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Study of the Singular Yamabe Problem in Some Bounded Domain of $\mathbb{R}^n$

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Abstract

In this paper, we extend the result of R. Mazzeo and F. Pacard in the following direction: Given Ω any bounded open regular subset of $\mathbb{R}^n$, $n \geq 3$ and given x_1, $x_2 \in \Omega$, we give a sufficient condition on x_1, x_2, for the problem

$$\Delta u + u^{\frac{n+2}{n-2}} = 0$$

to have a positive weak solution in Ω with 0 boundary data, which is singular at each x_i.

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1 Introduction

Recently R. Mazzeo and F. Pacard have developed a method to build positive solutions of

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{x_1, \cdots, x_k\} \tag{1.1}$$

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which are singular at every $x_i \in \mathbb{R}^n$ provided $n \geq 3$ and $k \geq 3$.

Our aim is to extend this method to study the following problem:
Given any Ω, a regular bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and given $x_1, \cdots x_k \in \Omega$, does there exist a positive weak solution of

$$\begin{cases} \Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

which is singular at each x_i?

In the particular case where $\Omega = \mathbb{R}^n$, some decay assumption has to be imposed at ∞, namely

$$u(x) = O(| x |^{2-n})$$

for $| x |$ large. This case has been studied by many authors. First of all, it is known since the result of L. A. Caffarelli, B. Gidas and J. Spruck in [1], that, when $k = 1$, the problem has no solution. Moreover, all solutions corresponding to $k = 2$ are known and constitute a $2(n+1)$ dimensional manifold. For at least 3 singularities, existence results of solutions have been given first by R. Schoen [6] and later on by R. Mazzeo and F. Pacard [2], the two proofs giving rise to two qualitatively different types of solutions. Solutions with even numbers of singularities have been found by R. Mazzeo, D. Pollack and K. Uhlenbeck [4]. Moreover, the same authors have proved in [3] some results about the structure of the set of all solutions of (1.2) itself when $\Omega = \mathbb{R}^n$. In particular, they have proved that the dimension of this set of solutions is equal to $k(n+1)$ near any "non-degenerate solution". Finally let us mention the result of D. Pollack which concerns the compactness properties of the space of all solutions which are singular at finitely many points [5].
To our knowledge, very few results are known when Ω is a bounded regular open subset of $\mathbb{R}^n$. Our first result is an easy nonexistence result :

Theorem 1.1 *If $\Omega \subset \mathbb{R}^n$ is a (bounded) regular domain which is starshaped with respect to the point x_0 then there are no solution of (1.2) with a non removable singularity at x_0.*

As a corollary, we get the nonexistence of solutions with one isolated singularity and 0 boundary data in any convex domain.
Before stating our result concerning the existence of singular solutions, we denote by $G(x, x')$, the Green's function in Ω and $H(x, x')$ the regular part of $G(x, x')$, namely

$$G(x, x') = | x - x' |^{2-n} + H(x, x'). \tag{1.3}$$

Our main result reads :

Theorem 1.2 *Given Ω a regular open subset of $\mathbb{R}^n$ with $n \geq 3$ and given $\Sigma = \{x_1, x_2\} \subset \Omega$. There exists a solution to the problem*

$$\begin{cases} \Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \quad in \quad \Omega \setminus \Sigma \\ u = 0 \quad on \quad \partial\Omega \\ u > 0 \end{cases} \tag{1.4}$$

singular at every point of Σ *provided*

$$G(x_1, x_2)^2 - H(x_1, x_1)H(x_2, x_2) > 0. \tag{1.5}$$

It is easy to see that the last condition is always full-filed provided the two points x_1 and x_2 are close enough.

The proof of this result relies on the proof of [2].

Remark 1.1 We don't know if there exists a solution with just one isolated singularity when the domain is not starshaped.

2 The building blocks : Delaunay type solutions

The Delaunay type solutions are defined to be singular radial solutions of

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0. \tag{2.1}$$

It is classical to look for such a solution in the form

$$u(x) = \mid x \mid^{\frac{2-n}{2}} v(-\log \mid x \mid). \tag{2.2}$$

Setting $t = -\log \mid x \mid$, we obtain the following autonomous ODE

$$\frac{d^2 v}{dt^2} - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0. \tag{2.3}$$

The trajectories of this ODE are easy to study since they are related to the level sets of the following Hamiltonian

$$\mathcal{H}(v, w) = w^2 - \frac{(n-2)^2}{4} v^2 + \frac{(n-2)^2}{4} v^{\frac{2n}{n-2}}. \tag{2.4}$$

Using this, it is proved in [2]:

Proposition 2.1 [2] *For any* $H_0 \in (-\frac{n-2}{2}(\frac{n-2}{n})^{\frac{n}{2}}, 0)$*, there exists a unique bounded solution of (2.3) satisfying* $\mathcal{H}(v, \dot{v}) = H_0$*,* $\dot{v}(0) = 0$ *and* $\ddot{v}(0) > 0$*. This solution is periodic and for all* $t \in \mathbb{R}$ *we have* $v(t) \in (0, 1)$*. This solution can be indexed by the parameter* $\varepsilon = v(0) \in (0, (\frac{n-2}{n})^{\frac{n-2}{4}})$*. When* $H_0 = -\frac{n-2}{2}(\frac{n-2}{n})^{\frac{n}{2}}$*, there is a unique bounded solution of (2.3), given by*

$$v(t) = (\frac{n-2}{n})^{\frac{n-2}{4}}.$$

Finally, if v *is a solution with* $H_0 = 0$*, then either* $v(t) \equiv 0$ *or* $v(t) = (\cosh(t - t_0))^{\frac{2-n}{2}}$ *for some* $t_0 \in \mathbb{R}$*.*

In this proposition, $\cdot$ denotes the derivative with respect to t.

These solutions are called Delaunay type solutions. Their behavior as ε tends to zero is explained in the following two propositions :

Proposition 2.2 [2] *Fix* $\varepsilon \in (0, (\frac{n-2}{n})^{\frac{n-2}{4}})$ *and let* v_ε *be the corresponding Delaunay type solution. Then the period* T_ε *of* v_ε *tends to infinity monotonically as* ε *tends to* 0 *and satisfies*

$$T_\varepsilon = -\left(\frac{4}{n-2} + o(1)\right)\log(\varepsilon).$$

In addition, for all $t \in \mathbb{R}$

$$\varepsilon \leq v_\varepsilon(t) \leq \varepsilon \cosh(\frac{n-2}{2}t).$$

And, more important for us, we also have :

Proposition 2.3 [2] *For any* $\varepsilon \in (0, (\frac{n-2}{n})^{\frac{n-2}{4}})$ *and for any* $t \in \mathbb{R}$*, the Delaunay type solution* v_ε *satisfies the estimates*

$$| \, v_\varepsilon(t) - \varepsilon \cosh(\frac{n-2}{2}t) \, | \leq c_n \varepsilon^{\frac{n+2}{n-2}} e^{\frac{n+2}{2}|t|},$$

$$| \, \dot{v}_\varepsilon(t) - \frac{n-2}{2}\varepsilon \sinh(\frac{n-2}{2}t) \, | \leq c_n \varepsilon^{\frac{n+2}{n-2}} e^{\frac{n+2}{2}|t|},$$

for some constant $c_n > 0$ *which depends only on* n.

For all $\varepsilon \in (0, (\frac{n-2}{n})^{\frac{n-2}{4}})$, for all $R > 0$ and for all $a \in \mathbb{R}^n$ we define

$$u_{\varepsilon,R,a}(x) = | \, x - a \, | \, x \, |^2 |^{\frac{2-n}{2}} \; v_\varepsilon(-2\log | \, x \, | + \log | \, x - a \, | \, x \, |^2 | + \log R). \qquad (2.5)$$

One easily checks that this is a solution of (2.1) in $\mathbb{R}^n \setminus \{0, \frac{a}{|a|^2}\}$. A simple proof of this fact is available in [2].

3 Construction of the approximate solutions

We define a family of approximate solutions to our problem using the Delaunay type solutions defined in the previous section. The approximate solutions are obtained by gluing together Delaunay type solutions centered at each x_i with a harmonic function. Let

$$\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in (\mathbb{R}_+^*)^2,$$

be a set of Delaunay parameters,

$$\bar{R} = (R_1, R_2) \in (\mathbb{R}_+^*)^2,$$

parameters corresponding to translations of Delaunay type solutions

$$\bar{\rho} = (\rho_1, \rho_2) \equiv (\varepsilon_1^{\frac{4}{n^2-4}}, \varepsilon_2^{\frac{4}{n^2-4}}) \in (\mathbb{R}_+^*)^2 \qquad (3.1)$$

and

$$\bar{a} = (a_1, a_2)$$

a set of vectors in $\mathbb{R}^n$.

We assume that the parameters ρ_i are sufficiently small so that the balls $B(x_i, \rho_i)$ do not intersect. We denote

$$\Omega_{\bar{\rho}} = \Omega \setminus \cup_{i=1}^{2} B(x_i, \rho_i). \tag{3.2}$$

We finally fix a pair of positive numbers q_1 and q_2 and assume that

$$\varepsilon_i = \varepsilon q_i \quad i = 1, 2, \tag{3.3}$$

or simply $\bar{\varepsilon} = \varepsilon \bar{q}$ for some $\varepsilon > 0$. In this case, for the sake of convenience we will denote

$$\rho = \varepsilon^{\frac{4}{n^2-4}}.$$

As we have already said, $G(x, x')$ denotes the Green's function in Ω and $H(x, x')$ the regular part of $G(x, x')$, namely

$$G(x, x') = \mid x - x' \mid^{2-n} + H(x, x'). \tag{3.4}$$

We have the following proposition:

Proposition 3.1 *Suppose that $\bar{\varepsilon} = \varepsilon(q_1, q_2)$ with $\varepsilon > 0$ small. Suppose that the parameters $R_i > 0$ and a_i satisfy the relations*

$$q_i R_i^{\frac{2-n}{2}} = q_i R_i^{\frac{n-2}{2}} H(x_i, x_i) + q_j R_j^{\frac{n-2}{2}} G(x_i, x_j) \tag{3.5}$$

and

$$q_i R_i^{\frac{2-n}{2}} (n-2) a_i = q_i R_i^{\frac{n-2}{2}} \nabla_x H(x_i, x_i) + q_j R_j^{\frac{n-2}{2}} \nabla_x G(x_i, x_j). \tag{3.6}$$

If $\bar{w}_{\bar{\varepsilon}, \bar{R}}(x)$ is defined on $\Omega_{\bar{\rho}}$ by

$$\bar{w}_{\bar{\varepsilon}, \bar{R}}(x) = \frac{\varepsilon_1}{2} R_1^{\frac{n-2}{2}} G(x, x_1) + \frac{\varepsilon_2}{2} R_2^{\frac{n-2}{2}} G(x, x_2). \tag{3.7}$$

Then, for $i = 1, 2$, we have the estimates

$$u_{\varepsilon_i, R_i, a_i}(x - x_i) - \bar{w}_{\bar{\varepsilon}, \bar{R}}(x) = O(\varepsilon \rho^2)$$

and

$$\nabla (u_{\varepsilon_i, R_i, a_i}(x - x_i) - \bar{w}_{\bar{\varepsilon}, \bar{R}}(x)) = O(\varepsilon \rho)$$

in $B(x_i, 2\rho_i) \setminus B(x_i, \rho_i)$.

Proof. In order to prove the result, it is sufficient to expand both $u_{\varepsilon_i, R_i, a_i}(x - x_i)$ and $\bar{w}_{\bar{\varepsilon}, \bar{R}}(x)$ in $B(x_i, 2\rho_i) \setminus B(x_i, \rho_i)$. First, we consider $u_{\varepsilon_i, R_i, a_i}(x - x_i)$, using Proposition 2.3 and the definition of $u_{\varepsilon, R, a}$, we get, if ρ is small enough

$$\begin{aligned} u_{\varepsilon_i, R_i, a_i}(x - x_i) &= \frac{\varepsilon_i}{2} \left(R_i^{\frac{n-2}{2}} \mid x - x_i \mid^{2-n} + R_i^{\frac{2-n}{2}} (1 + (n-2)(a_i \cdot (x - x_i))) \right) \\ &\quad + O(\varepsilon \mid x - x_i \mid^2) + O(\varepsilon^{\frac{n+2}{n-2}} \mid x - x_i \mid^{-n}). \end{aligned}$$

We now give the expansion of $\bar{w}_{\bar{\varepsilon},\bar{R}}(x)$ in the same set

$$\begin{aligned}\bar{w}_{\bar{\varepsilon},\bar{R}}(x) &= \frac{\varepsilon_i}{2} R_i^{\frac{n-2}{2}} \left[\mid x - x_i \mid^{2-n} + H(x_i, x_i) + \nabla_x H(x_i, x_i)(x - x_i) \right] \\ &\quad + \frac{\varepsilon_j}{2} R_j^{\frac{n-2}{2}} \left[G(x_i, x_j) + \nabla_x G(x_i, x_j)(x - x_i) \right] + O(\varepsilon \mid x - x_i \mid^2)\end{aligned}$$

and the result follows from the values given at the different parameters.

Let us now show that it is possible to find $\bar{q}$ and $\bar{R}$ satisfying (3.5). In fact, we have to be able to solve the following system

$$\left\{ \begin{array}{l} q_1 (R_1^{\frac{2-n}{2}} - R_1^{\frac{n-2}{2}} H(x_1, x_1)) = q_2 R_2^{\frac{n-2}{2}} G(x_1, x_2) \\ q_2 (R_2^{\frac{2-n}{2}} - R_2^{\frac{n-2}{2}} H(x_2, x_2)) = q_1 R_1^{\frac{n-2}{2}} G(x_2, x_1). \end{array} \right. \tag{3.8}$$

The problem reduces to find q_1, q_2, R_1 and R_2 all positive, which are solutions of (3.8). First observe that $H(x_1, x_1) < 0$, $H(x_2, x_2) < 0$ and $G(x_1, x_2) > 0$. Therefore, if the above system has a nontrivial solution it has a solution (q_1, q_2) whose components are all > 0. Thus, it is enough to find R_1, $R_2 > 0$ such that

$$(R_1^{\frac{2-n}{2}} - R_1^{\frac{n-2}{2}} H(x_1, x_1))(R_2^{\frac{2-n}{2}} - R_2^{\frac{n-2}{2}} H(x_2, x_2)) = R_1^{\frac{n-2}{2}} R_2^{\frac{n-2}{2}} G(x_1, x_2)^2.$$

But this is equivalent to say that we can find a solution X_1, $X_2 > 0$ to

$$(1 - X_1 H(x_1, x_1))(1 - X_2 H(x_2, x_2)) = X_1 X_2 G(x_1, x_2)^2.$$

This is always possible provided

$$G(x_1, x_2)^2 - H(x_1, x_1) H(x_2, x_2) > 0.$$

Let χ_i be a smooth radial function defined as follows

$$\chi_i(x) = \left\{ \begin{array}{lll} 1 & \text{if} & \mid x \mid \leq \rho_i \\ 0 & \text{if} & \mid x \mid \geq 2\rho_i \end{array} \right. \tag{3.9}$$

and

$$\left\{ \begin{array}{lll} \mid \partial_r \chi_i(x) \mid & \leq & c\rho_i^{-1} \\ \mid \partial_r^2 \chi_i(x) \mid & \leq & c\rho_i^{-2}. \end{array} \right.$$

we define the approximate solution as follows (regardless of whether the relations (1.5), (3.5) and (3.6) are satisfied.)

$$\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x) = \sum_{i=1}^{2} \chi_i(x - x_i) u_{\varepsilon_i, R_i, a_i}(x - x_i) + \bar{w}_{\bar{\varepsilon},\bar{R}}(x) \left(1 - \sum_{i=1}^{2} \chi_i(x - x_i) \right). \tag{3.10}$$

4 Definition of the Jacobi fields

The nonlinear operator we deal with is defined by

$$\mathcal{N}(u) \equiv \Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}.$$

The linearization of the operator $\mathcal{N}$ at a fixed Delaunay type solution $u_{\varepsilon,R,a}$ is defined by

$$L_{\varepsilon,R,a} v = \Delta v + \frac{n(n+2)}{4} u_{\varepsilon,R,a}^{\frac{4}{n-2}} v. \tag{4.1}$$

Varying the parameters in any one of the families of Delaunay type solutions leads to solutions of $L_{\varepsilon,R,a}\psi = 0$. Solutions of this homogeneous problem are called Jacobi fields. We will denote by λ_j the eigenvalues of the Laplacian on S^{n-1} and $\phi_j(\theta)$ the corresponding eigenfunctions.

For the time being, we set $R = 1$ and $a = 0$. To $\lambda_0 = 0$ correspond the following Jacobi fields

$$\begin{aligned}
\psi_\varepsilon^{0,-}(x) &= \frac{\partial u_\varepsilon}{\partial \varepsilon}(x) = \mid x \mid^{\frac{2-n}{2}} \frac{\partial v_\varepsilon}{\partial \varepsilon}(-\log \mid x \mid) \equiv \mid x \mid^{\frac{2-n}{2}} \Phi_\varepsilon^{0,-}(-\log \mid x \mid), \\
\psi_\varepsilon^{0,+}(x) &= - \mid x \mid \frac{\partial u_\varepsilon}{\partial r}(x) - \frac{n-2}{2} u_\varepsilon(x) = \mid x \mid^{\frac{2-n}{2}} \frac{\partial v_\varepsilon}{\partial t}(-\log \mid x \mid) \\
&\equiv \mid x \mid^{\frac{2-n}{2}} \Phi_\varepsilon^{0,+}(-\log \mid x \mid).
\end{aligned}$$

The Jacobi fields corresponding to the next set of eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_n = n-1$ are given by

$$\begin{aligned}
\psi_\varepsilon^{j,+}(x) &= \mid x \mid^{\frac{4-n}{2}} \left(\tfrac{n-2}{2} v_\varepsilon(-\log \mid x \mid) - \frac{\partial v_\varepsilon}{\partial t}(-\log \mid x \mid) \right) \phi_j(\theta) \\
&\equiv \mid x \mid^{\frac{2-n}{2}} \Phi_\varepsilon^{1,+}(-\log \mid x \mid) \phi_j(\theta) \\
\psi_\varepsilon^{j,-}(x) &= \frac{\partial u_\varepsilon}{\partial x_j}(x) \\
&= \mid x \mid^{-\frac{n}{2}} \left(\tfrac{2-n}{2} v_\varepsilon(-\log \mid x \mid) - \frac{\partial v_\varepsilon}{\partial t}(-\log \mid x \mid) \right) \phi_j(\theta) \\
&\equiv \mid x \mid^{\frac{2-n}{2}} \Phi_\varepsilon^{1,-}(-\log \mid x \mid) \phi_j(\theta).
\end{aligned}$$

We also have (cf. [2])

$$\begin{aligned}
\Phi_\varepsilon^{j,+}(t) &= e^{-t}\left(\tfrac{n-2}{2} v_\varepsilon(t) - \tfrac{\partial v_\varepsilon}{\partial t}(t)\right) \\
&= \varepsilon^{\frac{n-2}{2}} e^{-\frac{n}{2}t} + O(\varepsilon^{\frac{n+2}{n-2}} e^{\frac{n}{2}t})
\end{aligned} \tag{4.2}$$

and differentiating the equality $v_\varepsilon(t + T_\varepsilon) = v_\varepsilon(t)$ with respect to ε, we have

$$\Phi_\varepsilon^{0,-}(t + T_\varepsilon) + \frac{dT_\varepsilon}{d\varepsilon} \Phi_\varepsilon^{0,+}(t + T_\varepsilon) = \Phi_\varepsilon^{0,-}(t).$$

This last equation shows that $\Phi_\varepsilon^{0,-}$ is a linearly growing function of t.

Later, we shall use the Jacobi fields corresponding to differentiating the family $u_{\varepsilon,R,a}(x)$ and evaluating not necessarily at $R=1$, $a=0$. We shall denote these Jacobi fields by

$$\psi^{j,\pm}_{\varepsilon,R,a}(x) \equiv \mid x \mid^{\frac{2-n}{2}} \Phi^{j,\pm}_{\varepsilon,R,a}(-\log \mid x \mid), \quad j=0,\cdots,n.$$

5 Function spaces

We will use weighted Hölder spaces defind as follows.

5.1 Function spaces on the ball $B(0,R)$

For each $k \in \mathbb{N}$, $0<\alpha<1$ and $\sigma \in \mathbb{R}^+$, we set

$$\begin{aligned} \mid u \mid_{k,\alpha,[\sigma,2\sigma]} &= \sup_{|x|\in[\sigma,2\sigma]} \left(\sum_{j=0}^{k} \sigma^j \mid \nabla^j u(x) \mid \right) \\ &\quad + \sigma^{k+\alpha} \sup_{|x|,|y|\in[\sigma,2\sigma],x\neq y} \left(\frac{\mid \nabla^k u(x) - \nabla^k u(y) \mid}{\mid x-y \mid^\alpha} \right). \end{aligned} \tag{5.1.1}$$

Then, for any $\mu \in \mathbb{R}$, we define the weighted Hölder space on $B(0,R)\setminus\{0\}$ as follows

$$\mathcal{C}^{k,\alpha}_\mu(B(0,R)\setminus\{0\} = \left\{ u \in \mathcal{C}^{k,\alpha}_{loc}(B(0,R)\setminus\{0\}) \quad | \quad \| u \|_{k,\alpha,\mu} < \infty \right\}, \tag{5.1.2}$$

where

$$\| u \|_{k,\alpha,\mu} = \sup_{\sigma \le \frac{R}{2}} \sigma^{-\mu} \mid u \mid_{k,\alpha,[\sigma,2\sigma]} . \tag{5.1.3}$$

Notice that, the radial function r^μ is in $\mathcal{C}^{k,\alpha}_\mu(B(0,R)\setminus\{0\})$ for any k, α and μ.

5.2 Function spaces on $\Omega\setminus\Sigma$

We define analogous function spaces on $\Omega\setminus\Sigma$. For any $\nu \in \mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_\nu(\Omega\setminus\Sigma)$ as follows

$$\mathcal{C}^{k,\alpha}_\nu(\Omega\setminus\Sigma) = \left\{ u \in \mathcal{C}^{k,\alpha}(\Omega\setminus\Sigma) \mid \| u \|_{k,\alpha,\nu} < \infty \right\}, \tag{5.2.1}$$

where the norm is given by

$$\| u \|_{k,\alpha,\nu} = \| u \|_{k,\alpha,\nu,B(x_1,1)\setminus\{x_1\}} + \| u \|_{k,\alpha,\Omega\setminus\cup_i B(x_i,1)} + \| u \|_{k,\alpha,\nu,B(x_2,1)\setminus\{x_2\}} \tag{5.2.2}$$

and $\| f \|_{k,\alpha,\Omega\setminus\cup_i B(x_i,1)}$ denotes the usual Hölder norm of the function over the set $\Omega\setminus\cup_i B(x_i,1)$.

6 The linearized operator on the unit ball

We study the Dirichlet problem in the unit ball $B(0,1)$ for the linearization about one of the radial Delaunay solutions $u_{\varepsilon,R,0} \equiv u_{\varepsilon,R}$. The reason why we restrict our attention to the linearized operator around the radial Delaunay type solution is because it can be studied using separation of variables. Set

$$L_{\varepsilon,R,0} = L_{\varepsilon,R} \quad \text{and} \quad L_{\varepsilon,1,0} = L_{\varepsilon}.$$

We want to study the problem

$$\begin{cases} L_{\varepsilon,R} w &= f \quad \text{in} \quad B(0,1) \setminus \{0\} \\ w &= 0 \quad \text{on} \quad \partial B(0,1). \end{cases} \tag{6.1}$$

We will study the properties of $L_{\varepsilon,R}$ on the weighted Hölder spaces defined in the previous section.

We know that the Jacobi fields satisfy

$$\psi^{0,+}_{\varepsilon,R,a} \in \mathcal{C}^{2,\alpha}_{\mu} \quad \text{and} \quad \psi^{0,-}_{\varepsilon,R,a} \in \mathcal{C}^{2,\alpha}_{\mu} \quad \text{for all} \quad \mu \leq \frac{2-n}{2} \tag{6.2}$$

and for $j = 1, \cdots, n$,

$$\psi^{j,+}_{\varepsilon,R,a} \in \mathcal{C}^{2,\alpha}_{\mu} \quad \text{for all} \quad \mu \leq \frac{4-n}{2} \quad \text{and} \quad \psi^{j,-}_{\varepsilon,R,a} \in \mathcal{C}^{2,\alpha}_{\mu} \quad \text{for all} \quad \mu \leq -\frac{n}{2}. \tag{6.3}$$

We introduce $\{\mu_j^{\pm}(\varepsilon)\}_j$ a discrete set of reals determined by the ordinary differential equation induced by $L_{\varepsilon,R}$ on the j^{th} eigenspace of the Laplacian on S^{n-1}. These values play the same role as the indicial roots of a regular singular problem. They are defined as the rates of growth or decay of solutions of $L_{\varepsilon,R} w = 0$ and are usually impossible to compute. We notice that $\mu_0^{\pm}(\varepsilon)$ and $\mu_j^{\pm}(\varepsilon)$ for $j = 1, \cdots, n$ do not depend on ε, precisely we have

$$\mu_0^{\pm} = \frac{2-n}{2} \quad \text{and} \quad \mu_j^{+} = \frac{4-n}{2}, \mu_j^{-} = \frac{-n}{2}, \quad \text{for} \quad j = 1, \cdots, n.$$

We can prove as what is done in [4], that the operator

$$L_{\varepsilon,R} : \mathcal{C}^{2,\alpha}_{\mu,\mathcal{D}}(B(0,1) \setminus \{0\}) \longrightarrow \mathcal{C}^{0,\alpha}_{\mu-2}(B(0,1) \setminus \{0\})$$

is Fredholm for all $\mu \in \mathbb{R}$, $\mu \notin \{\mu_j^{\pm}(\varepsilon)\}$.

We have the following proposition

Proposition 6.1 *Fix $R \in \mathbb{R}_+^*$ and $\mu \in (\frac{4-n}{2}, 2)$. Let us assume that $\mathcal{A}$ and $\mathcal{B}$ are reals satisfying*

$$\mathcal{A} \cosh(\frac{n-2}{2} \log R) + \frac{n-2}{2} \mathcal{B} \sinh(\frac{n-2}{2} \log R) = 1 \tag{6.4}$$

and define the deficiency space

$$W \equiv Span\{\mathcal{A}\psi^{0,-}_{\varepsilon,R} + \frac{\mathcal{B}}{\varepsilon}\psi^{0,+}_{\varepsilon,R},\ \psi^{j,+}_{\varepsilon,R} \quad with \quad j = 1, \cdots, n\}. \tag{6.5}$$

Then, there exists an $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$, *the operator*

$$L_{\varepsilon,R} : [\mathcal{C}^{2,\alpha}_{\mu}(B(0,1) \setminus \{0\}) \oplus W]_{\mathcal{D}} \longrightarrow \mathcal{C}^{0,\alpha}_{\mu-2}(B(0,1) \setminus \{0\}) \tag{6.6}$$

is an isomorphism. The inverse map will be denoted $L^{-1}_{\varepsilon,R}$. *In particular, for* $f \in \mathcal{C}^{0,\alpha}_{\mu-2}(B(0,1) \setminus \{0\})$ *there exists a unique solution* w *of (6.1) which has a decomposition*

$$w(x) = G_{\varepsilon,R}(f)(x) + K^0_{\varepsilon,R}(f)\left(\mathcal{A}\psi^{0,-}_{\varepsilon}(\frac{x}{R}) + \frac{\mathcal{B}}{\varepsilon}\psi^{0,+}_{\varepsilon}(\frac{x}{R})\right) + \frac{1}{\varepsilon}\sum_{j=1}^{n} K^j_{\varepsilon,R}(f)\psi^{j,+}_{\varepsilon}(\frac{x}{R}).$$

The operator

$$G_{\varepsilon,R} : \mathcal{C}^{0,\alpha}_{\mu-2}(B(0,1) \setminus \{0\}) \longrightarrow \mathcal{C}^{0,\alpha}_{\mu}(B(0,1) \setminus \{0\})$$

and the functionnals $K^j_{\varepsilon,R}$ *for* $j = 0, 1, \cdots, n$ *defined from* $\mathcal{C}^{0,\alpha}_{\mu-2}(B(0,1) \setminus \{0\})$ *onto* $\mathbb{R}$ *satisfy the following properties:*

- $G_{\varepsilon,R}$ *is defined for all* $\mu \in (\frac{4-n}{2}, 2)$ *and is bounded independently of* ε *when* $1 < \mu < 2$.
- *The restriction of* $G_{\varepsilon,R}$ *to the space of functions* $h(r,\theta)$ *with eigencomponents* $h_j(r)$ *vanishing for* $j = 0, \cdots, n$, *is defined and bounded independently of* ε *when* $-n < \mu < 2$.
- *The functional* $K^0_{\varepsilon,R}$ *is defined for all* $\mu > \frac{2-n}{2}$ *and is bounded independently of* ε *when* $0 < \mu$.
- *The functionals* $K^j_{\varepsilon,R}$, $j = 1, \cdots, n$, *are defined for all* $\mu > \frac{4-n}{2}$ *and are bounded independently of* ε *when* $1 < \mu$.

Proof. The proof will be decomposed into three steps. The solution is constructed by separation of variables, restricting the problem to each eigenspace of the Laplacian on the sphere. Replacing $f(x)$ by $R^{-2}f(\frac{x}{R})$ and $w(x)$ by $w(\frac{x}{R})$, we remark that (6.1) is equivalent to

$$\left\{\begin{array}{rcll} L_\varepsilon w & = & f & \text{in} \quad B(0,\frac{1}{R}) \setminus \{0\} \\ w & = & 0 & \text{on} \quad \partial B(0,\frac{1}{R}). \end{array}\right. \tag{6.7}$$

Setting $t = -\log \mid x \mid$, this rescaling has the effect of replacing $v_{\varepsilon,R}(t) = v_\varepsilon(t + \log R)$ by $v_\varepsilon(t)$.

Replacing, in (6.7), w and f by their eigenfunction decompositions

$$w(x) = \mid x \mid^{-\frac{n-2}{2}} \sum_{j=0}^{+\infty} w_j(-\log \mid x \mid)\phi_j(\theta)$$

and

$$f(x) = \mid x \mid^{-\frac{n+2}{2}} \sum_{j=0}^{+\infty} f_j(-\log \mid x \mid)\phi_j(\theta),$$

we get the following series of ODE satisfied by w_j

$$\begin{cases} \mathbb{L}_{\varepsilon,j} w_j(t) &= \dfrac{d^2 w_j}{dt^2} - (\dfrac{n-2}{2})^2 w_j - \lambda_j w_j + \dfrac{n(n+2)}{4} v_\varepsilon^{\frac{4}{n-2}} w_j = f_j \\ w_j(\log R) &= 0, \end{cases}$$

where, for all $j \geq 0$, $\mathbb{L}_{\varepsilon,j}$ is the restriction to the eigenspace $Span\{\phi_j\}$ of $\Delta_{S^{n-1}}$ of the operator

$$\mathbb{L}_\varepsilon w(t,\theta) = \frac{d^2 w}{dt^2} - (\frac{n-2}{2})^2 w + \Delta_{S^{n-1}} w + \frac{n(n+2)}{4} v_\varepsilon^{\frac{4}{n-2}} w.$$

We assume, after multiplying by a suitable factor, that $\| f \|_{0,\alpha,\mu-2} = 1$. We will study the operators $\mathbb{L}_{\varepsilon,j}$ for $j > n$, then for $j = 0$ and finally for $j = 1, \cdots, n$.

1^{st} **step :** $j > n$. We assume that $-n < \mu < 2$ and $j > n$. We denote by $\bar{f}$ the projection of f onto $\oplus_{j>n} Span\{\phi_j\}$. Thus the problem we study is the following

$$\begin{cases} \mathbb{L}_\varepsilon \bar{w}(t,\theta) &= \bar{f}(t,\theta) \quad \text{in} \quad (\log R, +\infty) \times S^{n-1} \\ \bar{w}(\log R, \theta) &= 0. \end{cases} \tag{6.8}$$

It has a unique solution $\bar{w}$ which is uniformly bounded by $ce^{-\delta t}$ for $t \geq 0$ where $\delta = \frac{n-2}{2} + \mu$. We refer to [2] for the details.

2^{nd} **step :** $j = 0$. We assume that $\mu > 0$. We want to solve

$$\begin{cases} \mathbb{L}_{\varepsilon,0} w_0(t) &= f_0 \quad \text{in} \quad (\log R, +\infty) \\ w_0(\log R) &= 0. \end{cases} \tag{6.9}$$

We normalize f_0 so that $\| f_0 \|_{0,\alpha,\mu-2} = 1$ and choose an extension $\tilde{f}_0$ of f_0 to $\mathbb{R}$ satisfying $\| \tilde{f}_0 \|_{0,\alpha,\mu-2} \leq 2$. For each $T > \log R$, we denote by w_T the unique solution of

$$\mathbb{L}_{\varepsilon,0} w_T = \tilde{f}_0, \; w_T(T) = 0, \; \dot{w}_T(T) = 0.$$

We have the following result proved in [2].

Lemma 6.1 [2] *For $\mu > 0$, let $\delta = \frac{n-2}{2} + \mu$. Then for all $R > 0$, there are constants $\varepsilon_0 > 0$ and $C > 0$ independent of T, such that for all $\varepsilon \in (0, \varepsilon_0]$ we have*

$$\sup_{t \in [\log R, T]} e^{\delta t} \mid \bar{w}_T(t) \mid \leq C.$$

Letting T tend to infinity, we get a unique solution $\bar{w}$ to (6.9) satisfying

$$\sup_{t\in[\log R,+\infty]} e^{\delta t} \mid \bar{w}(t) \mid\leq C.$$

We must add an additional term to correct the boundary data. We define the solution w_0 by

$$w_0(t) = \bar{w}(t) - c[\mathcal{A}\Phi_\varepsilon^{0,-}(t) + \mathcal{B}\Phi_\varepsilon^{0,+}(t)],$$

where the constant $c \in \mathbb{R}$ is chosen so that $w_0(\log R) = 0$. Thanks to the result of Proposition 2.3, it is easy to see that

$$\lim_{\varepsilon\to 0}(\mathcal{A}\Phi_\varepsilon^{0,-}(\log R)+\mathcal{B}\Phi_\varepsilon^{0,+}(\log R)) = \mathcal{A}\cosh(\frac{n-2}{2}\log R)+\frac{n-2}{2}\mathcal{B}\sinh(\frac{n-2}{2}\log R)$$

which by assumption is equal to 1. Therefore, the choice of c is always possible provided ε is chosen small enough. Moreover the constant c is bounded by a constant.

Knowing that the functions $\Phi_\varepsilon^{0,\pm}$ are solutions to $\mathbb{L}_{\varepsilon,0}w = 0$, we conclude that w_0 satisfies the equation $\mathbb{L}_{\varepsilon,0}w_0(t) = f_0$ as well as the correct boundary conditions.

3^{rd} **step :** $1 \leq j \leq n$. We suppose that $\mu > 1$. We want to solve the problem

$$\begin{cases} \mathbb{L}_{\varepsilon,j}w_j(t) & = & f_j \quad \text{in} \quad (\log R, +\infty) \\ w_j(\log R) & = & 0. \end{cases} \tag{6.10}$$

We define an extension $\tilde{f}_j$ of f_j to all of $\mathbb{R}$ and find the unique solution $\bar{w}_T$ of $\mathbb{L}_{\varepsilon,j}\bar{w}_T = \tilde{f}_j$, $\bar{w}_T(T) = \dot{\bar{w}}_T(T) = 0$. We have the uniform bound of $e^{\delta t} \mid \bar{w}_T \mid$ on $(-\infty, T]$. Letting T tend to $+\infty$, we get a suitably bounded solution $\bar{w}$ to (6.10). To correct the boundary data, we define the solution w_j as follows

$$w_j(t) = \bar{w}(t) - \bar{w}(\log R)(\Phi_\varepsilon^{j,+}(\log R))^{-1}\Phi_\varepsilon^{j,+}(t).$$

Still using Proposition 2.3, we see that, for $j = 1, \cdots, n$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Phi_\varepsilon^{j,+}(\log R) = \frac{n-2}{2}R^{\frac{n}{2}}$$

and the result follows in this case too.

7 A model for the linearization about $\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x)$

We study here the linearization of $\mathcal{N}$ about the approximate solution $\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x)$.

Using the dilation invariance of our problem, we may always assume that for $i = 1, 2$, $B(x_i, 2)$ are included in Ω and moreover that $B(x_1, 2) \cap B(x_2, 2) = \emptyset$.

Let $\tilde{\chi}$ de a cutoff function defined by

$$\begin{cases} \tilde{\chi} \equiv 1 \quad \text{in} \quad B(0,1) \\ Supp(\tilde{\chi}) \subset B(0,2) \\ 0 \leq \tilde{\chi} \leq 1 \quad \text{in} \quad B(0,2) \setminus B(0,1). \end{cases}$$

We first study the simpler operator

$$\mathcal{L}_{\bar{\varepsilon},\bar{R}} w = \Delta w + \frac{n(n+2)}{4} \sum_{i=1}^{2} \tilde{\chi}(x - x_i) u_{\varepsilon_i, R_i}^{\frac{4}{n-2}}(x - x_i) w, \tag{7.1}$$

where the parameters a_i have all been set to zero and then will treat the true linearization as a perturbation of $\mathcal{L}_{\bar{\varepsilon},\bar{R}}$. The operator $\mathcal{L}_{\bar{\varepsilon},\bar{R}}$ is simple to study thanks to the fact that the term of order zero is radial in each $B(x_i, 1)$. We will construct an inverse of $\mathcal{L}_{\bar{\varepsilon},\bar{R}}$ resolving the equation

$$\mathcal{L}_{\bar{\varepsilon},\bar{R}} w = f \quad \text{in} \quad \Omega \setminus \Sigma. \tag{7.2}$$

To this end, we will first solve the homogeneous Dirichlet problem outside the balls $B(x_i, 1)$, then solve the homogeneous Dirichlet problem in each of the balls and finally show that the sum of these solutions can be modified using the Dirichlet to Neumann maps on the boundary of these balls. We have the following result

Proposition 7.1 *Let $\bar{\varepsilon} = \varepsilon \bar{q}$ be a set of Delaunay parameters. Suppose that $\bar{R} = (R_1, R_2)$ satisfies the relation (3.5). Suppose also that $1 < \nu < 2$. In addition assume that, for $i = 1, 2$, we have chosen $\mathcal{A}_i$ and $\mathcal{B}_i$ such that*

$$\mathcal{A}_i \cosh(\frac{n-2}{2} \log R_i) + \frac{n-2}{2} \mathcal{B}_i \sinh(\frac{n-2}{2} \log R_i) = 1 \tag{7.3}$$

and also such that

$$\begin{aligned} &\left(2 + (1 + H(x_1, x_1))\left(\mathcal{A}_1 \sinh(\tfrac{n-2}{2} \log R_1) + \tfrac{n-2}{2} \mathcal{B}_1 \cosh(\tfrac{n-2}{2} \log R_1)\right)\right) \\ &\times \left(2 + (1 + H(x_2, x_2))\left(\mathcal{A}_2 \sinh(\tfrac{n-2}{2} \log R_2) + \tfrac{n-2}{2} \mathcal{B}_2 \cosh(\tfrac{n-2}{2} \log R_2)\right)\right) \\ &\neq \left(\mathcal{A}_1 \sinh(\tfrac{n-2}{2} \log R_1) + \tfrac{n-2}{2} \mathcal{B}_1 \cosh(\tfrac{n-2}{2} \log R_1)\right) \\ &\times \left(\mathcal{A}_2 \sinh(\tfrac{n-2}{2} \log R_2) + \tfrac{n-2}{2} \mathcal{B}_2 \cosh(\tfrac{n-2}{2} \log R_2)\right) G(x_1, x_2)^2. \end{aligned} \tag{7.4}$$

We define the deficiency space

$$\mathcal{W}_0 \equiv Span \left\{ \tilde{\chi}(\cdot - x_i) \left(\mathcal{A}_i \psi_{\varepsilon_i}^{0,-}(\frac{\cdot - x_i}{R_i}) + \frac{\mathcal{B}_i}{\varepsilon_i} \psi_{\varepsilon_i}^{0,+}(\frac{\cdot - x_i}{R_i}) \right), \cdots, \right.$$

$$\left. \tilde{\chi}(\cdot - x_i) \psi_{\varepsilon_i}^{j,+}(\frac{\cdot - x_i}{R_i}), \quad \textit{with} \quad j = 1, \cdots, n \quad \textit{and} \quad i = 1, 2 \right\}.$$

Then for ε sufficiently small, the operator

$$\mathcal{L}_{\bar{\varepsilon},\bar{R}} : \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma) \oplus \mathcal{W}_0 \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma) \tag{7.5}$$

is an isomorphism. In particular, for each $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma)$ there exists a unique solution of (7.2) which has a decomposition

$$\omega(x) = \mathcal{G}_{\bar{\varepsilon},\bar{R}}(f)(x) + \sum_{i=1}^{2} \tilde{\chi}(x - x_i) \left(\mathcal{K}^0_{\bar{\varepsilon},\bar{R},i}(f) \left[\mathcal{A}_i \psi^{0,-}_{\varepsilon_i}(\frac{x - x_i}{R_i}) + \frac{\mathcal{B}_i}{\varepsilon_i} \psi^{0,+}_{\varepsilon_i}(\frac{x - x_i}{R_i}) \right] \right.$$
$$\left. + \sum_{j=1}^{n} \mathcal{K}^j_{\bar{\varepsilon},\bar{R},i}(f) \frac{1}{\varepsilon_i} \psi^{j,+}_{\varepsilon_i}(\frac{x - x_i}{R_i}) \right).$$

The operator

$$\mathcal{G}_{\bar{\varepsilon},\bar{R}} : \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma) \longrightarrow \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma)$$

and functionals

$$\mathcal{K}^j_{\bar{\varepsilon},\bar{R},i} : \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma) \longrightarrow \mathbb{R}, \quad \textit{for} \quad j = 0, \cdots, n \quad \textit{and} \quad i = 1, 2$$

are bounded independently of ε.

Remark 7.1 The condition (7.4) is not restrictive.

Proof. The proof will be done in five steps.

1^{st} step: The exterior problem

We have the following proposition similar to a result in [2]:

Proposition 7.2 *Let $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma)$. Then for $\varepsilon_0 > 0$ sufficiently small, there exists a unique solution $\omega \in \mathcal{C}^{2,\alpha}_{\nu}(\tilde{\Omega})$ to*

$$\left\{ \begin{array}{lcll} \mathcal{L}_{\bar{\varepsilon},\bar{R}} \omega & = & \bar{f} & \textit{in} \quad \tilde{\Omega} \\ \omega & = & 0 & \textit{on} \quad \partial\tilde{\Omega} \end{array} \right. \tag{7.6}$$

where $\tilde{\Omega} = \Omega \setminus \cup_{i=1}^{2} B(x_i, 1)$ and $\bar{f}$ is the restriction of f to $\tilde{\Omega}$. This solution satisfies the estimate

$$\| \omega \|_{2,\alpha,\nu} \leq c \, \| \bar{f} \|_{0,\alpha,\nu-2},$$

with some constant $c > 0$ independent of $\varepsilon_i \in (0, \varepsilon_0]$. The norms here are taken in $\tilde{\Omega}$.

The proof of this result is straightforward once we have noticed that, in $\tilde{\Omega}$, the potential in $\mathcal{L}_{\bar{\varepsilon},\bar{R}}$ tends to 0 like a constant times ε.

2^{nd} step: The exterior Dirichlet to Neumann map

Let $\Psi(\theta) = \{\psi_1(\theta), \psi_2(\theta)\} \in \oplus_{i=1}^{2} \mathcal{C}^{2,\alpha}(\partial B(x_i, 1))$ be a set of boundary values. Thanks to the previous proposition, we know that there exists a unique solution $\omega \in \mathcal{C}^{2,\alpha}_\nu(\tilde\Omega)$ of the homogeneous problem

$$\begin{cases} \mathcal{L}_{\bar\varepsilon,\bar R}\omega = 0 & \text{in} \quad \tilde\Omega \\ \omega = \psi_i & \text{on} \quad \partial B(x_i,1), \quad \text{for} \quad i=1,2 \\ \omega = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{7.7}$$

which we denote by ω_Ψ. We define $S_{\bar\varepsilon,\bar R}$ by

$$S_{\bar\varepsilon,\bar R}(\Psi) = (\partial_{r_1}\omega_{\Psi_{|r_1=1}}, \partial_{r_2}\omega_{\Psi_{|r_2=1}}) \in \oplus_{i=1}^2 \mathcal{C}^{1,\alpha}(\partial B(x_i,1)),\ r_i = \mid x - x_i \mid .$$

This is the Dirichlet to Neumann map for the operator $\mathcal{L}_{\bar\varepsilon,\bar R}$ on $\tilde\Omega$ (with 0 boundary conditions on $\partial\Omega$). We have the following result

Lemma 7.1 *The norms of the mappings*

$$\Psi \in \oplus_{i=1}^2 \mathcal{C}^{2,\alpha}(\partial B(x_i,1)) \longrightarrow \omega_\Psi \in \mathcal{C}^{2,\alpha}_\nu(\tilde\Omega)$$

and

$$S_{\bar\varepsilon,\bar R} : \oplus_{i=1}^2 \mathcal{C}^{2,\alpha}(\partial B(x_i,1)) \longrightarrow \oplus_{i=1}^2 \mathcal{C}^{1,\alpha}(\partial B(x_i,1))$$

are bounded independently of $\bar\varepsilon$ provided each $\varepsilon_i < \varepsilon_0$ and ε_0 is sufficiently small. Furthermore, if all ε_i tend to zero, $S_{\bar\varepsilon,\bar R}$ converges to a limiting operator S_0 which is the Dirichlet to Neumann map for the Laplacian on $\tilde\Omega$ (with 0 boundary conditions on $\partial\Omega$).

The proof of the Lemma also follows from the fact that, in $\tilde\Omega$, the potential in $\mathcal{L}_{\bar\varepsilon,\bar R}$ tends to 0 like a constant times ε.

3^{rd} step: The interior Dirichlet to Neumann map
Given $\Psi(\theta) = \{\psi_1(\theta), \psi_2(\theta)\} \in \oplus_{i=1}^2 \mathcal{C}^{2,\alpha}(\partial B(x_i,1))$, a set of boundary values, we know from Proposition 6.1 that there exists a unique solution to the homogeneous problem

$$\begin{cases} \mathcal{L}_{\bar\varepsilon,\bar R}\omega = 0 & \text{in} \quad B(x_i,1) \\ \omega = \psi_i & \text{on} \quad \partial B(x_i,1), \end{cases} \tag{7.8}$$

such that

$$\omega = \omega_{\psi_i} \in \mathcal{C}^{2,\alpha}_\nu(B(x_i,1)\setminus\{x_i\}) \oplus Span\left\{\mathcal{A}_i\psi^{0,-}_{\varepsilon_i}(\frac{\cdot - x_i}{R_i}) + \frac{\mathcal{B}_i}{\varepsilon_i}\psi^{0,+}_{\varepsilon_i}(\frac{\cdot - x_i}{R_i})\right\}$$
$$\oplus_{j=1}^n Span\left\{\frac{1}{\varepsilon_i}\psi^{j,+}_{\varepsilon_i}(\frac{\cdot - x_i}{R_i})\right\}.$$

This defines in $\oplus_{i=1}^2 \mathcal{C}^{2,\alpha}(\partial B(x_i,1))$ a continuous

$$\mathcal{T}_{\bar\varepsilon,\bar R}(\Psi) = (\mathcal{T}^1_{\bar\varepsilon,\bar R}(\psi_1), \mathcal{T}^2_{\bar\varepsilon,\bar R}(\psi_2)) = (\partial_{r_1}\omega_{\psi_{1|r_1=1}}, \partial_{r_2}\omega_{\psi_{2|r_2=1}}) \in \oplus_{i=1}^2 \mathcal{C}^{1,\alpha}(\partial B(x_i,1)),$$

where $r_i = \mid x - x_i \mid$. We have the following lemma

Lemma 7.2 *The norms of the mappings*

$$\psi_i \in \mathcal{C}^{2,\alpha}(\partial B(x_i,1)) \longrightarrow \omega_{\psi_i} \in \mathcal{C}^{2,\alpha}_\nu(B(x_i,1)\setminus\{x_i\})$$

$$\oplus Span\left\{\mathcal{A}_i\psi^{0,-}_{\varepsilon_i}(\frac{\cdot - x_i}{R_i}) + \frac{\mathcal{B}_i}{\varepsilon_i}\psi^{0,+}_{\varepsilon_i}(\frac{\cdot - x_i}{R_i})\right\} \oplus_{j=1}^n Span\left\{\frac{1}{\varepsilon_i}\psi^{j,+}_{\varepsilon_i}(\frac{\cdot - x_i}{R_i})\right\}.$$

and

$$\mathcal{T}_{\bar\varepsilon,\bar R} : \oplus_{i=1}^2\mathcal{C}^{2,\alpha}(\partial B(x_i,1)) \longrightarrow \oplus_{i=1}^2\mathcal{C}^{1,\alpha}(\partial B(x_i,1))$$

are bounded independently of ε_i provided all ε_i are sufficiently small. The mappings $\mathcal{T}_{\bar\varepsilon,\bar R}$ converge as $\bar\varepsilon$ tends to zero to a limit operator $\mathcal{T}_0 \equiv (\mathcal{T}_0^1, \mathcal{T}_0^2)$ where $\mathcal{T}_0^i$ is defined as follows

$$\mathcal{T}_0^i(\phi_j) = \left(\frac{2-n}{2} + (\frac{(n-2)^2}{4} + \lambda_j)^{\frac12}\right)\phi_j \quad \textit{for all} \quad j \geq 1$$

$$\mathcal{T}_0^i(\phi_0) = \frac{(n-2)}{2}\left(\mathcal{A}_i \sinh(\frac{n-2}{2}\log R_i) + \frac{n-2}{2}\mathcal{B}_i\cosh(\frac{n-2}{2}\log R_i)\right)\phi_0,$$

Proof. The proof is almost similar to the one done in [2]. The only difference being on the limit of $\mathcal{T}^i_{\bar\varepsilon,\bar R}(\phi_0)$. Therefore, we refer to [2] for the proof of the relation giving $\mathcal{T}_0^i(\phi_j)$ for $j \geq 1$. When $j = 0$ and ψ_i is the eigenfunction ϕ_0, an explicit solution of (7.8) is given by

$$c_{\varepsilon_i}\left[\mathcal{A}_i\psi^{0,-}_{\varepsilon_i}(\frac{|\, x - x_i \,|}{R_i}) + \frac{\mathcal{B}_i}{\varepsilon_i}\psi^{0,+}_{\varepsilon_i}(\frac{|\, x - x_i\,|}{R_i})\right]\phi_0(\theta),$$

where the constant c_{ε_i} tends to 1 as ε_i tends to 0. Therefore, we have

$$\mathcal{T}_0^i(\phi_0) = \lim_{\varepsilon_i\to 0}\left(\mathcal{A}_i\dot\Phi^{0,-}_{\varepsilon_i}(\log R_i) + \mathcal{B}_i\dot\Phi^{0,+}_{\varepsilon_i}(\log R_i)\right)\phi_0,$$

where $\dot\Phi$ is the derivative of Φ with respect to t. We recall the following expansions

$$\begin{aligned}\Phi^{0,-}_\varepsilon(t) &= \cosh(\tfrac{n-2}{2}t) + O(\varepsilon^{\frac{4}{n-2}}e^{-\frac{n+2}{2}|t|})\\ \Phi^{0,+}_\varepsilon(t) &= \tfrac{n-2}{2}\varepsilon\sinh(\tfrac{n-2}{2}t) + O(\varepsilon^{\frac{n+2}{n-2}}e^{n|t|}).\end{aligned}$$

Using these expansions, we get

$$\mathcal{T}_0^i(\phi_0) = \frac{(n-2)}{2}\left(\mathcal{A}_i \sinh(\frac{n-2}{2}\log R_i) + \frac{n-2}{2}\mathcal{B}_i\cosh(\frac{n-2}{2}\log R_i)\right)\phi_0,$$

which ends the proof of the lemma.

4^{th} step: Invertibility of the difference of Dirichlet to Neumann maps

In order to glue the interior and exterior solutions together, we must add correction terms to these solutions. They are solutions of the homogeneous problem chosen such that the Cauchy data from inside and outside match up. For that, we must prove that the difference $S_{\bar\varepsilon,\bar R} - \mathcal{T}_{\bar\varepsilon,\bar R}$ is invertible when all ε_i are sufficiently small. This is the aim of the following proposition :

Proposition 7.3 *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon_i < \varepsilon_0$,*

$$S_{\bar{\varepsilon},\bar{R}} - \mathcal{T}_{\bar{\varepsilon},\bar{R}} : \oplus_{i=1}^{2} \mathcal{C}^{2,\alpha}(\partial B(x_i, 1)) \longrightarrow \oplus_{i=1}^{2} \mathcal{C}^{1,\alpha}(\partial B(x_i, 1))$$

is invertible. The norm of its inverse is bounded by some constant $C > 0$ independent of ε_i for all $i = 1, 2$.

Proof. It is proved in [2] that, it is sufficient to show that the limit operator $S_0 - \mathcal{T}_0$ is injective. To do so, we argue by contradiction assuming that $S_0 - \mathcal{T}_0$ is not injective. Then there exists a function $\Psi_0 \in \oplus_{i=1}^{2} \mathcal{C}^{\infty}(\partial B(x_i, 1))$ satisfying $(S_0 - \mathcal{T}_0)(\Psi_0) = 0$. We extend the Dirichlet data Ψ_0 to a harmonic function, w', on $\Omega \setminus \cup_{i=1}^{2} B(x_i, 1)$, and to another harmonic function, w'', in the interior of the balls $B(x_i, 1) \setminus \{x_i\}$. On $\partial B(x_i, 1)$, these functions have the same Dirichlet data equal to $\Psi_{0,i}$ and by assumption, all eigencomponents with index $j \geq 1$ of their Neumann data agree since S_0 is the Dirichlet to Neumann map for the Laplacian on the exterior region and $\mathcal{T}_0$ is the Dirichlet to Neumann map for the Laplacian on the union of the balls $B(x_i, 1)$ only for the spherical eigencomponents with index $j \geq 1$.

We call ω the function defined on $\Omega \setminus \Sigma$. Given the definition of $\mathcal{T}_0^i(\phi_j)$ for $j \geq 1$, it is easy to see that there exist two reals p_1 and p_2 such that

$$\omega(x) = \sum_{i=1}^{2} p_i G(x, x_i),$$

or

$$\omega(x) = \sum_{i=1}^{2} p_i (| \, x - x_i \, |^{2-n} + H(x, x_i)).$$

Since ω is harmonic, it is easy to see that the radial part of ω in $B(x_1, 1)$ is given by

$$\omega_0^1(x) = p_1 \left(| \, x - x_1 \, |^{2-n} + H(x_1, x_1) \right) + p_2 G(x_1, x_2).$$

The normal derivative of this function at $\partial B(x_1, 1)$ is just $(2 - n)p_1$, it must agree with $\mathcal{T}_0^1(\omega_0^1)$ which gives the relation

$$(2-n)p_1 = \frac{(n-2)}{2} \left(\mathcal{A}_1 \sinh(\frac{n-2}{2} \log R_1) + \frac{n-2}{2} \mathcal{B}_1 \cosh(\frac{n-2}{2} \log R_1) \right) \times \left(\quad p_1(1 + H(x_1, x_1)) + p_2 G(x_1, x_2) \quad \right).$$

Similarly

$$(2-n)p_2 = \frac{(n-2)}{2} \left(\mathcal{A}_2 \sinh(\frac{n-2}{2} \log R_2) + \frac{n-2}{2} \mathcal{B}_2 \cosh(\frac{n-2}{2} \log R_2) \right) \times \left(\quad p_2(1 + H(x_2, x_2)) + p_1 G(x_2, x_1) \quad \right).$$

This system in p_1 and p_2 only has nontrivial solutions if and only if its determinant is 0. But it is always possible to choose $\mathcal{A}_i$ and $\mathcal{B}_i$ satisfying (7.3) in such a way that the determinant is nonzero. Namely

$$\left(2+(1+H(x_1,x_1))\left(\mathcal{A}_1\sinh(\frac{n-2}{2}\log R_1)+\frac{n-2}{2}\mathcal{B}_1\cosh(\frac{n-2}{2}\log R_1)\right)\right)$$
$$\times\left(2+(1+H(x_2,x_2))\left(\mathcal{A}_2\sinh(\frac{n-2}{2}\log R_2)+\frac{n-2}{2}\mathcal{B}_2\cosh(\frac{n-2}{2}\log R_2)\right)\right)$$
$$\neq\left(\mathcal{A}_1\sinh(\frac{n-2}{2}\log R_1)+\frac{n-2}{2}\mathcal{B}_1\cosh(\frac{n-2}{2}\log R_1)\right)$$
$$\times\left(\mathcal{A}_2\sinh(\frac{n-2}{2}\log R_2)+\frac{n-2}{2}\mathcal{B}_2\cosh(\frac{n-2}{2}\log R_2)\right)G(x_1,x_2)^2.$$

5^{th} **step: The correction term**

We consider $f\in\mathcal{C}^{0,\alpha}_{\nu-2}(\Omega\setminus\Sigma)$. Thanks to the previous propositions, there exist functions ω_{ext} and $\omega_{int,i}$ for $i=1,2$, such that

$$\begin{cases}\mathcal{L}_{\bar\varepsilon,\bar R}\omega_{ext} & = & f & \text{in} & \tilde\Omega\\ \omega_{ext} & = & 0 & \text{on} & \partial\tilde\Omega\end{cases}$$

and

$$\begin{cases}\mathcal{L}_{\bar\varepsilon,\bar R}\omega_{int,i} & = & f & \text{in} & B(x_i,1)\\ \omega_{int,i} & = & 0 & \text{on} & \partial B(x_i,1).\end{cases}$$

We also know that $\omega_{int,i}$ has the following decomposition in $B(x_i,1)$

$$\omega_{int,i}(x)=G_{\varepsilon_i,R_i}(f_i)(x)+K^0_{\varepsilon_i,R_i}(f_i)\left(\mathcal{A}_i\psi^{0,-}_{\varepsilon_i}+\frac{\mathcal{B}_i}{\varepsilon_i}\psi^{0,+}_{\varepsilon_i}\right)(\frac{x-x_i}{R_i})$$
$$+\sum_{j=1}^{n}K^j_{\varepsilon_i,R_i}(f_i)\frac{1}{\varepsilon_i}\psi^{j,+}_{\varepsilon_i}(\frac{x-x_i}{R_i}),$$

where f_i is the restriction of f to the ball $B(x_i,1)$. Moreover

$$\| G_{\varepsilon_i,R_i}(f_i)\|_{2,\alpha,\nu}+\sum_{j=0}^{n}| K^j_{\varepsilon_i,R_i}(f_i)|\le c\| f\|_{0,\alpha,\nu-2}$$

and

$$\|\omega_{ext}\|_{2,\alpha,\nu}\le c\| f\|_{0,\alpha,\nu-2}\ .$$

Let us find a function $\omega_{ker}\in\Omega\setminus\{\cup_{i=1}^{2}\partial B(x_i,1)\cup\Sigma\}$, continuous across $\partial B(x_i,1)$, such that the function ω defined by

$$\omega(x)=\begin{cases}\omega_{ext}(x)+\omega_{ker}(x) & \text{in}\quad\tilde\Omega\\ \omega_{int,i}(x)+\omega_{ker}(x) & \text{in}\quad B(x_i,1)\end{cases}$$

satisfies $\mathcal{L}_{\bar{\varepsilon},\bar{R}}(\omega) = f$ in $\Omega \setminus \Sigma$. In order for this to be true, the function ω_{ker} must satisfy $\mathcal{L}_{\bar{\varepsilon},\bar{R}}(\omega_{ker}) = 0$ in $\Omega \setminus \{\cup_{i=1}^{2} \partial B(x_i, 1) \cup \Sigma\}$ and the jump of $\partial_{r_i}\omega_{ker}$ across $\partial B(x_i, 1)$ should be equal to $\partial_{r_i}\omega_{int,i} - \partial_{r_i}\omega_{ext}$. This is equivalent to finding Ψ solution of

$$(S_{\bar{\varepsilon},\bar{R}} - \mathcal{T}_{\bar{\varepsilon},\bar{R}})(\Psi) = (\partial_{r_1}\omega_{int,1} - \partial_{r_1}\omega_{ext}, \partial_{r_2}\omega_{int,2} - \partial_{r_2}\omega_{ext}) \in \oplus_{i=1}^{2} \mathcal{C}^{1,\alpha}(\partial B(x_i, 1)).$$

This equation has a solution thanks to Proposition 7.3. Using the estimates of Lemma 7.1 and 7.2, we get the estimates of the Proposition which ends the proof.

8 The true linearization

We analyze in this section the linearization of $\mathcal{N}$ about $\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x)$, namely

$$\Lambda_{\bar{\varepsilon},\bar{R},\bar{a}} = \Delta + \frac{n(n+2)}{4} (\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}})^{\frac{4}{n-2}}. \tag{8.1}$$

Notice that

$$\bar{\psi}^{0,-}_{\bar{\varepsilon},\bar{R},\bar{a},i}(x) = \partial_{\varepsilon_i} \bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x) \quad \text{for} \quad i = 1, 2,$$

$$\bar{\psi}^{0,+}_{\bar{\varepsilon},\bar{R},\bar{a},i}(x) = \partial_{R_i} \bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x) \quad \text{for} \quad i = 1, 2,$$

and

$$\bar{\psi}^{j,+}_{\bar{\varepsilon},\bar{R},\bar{a},i}(x) = \partial_{a_i^j} \bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x) \quad \text{for} \quad j = 1, \cdots, n \quad \text{and} \quad i = 1, 2,$$

which are defined by differentiating $\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x)$ with respect to ε_i, R_i and a_i^j respectively, satisfy

$$\Lambda_{\bar{\varepsilon},\bar{R},\bar{a}} \bar{\psi}^{j,+}_{\bar{\varepsilon},\bar{R},\bar{a},i}(x) = 0 \quad \text{near each} \quad x_i.$$

In the remaining, we will denote

$$\begin{aligned} \varepsilon_i \mathcal{A}_i \bar{\psi}^{0,-}_{\bar{\varepsilon},\bar{R},\bar{a},i}(x) + \mathcal{B}_i \bar{\psi}^{0,+}_{\bar{\varepsilon},\bar{R},\bar{a},i}(x) &= \mu_i(\bar{\varepsilon}, \bar{R}, \bar{a}, x) \\ \bar{\psi}^{j,+}_{\bar{\varepsilon},\bar{R},\bar{a},i}(x) &= \gamma_i^j(\bar{\varepsilon}, \bar{R}, \bar{a}, x) \quad \text{for} \quad j = 1, \cdots, n \end{aligned} \tag{8.2}$$

We assume that $\mathcal{A}_i$ and $\mathcal{B}_i$ satisfy the conditions of Proposition 7.1.

We define the space

$$\tilde{\mathcal{M}} = \left\{ (v, \bar{S}, \bar{\beta}) \in \mathcal{C}^{2,\alpha}_{\nu,\mathcal{D}}(\Omega \setminus \Sigma) \oplus \mathbb{R}^2 \oplus (\mathbb{R}^n)^2 \right\},$$

with norm

$$\| (v, \bar{S}, \bar{\beta}) \|_{\tilde{\mathcal{M}}} = \rho^{\nu} \| v \|_{2,\alpha,\nu} + \varepsilon \mid \bar{S} \mid + \rho\varepsilon \mid \bar{\beta} \mid .$$

Moreover, we define the linear mapping

$$\iota : (v, \bar{S}, \bar{\beta}) \longmapsto v(x) + \sum_{i=1}^{2} \left(S_i \mu_i(\bar{\varepsilon}, \bar{R}, \bar{a}, x) + \sum_{j=1}^{n} \beta_i^j \gamma_i^j(\bar{\varepsilon}, \bar{R}, \bar{a}, x) \right).$$

Finally, we define

$$\tilde{\Lambda}_{\bar{\varepsilon},\bar{R},\bar{a}} = \Lambda_{\bar{\varepsilon},\bar{R},\bar{a}} \circ \iota : \tilde{\mathcal{M}} \longrightarrow \mathcal{C}^{0;\alpha}_{\nu-2}(\Omega \setminus \Sigma).$$

We consider the problem

$$\tilde{\Lambda}_{\bar{\varepsilon},\bar{R},\bar{a}}\omega = f \quad \text{for} \quad f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma).$$

We will prove that we can find a solution with optimal bounds in the space $\tilde{\mathcal{M}}$.

As in [2], we have the following proposition:

Proposition 8.1 *Let $\bar{\varepsilon} = \varepsilon\bar{q}$ be a set of Delaunay parameters and suppose that $R_i > 0$ and $a_i \in \mathbb{R}^n$ are a collection of numbers and vectors satisfying (3.5) and (3.6). Then*

$$\tilde{\Lambda}_{\bar{\varepsilon},\bar{R},\bar{a}} : \tilde{\mathcal{M}} \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Sigma)$$

is an isomorphism, provided all ε_i are less than a sufficiently small numbers ε_0. Furthermore, for $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega\setminus\Sigma)$, the solution $w = (v, \bar{S}, \bar{\beta})$ of the equation $\tilde{\Lambda}_{\bar{\varepsilon},\bar{R},\bar{a}}w = f$ satisfies

$$\rho^{\nu} \parallel v \parallel_{2,\alpha,\nu} \leq c\left(\rho^{\nu} \parallel f \parallel_{0,\alpha,\nu-2} + \sup_i \mid \mathcal{K}^0_{\bar{\varepsilon},\bar{R},i}(f) \mid + \rho \sup_{i,j} \mid \mathcal{K}^j_{\bar{\varepsilon},\bar{R},i}(f) \mid\right),$$

$$\varepsilon \mid \bar{S} \mid \leq c\left(\rho^{\nu+1} \parallel f \parallel_{0,\alpha,\nu-2,-2} + \sup_i \mid \mathcal{K}^0_{\bar{\varepsilon},\bar{R},i}(f) \mid + \rho^2 \sup_{i,j} \mid \mathcal{K}^j_{\bar{\varepsilon},\bar{R},i}(f) \mid\right)$$

and

$$\varepsilon\rho \mid \bar{\beta} \mid \leq c\left(\rho^{\nu+1} \parallel f \parallel_{0,\alpha,\nu-2,-2} + \rho \sup_i \mid \mathcal{K}^0_{\bar{\varepsilon},\bar{R},i}(f) \mid + \rho \sup_{i,j} \mid \mathcal{K}^j_{\bar{\varepsilon},\bar{R},i}(f) \mid\right).$$

Proof. The proof being completely identical to the one in [2], we will omit it.

9 Estimates on the error term in the approximate solution

Thanks to our construction of the approximate solution, we have a result similar to the one proved in [2]. Its proof being readily the same, we will refer to [2] for it.
In the following, we recall the result

Lemma 9.1 [2] *Suppose that the parameters $R_i > 0$ and $a_i \in \mathbb{R}^n$ satisfy the relations (3.5) and (3.6) of Proposition 3.1. Define $\zeta \equiv \Delta\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}} + \frac{n(n-2)}{4}\bar{u}^{\frac{n+2}{n-2}}_{\bar{\varepsilon},\bar{R},\bar{a}}$ and set $w = \tilde{\Lambda}^{-1}_{\bar{\varepsilon},\bar{R},\bar{a}}\zeta$. Then, for $\nu \in (1,2)$ and for some constant C independent of ε,*

$$\parallel w \parallel_{\tilde{\mathcal{M}}} \leq C_0\varepsilon\rho^2.$$

10 The nonlinear fixed point argument

In this section, we will find a solution of the nonlinear problem. We consider an approximate solution as constructed in (3.10) in section 3, $\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x)$ associated to the data $\bar{q}$, $\bar{R}$ and $\bar{a}$ satisfying (3.5) and (3.6) of the Proposition 3.1. The perturbation of $\bar{u}_{\bar{\varepsilon},\bar{R},\bar{a}}(x)$ to an exact solution will involve not only a decaying term in the space $\mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Sigma)$, but also a slight adjustment of the parameters $\bar{\varepsilon}$, $\bar{R}$ and $\bar{a}$.
We recall that an element $w \in \tilde{\mathcal{M}}$ has components $(v, \bar{S}, \bar{\beta})$. Given $\mathcal{A}_i$ and $\mathcal{B}_i$ satisfying (7.3), we denote

$$\bar{\varepsilon}(1 + \bar{\mathcal{A}} \times \bar{S}) \equiv (\varepsilon_1(1 + \mathcal{A}_1 S_1), \varepsilon_2(1 + \mathcal{A}_2 S_2)).$$

Similarly, we will denote

$$\bar{R} + \bar{\mathcal{B}} \times \bar{S} \equiv (R_1 + \mathcal{B}_1, R_2 + \mathcal{B}_2 S_2).$$

We want to solve the equation $\tilde{\mathcal{N}}(w) = 0$ where

$$\begin{aligned} \tilde{\mathcal{N}}(w) \quad \equiv \quad & \Delta(\bar{u}_{\bar{\varepsilon}(1+\bar{\mathcal{A}}\times\bar{S}),\bar{R}+\bar{\mathcal{B}}\times\bar{S},\bar{a}+\bar{\beta}} + v) \\ & + \tfrac{n(n-2)}{4}(\bar{u}_{\bar{\varepsilon}(1+\bar{\mathcal{A}}\times\bar{S}),\bar{R}+\bar{\mathcal{B}}\times\bar{S},\bar{a}+\bar{\beta}} + v)^{\frac{n+2}{n-2}}. \end{aligned} \tag{10.1}$$

We see that $\tilde{\Lambda}_{\bar{\varepsilon},\bar{R},\bar{a}}$ is just the linearization of $\tilde{\mathcal{N}}$ at $w = 0$. Using Taylor expansion, (10.1) becomes

$$\tilde{\mathcal{N}}(w) = \tilde{\mathcal{N}}(0) + \tilde{\Lambda}_{\bar{\varepsilon},\bar{R},\bar{a}} w + \int_0^1 (D\tilde{\mathcal{N}}_{|(tv,\bar{S},\bar{\beta})} - D\tilde{\mathcal{N}}_{|(0,\bar{S},\bar{\beta})})(v, 0, 0)dt$$

$$+(D\tilde{\mathcal{N}}_{|(0,\bar{S},\bar{\beta})} - D\tilde{\mathcal{N}}_{|(0,0,0)})(v, 0, 0) + \int_0^1 (D\tilde{\mathcal{N}}_{|(0,t\bar{S},t\bar{\beta})} - D\tilde{\mathcal{N}}_{|(0,0,0)})(0, \bar{S}, \bar{\beta})dt.$$

Then, the equation $\tilde{\mathcal{N}}(w) = 0$ becomes

$$w = \mathcal{P}(w),$$

where

$$\mathcal{P}(w) = -\tilde{\Lambda}^{-1}_{\bar{\varepsilon},\bar{R},\bar{a}} \tilde{\mathcal{N}}(0) - \tilde{\Lambda}^{-1}_{\bar{\varepsilon},\bar{R},\bar{a}} \left(\int_0^1 (D\tilde{\mathcal{N}}_{|(tv,\bar{S},\bar{\beta})} - D\tilde{\mathcal{N}}_{|(0,\bar{S},\bar{\beta})})(v, 0, 0)dt \right.$$

$$\left. +(D\tilde{\mathcal{N}}_{|(0,\bar{S},\bar{\beta})} - D\tilde{\mathcal{N}}_{|(0,0,0)})(v, 0, 0) + \int_0^1 (D\tilde{\mathcal{N}}_{|(0,t\bar{S},t\bar{\beta})} - D\tilde{\mathcal{N}}_{|(0,0,0)})(0, \bar{S}, \bar{\beta})dt \right).$$

To prove the existence of a solution of $\tilde{\mathcal{N}}(w) = 0$ in the space $\tilde{\mathcal{M}}$, we prove as in [2] that for ε small enough, $\mathcal{P}$ is a contraction in a small ball in $\tilde{\mathcal{M}}$ of radius $C_1 \varepsilon \rho^2$, for a constant C_1 equal to twice the constant C_0 appearing in Lemma 9.1.

More precisely, if w_1, w_2 are two elements in this ball, we have, as in [2],

$$\| \mathcal{P}(w_2) - \mathcal{P}(w_1) \|_{\tilde{\mathcal{M}}} \leq c\rho^{2-\nu} \| w_2 - w_1 \|_{\tilde{\mathcal{M}}} .$$

Taking ε sufficiently small, we conclude that the map $\mathcal{P}$ is a contraction on the ball of radius $C_0\varepsilon\rho^2$ in $\tilde{\mathcal{M}}$. Therefore, thanks to the fixed point theorem, we deduce that it has a (unique) fixed point w which we can write $w = (v, \bar{S}, \bar{\beta})$ and which is a solution of the equation $\tilde{\mathcal{N}}(w) = 0$. It is clear that $\bar{u}_{\bar{\varepsilon}(1+\bar{\mathcal{A}}\times\bar{S}),\bar{R}+\bar{\mathcal{B}}\times\bar{S},\bar{a}+\bar{\beta}} + v$ is positive near the points $x_i \in \Sigma$ thanks to the behavior of $\bar{u}_{\bar{\varepsilon}(1+\bar{\mathcal{A}}\times\bar{S}),\bar{R}+\bar{\mathcal{B}}\times\bar{S},\bar{a}+\bar{\beta}}$ near x_i and also thanks to the fact that we have $v \in \mathcal{C}^{2,\alpha}_\nu(\Omega \setminus \Sigma)$ with $\nu > 1$. Then by the maximum principle, it is positive everywhere. This completes the proof of Theorem 1.2.

11 Nonexistence results in the case of a starshaped domain

We give here a proof of Theorem 1.1. Let us assume that $\Omega \subset \mathbb{R}^n$ is a bounded regular domain which is starshaped with respect to x_0. Assume that we have a solution of

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0$$

defined in $\Omega \setminus \{x_0\}$ and having 0 boundary data, with a non-removable singularity at x_0. Up to a translation, we may assume that $x_0 = 0$.

Thanks to the result of L. A. Caffarelli, B. Gidas and J. Spruck [1], we know that u near x_0 is asymptotically equivalent to one of the Delaunay type solution. More precisely, there exist $\varepsilon > 0$ and $R > 0$ such that

$$u(x) = | x |^{\frac{2-n}{2}} \; v_\varepsilon(-\log | x | + \log R) \, (1 + o(1)) \tag{11.1}$$

near 0.

Now let us use the Pohozaev formula: we multiply the equation satisfied by u by the quantity

$$\sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}.$$

We obtain

$$\sum_{i=1}^n \sum_{j=1}^n x_i \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) - x_i \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_j} + \frac{(n-2)^2}{8} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (u^{\frac{2n}{n-2}}) = 0,$$

which can be written

$$\begin{aligned} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n \left(x_i \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) - \frac{1}{2} x_i \, | \nabla u |^2 + \frac{(n-2)^2}{8} x_i u^{\frac{2n}{n-2}} \right) \\ + \tfrac{n-2}{2} \left(| \nabla u |^2 - \tfrac{n(n-2)}{4} u^{\frac{2n}{n-2}} \right) = 0. \end{aligned} \tag{11.2}$$

Using once more the equation satisfied by u which we multiply this time by u itself, we get

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(u \frac{\partial u}{\partial x_i} \right) = | \nabla u |^2 - \frac{n(n-2)}{4} u^{\frac{2n}{n-2}}. \tag{11.3}$$

Using (11.2) and (11.3), we conclude that

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \left(x_j \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) - \frac{1}{2} x_i \mid \nabla u \mid^2 + \frac{(n-2)^2}{8} x_i u^{\frac{2n}{n-2}} + \frac{n-2}{2} u \frac{\partial u}{\partial x_i} \right) = 0. \tag{11.4}$$

Since $u = 0$ on $\partial\Omega$, we see that

$$(x \cdot \nabla u)(\nabla u \cdot \nu) = \sum_{i,j=1}^{n} \left(x_j \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) \nu_j = | \nabla u |^2 \, (x \cdot \nu),$$

where ν denotes the unit normal vector on $\partial\Omega$.

Integrating the equality (11.4) in $\Omega \setminus B(0, \rho)$, for some ρ small we find

$$\frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) d\sigma$$

$$= \int_{\partial B(0,\rho)} \left(\rho (\frac{\partial u}{\partial r})^2 - \frac{1}{2} \rho \mid \nabla u \mid^2 + \frac{(n-2)^2}{8} \rho u^{\frac{2n}{n-2}} + \frac{n-2}{2} u \frac{\partial u}{\partial r} \right) d\sigma.$$

The left hand side of this equality is clearly ≥ 0 since we have assumed that Ω is starshaped with respect to 0. Now, letting ρ tend to 0 and using (11.1), we find that

$$\lim_{\rho \to 0} \int_{\partial B(0,\rho)} \left(\rho (\frac{\partial u}{\partial r})^2 - \frac{1}{2} \rho \mid \nabla u \mid^2 + \frac{(n-2)^2}{8} \rho u^{\frac{2n}{n-2}} + \frac{n-2}{2} u \frac{\partial u}{\partial r} \right) d\sigma$$

$$= \quad | S^{n-1} | \, \mathcal{H}(v_\varepsilon, \dot{v}_\varepsilon)$$

which is < 0 as we have seen in Proposition 2.1. We therefore get the desired contradiction. This ends the proof of Theorem 1.1.

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