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Unique Strong Solutions and V-Attractors of a Three Dimensional System of Globally Modified Navier-Stokes Equations

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Abstract

We prove the existence and uniqueness of strong solutions of a three dimensional system of globally modified Navier-Stokes equations. The flattening property is used to establish the existence of global V-attractors and a limiting argument is then used to obtain the existence of bounded entire weak solutions of the three dimensional Navier-Stokes equations with time independent forcing.

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary Γ, and consider the Navier-Stokes equations (NSE) on Ω with a homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) \ \text{ in } \ (0,+\infty)\times\Omega, \\ \nabla\cdot u \ = 0 \ \text{ in } \ (0,+\infty)\times\Omega, \\ u = 0 \ \text{ on } (0,+\infty)\times\Gamma, \\ u(0,x) = u_0(x), \quad x\in\Omega, \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, u_0 the initial velocity field, and $f(t)$ a given external force field.

We define $F_N : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$F_N(r) := \min\left\{1, \frac{N}{r}\right\}, \quad r\in\mathbb{R}^+,$$

for some $N \in \mathbb{R}^+$ and will consider the following system of globally modified Navier-Stokes equations (GMNSE)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|u\|)\left[(u\cdot\nabla)u\right] + \nabla p = f(t) \ \text{ in } \ (0,+\infty)\times\Omega, \\ \nabla\cdot u \ = 0 \ \text{ in } \ (0,+\infty)\times\Omega, \\ u = 0 \ \text{ on } (0,+\infty)\times\Gamma, \\ u(0,x) = u_0(x), \quad x\in\Omega, \end{cases} \tag{1.2}$$

The GMNSE (1.2) are indeed globally modified – the modifying factor $F_N(\|u\|)$ depends on the norm $\|u\| = \|\nabla u\|_{(L^2(\Omega))^{3\times 3}}$, which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x\in\Omega$ under consideration. Essentially, it prevents large gradients dominating the dynamics and leading to explosions. It violates the basic laws of mechanics, but mathematically the GMNSE (1.2) are a well defined system of equations, just like the modified versions of the NSE of Leray and others with other mollifications of the nonlinear term, see the review paper of Constantin [1]. (In passing, we mention that Flandoli and Maslowski [3] used a global cut off function involving the $D(A^{1/4})$ norm for the two dimensional stochastic Navier-Stokes equations).

In the next section we formulate the GMNSE (1.2) in the abstract framework that is usually used for the NSE (1.1) and show that a strong solution of the GMNSE is unique in the class of weak solutions. In section 3 we then establish the existence and uniqueness of strong solutions of the GMNSE in three dimensions and their continuous dependence on N and on the initial value in the space V. In addition, we investigate the relationship between the Galerkin approximations of the GMNSE and the NSE for a fixed finite dimension. In section 4 we verify the flattening property for the GMNSE and use it to conclude the existence of global V-attractors. Then in section 5 we show that a subsequence of solutions of the GMNSE with a convergent sequence of initial values converges to a weak solution of the NSE with the same initial value on any finite time interval. Finally, for time independent forcing, we use the previous result for initial values in global attractors to establish the existence of bounded entire weak solutions of the three dimensional Navier-Stokes equations. Such solutions would belong to a global attractor of the NSE, if such an attractor were to exist.

2 Preliminaries

To set our problem in the abstract framework, we consider the following usual abstract spaces (see Lions [7] and Temam [10, 11]):

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u \ = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$, where for $u, v \in (L^2(\Omega))^3$,

$$(u, v) = \sum_{j=1}^{3} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V \ =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$, where for $u, v \in (H_0^1(\Omega))^3$,

$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. Finally, we will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot,\cdot \rangle$ for the duality pairing between V and V'.

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x, \quad \forall\, u, v, w \in V,$$

and we denote

$$b_N(u, v, w) = F_N(\|v\|) b(u, v, w), \quad \forall\, u, v, w \in V.$$

The form b_N is linear in u and w, but it is nonlinear in v. Evidently we have $b_N(u, v, v) = 0$, for all $u, v \in V$. Moreover, from the properties of b (see [9] or [10]), and the definition of F_N, one easily obtains the existence of a constant $C_1 > 0$ only dependent on Ω such that

$$|b_N(u, v, w)| \le C_1 \, |u|^{1/4} \|u\|^{3/4} |v|^{1/4} \|v\|^{3/4} \|w\|, \quad \forall\, u, v, w \in V,$$

$$|b_N(u, v, w)| \le N C_1 \, \|u\| \|w\|, \quad \forall\, u, v, w \in V.$$

Thus, if we denote

$$\langle B_N(u,v),w\rangle = b_N(u,v,w), \quad \forall\, u,v,w \in V,$$

we have

$$\|B_N(u,v)\|_* \le C_1|u|^{1/4}\|u\|^{3/4}|v|^{1/4}\|v\|^{3/4}, \quad \forall\, u,v \in V, \tag{2.3}$$

$$\|B_N(u,v)\|_* \le NC_1\,\|u\|, \quad \forall\, u,v \in V. \tag{2.4}$$

We also consider $A : V \to V'$ defined by $\langle Au,v\rangle = ((u,v))$. Denoting $D(A) = (H^2(\Omega))^3 \cap V$, then $Au = -P\Delta u, \forall\, u \in D(A)$, is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H).

We recall (see [10]) that there exists a constant $C_2 > 0$ depending only on Ω such that

$$\|u\|_{(L^\infty(\Omega))^3} \le C_2|Au|, \quad \forall\, u \in D(A), \tag{2.5}$$

$$|b(u,v,w)| \le C_2|u|^{1/4}|Au|^{3/4}\|v\||w|, \quad \forall\, u \in D(A), v \in V, w \in H, \tag{2.6}$$

$$|b(u,v,w)| \le C_2\|u\|^{1/2}|Au|^{1/2}\|v\||w|, \quad \forall\, u \in D(A), v \in V, w \in H. \tag{2.7}$$

Definition 1 Let $u_0 \in H$ and $f \in L^2(0,T;(L^2(\Omega))^3)$ for all $T > 0$. A *weak solution* of (1.2) is any $u \in L^2(0,T;V)$ for all $T > 0$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t) + \nu Au(t) + B_N(u(t),u(t)) = f(t) \text{ in } \mathcal{D}'(0,+\infty;V'), \\ u(0) = u_0, \end{cases}$$

or equivalently

$$(u(t),w) + \nu\int_0^t ((u(s),w))\,ds + \int_0^t b_N\,(u(s),u(s),w)\;ds = (u_0,w) + \int_0^t (f(s),w)\;ds, \tag{2.8}$$

for all $t \ge 0$ and all $w \in V$.

Remark 2 Observe that if $u \in L^2(0,T;V)$ for all $T > 0$ and satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) + \nu Au(t) + B_N(u(t),u(t)) = f(t) \quad \text{in } \mathcal{D}'(0,+\infty;V'),$$

then, as a consequence of (2.4), $\dfrac{\mathrm{d}}{\mathrm{d}t}u(t) \in L^2(0,T;V')$, and consequently (see [11]) $u \in C([0,+\infty);H)$ and satisfies the energy equality

$$|u(t)|^2 - |u(s)|^2 + 2\nu\int_s^t \|u(r)\|^2\,dr = 2\int_s^t (f(r),u(r))\,dr \quad \text{for all } 0 \le s \le t. \tag{2.9}$$

We will prove the existence of (strong) solutions in the next section. First, we consider the counterpart of Serrin's classical theorem on the three-dimensional NSE which says that a strong solution, if it exists, is unique in the class of weak solutions. We remark that strong solutions, to be defined later, are examples of weak solutions of the kind in the following theorem.

Theorem 3 *If there exists a weak solution u of (1.2) such that $u \in L^2(0,T;D(A))$ for all $T > 0$, then u is the unique weak solution of (1.2).*

The proof is similar to, but a bit more complicated than in the NSE case and depends on the following Lemmata.

Lemma 4

$$|F_N(p) - F_N(r)| \leq \frac{|p-r|}{r}$$

for all $p,\ r \in \mathbb{R}^+$.

Proof. Define

$$\overline{F}_N(p,r) := |F_N(p) - F_N(r)| = \left|\min\left\{1, \frac{N}{p}\right\} - \min\left\{1, \frac{N}{r}\right\}\right|.$$

Taking into account that $\overline{F}_N(r,p) = \overline{F}_N(p,r)$, it is enough to consider three cases.

If $p,\ r \geq N$, then

$$\overline{F}_N(p,r) = \left|\frac{N}{p} - \frac{N}{r}\right| = N\,\frac{|p-r|}{pr} \leq \frac{|p-r|}{r}.$$

If $p,\ r \leq N$, then

$$\overline{F}_N(p,r) = |1-1| = 0 \leq \frac{|p-r|}{r}.$$

Finally, if $r \geq N \geq p$, then

$$\overline{F}_N(p,r) = \left|1 - \frac{N}{r}\right| = \frac{r-N}{r} \leq \frac{r-p}{r} = \frac{|p-r|}{r}.$$

■

Lemma 5

$$|F_N(\|u\|) - F_N(\|v\|)| \leq \frac{\|u-v\|}{\|v\|}$$

for all $u,\ v \in V$ with $v \neq 0$.

Proof. We apply Lemma 4 with $p = \|u\|$ and $r = \|v\|$, using the fact that

$$|\|u\| - \|v\|| \leq \|u-v\|.$$

■

Proof of Theorem 3. Let v be any weak solution with the same initial condition as the weak solution u in the statement of the theorem and write $w = v - u$. Then, using the energy equality, after rearrangement and using the fact that $b(w,w,w) = b(u,w,w) = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + F_N(\|v\|)b(w,u,w) + (F_N(\|v\|) - F_N(\|u\|)]\, b(u,u,w) = 0.$$

From (2.5), we have that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}|F_N(\|v\|)b(w,u,w)| &\leq |b(w,u,w)| \\ &= |b(w,w,u)| \\ &\leq C_3|w|\,\|w\||Au| \\ &\leq \frac{\nu}{4}\|w\|^2 + \frac{C_3^2}{\nu}|Au|^2|w|^2.\end{aligned}$$

Using Lemma 5 and (2.7), we have

$$\begin{aligned}|[F_N(\|v\|) - F_N(\|u\|)]\, b(u,u,w)| &\leq \frac{\|v-u\|}{\|u\|}\, C_2\|u\|^{3/2}|Au|^{1/2}|w| \\ &= C_2\|w\|\|u\|^{1/2}|Au|^{1/2}|w| \\ &\leq \frac{\nu}{4}\|w\|^2 + \frac{C_2^2}{\nu}\|u\||Au||w|^2.\end{aligned}$$

Thus we obtain

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 \leq C\left(1 + \|u\|^2 + |Au|^2\right)|w|^2$$

for an appropriate constant C, and the result follows from Gronwall's lemma, since $|w(0)|^2 = 0$. ■

Later we will need the following generalization of Lemma 4.

Lemma 6

$$|F_M(p) - F_N(r)| \leq \frac{|M-N|}{r} + \frac{|p-r|}{r}$$

for all M, N, p, $r \in \mathbb{R}^+$.

Proof.

$$\begin{aligned}|F_M(p) - F_N(r)| &\leq |F_M(p) - F_M(r)| + |F_M(r) - F_N(r)| \\ &\leq |F_M(r) - F_N(r)| + \frac{|p-r|}{r}\end{aligned}$$

by Lemma 4. To show that

$$|F_M(r) - F_N(r)| \leq \frac{|M-N|}{r}$$

for all M, N, $r \in \mathbb{R}^+$, we consider three cases.

If $r \geq M$, N, then

$$|F_M(r) - F_N(r)| = \left|\frac{M}{r} - \frac{N}{r}\right| = \frac{|M-N|}{r}.$$

If $r \leq M$, N, then

$$|F_M(r) - F_N(r)| = |1-1| = 0 \leq \frac{|M-N|}{r}.$$

Finally, if $M \leq r \leq N$, then

$$|F_M(r) - F_N(r)| = \left|\frac{M}{r} - 1\right| = \frac{M-r}{r} \leq \frac{|M-N|}{r}.$$

■

3 Existence and uniqueness of strong solutions

Here we prove the following existence and regularity result for the system GMNSE.

Theorem 7 *Suppose $f \in L^2(0,T;(L^2(\Omega))^3)$ for all $T > 0$, and let $u_0 \in H$ be given. Then,*

(a) If $u_0 \in V$, there exists a unique weak solution u of (1.2), and in fact u is a strong solution, i.e., satisfies

$$u \in C([0,T];V) \cap L^2(0,T;D(A)) \quad \textit{for all } T > 0. \tag{3.10}$$

(b) If $u_0 \notin V$, every weak solution u of (1.2) is a strong solution, in the sense that

$$u \in C([\varepsilon,T];V) \cap L^2(\varepsilon,T;D(A)) \quad \textit{for all } T > \varepsilon > 0. \tag{3.11}$$

(c) If $u_0 \notin V$ but $f \in L^\infty(0,\delta;(L^2(\Omega))^3)$ for some $\delta > 0$, then there exists at least one weak solution u of (1.2).

Proof. The assertion (b) is a consequence of the assertion (a). If $u_0 \notin V$ and u is a weak solution of (1.2), then $u \in C([0,T];H) \cap L^2(0,T;V)$ for all $T > 0$, and consequently, for any $\varepsilon > 0$ there exists $0 < t_0 < \varepsilon$ such that $u(t_0) \in V$. Then, the function $v(t) := u(t+t_0)$ is a weak solution of system (1.2) with $v(0) = u(t_0) \in V$ and forcing term $f(t+t_0)$, and by assertion (a) $v \in C([0,T];V) \cap L^2(0,T;D(A))$ for all $T > 0$, and in particular u satisfies (3.11).

Thus, we must prove assertions (a) and (c). Consider the Galerkin approximations for the GMNSE, given by

$$\frac{du_m}{dt} + \nu A u_m + P_m B_N(u_m, u_m) = P_m f, \quad u_m(0) = P_m u_0, \tag{3.12}$$

where $u_m = \sum_{j=1}^m u_{m,j}\phi_j$, $Au_m = \sum_{j=1}^m \lambda_j u_{m,j}\phi_j$. Here the λ_j and ϕ_j are the corresponding eigenvalues and orthonormal eigenfunctions of the operator A and P_m is the projection onto the subspace of H spanned by $\{\phi_1, \ldots, \phi_m\}$. Then

$$\|u_m\|^2 = \sum_{j=1}^m \lambda_j u_{m,j}^2, \qquad |Au_m|^2 = \sum_{j=1}^m \lambda_j^2 u_{m,j}^2.$$

In addition

$$|u_m|^2 = \sum_{j=1}^m u_{m,j}^2,$$

which can be interpreted as either the Euclidean norm of $u_m \in \mathbb{R}^m$ or the L^2-norm of $u_m \in H$.

The existence of local solution of (3.12) is a consequence of Peano's theorem, the uniqueness of solution of (3.12) can be proved as in Theorem 3, and the fact that the local solution is a global one is a consequence of the estimate (3.13) below.

Indeed, it is standard that if we take the inner product of the Galerkin ODE (3.12) with u_m and use that $b(u_m, u_m, u_m) = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m|^2 + \nu\|u_m\|^2 = (f, u_m),$$

which is exactly the same as in the NSE, and taking into account that $\lambda_1|u_m|^2 \le \|u_m\|^2$, easily gives the inequality

$$\frac{d}{dt}|u_m|^2 + \nu\|u_m\|^2 \le \frac{|f|^2}{\nu\lambda_1}, \tag{3.13}$$

and consequently

$$|u_m(t)|^2 + \nu\int_0^t \|u_m(s)\|^2\, ds \le |u_0|^2 + \int_0^t \frac{|f(s)|^2}{\nu\lambda_1}\, ds \quad \text{for all } t \ge 0. \tag{3.14}$$

Also, from (3.13) we have

$$\frac{d}{dt}|u_m|^2 + \nu\lambda_1|u_m|^2 \le \frac{|f|^2}{\nu\lambda_1},$$

thus

$$\frac{d}{dt}\left(e^{\nu\lambda_1 t}|u_m(t)|^2\right) \le \frac{|f|^2}{\nu\lambda_1} e^{\nu\lambda_1 t},$$

and integrating one easily obtains

$$|u_m(t)|^2 \leq |u_0|^2 e^{-\nu\lambda_1 t} + \int_0^t \frac{|f(s)|^2}{\nu\lambda_1} e^{-\nu\lambda_1(t-s)}\, ds \quad \text{for all } t \geq 0. \tag{3.15}$$

From (3.14), exactly as with the NSE, one determines the existence of a

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

and a subsequence of $\{u_m\}_{m\in\mathbb{N}}$ which converges weak-star to u in $L^\infty(0,T;H)$ and weakly to u in $L^2(0,T;V)$. By the compactness theorem 5.1 in Chapter 1 of [7], it follows that a subsequence in fact converges strongly to u in $L^2(0,T;H)$ and a.e. in $(0,T) \times \Omega$. But the weak convergence in $L^2(0,T;V)$ is not enough to ensure that

$$\|u_m\| \to \|u\| \qquad \text{or at least} \qquad F_N(\|u_m(t)\|) \to F_N(\|u(t)\|) \quad \text{for a.a. } t.$$

Thus we go one step further and find a stronger estimate. We now take the inner product of the Galerkin ODE (3.12) with Au_m and obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 + b_N(u_m,u_m,Au_m) = (f,Au_m). \tag{3.16}$$

Evidently,

$$|(f,Au_m)| \leq \frac{\nu}{4}|Au_m|^2 + \frac{|f|^2}{\nu}$$

In addition, by (2.7) and Young's inequality, one obtains

$$\begin{aligned}|b_N(u_m,u_m,Au_m)| &\leq \frac{N}{\|u_m\|}\, C_2\|u_m\|^{3/2}|Au_m|^{3/2} \\ &= NC_2\|u_m\|^{1/2}|Au_m|^{3/2} \\ &\leq \frac{\nu}{4}|Au_m|^2 + C_N\|u_m\|^2,\end{aligned}$$

with $C_N = \dfrac{27(NC_2)^4}{4\nu^3}$.

Thus (3.16) simplifies to

$$\frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 \leq \frac{2}{\nu}|f|^2 + 2C_N\|u_m\|^2. \tag{3.17}$$

Now we distinguish three cases:

Case 1. when $u_0 \in V$.

In this case, from (3.17) and the fact that

$$\|u_m(0)\| = \|P_m u_0\| \leq \|u_0\|$$

by the choice of the basis $\{\phi_j\}$ of H, one easily obtains that the sequence $\{u_m\}$ is bounded in $L^\infty(0,T;V)$ and in $L^2(0,T;D(A))$ for all $T>0$.

Then, observe that for any $w \in H$, $|b_N(u_m,u_m,w)| \le NC_3|Au_m||w|$, and in consequence, the sequence $\{P_m B_N(u_m,u_m)\}$ is bounded in $L^2(0,T;H)$ for all $T>0$.

Therefore, from the equation $\dfrac{du_m}{dt} = -\nu Au_m - P_m B_N(u_m,u_m) + P_m f$, one has that the sequence $\left\{\dfrac{du_m}{dt}\right\}$ is also bounded in $L^2(0,T;H)$.

Consequently, as $D(A) \subset V \subset H$ with compact injection, by Theorem 5.1 in Chapter 1 of [7] there exists an element $u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$ for all $T>0$, and a subsequence of $\{u_m\}$, that we will also denote by $\{u_m\}$, such that

$$\begin{cases} u_m \to u \quad \text{strong in } \ L^2(0,T;V), \\ u_m \to u \quad \text{a.e. in } \ (0,T)\times\Omega, \\ u_m \rightharpoonup u \quad \text{weak in } \ L^2(0,T;D(A)), \\ u_m \overset{*}{\rightharpoonup} u \quad \text{weak-star in } \ L^\infty(0,T;V), \\ \dfrac{du_m}{dt} \rightharpoonup \dfrac{du}{dt} \quad \text{weak in } \ L^2(0,T;H), \end{cases} \tag{3.18}$$

for all $T>0$.

Also, as u_m converges to u in $L^2(0,T;V)$ for all $T>0$, we can assume, possibly extracting a subsequence, that

$$\|u_m(t)\| \to \|u(t)\| \quad \text{a.e. in } (0,+\infty),$$

and therefore

$$F_N(\|u_m(t)\|) \to F(\|u(t)\|) \quad \text{a.e. in } (0,+\infty). \tag{3.19}$$

From (3.18) and (3.19) we can take limits in (3.12) and we obtain that u is a solution of (1.2) satisfying (3.10). In fact, this can be done reasoning as in [7] for the case of the Navier-Stokes system. The only slight difference is for the nonlinear term, and for the sake of completeness we include the arguments here.

First, by linearity, density, and the properties of b, we only need to prove that

$$\lim_{m\to\infty} \int_0^t F_N(\|u_m(s)\|) b(u_m(s),\phi_j,u_m(s))\,ds = \int_0^t F_N(\|u(s)\|) b(u(s),\phi_j,u(s))\,ds \tag{3.20}$$

for all $j \ge 1$, $t \ge 0$.

Let $T>0$ be fixed. According to [7], as u_m is bounded in $L^\infty(0,T;H)\cap L^2(0,T;V)$, for each $1 \le i,k \le 3$, the product of the components $u_m^{(i)}u_m^{(k)}$ is bounded in $L^2(0,T;L^{3/2}(\Omega))$, and therefore, as $0 \le F_N \le 1$, the product $F_N(\|u_m(s)\|)u_m^{(i)}u_m^{(k)}$ is also bounded in $L^2(0,T;L^{3/2}(\Omega))$. But then, eventually extracting a subsequence, we can assume that

$$F_N(\|u_m(s)\|)u_m^{(i)}u_m^{(k)} \rightharpoonup \chi_{i,k} \quad \text{weak in } \ L^2(0,T;L^{3/2}(\Omega)), \quad 1 \le i,k \le 3. \tag{3.21}$$

But, $u_m \to u$ a.e. in $(0,T)\times\Omega$, and therefore, by (3.19),

$$F_N(\|u_m(s)\|)u_m^{(i)}u_m^{(k)} \to F_N(\|u(s)\|)u^{(i)}u^{(k)} \quad \text{a.e. in } (0,T)\times\Omega, \tag{3.22}$$

for $1\le i,k\le 3$. The sequence $F_N(\|u_m(s)\|)u_m^{(i)}u_m^{(k)}$ is also bounded in $L^{3/2}((0,T)\times\Omega)$, and thus, by (3.22) and Lemma 1.3 in [7],

$$F_N(\|u_m(s)\|)u_m^{(i)}u_m^{(k)} \rightharpoonup F_N(\|u(s)\|)u^{(i)}u^{(k)} \text{ weak in } L^{3/2}((0,T)\times\Omega), \quad 1\le i,k\le 3. \tag{3.23}$$

Evidently, (3.21) and (3.23) imply that $\chi_{i,k} = F_N(\|u(s)\|)u^{(i)}u^{(k)}$, and then (3.20) is obtained exactly as in [7].

Case 2. When $u_0 \notin V$ but $f\in L^\infty(0,+\infty;(L^2(\Omega))^3)$.

In this case, we must go further with the previous estimates. We denote $|f|_\infty = \|f\|_{L^\infty(0,+\infty;(L^2(\Omega))^3)}$, and we argue as in [2]. From (3.17) we have in this case

$$\|u_m(t)\|^2 \le \left[\|u_m(t_0)\|^2 + \frac{2|f|_\infty^2}{\nu}(t-t_0)\right]e^{2C_N(t-t_0)} \quad \text{for all } 0\le t_0\le t. \tag{3.24}$$

Now, observe that (3.15) implies

$$|u_m(t)|^2 \le |u_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{(\nu\lambda_1)^2} \quad \text{for all } t\ge 0. \tag{3.25}$$

Also, integrating (3.13) between t and $t+\tau$, we have

$$\nu\int_t^{t+\tau}\|u_m(s)\|^2\,ds \le |u_m(t)|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}\tau \quad \text{for all } 0\le t< t+\tau,$$

and then, from (3.25)

$$\nu\int_t^{t+\tau}\|u_m(s)\|^2\,ds \le |u_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|_\infty^2}{\nu\lambda_1}\left[\tau + \frac{1}{\nu\lambda_1}\right] \quad \text{for all } 0\le t<t+\tau. \tag{3.26}$$

For $t\ge 0$ and $\tau>0$ given, let us define $\rho>0$ by

$$\rho^2 = \frac{2}{\nu\tau}\left\{|u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}\left[\tau+\frac{1}{\nu\lambda_1}\right]\right\},$$

consider the sets

$$D_m = \{s\in[t,t+\tau] : \ \|u_m(s)\|\ge\rho\},$$

and let us denote $|D_m|$ the Lebesgue measure of D_m. Evidently, by (3.26),

$$\begin{aligned}\rho^2|D_m| &\leq \int_{D_m} \|u_m(s)\|^2\,ds \\ &\leq \int_t^{t+\tau} \|u_m(s)\|^2\,ds \\ &\leq \frac{1}{\nu}\left\{|u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}\left[\tau + \frac{1}{\nu\lambda_1}\right]\right\} = \frac{\tau\rho^2}{2},\end{aligned}$$

and thus $|D_m| \leq \tau/2$.

Consequently we can ensure that for any $t \geq 0$, $\tau > 0$ and $m \geq 1$ there exists a $t_0 \in (t, t+\tau)$ such that

$$\|u_m(t_0)\|^2 \leq \frac{2}{\nu\tau}\left\{|u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}\left[\tau + \frac{1}{\nu\lambda_1}\right]\right\}.$$

From this property, we have that for any given $\varepsilon > 0$ and any $t \geq \varepsilon$, there exists a $t_0 \in (t-\varepsilon, t)$ such that

$$\|u_m(t_0)\|^2 \leq \frac{2}{\nu\varepsilon}\left\{|u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}\left[\varepsilon + \frac{1}{\nu\lambda_1}\right]\right\},$$

and hence, from (3.24) we deduce that

$$\|u_m(t)\|^2 \leq \left[\frac{2}{\nu\varepsilon}\left\{|u_0|^2 + \frac{|f|_\infty^2}{\nu\lambda_1}\left[\varepsilon + \frac{1}{\nu\lambda_1}\right]\right\} + \frac{2|f|_\infty^2}{\nu}\varepsilon\right] e^{2C_N\varepsilon} \quad \text{for all } t \geq \varepsilon. \tag{3.27}$$

From (3.14), (3.27) and (3.17), we immediately obtain that the sequence $\{u_m\}$ is bounded in $L^\infty(0,T;H)$, in $L^2(0,T,V)$, in $L^\infty(\varepsilon,T;V)$, and in $L^2(\varepsilon,T;D(A))$, for all $T > \varepsilon > 0$.

Reasoning as in Case 1, we see that the sequence $\left\{\dfrac{du_m}{dt}\right\}$ is also bounded in $L^2(\varepsilon,T;H)$ for all $T > \varepsilon > 0$. Hence, there exists an element

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^\infty(\varepsilon,T;V) \cap L^2(\varepsilon,T;D(A))$$

for all $T > \varepsilon > 0$, and a subsequence of $\{u_m\}$, that we will also denote by $\{u_m\}$, such that

$$\begin{cases} u_m \rightharpoonup u \quad \text{weak in } L^2(0,T;V), \\ u_m \overset{*}{\rightharpoonup} u \quad \text{weak-star in } L^\infty(0,T;H), \\ u_m \to u \quad \text{strong in } L^2(0,T;H), \\ u_m \to u \quad \text{a.e. in } (0,T)\times\Omega, \\ u_m \to u \quad \text{strong in } L^2(\varepsilon,T;V), \\ u_m \rightharpoonup u \quad \text{weak in } L^2(\varepsilon,T;D(A)), \\ u_m \overset{*}{\rightharpoonup} u \quad \text{weak-star in } L^\infty(\varepsilon,T;V), \\ \dfrac{du_m}{dt} \rightharpoonup \dfrac{du}{dt} \quad \text{weak in } L^2(\varepsilon,T;H), \end{cases} \tag{3.28}$$

for all $T > \varepsilon > 0$.

Also, as u_m converges to u in $L^2(\varepsilon,T;V)$ for all $T > \varepsilon > 0$, we can assume, eventually extracting a subsequence, that (3.19) is also satisfied in this case. From (3.28) and (3.19) we can take limits in (3.12) as in Case 1 above and we obtain that u is a solution of (1.2) satisfying (3.11).

Case 3. When $u_0 \notin V$ but $f \in L^\infty(0,\delta;(L^2(\Omega))^3)$ for some $\delta > 0$.

In this case, considering the prolongation of f as zero to $(\delta,+\infty)$, by the Case 2 we can ensure that there exists a function u^1 such that, in particular,

$$u^1 \in C([0,\delta];H) \cap L^2(0,\delta;V) \cap C([\varepsilon,\delta];V) \cap L^2(\varepsilon,\delta;D(A)) \quad \text{for all } 0 < \varepsilon < \delta,$$

and

$$\left(u^1(t),w\right) + \nu \int_0^t ((u^1(s),w))\,ds + \int_0^t b_N\left(u^1(s),u^1(s),w\right)\,ds$$

$$= (u_0,w) + \int_0^t (f(s),w)\,ds$$

for all $t \in [0,\delta]$ and all $w \in V$.

Now, considering the problem (1.2) with initial datum $u^1(\delta)$ and forcing term $f(t+\delta)$, it is immediate from Case 1 that there exists a function u^2 such that

$$u^2 \in C([\delta,T];V) \cap L^2(\delta,T;D(A)) \quad \text{for all } T > \delta,$$

and

$$\left(u^2(t),w\right) + \nu \int_\delta^t ((u^2(s),w))ds + \int_\delta^t b_N\left(u^2(s),u^2(s),w\right)ds$$

$$= \left(u^1(\delta),w\right) + \int_\delta^t (f(s),w)\,ds$$

for all $t \in [\delta, +\infty)$ and all $w \in V$.

Thus, the function

$$u(t) = \begin{cases} u^1(t) & \text{if } t \in [0, \delta], \\ u^2(t) & \text{if } t \in [\delta, +\infty), \end{cases}$$

is a solution of (1.2) satisfying (3.11). ■

3.1 Continuous dependence on initial values and N

We prove now a result which shows, in particular, that the semi flows generated by the solutions $u^{(N)}(t, u_0)$ of the GMNSE (1.2) with parameter N, depend continuously on the parameter N as well as on the initial value u_0. First we prove a preliminary estimate.

Theorem 8 *Suppose that $f \in L^2(0, T; (L^2(\Omega))^3)$ for all $T > 0$, and let $N, M > 0$, and $u_0, v_0 \in V$ be given. Let us denote by $u(t)$ (respectively, $v(t)$) the solution of (1.2) corresponding to the parameter N and the initial value u_0 (respectively, to the parameter M and the initial condition v_0). Then, there exists a positive constant $C > 0$ depending only on Ω and ν such that*

$$||v(t) - u(t)||^2 \leq [||v_0 - u_0||^2 + C(M - N)^2 \int_0^t |Au(s)|^2 ds] \times \tag{3.29}$$
$$\times \exp\left(C\left(M^4 t + \int_0^t |Au(s)|^2 ds \right)\right), \quad \text{for all} \ \ t \geq 0,$$

and

$$\nu \int_0^t |Av(s) - Au(s)|^2 ds \leq [||v_0 - u_0||^2 + C(M - N)^2 \int_0^t |Au(s)|^2 ds] \times \tag{3.30}$$
$$\left[1 + \left(C \int_0^t \left(|Au(s)|^2 + M^4 \right) ds \right) \times \exp\left(C\left(M^4 t + \int_0^t |Au(s)|^2 ds \right)\right)\right],$$

for all $t \geq 0$.

Proof. Denote $w(t) = v(t) - u(t)$ where $v(t) = u^{(M)}(t, u_0)$ and $u(t) = u^{(N)}(t, u_0)$. Taking into account that $w \in C([0, T]; V) \cap L^2(0, T; D(A))$ for all $T > 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) + \nu A w(t) + B_M(v(t), v(t)) - B_N(u(t), u(t)) = 0,$$

we obtain the energy equality

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu |Aw|^2 + b_M(v, v, Aw) - b_N(u, u, Aw) = 0. \tag{3.31}$$

Writing $\pm X$ for $+X - X$ $(= 0)$ for an expression X, we have

$$b_M(v, v, Aw) - b_N(u, u, Aw)$$

$$
\begin{aligned}
&= F_M(\|v\|)b(v,v,Aw) - F_N(\|u\|)b(u,u,Aw)\\
&= F_M(\|v\|)b(v,v,Aw) \pm F_M(\|v\|)b(u,v,Aw) - F_N(\|u\|)b(u,u,Aw)\\
&= F_M(\|v\|)b(w,v,Aw) + F_M(\|v\|)b(u,v,Aw) - F_N(\|u\|)b(u,u,Aw)\\
&= F_M(\|v\|)b(w,v,Aw) + F_M(\|v\|)b(u,v,Aw) \pm F_N(\|u\|)b(u,v,Aw)\\
&- F_N(\|u\|)b(u,u,Aw)\\
&= F_M(\|v\|)\,b(w,v,Aw) + (F_M(\|v\|) - F_N(\|u\|))\; b(u,v,Aw) + F_N(\|u\|)\,b(u,w,Aw).
\end{aligned}
$$

We then estimate the individual terms:

$$
\begin{aligned}
F_M(\|v\|)\,|b(w,v,Aw)| &\leq C_2\frac{M}{\|v\|}\|w\|^{1/2}\|v\||Aw|^{3/2}\\
&\leq \frac{\nu}{8}|Aw|^2 + C'M^4\|w\|^2,
\end{aligned}
$$

by Young's inequality. On the other hand,

$$
\begin{aligned}
F_N(\|u\|)\,|b(u,w,Aw)| &\leq |b(u,w,Aw)|\\
&\leq C|Au|\|w\||Aw|\\
&\leq \frac{\nu}{8}|Aw|^2 + C''|Au|^2\|w\|^2.
\end{aligned}
$$

Also, by Lemma 6

$$
\begin{aligned}
|F_M(\|v\|) - F_N(\|u\|)|\;|b(u,v,Aw)| &\leq \left(\frac{||w||}{||v||} + \frac{|M-N|}{||v||}\right) C|Au|||v|||Aw|\\
&\leq \frac{\nu}{4}|Aw|^2 + C'''|Au|^2\left(||w||^2 + (M-N)^2\right).
\end{aligned}
$$

Thus,

$$
\frac{d}{dt}\|w(t)\|^2 + \nu|Aw(t)|^2 \leq C\left(|Au(t)|^2 + M^4\right)\,\|w(t)\|^2 + C|Au(t)|^2\,(M-N)^2\,,\;\, t\geq 0,
$$

whence

$$
\begin{aligned}
||w(t)||^2 + \nu\int_0^t |Aw(s)|^2 ds &\leq ||w(0)||^2 + C(M-N)^2\int_0^t |Au(s)|^2 ds \qquad (3.32)\\
&+ C\int_0^t \left(|Au(s)|^2 + M^4\right) ||w(s)||^2 ds, \quad \text{for all } \; t\geq 0,
\end{aligned}
$$

and

$$||w(t)||^2 + \nu \int_0^t |Aw(s)|^2 ds \leq ||w(0)||^2 + C(M-N)^2 \int_0^T |Au(s)|^2 ds \tag{3.33}$$
$$+ C \int_0^t \left(|Au(s)|^2 + M^4\right) ||w(s)||^2 ds, \quad \text{for all } t \in [0,T].$$

Consequently,

$$||w(T)||^2 \leq$$
$$\left[||w(0)||^2 + C(M-N)^2 \int_0^T |Au(s)|^2 ds \right] \exp\left(C \int_0^T \left(|Au(s)|^2 + M^4\right) ds \right).$$

Taking into account that $T > 0$ is arbitrary, then (3.29) follows immediately. As for (3.30), it is a straightforward consequence of (3.32) and (3.29). ■

As a consequence of the previous theorem, we have continuous dependence on the initial value and on N, being also the convergence in $L^2(0,T;D(A))$. These properties are stated in the following corollary.

Corollary 9 *Suppose that $f \in L^2(0,T;(L^2(\Omega))^3)$ for all $T > 0$. Then, for any $u_0 \in V$ and $N > 0$ given,*

$$u^{(M)}(\cdot, v_0) \to u^{(N)}(\cdot, u_0) \quad \textit{in} \quad C([0,T];V) \cap L^2(0,T;D(A))$$

as $(M, v_0) \to (N, u_0)$ in $\mathbb{R}^+ \times V$, for all $T > 0$.

Proof. The proof follows from estimates (3.29) and (3.30). ■

3.2 Comparison of Galerkin solutions of the GMSE and NSE

We suppose here that $f \in L^\infty(0,T;(L^2(\Omega))^3)$ and consider the Galerkin approximations of the GMNSE and NSE of fixed dimension m for the same initial value u_0 over the time interval $[0,T]$. In particular, we want to discover what happens as $N \to \infty$.

The Galerkin ODE for the GMNSE with parameter N is

$$\frac{du_m^{(N)}}{dt} + \nu A u_m^{(N)} + P_m B_N(u_m^{(N)}, u_m^{(N)}) = P_m f \tag{3.34}$$

The energy inequality for this ODE combined with $\lambda_1 |u|^2 \leq \|u\|^2$ reads

$$\frac{d}{dt}|u_m^{(N)}|^2 + \nu\lambda_1 |u_m^{(N)}|^2 \leq \frac{|f|^2}{\nu\lambda_1}$$

and leads to the boundedness of solutions uniformly in N (and m),

$$|u_m^{(N)}(t)|^2 \leq |u_0|^2 + \frac{1}{\nu^2\lambda_1^2} M^2 T =: K, \qquad \forall t \in [0,T], \tag{3.35}$$

where we have denoted $M := \|f\|_{L^\infty(0,T;(L^2(\Omega))^3)}$.

In addition, from (3.34) we obtain that a.e. in $(0,T)$,

$$\left|\frac{du_m^{(N)}}{dt}\right| \leq \nu\left|A_m u_m^{(N)}\right| + F_N\left(\|u_m^{(N)}\|\right)\left|P_m B(u_m^{(N)}, u_m^{(N)})\right| + |P_m f|$$

$$\leq \nu\lambda_m\left|u_m^{(N)}\right| + \left|B(u_m^{(N)}, u_m^{(N)})\right| + |f|$$

$$\leq \nu\lambda_m\left|u_m^{(N)}\right| + C_2\left\|u_m^{(N)}\right\|^{3/2}\left|Au_m^{(N)})\right|^{1/2} + M$$

$$\leq \nu\lambda_m\left|u_m^{(N)}\right| + C_2\lambda_m^{5/4}\left|u_m^{(N)}\right|^2 + M$$

using the facts that

$$0 \leq F_N\left(\|u_m^{(N)}\|\right) \leq 1$$

by definition, inequality (2.7), and that

$$\|u_m\| \leq \lambda_m^{1/2}|u_m|, \qquad |Au_m| \leq \lambda_m|u_m|. \tag{3.36}$$

Using the uniform boundedness (in both N and m) of $\left|u_m^{(N)}\right|$ on the interval $[0,T]$, we see that its derivative is also a.e. uniformly bounded in N on the interval $(0,T)$. It follows then by the Ascoli theorem that there is a subsequence $u_m^{(N_j)}$ which converges uniformly in $C([0,T],\mathbb{R}^3)$ to a function $u_m^{(\infty)}$ in $C([0,T],\mathbb{R}^3)$.

This function is in fact the corresponding solution of the m-dimensional Galerkin ODE for the NSE. This follows by the uniqueness of solutions of the Galerkin ODE for a given initial value and the fact that

$$1 \geq F_N\left(\|u_m^{(N)}\|\right) = \min\left\{1, \frac{N}{\|u_m^{(N)}\|}\right\} \geq \min\left\{1, \frac{N}{\lambda_m^{1/2}|u_m^{(N)}|}\right\}$$

$$\geq \min\left\{1, \frac{N}{\lambda_m^{1/2}K}\right\},$$

so

$$F_N\left(\|\nabla u_m^{(N)}\|\right) = 1 \qquad \text{for} \quad N \geq \lambda_m^{1/2}K$$

which means that the Galerkin ODE (3.34) for the GMNSE with parameter N is the same as the Galerkin ODE for the NSE for $N \geq \lambda_m^{1/2}K$, i.e. $u_m^{(N)} \equiv u_m^{(\infty)}$ for all such N.

4 Existence of global attractor in V of the GMNSE

We now assume that the forcing term f does not depend on time. Then, for each $u_0 \in V$, we denote $S^{(N)}(t)u_0 = u^{(N)}(t,u_0)$, where $u^{(N)}(t,u_0)$ is the unique strong solution $u^{(N)}(t)$ of (1.2).

In the sequel, we consider the GMNSE for a given N, which is kept fixed, and we omit the superscript (N) when no confusion in possible.

Then, from Theorems 7 and 8, it is immediate that $\{S(t)\}_{t\geq 0}$ is a C^0 semigroup in V, i.e., $S(t) : V \to V$ for each $t \geq 0$, and satisfies

(a) $S(0) = I$, the identity map on V,

(b) $S(t+s) = S(t)S(s)$ for all $s, t \geq 0$,

(c) The function $(t, u_0) \in [0, +\infty) \times V \mapsto S(t)u_0 \in V$ is continuous.

Let $u(t) = S(t)u_0$ with $u_0 \in V$. With the same arguments used to obtain (3.13), we obtain the inequality

$$\frac{d}{dt}|u|^2 + \nu\lambda_1|u|^2 \leq \frac{1}{\nu\lambda_1}|f|^2. \tag{4.37}$$

Similarly, reasoning as in the derivation of (3.17), and taking into account that $\lambda_1\|u\|^2 \leq |Au|^2$ and that, by (2.6), $|b_N(u, u, Au)| \leq NC_2|u|^{1/4}|Au|^{7/4}$ if $u \in D(A)$, we obtain the inequality

$$\frac{d}{dt}\|u\|^2 + \nu\lambda_1\|u\|^2 \leq \frac{2}{\nu}|f|^2 + C^{(N)}|u|^2, \tag{4.38}$$

with $C^{(N)} = \dfrac{(NC_2)^8 7^7}{2^9\nu^7}$.

Integrating the differential inequality (4.37) we obtain

$$|u(t)|^2 \leq |u_0|^2 e^{-\nu\lambda_1 t} + \frac{1}{\nu^2\lambda_1^2}|f|^2\left(1 - e^{-\nu\lambda_1 t}\right),$$

from which we see that $|u(t)|^2$ remains bounded for all future time, i.e.

$$|u(t)|^2 \leq |u_0|^2 e^{-\nu\lambda_1 t} + \frac{1}{\nu^2\lambda_1^2}|f|^2, \qquad \forall t \geq 0, \tag{4.39}$$

and thus $S(t)$ has an absorbing set $\mathcal{B}_H$ in H (which absorbs bounded sets of V) given by

$$\mathcal{B}_H = \left\{ u \in H \ : |u|^2 \leq 1 + \frac{1}{\nu^2\lambda_1^2}|f|^2 \right\}. \tag{4.40}$$

Substituting the bound (4.39) for $|u(t)|^2$ in the differential inequality (4.38) gives

$$\frac{d}{dt}\|u\|^2 + \nu\lambda_1\|u\|^2 \leq C^{(N)}|u_0|^2 e^{-\nu\lambda_1 t} + \frac{|f|^2}{\nu}\left(2 + \frac{C^{(N)}}{\nu\lambda_1^2}\right).$$

Integrating this inequality then gives the solution estimate

$$\|u(t)\|^2 \leq (\|u_0\|^2 + C^{(N)}t|u_0|^2)e^{-\nu\lambda_1 t} + \frac{|f|^2}{\nu^2\lambda_1}\left(2 + \frac{C^{(N)}}{\nu\lambda_1^2}\right), \qquad \forall t \geq 0, \tag{4.41}$$

from which we deduce that $S(t)$ has an absorbing set $\mathcal{B}_V^{(N)}$ in V (which absorbs bounded sets of V) given by

$$\mathcal{B}_V^{(N)} = \left\{ u \in V \ : \|u\|^2 \leq 1 + \frac{|f|^2}{\nu^2 \lambda_1} \left(2 + \frac{C^{(N)}}{\nu \lambda_1^2} \right) \right\}. \tag{4.42}$$

As we will see in the next subsection, the semigroup in V here is asymptotically compact, from which we conclude that the GMNSE system (1.2) has a global attractor $\mathcal{A}_N$ in V for each N. Moreover, $\mathcal{A}_N \subset \mathcal{B}_V^{(N)}$ for each N and $\mathcal{B}_V^{(N)} \subset \mathcal{B}_V^{(N^*)}$ for $N \leq N^*$ in view of the definition of the constant $C^{(N)}$. The upper semi continuous dependence of the global attractors $\mathcal{A}_N$ in N follows by standard theorems in dynamical systems theory in view of the continuity of the semi groups $S^{(N)}$ in N already established in Corollary 9, see e.g. [9], Theorem 10.16.

Theorem 10 *If $f \in (L^2(\Omega))^3$ then the GMNSE (1.2) has a global attractor $\mathcal{A}_N$ in V for each $N > 0$. Moreover the set-valued mapping $N \mapsto \mathcal{A}_N$ is upper semi continuous, i.e.*

$$dist_V \left(\mathcal{A}_M, \mathcal{A}_N \right) \to 0 \quad as \quad M \to N, \tag{4.43}$$

where $dist_V$ is the Hausdorff semi distance on V.

4.1 Asymptotic compactness of the semigroup in V

To show that the semigroup is asymptotically compact, it is enough to verify the following condition which was called Condition (C) by Ma, Wang and Zhong [8] and later called the flattening property by Kloeden and Langa [6], who investigated its relationship to the better known squeezing property.

Flattening property For any bounded set B of V and for any $\varepsilon > 0$, there exists $T_\varepsilon(B) > 0$ and a finite dimensional subspace V_ε of V, such that $\{P_\varepsilon S(t)B, t \geq T_\varepsilon(B)\}$ is bounded and

$$\|(I - P_\varepsilon)S(t)u_0\| < \varepsilon \quad \text{for} \quad t \geq T_\varepsilon(B), u_0 \in B, \tag{4.44}$$

where $P_\varepsilon : V \to V_\varepsilon$ is the projection operator.

Without loss of generality, we can restrict our attention to $B = \mathcal{B}_V^{(N)}$, the absorbing set in V defined above by (4.42).

Given $\varepsilon > 0$ we will find an appropriate integer N_ε such that the flattening property is satisfied for the N_ε-dimensional subspace V_ε of V which is spanned by the first N_ε eigenfunctions $\{\phi_1, \dots, \phi_{N_\varepsilon}\}$, with P_ε the projection onto this subspace.

The first boundedness condition in property clearly holds from the estimate (4.41) and the fact that $\|P_\varepsilon v\| \leq \|v\|$ for all $v \in V$.

To verify the condition (4.44), let $u_0 \in \mathcal{B}_V^{(N)}$, and $u(t) = S(t)u_0$. Observe first that for any $\lambda \in \mathbb{R}$ one has, arguing as in deriving (3.17),

$$\frac{d}{dt}\left(e^{\nu\lambda t}\|u\|^2\right) + \nu e^{\nu\lambda t}|Au|^2 \le \frac{2}{\nu}e^{\nu\lambda t}|f|^2 + (2C^{(N)} + \nu\lambda)e^{\nu\lambda t}\|u\|^2. \tag{4.45}$$

Then integrating one easily obtains

$$e^{-\nu\lambda t}\int_0^t e^{\nu\lambda s}|Au(s)|^2\,ds \le \frac{\|u_0\|^2}{\nu} + \frac{2}{\nu^3\lambda}|f|^2 + \frac{2C^{(N)} + \nu\lambda}{\nu}e^{-\nu\lambda t}\int_0^t e^{\nu\lambda s}\|u(s)\|^2\,ds. \tag{4.46}$$

Now, define $q_\varepsilon(t) := (I - P_\varepsilon)u(t, u_0) = (I - P_\varepsilon)S(t)u_0$. Then $\|q_\varepsilon(t)\|$ satisfies the following equation which is obtained taking the L^2 inner product of the GMNSE with Aq_ε $(= Au - AP_\varepsilon u = Au - P_\varepsilon Au = A_\varepsilon u)$:

$$\frac{1}{2}\frac{d}{dt}\|q_\varepsilon\|^2 + \nu|Aq_\varepsilon|^2 + b_N(u, u, Aq_\varepsilon) = (f, Aq_\varepsilon). \tag{4.47}$$

Using the estimates

$$\begin{aligned}
|b_N(u, u, Aq_\varepsilon)| &\le \frac{N}{\|u\|}\,C_2\|u\|^{3/2}|Au|^{1/2}|Aq_\varepsilon| \\
&= C_2N\|u\|^{1/2}|Au|^{1/2}|Aq_\varepsilon| \\
&\le \frac{\nu}{4}|Aq_\varepsilon|^2 + \frac{1}{\nu}C_2^2N^2\|u\||Au| \\
&\le \frac{\nu}{4}|Aq_\varepsilon|^2 + C|Au|
\end{aligned}$$

for an approriate constant C, using the fact that $\|u(t)\|$ is bounded for all future time, and

$$(f, Aq_\varepsilon) \le \frac{\nu}{4}|Aq_\varepsilon|^2 + \frac{1}{\nu}|f|^2,$$

we obtain

$$\frac{d}{dt}\|q_\varepsilon\|^2 + \nu|Aq_\varepsilon|^2 \le 2C|Au| + \frac{2}{\nu}|f|^2$$

and hence

$$\frac{d}{dt}\|q_\varepsilon\|^2 + \nu\lambda_{N_\varepsilon}\|q_\varepsilon\|^2 \le 2C|Au| + \frac{2}{\nu}|f|^2, \tag{4.48}$$

since $\lambda_{N_\varepsilon}\|q_\varepsilon\|^2 \le |Aq_\varepsilon|^2$.

We integrate (4.48) and obtain

$$\|q_\varepsilon(t)\|^2 \leq \|q_\varepsilon(0)\|^2 e^{-\nu\lambda_{N_\varepsilon} t} + \frac{2}{\nu^2\lambda_{N_\varepsilon}}|f|^2\left(1 - e^{-\nu\lambda_{N_\varepsilon} t}\right)$$

$$+ 2Ce^{-\nu\lambda_{N_\varepsilon} t}\int_0^t e^{\nu\lambda_{N_\varepsilon} s}|Au(s)|\,ds$$

$$\leq \|u_0\|^2 e^{-\nu\lambda_{N_\varepsilon} t} + \frac{2}{\nu^2\lambda_{N_\varepsilon}}|f|^2$$

$$+ 2Ce^{-\nu\lambda_{N_\varepsilon} t}\left(\int_0^t e^{\nu\lambda_{N_\varepsilon} s}\,ds\right)^{1/2}\left(\int_0^t |Au(s)|^2 e^{\nu\lambda_{N_\varepsilon} s}\,ds\right)^{1/2}$$

$$\leq \|u_0\|^2 e^{-\nu\lambda_{N_\varepsilon} t} + \frac{2}{\nu^2\lambda_{N_\varepsilon}}|f|^2 + \frac{\widetilde{C}}{\sqrt{2\nu\lambda_{N_\varepsilon}}},$$

using the fact that by (4.46) the expression

$$e^{-\nu\lambda_{N_\varepsilon} t}\int_0^t \|Au(s)\|^2 e^{\nu\lambda_{N_\varepsilon} s}\,ds$$

remains bounded for all future time.

Thus, taking N_ε and t large enough so that the sum of the terms involving λ_{N_ε} in the denominator is smaller than $\varepsilon/2$ and so that the initial condition term becomes smaller than $\varepsilon/2$, we obtain

$$\|q_\varepsilon(t)\|^2 \leq \varepsilon,$$

which is precisely the second requirement of the flattening property.

Remark 11 Applying the results of Ma, Wang and Zhong [8], we have thus proved that the semigroup $S_N(t)$ in V has a global attractor $\mathcal{A}_N$ in V for each N which attracts bounded subsets of V. In fact, since by Theorem 7, any weak solution starting at a $u_0 \in H$ immediately becomes a strong solution, this global attractor also attracts all weak solutions. Although the weak solutions may not be unique, one can show that $\mathcal{A}_N$ is also the global attractor in H. The details will be presented elsewhere.

5 Convergence to weak solutions of Navier-Stokes Equations

Suppose that $f \in L^2(0,T;(L^2(\Omega))^3)$ for each $T > 0$ and let $u^{(N)}(t)$ be a weak solution of the GMNSE (1.2) with the initial value $u_0^{(N)} \in H$, where $u_0^{(N)} \rightharpoonup u_0$ weakly in H as $N \to \infty$. Each of these weak solutions satisfies the energy equality

$$\frac{1}{2}\frac{d}{dt}|u^{(N)}|^2 + \nu\|u^{(N)}\|^2 = (f, u^{(N)}), \tag{5.49}$$

from which it follows that

$$\frac{d}{dt}|u^{(N)}|^2 + \nu\|u^{(N)}\|^2 \leq \frac{1}{\nu\lambda_1}|f|^2 \tag{5.50}$$

uniformly in $N > 0$.

From this inequality and the boundedness of $|u_0^{(N)}|$, we obtain that the sequence $u^{(N)}$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$, and then, by (2.3), the sequence $\frac{d}{dt}u^{(N)}$ is bounded in $L^{4/3}(0,T;V^*)$ for all $T > 0$. Thus, by a diagonal argument, there exists a subsequence $u^{(N_j)}$ of the $u^{(N)}$ which converges to a funtion

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

weak-star in $L^\infty(0,T;H)$, weakly in $L^2(0,T;V)$ and strongly in $L^2(0,T;H)$ for all $T > 0$.

This limiting function, as we will see, is a weak solution of the NSE (1.1) for the given initial condition u_0, i.e satisfies the variational equation

$$(u(t), w) + \nu\int_0^t ((u(s), w))\,ds + \int_0^t b\,(u(s), u(s), w)\;ds$$

$$= (u_0, w) + \int_0^t (f(s), w)\;ds \quad t \geq 0, \tag{5.51}$$

for all $w \in V$, which differs from the variational equation (2.8) for the GMNSE by the absence of the $F_N(\|u(s)\|)$ factor multiplying the nonlinear term b. The convergence of the corresponding terms as $N_j \to \infty$ follows exactly the same as the convergence of Galerkin approximations, with the exception of the nonlinear term for which we are going to show that

$$\int_0^t F_{N_j}\left(\|u^{(N_j)}(s)\|\right) b\left(u^{(N_j)}(s), u^{(N_j)}(s), w\right)\,ds \to \int_0^t b\,(u(s), u(s), w)\;ds \tag{5.52}$$

for all $t \in [0,T]$ and $w \in D(A)$. To prove this we need the following lemma.

Lemma 12

$$F_N\left(\|u^{(N)}(s)\|\right) \to 1 \qquad in \quad L^p(0,T;\mathbb{R})$$

as $N \to \infty$ for each $p \geq 1$.

Proof. From (5.50) we see that

$$\int_0^T \|u^{(N)}(s)\|^2\,ds \leq K_T := \frac{C^2}{\nu} + \frac{1}{\nu^2\lambda_1}\int_0^T |f(s)|\,ds,$$

where $C > 0$ is an upper bound of $|u_0^{(N)}|$.

Let us denote

$$O_N = \{s \in (0,T) : \ \|u^{(N)}(s)\| \geq N\},$$

and $|O_N|$ the Lebesgue measure of O_N.

Then

$$|O_N| N^2 \leq \int_0^T \|u^{(N)}(s)\|^2 \, ds \leq K_T,$$

and so

$$|O_N| \leq \frac{K_T}{N^2} \to 0 \qquad N \to \infty,$$

Now, as

$$\begin{aligned} T - |O_N| &= \int_{[0,T]\setminus O_N} F_N\left(\|u^{(N)}(s)\|\right) \, ds \\ &\leq \int_0^T F_N\left(\|u^{(N)}(s)\|\right) \, ds \leq T, \end{aligned}$$

we see that

$$\int_0^T F_N\left(\|u^{(N)}(s)\|\right) \, ds \to \int_0^T 1 \, ds \qquad \text{as} \quad N \to \infty.$$

But $0 \leq F_N\left(\|u^{(N)}(s)\|\right) \leq 1$, so

$$\int_0^T \left|1 - F_N\left(\|u^{(N)}(s)\|\right)\right| \, ds = \int_0^T \left(1 - F_N\left(\|u^{(N)}(s)\|\right)\right) \, ds \to 0 \qquad \text{as} \quad N \to \infty.$$

Finally, since $\left|1 - F_N\left(\|u^{(N)}(s)\|\right)\right| \leq 1$, we have

$$\begin{aligned} \int_0^T \left|1 - F_N\left(\|u^{(N)}(s)\|\right)\right|^p \, ds &= \int_0^T \left|1 - F_N\left(\|u^{(N)}(s)\|\right)\right|^{p-1} \left|1 - F_N\left(\|u^{(N)}(s)\|\right)\right| \, ds \\ &\leq \int_0^T \left|1 - F_N\left(\|u^{(N)}(s)\|\right)\right| \, ds \to 0 \qquad \text{as} \quad N \to \infty \end{aligned}$$

for any $p > 1$. ■

Reasoning as in the proof of the existence of weak solutions of the NSE we have

$$\int_0^t b\left(u^{(N_j)}(s), u^{(N_j)}(s), w\right) \, ds \to \int_0^t b\left(u(s), u(s), w\right) \, ds$$

for all $t \in [0,T]$ and $w \in V$.

Let us fix $w \in D(A)$. Introduce the abbreviations

$$F^{(N_j)}(s) = F_{N_j}\left(\|u^{(N_j)}(s)\|\right), \quad b^{(N_j)}(s) = b\left(u^{(N_j)}(s), u^{(N_j)}(s), w\right),$$

$$b(s) = b\left(u(s), u(s), w\right).$$

We want to prove that

$$\int_0^T F^{(N_j)}(s) b^{(N_j)}(s)\, ds \to \int_0^T b(s)\, ds \qquad \text{as} \quad N \to \infty.$$

We rearrange this as

$$\int_0^T \left(F^{(N_j)}(s) b^{(N_j)}(s) - b(s)\right) ds = \int_0^T \left(F^{(N_j)}(s) - 1\right) b^{(N_j)}(s)\, ds + \int_0^T \left(b^{(N_j)}(s) - b(s)\right) ds,$$

where, as seen above, the second integral converges to zero. The first integral also converges to zero from Lemma 12, the estimate

$$\left|\int_0^T \left(F^{(N_j)}(s) b^{(N_j)}(s) - b^{(N_j)}(s)\right) ds\right|^2 \leq \int_0^T \left|F^{(N_j)}(s) - 1\right|^2 ds \int_0^T \left|b^{(N_j)}(s)\right|^2 ds$$

and the uniform boundedness in N_j of the second integral here, namely

$$\int_0^T \left|b^{(N_j)}(s)\right|^2 ds \leq C_2^2 \|w\| |Aw| \int_0^T |u^{(N_j)}(s)|^2 \|u^{(N_j)}(s)\|^2\, ds$$

$$\leq C_2^2 C_T \|w\| |Aw| < \infty,$$

using the uniform boundedness of the sequence $u^{(N_j)}$ in $L^\infty(0,T;H) \cap L^2(0,T;V)$.

The desired convergence (5.52) and thus the equation (5.51) holds for all $w \in D(A)$, and then, by density, for all $w \in V$.

We have thus proved the following theorem.

Theorem 13 *Suppose that $f \in L^2(0,T;(L^2(\Omega))^3)$ for each $T > 0$ and let $u^{(N)}(t)$ be a weak solution of the GMNSE (1.2) with the initial value $u_0^{(N)} \in H$, where $u_0^{(N)} \rightharpoonup u_0$ weakly in H as $N \to \infty$. Then, there exists a subsequence $\{u^{(N_j)}(t)\}$ which converges as $N_j \to \infty$, weak-star in $L^\infty(0,T;H)$, weakly in $L^2(0,T;V)$ and strongly in $L^2(0,T;H)$, to a weak solution $u(t)$ on the interval $[0,T]$ of the NSE (1.1) with initial condition u_0, for every $T > 0$.*

5.1 Existence of bounded entire weak solutions of Navier-Stokes Equations

When the forcing term $f \in (L^2(\Omega))^3$ is independent of time, we can use of Theorem 13 and the existence of a global attractor $\mathcal{A}_N$ of the GMNSE (1.2) for each N to show that the NSE (1.1) have bounded entire weak solutions, that is, weak solutions which exist and are bounded for all $t \in \mathbb{R}$. Such solutions are interesting as they would belong to a global attractor of the 3-dimensional NSE, if such an attractor were to exist.

Consider a sequence of $\{u_0^{(N)}\}$ in V with $u_0^{(N)} \in \mathcal{A}_N$ for each N. The global attractor is invariant under the semi flow, i.e. $S^{(N)}(t)\mathcal{A}_N = \mathcal{A}_N$ for all $t \geq 0$. It follows that there exists an entire strong solution of the GMNSE (1.2) $\bar{\phi}^{(N)} : \mathbb{R} \to V$ with $\bar{\phi}^{(N)}(0) = u_0^{(N)}$ and $\bar{\phi}^{(N)}(t) \in \mathcal{A}_N$ for all $t \in \mathbb{R}$ for each N. To see this, first note that for any $u_0^{(N)} \in \mathcal{A}_N$ there exists a unique strong solution $S^{(N)}(t)u_0^{(N)} \in \mathcal{A}_N$ for all $t \in \mathbb{R}^+$. There also exists $u_{-1}^{(N)} \in \mathcal{A}_N$ such that $S^{(N)}(1)u_{-1}^{(N)} = u_0^{(N)}$ and $S^{(N)}(t+1)u_{-1}^{(N)} = S^{(N)}(t)u_0^{(N)}$ for all $t \in \mathbb{R}^+$. Repeating this argument inductively we find an $u_{-k-1}^{(N)} \in \mathcal{A}_N$ such that $S^{(N)}(1)u_{-k-1}^{(N)} = u_{-k}^{(N)}$ and $S^{(N)}(t+1)u_{-k-1}^{(N)} = S^{(N)}(t)u_{-k}^{(N)}$ for all $t \in \mathbb{R}^+$ for each $k = 1, 2, \ldots$. In this way, defining

$$\bar{\phi}^{(N)}(t) = \begin{cases} S^{(N)}(t)u_0^{(N)}, & \text{for } t \geq 0, \\ S^{(N)}(t+k)u_{-k}, & \text{for } t \in [-k, -k+1], \ k = 1, 2, \ldots \end{cases}$$

we have an entire strong solution $\bar{\phi}^{(N)} : \mathbb{R} \to \mathcal{A}_N$ with $\bar{\phi}^{(N)}(0) = u_0^{(N)}$. We note that this entire solution need not be unique, but it has been shown in [5] that the totality of backward extensions satisfies the properties of a compact setvalued semi group in reversed time.

Now the attractor $\mathcal{A}_N \subset \mathcal{B}_H$ for each N, where the closed and bounded subset $\mathcal{B}_H$ of H, which is defined in (4.40), is independent of N. This means the entire solutions $\bar{\phi}^{(N)}$ takes values in $\mathcal{B}_H$ for all N. Therefore, by Theorem 13, there exists a subsequence $\bar{\phi}^{(N_1)}$ of the $\bar{\phi}^{(N)}$ which converges to a function $\bar{\phi} \in L^\infty(0,T;H) \cap L^2(0,T;V)$ weak-star in $L^\infty(0,T;H)$, weakly in $L^2(0,T;V)$ and strongly in $L^2(0,T;H)$ for all $T > 0$ such that $\bar{\phi}$ is a weak solution of the NSE on the interval $[0,T]$ for every $T > 0$. Moreover, by the weak-star lower semicontinuity of the norm in $L^\infty(0,T;H)$, the function $\bar{\phi}$ takes values in $\mathcal{B}_H$.

To extend this weak solution backwards in time, we start with the subsequence $\bar{\phi}^{(N_1)}$ and repeat the argument to obtain a further subsequence $\bar{\phi}^{(N_2)}$ of the entire solutions which converges to a weak solution on $[-1,T]$ for any $T > 0$, with values in $\mathcal{B}_H$, and that coincides with the previous weak solution on $[0,+\infty)$. We repeat this arguments inductively with the corresponding subsequence of the $\bar{\phi}^{N_1}$ and extend the weak solution backwards to exist on $[-k,+\infty)$ for each $k = 0, 1, 2, \ldots$. In this way we obtain an entire weak solution $\bar{\phi}$ of the NSE with values in $\mathcal{B}_H$.

We have thus proved, in particular, the following theorem.

Theorem 14 *There exists a bounded entire weak solution of the NSE (1.1). More exactly, there exists a bounded entire weak solution of the NSE (1.1) with initial value u_0 for each $u_0 \in \mathcal{U}_0$, where $\mathcal{U}_0$ is the subset in H consisting of the weak H-cluster points of sequences $u_0^{(N)} \in \mathcal{A}_N$ for $N \to \infty$.*

The set $\mathcal{U}_0$ here is obviously a non-empty subset of the closed and bounded subset $\mathcal{B}_H$ of H.

This result improves on a similar one of Kapustyan and Valero [4], who required the forcing term to be in the space $(L^4(\Omega))^3 \cap H$.

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