ is positive and increasing, while the function $B(r)$ is positive on $(0,1)$. Assume also that the function $\frac{rb'(r)}{b(r)}$ is decreasing on $(0,1)$. Then the problem (7.14) has a unique positive solution. Moreover, the Morse index of the solution is equal to one.*

Proof. We imbed the problem (7.14) into a two-parameter family of problems, with parameters $\theta \in [0,1]$ and $\lambda \in [0,1]$,

$$\Delta u - \lambda a(r)u + b^\theta(r)u^p = 0, \quad r \in (0,1), \tag{7.16}$$
$$u = 0 \text{ for } r = 1.$$

At $\lambda = 0$ and $\theta = 0$ the problem has unique solution of Morse index one. This is known for general domains, see e.g. K.C. Chang [3], or a simple proof for the case of a ball can be found in [10]. At $\lambda = 1$ and $\theta = 1$ we have the original problem (7.14).

We check that the Theorem 7.1 applies for all $\theta \in [0,1]$ and all $\lambda \in [0,1]$. Compute

$$(p-1)\alpha(r) = -\lambda \frac{A(r)}{b^\theta(r)u^{p-1}(r)} + 2 + \theta \frac{rb'(r)}{b(r)}.$$

In view of our assumption, $\alpha(r)$ is a decreasing function, verifying (7.6). Compute

$$uf(u) - 2F = \frac{p-1}{p+1} b^\theta(r)u^{p+1} > 0,$$

verifying (7.7).

Denote $B_\theta(r) = \left(\frac{n}{p+1} - \frac{n-2}{2}\right) b^\theta(r) + \theta \frac{rb^{\theta-1}(r)b'(r)}{p+1}$. Now, $B_\theta(r) > b^{\theta-1}(r)B(r) > 0$. Writing $B_\theta(r) = b^\theta(r)\left[\frac{n}{p+1} - \frac{n-2}{2} + \frac{\theta}{p+1}\frac{rb'(r)}{b(r)}\right]$, we see that the quantity in the square bracket is positive and decreasing on $(0,1)$, and hence $B_\theta(r)$ is positive and decreasing on $(0,1)$. Compute

$$I(r) = -\lambda A(r)u^2 + 2B_\theta(r)u^{p+1} = 2B_\theta(r)u^2\left[-\frac{\lambda A(r)}{2B_\theta(r)} + u^{p-1}\right].$$

The quantity in the square bracket is a decreasing function, which is negative near $r = 1$. Hence, either $I(r)$ is negative over $(0,1)$, or it changes sign exactly once, thus verifying the conditions of Lemma 2.1.

As we vary λ and θ (along any curve from $(0,0)$ to $(1,1)$), we can apply the implicit function theorem to continue the solution, since by Theorem 7.1 all solutions of (7.16) are non-degenerate. By the a priori estimates of B. Gidas and J. Spruck [9] the solutions stay bounded, and hence they can be continued for all λ and θ. At $\lambda = 1$ and $\theta = 1$ we conclude existence and uniqueness of positive solutions for our problem. (If there were more than one solution at $\lambda = 1$ and $\theta = 1$, we would have more than one solution at $\lambda = 0$ and $\theta = 0$, a contradiction.) Moreover, the Morse index of this solution is one, since eigenvalues of the linearized problem for (7.14) change continuously, and they cannot cross zero, since solutions are non-degenerate (and so the number of negative eigenvalues of the linearized problem at $\lambda = 1$ and $\theta = 1$, is the same as at $\lambda = 0$ and $\theta = 0$, i.e. one). ♢

Remark According to Lemma 7.1 the conditions (7.2) and (7.5) imply that $f(r, u(r))$ changes sign at most once on $[0, 1)$. When one adds the condition (7.6) (i.e. under the conditions of the Theorem 7.1), then one can typically expect that $f(r, u(r))$ changes sign exactly once, the way it happens for the model example $f(r, u) = -a(r)u + b(r)u^p$, considered above. We shall prove that for an important special case. Namely, assume that the conditions of the Theorem 7.1 hold, $f = f(u)$ and $f'(0) \neq 0$. We claim that $f(u)$ is negative for small u. Indeed, assuming that on the contrary $f(u) > 0$, we conclude from (7.5) that $K(u) \equiv \frac{uf'(u)}{f(u)} > 1$ for all $u > 0$. On the other hand, from (7.6) $K(u)$ is a non-increasing function, and by L'Hospital, $\lim_{u \to 0} K(u) = 1$, implying that $K(u) \leq 1$ for small u, a contradiction.

References

[1] Adimurthi and S. L. Yadava, *An elementary proof of the uniqueness of positive radial solutions of a quasilinear Dirichlet problem*, Arch. Rational Mech. Anal. **127**, no. 3, 219-229 (1994).

[2] A. Aftalion and F. Pacella, *Uniqueness and nondegeneracy for some nonlinear elliptic problems in a ball*, J. Differential Equations **195**, 389-397 (2003).

[3] K.C. Chang, *Infinite-dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications, **6** Birkhauser Boston, Inc., Boston, MA, (1993).

[4] M. Chaves and J. García-Azorero, *On uniqueness of positive solutions of semilinear elliptic equations with singular potential*, Adv. Nonlinear Stud. **3**, no. 2, 273-288 (2003).

[5] M.G. Crandall and P.H. Rabinowitz, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal. **52**, 161-180 (1973).

[6] L. Damascelli, M. Grossi and F. Pacella, *Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle*, Ann. Inst. H. Poincare Anal. Non Lineaire **16**, no. 5, 631-652 (1999).

[7] B. Franchi, E. Lanconelli and J. Serrin, *Existence and uniqueness of nonnegative solutions of quasilinear equations in* R^n, Adv. Math. **118**, no. 2, 177-243 (1996).

[8] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. **68**, 209-243 (1979).

[9] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. in PDE **6(8)**, 883-901 (1981).

[10] P. Korman, A global solution curve for a class of semilinear equations, Differential Equations and Computational Simulations III, J. Graef, R. Shivaji, B. Soni & J. Zhu, Eds, EJDE, Conf. 01 119-127 (1997).

[11] P. Korman, *Curves of sign-changing solutions for semilinear equations*, Nonlinear Anal. TMA **51**, no. 5, 801-820 (2002).

[12] P. Korman, Y. Li and T. Ouyang, *An exact multiplicity result for a class of semilinear equations*, Commun. PDE. **22** (3&4), 661-684 (1997).

[13] P. Korman and T. Ouyang, *Positivity for the linearized problem for semilinear equations*, To appear in Topological Methods in Nonlinear Analysis.

[14] M.K. Kwong and Y. Li, *Uniqueness of radial solutions of semilinear elliptic equations*, Trans. Amer. Math. Soc. **333(1)**, 339-363, (1992).

[15] C.S. Lin and W.-M. Ni, *A counterexample to the nodal domain conjecture and related semilinear equation*, Proc. Amer. Math. Soc. **102**, 271-277 (1988).

[16] W.-M. Ni and J. Serrin, *Nonexistence theorems for singular solutions of quasilinear partial differential equations*, Comm. Pure Appl. Math. **39**, no. 3, 379-399 (1986).

[17] T. Ouyang and J. Shi, *Exact multiplicity of positive solutions for a class of semilinear problems*, J. Differential Equations **146**, no. 1, 121-156 (1998).

[18] T. Ouyang and J. Shi, *Exact multiplicity of positive solutions for a class of semilinear problem*, II J. Differential Equations **158** no. 1, 94-151 (1999).

[19] P. H. Rabinowitz, *A note on a nonlinear eigenvalue problem for a class of differential equations*, J. Differential Equations **9**, 536-548 (1971).

[20] M. Tang, *Uniqueness of positive radial solutions for* $\Delta u - u + u^p = 0$ *on an annulus.* J. Differential Equations **189**, no. 1, 148-160 (2003).

[21] E. Yanagida, *Uniqueness of positive radial solutions of* $\Delta u + g(r)u + h(r)u^p = 0$ *in* R^n, Arch. Rational Mech. Anal. **115**, no. 3, 257-274 (1991).