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Some Notes on Function Spaces and Dirichlet Forms on Self-Similar Sets

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Abstract

Grigor'yan, Hu and Lau [10] introduced sub-Gaussian heat kernels on general metric measure spaces and defined a family of function spaces to characterize the domain of associated Dirichlet forms. In this paper, we will improve their results about norm equivalence. As an application, we construct self-similar Dirichlet forms on a class of self-similar sets containing the Sierpiński gaskets and carpets. Then we prove the Poincaré inequality and give effective resistance estimates by the self-similarity. Consequently, we have a new equivalent characterization of heat kernel estimates through function spaces with strong recurrent condition.

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1 Introduction

It is well known that the Gaussian kernel $p_t(x,y)$ in $\mathbb{R}^n$ given by

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

plays an important role in the studies of heat equation $\frac{\partial u}{\partial t} = \Delta u$. To introduce a Laplacian on the Sierpiński gasket(SG) in $\mathbb{R}^2$, Barlow-Perkins [4] showed the existence of a heat kernel $p_t(x,y)$ via simple random walks such that

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x,y)}{Ct}\right)^{\frac{1}{\beta-1}}\right) \qquad (HK(\beta))$$

for $0 < t < 1$ and a constant $C > 0$ (the sign $\asymp$ means both inequality signs $\leqslant$ and $\geqslant$ are admitted instead but the constant C may be different in upper and lower bounds). Here $\alpha = \log_2 3$ and $\beta = \log_2 5$ denote the Hausdorff dimension and walk dimension of SG. Then Barlow-Bass [2] constructed diffusion processes on the Sierpiński carpet(SC) and its higher dimensional analogues, and obtained similar estimates of the associated heat kernels as $(HK(\beta))$. Note that SC is much more complicated than SG since it is an infinitely ramified fractal.

More generally, Grigor'yan, Hu and Lau [10] introduced the sub-Gaussian heat kernel $p_t(x,y)$ on metric measure space(MMS) (M,d,μ) satisfying

$$\frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right) \leqslant p_t(x,y) \leqslant \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right) \qquad (1.1)$$

for μ-almost all $x,y \in M$ and $t > 0$. Here α,β are positive indexes, and Φ_1,Φ_2 are priori non-negative monotone decreasing functions on $[0,+\infty)$. The heat kernel p_t is said to fulfill hypothesis $\mathcal{H}(\theta)$, if (1.1) holds, $\Phi_1(1) > 0$, and

$$\int_0^\infty s^\theta \Phi_2(s)\frac{ds}{s} < \infty.$$

For example, take

$$\Phi_i(s) \asymp C\exp\left(-Cs^{\frac{\beta}{\beta-1}}\right), \qquad i = 1,2,$$

then it is easily checked that $(HK(\beta))$ implies $\mathcal{H}(\theta)$ for all $\theta > 0$.

It is known that $p_t(x,y)$ determines a Dirichlet form $(\mathcal{E},\mathcal{F})$ on M, where $\mathcal{F}$ is a dense linear subspace in $L^2(M,\mu)$ and $\mathcal{E}$ is a non-negative symmetric bilinear functional defined on $\mathcal{F}$. See Fukushima [8] for the details. How to characterize $\mathcal{F}$ is a basic problem in the studies of Dirichlet forms. For this aim, Grigor'yan [9] introduced a family of function spaces as the generalization of Besov spaces in $\mathbb{R}^n$. Fix α in (1.1), for any $\sigma > 0$, define functionals on $L^2(M,\mu)$ by

$$D(f,r) := \int\int_{x,y\in M,\ d(x,y)<r} |f(x)-f(y)|^2 d\mu(y)d\mu(x),$$

$$N_{\sigma,\infty}(f) := \sup_{0<r\leqslant 1} \frac{D(f,r)}{r^{\alpha+2\sigma}},$$

and define Besov spaces with norms by

$$\Lambda^{\sigma}_{2,\infty} := \{f \in L^2 : N_{\sigma,\infty}(f) < \infty\}, \quad ||f||_{\Lambda^{\sigma}_{2,\infty}} := \left(||f||_2^2 + N_{\sigma,\infty}(f)\right)^{1/2},$$

where $||\cdot||_2$ denotes the L^2-norm.

Abbreviate $\mathcal{E}[f] = \mathcal{E}(f,f)$. Domain $\mathcal{F}$ can be characterized in the following way (see Theorem 4.2 in [10] and Theorem 5.1 in [9]).

Proposition 1.1. *Let p_t be a heat kernel on M satisfying hypothesis $\mathcal{H}(\alpha+\beta)$ for $0 < t \leqslant 1$. Then $\mathcal{F} = \Lambda^{\beta/2}_{2,\infty}$, and for all any $f \in \mathcal{F}$*

$$\mathcal{E}[f] \simeq N_{\beta/2,\infty}(f), \tag{1.2}$$

moreover,

$$\mathcal{E}[f] \simeq \limsup_{r\to 0+} \frac{D(f,r)}{r^{\alpha+\beta}}. \tag{1.3}$$

Remark 1.1. Let S and T be two function(al)s, $S \simeq T$ means that there is an independent constant $C > 0$ such that $C^{-1}S(\cdot) \leqslant T(\cdot) \leqslant CS(\cdot)$.

The conclusion of Proposition 1.1 was first obtained by Jonsson [11] for the Sierpiński gaskets, and he called $\Lambda^{\beta/2}_{2,\infty}$ the Lipschitz space. Then Pietruska-Pałuba [19] extended the result to a class of simple nested fractals.

As the first result of this paper, we will give a stronger conclusion than (1.3).

Theorem 1.1. *With the same assumptions in Proposition 1.1, for all $f \in \mathcal{F}$*

$$\mathcal{E}[f] \simeq \liminf_{r\to 0+} \frac{D(f,r)}{r^{\alpha+\beta}}. \tag{1.4}$$

Remark 1.2. Though the right-hand term above is not exactly a semi-norm, we still call (1.4) a norm equivalence.

Remark 1.3. To construct a local Dirichlet form on MMS, Kumagai and Sturm made an assumption((A2) in [15]) as follows: there is a $\delta > 0$ such that for all $u \in L^2$ and all $(u_n)_n \in L^2$ with $u_n \to u$ in L^2

$$\liminf_{r_n\to 0+} \mathcal{E}^{r_n}[u_n] \geqslant \delta \limsup_{r_n\to 0+} \mathcal{E}^{r_n}[u], \tag{1.5}$$

where $\mathcal{E}^{r_n}[u] = \int_M \int_M |u(x) - u(y)|^2 K_{r_n}(x,y)d\mu(y)d\mu(x)$ is called approximating Dirichlet form. But we don't know whether (1.5) holds for general cases. Take now $K_r(x,y) = 1_{B(x,r)}(y)/r^{\alpha+\beta}$, by norm equivalences (1.3) and (1.4), we see that the assumption (1.5) holds really.

By Proposition 1.1, Pietruska-Pałuba [20] showed the function space

$$B^{\beta/2}_{2,2} := \left\{ f \in L^2 : \int_M \int_M \frac{|f(y)-f(x)|^p}{d(x,y)^{\alpha+\beta}} d\mu(x) d\mu(y) < \infty \right\}$$

is trivial, i.e. all elements in the space are constants. As an application of (1.4), we will give a short proof of the triviality.

Let K be a self-similar set generated by the similitudes $\psi_i (1 \leqslant i \leqslant N)$ and $(\mathcal{E}^*, \mathcal{F}^*)$ be a Dirichlet form on K. Say $(\mathcal{E}^*, \mathcal{F}^*)$ is a self-similar Dirichlet form(SSDF), if there exist positive constants $r_i (1 \leqslant i \leqslant N)$ such that for all $f, g \in \mathcal{F}^*$,

$$\mathcal{E}^*(f,g) = \sum_{i=1}^{N} \frac{1}{r_i} \mathcal{E}^*(f \circ \psi_i, g \circ \psi_i).$$

Kigami [12] defined a SSDF on SG (or p.c.f. sets) as a limit of discrete analytical approximation via regular harmonic structure. Kusuoka and Zhou [16] constructed a SSDF on SC through an averaging method. Both ideas seem hard to extend to higher dimensional carpets. Then a question arises naturally: given a self-similar set, is there any SSDF on it? Assuming the existence of sub-Gaussian heat kernels, we apply norm equivalences (1.2)-(1.4) to give a positive answer of above question for a class of arcwise connected self-similar sets containing the Sierpiński gaskets and carpets. It is the second main result of this paper. We denote this SSDF still by $(\mathcal{E}^*, \mathcal{F}^*)$.

Remark 1.4. [16] defined a SSDF on SC denoted here by $(\mathcal{E}', \mathcal{F}')$. By Theorem 5.6 in [3], we know that the heat kernel p'_t and p^*_t corresponding to $\mathcal{E}'$ and $\mathcal{E}^*$ fulfill (1.1) at the same time. Then the associated Besov space $\Lambda^{\beta'/2}_{2,\infty}$ coincides with $\Lambda^{\beta^*/2}_{2,\infty}$ by using (1.2) and (1.3), which yields that $\mathcal{E}' = \mathcal{E}^*$ up to a constant coefficient.

Without requiring any sub-Gaussian heat kernel on K, we directly assume the existence of a Besov space $\Lambda^{\beta/2}_{2,\infty}$ fulfilling:

(NE) norm equivalence: for all $f \in \Lambda^{\beta/2}_{2,\infty}$,

$$N_{\beta/2,\infty}(f) \simeq \liminf_{r\to 0} \frac{D(f,r)}{r^{\alpha+\beta}},$$

(DR) domain's regularity: $\mathcal{F} \cap C(K)$ is dense in $C(K)$ with uniform norm.

By using the SSDF constructed from $\Lambda^{\beta/2}_{2,\infty}$, the third result of this paper is to get the Poincaré inequality $PI(\beta)$ and effective resistance estimates $RES(\beta)$ on K with some more geometric assumptions.

The condition $\beta > \alpha$ is called the strong recurrent condition, as a consequence, we obtain the equivalent characterization below depending on Barlow [1] and Kumagai [14]:

$$HK(\beta) \Longleftrightarrow (NE) + (DR). \tag{1.6}$$

This paper is organized as follows. In Section 2, we shall prove (1.4) and give an application. Sections 3 is devoted to the construction of a SSDF. The Poincaré inequality, resistance estimates and (1.6) will be given in Section 4.

2 Norm equivalence

About preliminaries of heat kernels, we refer to [9]. Here we only introduce some more notations. Given a heat kernel p_t, denote

$$T_t f = \int_M p_t(x,y) f(y) d\mu(y)$$

for all $f \in L^2(M,\mu)$. For any $t > 0$, define a quadratic form $\mathcal{E}_t$ on L^2 by

$$\mathcal{E}_t[f] := \left(\frac{f - T_t f}{t}, f \right),$$

where $(\cdot,\cdot)$ is the inner product in L^2. Fix f, then $\mathcal{E}_t[f]$ increases when t tends to $0+$, which yields a Dirichlet form $(\mathcal{E},\mathcal{F})$ as follows:

$$\mathcal{E}[f] := \lim_{t\to 0+} \mathcal{E}_t[f], \qquad \mathcal{F} := \{f \in L^2 : \mathcal{E}[f] < \infty\}.$$

We see that $\mathcal{E}(f,g)$ is well defined by the polarization identity

$$\mathcal{E}(f,g) = \frac{1}{2}(\mathcal{E}[f+g] - \mathcal{E}[f-g]).$$

Abbreviate

$$\overline{I}_0(f) = \limsup_{r\to 0+} \frac{D(f,r)}{r^{\alpha+\beta}}, \qquad \underline{I}_0(f) = \liminf_{r\to 0+} \frac{D(f,r)}{r^{\alpha+\beta}}.$$

Proof of Theorem 1.1. Given $f \in \mathcal{F}$, $r = 1$ and $B(x,r) = \{y : d(x,y) < r\}$. For any $t > 0$, write

$$\mathcal{E}_t[f] = \frac{1}{2t} \int_M \int_M (f(x) - f(y))^2 p_t(x,y) d\mu(y) d\mu(x) = A(t) + B(t),$$

where

$$A(t) = \frac{1}{2t} \int_M \int_{M\setminus B(x,r)} (f(x) - f(y))^2 p_t(x,y) d\mu(y) d\mu(x),$$
$$B(t) = \frac{1}{2t} \int_M \int_{B(x,r)} (f(x) - f(y))^2 p_t(x,y) d\mu(y) d\mu(x).$$

Repeating the second part of the proof for Theorem 5.1 in [9], we obtain

$$\lim_{t\to 0+} A(t) = 0.$$

The quantity $B(t)$ is estimated by using (1.1) and $(\mathcal{H}(\alpha+\beta))$. Let $t = 2^{-\beta j}$, $r_k = 2^{-k}$, and $a_l = 2^{-l(\alpha+\beta)}\Phi_2(2^{-l})$ for any $j \in \mathbb{N}, k \in \mathbb{N}$ and $l \in \mathbb{Z}$, then

$$\begin{aligned} B(t) &= \frac{1}{2t}\sum_{k=1}^{\infty}\int_M\int_{B(x,r_{k-1})\setminus B(x,r_k)}(f(x)-f(y))^2 p_t(x,y)d\mu(y)d\mu(x) \\ &\leqslant \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_k}{t^{1/\beta}}\right)\int_M\int_{B(x,r_{k-1})}(f(x)-f(y))^2 d\mu(y)d\mu(x) \\ &\leqslant C\sum_{k=1}^{\infty}\left(\frac{r_k}{t^{1/\beta}}\right)^{\alpha+\beta}\Phi_2\left(\frac{r_k}{t^{1/\beta}}\right)\frac{D(f,r_{k-1})}{r_{k-1}^{\alpha+\beta}} \\ &= C\sum_{k=1}^{\infty} a_{k-j}\,\frac{D(f,r_{k-1})}{r_{k-1}^{\alpha+\beta}}\ . \end{aligned}$$

Since there is a constant $c > 0$ such that $\mathcal{E}[f] \geqslant cN_{\beta/2,\infty}(f)$ by Proposition 1.1, we obtain for big j (depending on f)

$$cN_{\beta/2,\infty}(f) \leqslant 2B(t) \leqslant 2C\sum_{k=1}^{\infty} a_{k-j}\frac{D(f,r_{k-1})}{r_{k-1}^{\alpha+\beta}}\ . \tag{2.1}$$

Recall that Φ_2 is decreasing and $\int_0^{\infty} s^{\alpha+\beta}\Phi_2(s)\frac{ds}{s} < \infty$, then there exist two positive constants ϱ and a such that

$$\sum_{|l|\geqslant\varrho} a_l \leqslant \frac{c}{4C}, \quad \text{and} \quad a_l \leqslant a, \quad \text{for all } l \in \mathbb{Z},$$

which implies

$$\begin{aligned} &\sum_{k=1}^{\infty} a_{k-j}\frac{D(f,r_{k-1})}{r_{k-1}^{\alpha+\beta}} \\ \leqslant\ &\sum_{|k-j|<\varrho} a_{k-j}\frac{D(f,r_{k-1})}{r_{k-1}^{\alpha+\beta}} + \frac{c}{4C}N_{\beta/2,\infty}(f) \\ \leqslant\ &2\varrho a\cdot\max_{|k-j|<\varrho}\frac{D(f,r_{k-1})}{r_{k-1}^{\alpha+\beta}} + \frac{c}{4C}N_{\beta/2,\infty}(f). \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we have (for big j)

$$N_{\beta/2,\infty}(f) \leqslant C\max_{|l|<\varrho}\frac{D(f,2^{-(l+j-1)})}{2^{-(l+j-1)(\alpha+\beta)}} \leqslant C\frac{D(f,2^{-(j-\varrho)})}{2^{-(j-\varrho)(\alpha+\beta)}}.$$

By letting $j \to \infty$ we get

$$N_{\beta/2,\infty}(f) \leqslant C\underline{I}_0(f),$$

which follows that $\mathcal{E}[f] \simeq \underline{I}_0(f)$. □

Another class of Besov spaces $B^\sigma_{p,p}$ in [9] is defined by

$$\left\{ f \in L^p : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)-f(x)|^p}{|y-x|^{n+p\sigma}} dx dy < \infty \right\}.$$

Replacing $\mathbb{R}^n$ by M, dx by $d\mu(x)$, n by α, and σ by $\beta/2$ respectively, we define $B^{\beta/2}_{2,2}$ formally as

$$\left\{ f \in L^2 : \int_M \int_M \frac{|f(y)-f(x)|^2}{d(x,y)^{\alpha+\beta}} d\mu(x) d\mu(y) < \infty \right\}.$$

With the same assumption in Theorem 1.1, we have

Corollary 2.1. *$B^{\beta/2}_{2,2}$ is trivial.*

Proof. Note that $B^{\beta/2}_{2,2} \subset \Lambda^{\beta/2}_{2,\infty}$ by the definitions. Given $r \in (0,1)$, which will be chosen later, let

$$B(f) := \int_M \int_M \frac{|f(y)-f(x)|^2}{d(x,y)^{\alpha+\beta}} d\mu(x) d\mu(y).$$

Then we have

$$\begin{aligned} B(f) &\geqslant \sum_{k=0}^{\infty} \frac{1}{r^{k(\alpha+\beta)}} \int_M \int_{B(x,r^k)\setminus B(x,r^{k+1})} |f(y)-f(x)|^2 d\mu(x) d\mu(y) \\ &= \sum_{k=0}^{\infty} \left(\frac{D(f,r^k)}{r^{k(\alpha+\beta)}} - r^{\alpha+\beta} \frac{D(f,r^{k+1})}{r^{(k+1)(\alpha+\beta)}} \right). \end{aligned} \tag{2.3}$$

By (1.4) in Theorem 1.1, for large k we have

$$\begin{aligned} & \frac{D(f,r^k)}{r^{k(\alpha+\beta)}} - r^{\alpha+\beta} \frac{D(f,r^{k+1})}{r^{(k+1)(\alpha+\beta)}} \\ \geqslant\ & \frac{1}{2} \underline{I}_0(f) - 2r^{\alpha+\beta} \overline{I}_0(f) \\ \geqslant\ & \frac{1}{2C} \overline{I}_0(f) - 2r^{\alpha+\beta} \overline{I}_0(f). \end{aligned}$$

If we take r so small that $(2C)^{-1} > 2r^{\alpha+\beta}$, then the series in (2.3) diverges. That is, $B^{\beta/2}_{2,2}$ is trivial. □

3 Self-similar Dirichlet form

We give first a quick review of self-similar sets (see [7] for example), then we construct a SSDF on a self-similar set admitting a sub-Gaussian heat kernel. Some geometric assumptions are required, which hold for the Sierpiński gaskets and carpets, at least.

3.1 Notations and definitions

Let $\Omega = \{1, \cdots, N\}$ be an alphabet of finite letters. For all $m \in \mathbb{N}$, define

$$\Omega_m = \{w_1 w_2 \cdots w_m : w_k \in \Omega, 1 \leqslant k \leqslant m\}, \qquad \Omega^* = \bigcup_{m \in \mathbb{N}} \Omega_m,$$

and the shift space

$$\Omega^\omega = \{w_1 w_2 \cdots : w_i \in \Omega, i \in \mathbb{N}\}.$$

For convenience we add $\emptyset$ into Ω^* as an empty word. For $w = w_1 w_2 \cdots w_m$ and $v = v_1 v_2 \cdots v_n$, denote $wv = w_1 w_2 \cdots w_m v_1 v_2 \cdots v_n$ and $w^* = w_1 w_2 \cdots w_{m-1}$. If $u = wv$, denote $w < u$. u and v are called incomparable if there is no w such that $u = vw$ or $v = uw$. Given $0 < r < 1$ and $w, v \in \Omega^\omega$ with $w \neq v$, define $\delta_r(w, v) = r^{s(w,v)}$, where $s(w, v) = \min\{m : w_m \neq v_m\} - 1$. Also define $\delta_r(w, v) = 0$ if $w = v$. Then δ_r is a metric on Ω^ω, and $(\Omega^\omega, \delta_r)$ becomes a compact metric space. Define a map $\sigma_i : \Omega^\omega \to \Omega^\omega$ by $\sigma_i(w_1 w_2 \cdots) = i w_1 w_2 \cdots$ for each $i \in \Omega$, and the shift map $\sigma : \Omega^\omega \to \Omega^\omega$ by $\sigma(w_1 w_2 w_3 \cdots) = w_2 w_3 \cdots$.

We consider $\mathbb{R}^n$ with the Euclidean metric $|\cdot|$. Let $K \subset \mathbb{R}^n$ be the self-similar set generated by an iterated function scheme $\{\psi_i\}_{i \in \Omega}$ such that

$$K = \bigcup_{i \in \Omega} \psi_i(K),$$

where each ψ_i is a similitude satisfying

$$|\psi_i(x) - \psi_i(y)| = c_i |x - y|, \qquad \forall x, y \in \mathbb{R}^n.$$

There exists a unique continuous map $\pi : \Omega^\omega \to K$ such that for all $i \in \Omega$,

$$\psi_i \circ \pi = \pi \circ \sigma_i \, .$$

Let $(p_1, p_2, \cdots, p_N)$ be a weight with $\sum\limits_{i \in \Omega} p_i = 1$ and $0 < p_i < 1$ for all $i \in \Omega$. Then there exists a unique probability measure μ on K such that for any Borel set $B \subset K$,

$$\mu(B) = \sum_{i \in \Omega} p_i \, \mu(\psi_i^{-1}(B)),$$

which is called a self-similar measure. Moreover, if K fulfills the open set condition(OSC), i.e. there is an open set O such that

$$\psi_i(O) \subset O, \quad \text{and} \quad \psi_i(O) \cap \psi_j(O) = \emptyset \quad \text{for } i \neq j \in \Omega,$$

then the Hausdorff dimension d_f of K satisfies $\sum_{i \in \Omega} c_i^{d_f} = 1$, and the self-similar measure μ associated to the weight $(p_i = c_i^{d_f})$ is the d_f-dimensional Hausdorff measure restricted to K normalized so that $\mu(K) = 1$.

Let d be the Euclidean metric restricted on K, then (K, d, μ) is a MMS. In the following, a ball $B(x, r)$ for any $x \in K$ and $r > 0$ means the subset $\{y \in K : d(x, y) < r\}$.

Given $F \subset K$, define $U(F,r) = \bigcup_{x\in F} B(x,r)$, and $N_r(F)$ to be the smallest number of balls centered in F and of diameter r which can cover $U(F, r/2)$. We abbreviate $\{B^F_{k,r}\}$ to be a collection of these $N_r(F)$ balls with centers $\{p_{k,r}\}$ respectively.

For any $w \in \Omega_m$, denote

$$\psi_w = \psi_{w_1} \circ \psi_{w_2} \cdots \circ \psi_{w_m}, \quad K_w = \psi_w(K), \quad c_w = c_{w_1} c_{w_2} \cdots c_{w_m}.$$

Say K_w is of size l if $c_w \leqslant l < c_{w^*}$ holds. By OSC, there exists a uniform bound C_K for the total number of K_w of size r that intersects with $B(x,r)$ for any $x \in K$ and $r > 0$. Consequently there is another constant $C_\mu > 0$ such that

$$C_\mu^{-1} r^\alpha \leqslant \mu(B(x,r)) \leqslant C_\mu r^\alpha.$$

Write σ^k to be the k times compositions of σ, define

$$D = \bigcup_{i\neq j} (K_i \cap K_j), \quad \mathcal{D} = \pi^{-1}(D), \quad \mathcal{P} = \bigcup_{k>0} \sigma^k(\mathcal{D}), \quad \partial K = \pi(\mathcal{P}).$$

∂K is called the boundary of K. If $\mathcal{P}$ is finite, then K is called a post critical finite(p.c.f.) set. Let $\partial K_w = \psi_w(\partial K)$ for any $w \in \Omega_m$.

Kigami [12] showed that (see Proposition 1.3.5 and 1.3.11)

Proposition 3.1. *$\partial K \subset \bigcup_{i\in\Omega} \partial K_i$, and $K_w \cap K_v = \psi_w(\partial K) \cap \psi_v(\partial K)$ for any incomparable words w and v.*

Let $F = \psi_i^{-1}(K_i \cap K_j)$ for any $i \neq j$. If F is not empty, we say F is a face. Let $\mathfrak{F}$ be a collection of all the faces. Let F' be the closure of $\partial K \setminus \bigcup_{F\in\mathfrak{F}} F$. If F' is not empty, we add it into $\mathfrak{F}$. For any $A, B \in \mathfrak{F}$, let $\mathfrak{F}' = \{A \cup B\} \bigcup (\mathfrak{F} \setminus \{A, B\})$. We say $\mathfrak{F}$ can be reduced to $\mathfrak{F}'$.

3.2 Construction

Let K be a self-similar set in $\mathbb{R}^n$ with OSC, α the Hausdorff dimension, d the inherited Euclidean metric and μ the self-similar measure with weight $p_i = c_i^\alpha$ for $i \in \Omega$. Let p_t be a heat kernel on (K, d, μ) satisfying hypothesis $\mathcal{H}(\alpha+\beta)$ for $0 < t \leqslant 1$, and write $(\mathcal{E}, \mathcal{F})$ to be the Dirichlet form associated to p_t with $\mathcal{F} = \Lambda^{\beta/2}_{2,\infty}$.

We impose two assumptions on the boundary ∂K:

$(B1)$ $\{\psi_i(F) \subsetneq D : i \in \Omega, F \in \mathfrak{F}\}$ can be reduced to $\mathfrak{F}$ by finite steps.

$(B2)$ If $\beta \leqslant \alpha$, then the Hausdorff dimension δ of ∂K fulfills $\delta < \alpha < \delta + \beta$ and

$$\mu(U(F \cap B(p,r), l)) \simeq r^\delta l^{\alpha-\delta}$$

for any $F \in \mathfrak{F}$, $p \in F$, $r > 0$ and $l \leqslant r$.

For example, the Euclidean cubes, Sierpiński gaskets and carpets satisfy these conditions.

Theorem 3.1. *With the above assumptions, there exists a SSDF $(\mathcal{E}^*, \mathcal{F})$ on K such that $\mathcal{E}^* \simeq \mathcal{E}$.*

The idea for the proof is to apply the Schauder's fixed point theorem. We first give a lemma which will be verified later.

Lemma 3.1. *There exists a functional $\overline{I}_0^* \geqslant \overline{I}_0$ on $\mathcal{F}$ such that for any $f \in \mathcal{F}$:*

1. *There is a constant $C_S > 0$ such that*

$$C_S^{-1}\, \overline{I}_0^*(f) \leqslant \mathcal{E}[f] \leqslant C_S\, \underline{I}_0(f).$$

2. *For $r_i = c_i^{\beta-\alpha}$ with $i \in \Omega$,*

$$\overline{I}_0^*(f) \leqslant \sum_{i\in\Omega} \frac{1}{r_i} \overline{I}_0^*(f \circ \psi_i).$$

Remark 3.1. $\overline{I}_0$ doesn't fulfill the second item, so we need some adjustments.

Now we recall some facts in functional analysis which can be found in [5] and [6]. Let $\mathbb{X}$ be a vector space composed of all bounded Dirichlet forms defined on $\mathcal{F}$, and endowed with the weak-star topology. More precisely, since $\mathcal{F}$ is a Banach space, the tensor product $\mathcal{F} \otimes \mathcal{F}$ is also a normed space with the norm defined as follows:

$$||h||_{\mathcal{F}\otimes\mathcal{F}} = \inf_{h=\sum\limits_{\text{finite } i} f_i \otimes g_i} \sum_i ||f_i||_{\mathcal{F}} ||g_i||_{\mathcal{F}}, \quad h \in \mathcal{F}\otimes\mathcal{F}.$$

Thus $\mathbb{X}$ is a subspace of the locally convex space $(\mathcal{F}\otimes\mathcal{F})^*$ endowed with the weak-star topology.

Define a subset $\mathbb{D} \subset \mathbb{X}$ composed of those Dirichlet forms W with

$$C_S^{-1}\, \overline{I}_0^*(f) \leqslant W[f] \leqslant C_S\, \underline{I}_0(f) \tag{3.1}$$

for all $f \in \mathcal{F}$, where $W[f] = W(f, f) = W(f \otimes f)$. Since $\mathcal{E} \in \mathbb{D}$, $\mathbb{D}$ is not empty.

Lemma 3.2. *$\mathbb{D}$ is a compact convex subset of $\mathbb{X}$.*

Proof. By the Banach-Alaoglu's theorem, it is sufficient to show that $\mathbb{D}$ is a bounded closed convex subset.

Given $W \in \mathbb{D}$, Let $h = \sum\limits_{i=1}^m f_i \otimes g_i$ in $\mathcal{F}\otimes\mathcal{F}$. Then,

$$W(h) = \sum_{i=1}^m W(f_i, g_i) \leqslant \sum_{i=1}^m \sqrt{W[f_i]}\sqrt{W[g_i]} \leqslant C_S \sum_{i=1}^m ||f_i||_{\mathcal{F}} ||g_i||_{\mathcal{F}},$$

which implies

$$W(h) \leqslant C_S ||h||_{\mathcal{F}\otimes\mathcal{F}}.$$

Namely, $\mathbb{D}$ is uniformly bounded.

Suppose that the sequence $\{W_n\} \subset \mathbb{D}$ weakly-star converges to W. That is, $W_n(f,g)$ converges to $W(f,g)$ for any given $f,g \in \mathcal{F}$. Thus W is also a Dirichlet form and satisfies (3.1), which means $W \in \mathbb{D}$. Namely, $\mathbb{D}$ is closed with respect to the weak-star topology.

Let $W, W' \in \mathbb{D}$, and $\lambda \in (0,1)$. Then, $\lambda W + (1-\lambda)W'$ is also a Dirichlet form satisfying (3.1). Namely, $\mathbb{D}$ is convex. □

Given $f, g \in \mathcal{F}$, for all $W \in \mathbb{X}$, define $\Psi : \mathbb{X} \to \mathbb{X}$ by

$$(\Psi(W))(f,g) = \sum_{i\in\Omega} \frac{1}{r_i} W(f\circ\psi_i, g\circ\psi_i).$$

Lemma 3.3. *Ψ is continuous on $\mathbb{X}$, and $\Psi(\mathbb{D}) \subset \mathbb{D}$.*

Proof. Ψ is continuous since it keeps the weakly-star convergence. More precisely, if W_n weakly-star converges to W, then $\Psi(W_n)$ weakly-star converges to $\Psi(W)$ by the definition of Ψ.

Given $W \in \mathbb{D}$ and $f \in \mathcal{F}$, we have by the integral transformation

$$\begin{aligned} \Psi(W)[f] &= \sum_{i\in\Omega} \frac{1}{r_i} W[f\circ\psi_i] \\ &\leqslant C_S \sum_{i\in\Omega} \frac{1}{r_i} \underline{I}_0(f\circ\psi_i) \\ &= C_S \sum_{i\in\Omega} \frac{1}{r_i} \liminf_{r\to 0} \frac{1}{r^{\alpha+\beta}} \int_K \int_{B(x,r)} |f\circ\psi_i(x) - f\circ\psi_i(y)|^2 d\mu(y) d\mu(x) \\ &= C_S \sum_{i\in\Omega} \liminf_{l\to 0} \frac{1}{l^{\alpha+\beta}} \int_{K_i} \int_{B(x,l)\bigcap K_i} |f(x) - f(y)|^2 d\mu(y) d\mu(x) \\ &\leqslant C_S\, \underline{I}_0(f). \end{aligned}$$

On the other hand, by Lemma 3.1 we have

$$\Psi(W)[f] = \sum_{i\in\Omega} \frac{1}{r_i} W[f\circ\psi_i] \geqslant C_S^{-1} \sum_{i\in\Omega} \frac{1}{r_i} \overline{I}_0^*(f\circ\psi_i) \geqslant C_S^{-1}\, \overline{I}_0^*(f),$$

which yields $\Psi(W) \in \mathbb{D}$. □

Proof of Theorem 3.1. By above discussions, all conditions of Schauder's fixed point theorem are fulfilled, so there exists a Dirichlet form $\mathcal{E}^* \in \mathbb{D}$ such that

$$\Psi(\mathcal{E}^*) = \mathcal{E}^*,$$

which implies that $\mathcal{E}^*$ is self-similar and $\mathcal{E}^* \simeq \mathcal{E}$. □

Now we go back to prove Lemma 3.1. Firstly we are going to define $\overline{I}_0^*$. Recall notations in Subsection 3.1 and assumptions in Theorem 3.1, let $E_m = U(F \cap B(p,r), 2^{-m}r)$ for given $F \in \mathfrak{F}$, $p \in F$, $r > 0$ and m a non-negative integer. Let f_A be the mean integral for any $f \in \mathcal{F}$ on A with positive μ-mass. Let $r_m = 2^{-m}r$, $B_{m,i} = B_{i,r_m}^{E_m}$ for $1 \leqslant i \leqslant N_{r_m}(E_m)$, and

$$\tilde{f}_{E_m} := \frac{1}{N_{r_m}(E_m)} \sum_{i=1}^{N_{r_m}(E_m)} f_{B_{m,i}}.$$

Lemma 3.4. *Suppose $\beta \leqslant \alpha$, then $\{\tilde{f}_{E_m}\}$ is a Cauchy sequence.*

Proof. For $A, B \subset K$, we have

$$f_A = \frac{1}{\mu(A)\mu(B)} \int_A \int_B f(\xi) d\mu(\eta) d\mu(\xi), \ \ f_B = \frac{1}{\mu(A)\mu(B)} \int_A \int_B f(\eta) d\mu(\eta) d\mu(\xi),$$

which yield that

$$|f_A - f_B| \leqslant \frac{1}{\mu(A)\mu(B)} \int_A \int_B |f(\xi) - f(\eta)| d\mu(\eta) d\mu(\xi). \tag{3.2}$$

Note that $\mu(B_{m,i}) \simeq r_m^\alpha$, and the number of $B_{m,j}$ intersecting with $B_{m,i}$ is bounded by a constant $P(n)$ from the Besicovitch covering lemma (see [17]). Thus $N_{r_m}(E_m) \simeq 2^{m\delta}$ by $(B2)$. Using (3.2) and the Hölder inequality, we obtain

$$\begin{aligned}
& |\tilde{f}_{E_m} - \tilde{f}_{E_{m+1}}|^2 \\
\leqslant \ & \left(\frac{C}{N_{r_m}(E_m)} \sum_{i=1}^{N_{r_m}(E_m)} \sum_{B_{m+1,j} \bigcap B_{m,i} \neq \emptyset} |f_{B_{m+1,j}} - f_{B_{m,i}}| \right)^2 \\
\leqslant \ & \left(\frac{C}{N_{r_m}(E_m)} \sum_{i=1}^{N_{r_m}(E_m)} \frac{1}{r_m^\alpha r_{m+1}^\alpha} \int_{B_{m,i}} \int_{B(\xi, r_m)} |f(\xi) - f(\eta)| d\mu(\eta) d\mu(\xi) \right)^2 \\
\leqslant \ & \left(\frac{C}{2^{m\delta} r_m^{2\alpha}} \int_{E_m} \int_{B(\xi, r_m)} |f(\xi) - f(\eta)| d\mu(\eta) d\mu(\xi) \right)^2 \\
\leqslant \ & \frac{C\mu(E_m) r_m^\alpha}{(2^{m\delta} r_m^{2\alpha})^2} \int_{E_m} \int_{B(\xi, r_m)} |f(\xi) - f(\eta)|^2 d\mu(\eta) d\mu(\xi) \\
\leqslant \ & \frac{C r^{\beta - \alpha}}{2^{m(\beta + \delta - \alpha)}} \left(\frac{1}{r_m^{\alpha + \beta}} \int_{E_m} \int_{B(\xi, r_m)} |f(\xi) - f(\eta)|^2 d\mu(\eta) d\mu(\xi) \right) \qquad (3.3) \\
\leqslant \ & \frac{C r^{\beta - \alpha} N_{\beta/2, \infty}(f)}{2^{m(\beta + \delta - \alpha)}}.
\end{aligned}$$

Due to $\beta + \delta > \alpha$, we get the convergence of $\{\tilde{f}_{E_m}\}$ and denote the limit by $\tilde{f}(p,r)$. □

Remark 3.2. The proof implies $\tilde{f}(p,r)$ is independent of the choice of $\{B_{m,i}\}$ covering E_m for all m.

By the same argument we extend above result to any $q \in K_i \cap K_j$ with $i, j \in \Omega$ and $i \neq j$. Let $E'_m = U(K_i \cap K_j \cap B(q,r), 2^{-m}r)$ and define $\tilde{f}_{E'_m}$ similarly.

Corollary 3.1. $\underbrace{\{\tilde{f}_{E'_m}\}}$ *is still a Cauchy sequence, whose limit, denoted also by* $\tilde{f}(q,r)$*, equals to* $\overbrace{f \circ \psi_i}(\psi_i^{-1}(q), c_i^{-1}r)$.

By this corollary, for any $p \in \partial K_i$ with $i \in \Omega$, set

$$f(p,r) = \begin{cases} f(p), & \text{if } \beta > \alpha; \\ \tilde{f}(p,r), & \text{if } \beta \leqslant \alpha. \end{cases} \tag{3.4}$$

It is easily to see that by Proposition 3.1, the definition includes points in ∂K .

Define functionals

$$A^F(f,r) = \sup_{\{B^F_{k,r}\}} \sum_{k=1}^{N_r(F)} \int_{B^F_{k,r}} |f(x) - f(p_{k,r}, r)|^2 d\mu(x), \qquad \text{for } F \in \mathfrak{F},$$

$$A(f,r) = C_0 \sum_{F \in \mathfrak{F}} A^F(f, 2r), \qquad \text{for } C_0 = 6 \cdot 2^{\alpha} \cdot C_{\mu},$$

$$\overline{I}_0^*(f) = \limsup_{r \to 0} \left(\frac{D(f,r)}{r^{\alpha+\beta}} + \frac{A(f,r)}{r^{\beta}} \right).$$

Proof of the first claim in Lemma 3.1. Given $F \in \mathfrak{F}$, let $B_k = B^F_{k,r}$, $p_k = p_{k,r}$ for $1 \leqslant k \leqslant N_r(F)$, and $r_m = 2^{-m}r$ for any integer $m \geqslant 0$. We divide the proof into two parts.

Part1: $\beta > \alpha$. For any $x \in K$, using (3.2) and the Hölder inequality we have

$$\begin{aligned} & |f_{B(x,r)} - f_{B(x,r/2)}|^2 \\ \leqslant\ & \frac{1}{\mu(B(x,r))\mu(B(x,r/2))} \int_{B(x,r)} \int_{B(x,r/2)} |f(\xi) - f(\eta)|^2 d\mu(\eta) d\mu(\xi) \\ \leqslant\ & C_{\mu}^2 r^{-2\alpha} \int_{B(x,r)} \int_{B(\xi,2r)} |f(\xi) - f(\eta)|^2 d\mu(\eta) d\mu(\xi) \\ \leqslant\ & C r^{\beta-\alpha} \left(\frac{1}{(2r)^{\alpha+\beta}} \int_{B(x,r)} \int_{B(\xi,2r)} |f(\xi) - f(\eta)|^2 d\mu(\eta) d\mu(\xi) \right). \end{aligned} \tag{3.5}$$

By the Lebesgue's density theorem, for μ-a.e. x, we have

$$|f_{B(x,r)} - f(x)| \leqslant \sum_{m \geqslant 0} |f_{B(x,r_m)} - f_{B(x,r_{m+1})}|. \tag{3.6}$$

Recalling the definition of $A^F(f,r)$, by Proposition 1.1 we obtain

$$\begin{aligned}
&\sum_{k=1}^{N_r(F)} \int_{B_k} |f(x)-f(p_k,r)|^2 d\mu(x) \\
\leqslant\ & \sum_{k=1}^{N_r(F)} \int_{B_k} 2|f(x)-f_{B_k}|^2 + 2|f_{B_k}-f(p_k,r)|^2 d\mu(x) \\
\leqslant\ & \sum_{k=1}^{N_r(F)} \frac{2C_\mu}{r^\alpha} \int_{B_k}\int_{B_k} |f(y)-f(x)|^2 d\mu(y)d\mu(x) + 2C_\mu r^\alpha |f_{B_k}-f(p_k,r)|^2 \\
\leqslant\ & 2C_\mu P(n)\frac{D(f,r)}{r^\alpha} + 2C_\mu r^\alpha \sum_{k=1}^{N_r(F)} |f_{B_k}-f(p_k,r)|^2, \qquad (3.7)
\end{aligned}$$

where $P(n)$ is the same constant in Lemma 3.4.

Note that $f(p_k,r)=f(p_k)$. Applying (3.6) and the following inequality

$$\left(\sum_k \left(\sum_m a_{m,k}\right)^2\right)^{1/2} \leqslant \sum_m \left(\sum_k a_{m,k}^2\right)^{1/2},$$

we see that the last summation of (3.7) is controlled by

$$2C_\mu r^\alpha \left(\sum_{m\in\mathbb{N}} \left(\sum_{k=1}^{N_r(F)} |f_{r_{m-1}}(p_k)-f_{r_m}(p_k)|^2\right)^{1/2}\right)^2,$$

which, using (3.5), is not greater than

$$2C_\mu r^\alpha \left(\sum_{m\in\mathbb{N}} \left(CP(n) r_{m-1}^{\beta-\alpha} N_{\beta/2,\infty}(f)\right)^{1/2}\right)^2.$$

Combining above estimates we have

$$\frac{A^F(f,r)}{r^\beta} \leqslant CN_{\beta/2,\infty}(f),$$

which yields $\overline{I}_0^*(f) \leqslant CN_{\beta/2,\infty}(f)$.

Part 2: $\beta\leqslant\alpha$. Similarly we have to estimate the summation in (3.7). For each p_k, change the notation E_m to $E_{k,m}$; thus $E_{k,0}=B_k$ and $\tilde{f}_{E_{k,0}}=f_{B_k}$. By Lemma 3.4 we obtain

$$|f_{B_k}-f(p_k,r)| \leqslant \sum_{m\geqslant 0} |\tilde{f}_{E_{k,m}}-\tilde{f}_{E_{k,m+1}}|. \qquad (3.8)$$

Then replacing (3.5) by (3.3) and (3.6) by (3.8), we get the same bound. □

Proof of the second claim in Lemma 3.1. Let $l = c_i r$, define two functionals by

$$\begin{aligned} & I_i(f,l) := \frac{1}{r_i} \cdot \frac{D(f\circ\psi_i, r)}{r^{\beta+\alpha}} \\ = \quad & \frac{1}{l^{\beta+\alpha}} \int_{K_i} \int_{B(x,l)\bigcap K_i} |f(x)-f(y)|^2 d\mu(y) d\mu(x) \\ \geqslant \quad & \frac{1}{l^{\beta+\alpha}} \int_{K_i \setminus U(\partial K_i, l)} \int_{B(x,l)} |f(x)-f(y)|^2 d\mu(y) d\mu(x), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} & \frac{A_i(f,l)}{l^\beta} := \frac{1}{r_i} \cdot \frac{A(f\circ\psi_i, r)}{r^\beta} \\ = \quad & \frac{C_0}{l^\beta} \sum_{F\in\mathfrak{F}} \sup_{\{B^{F_i}_{k,2l}\}} \sum_{k=1}^{N_{2l}(F_i)} \int_{B^{F_i}_{k,2l}\bigcap K_i} |f(x) - f(p_{k,2l}, 2l)|^2 d\mu(x), \end{aligned} \tag{3.10}$$

where $F_i = \psi_i(F)$ and $C_0 = 6\cdot 2^\alpha \cdot C_\mu$.

Let $B = B(q, 2l)$ for any $q \in K_i \cap K_j$ $(i \neq j)$ if no confusion occurs. Then

$$\begin{aligned} & 2C_\mu \left\{ \int_{B\bigcap K_i} + \int_{B\bigcap K_j} \right\} |f(x) - f(q,2l)|^2 d\mu(x) \\ \geqslant \quad & \frac{1}{(2l)^\alpha} \int_{B\bigcap K_i} \int_{B\bigcap K_j} |f(x)-f(y)|^2 d\mu(y) d\mu(x), \end{aligned} \tag{3.11}$$

and similarly

$$\begin{aligned} & 4C_\mu \int_{B\bigcap K_i} |f(x) - f(q,2l)|^2 d\mu(x) \\ \geqslant \quad & \frac{1}{(2l)^\alpha} \int_{B\bigcap K_i} \int_{B\bigcap K_i} |f(x)-f(y)|^2 d\mu(y) d\mu(x). \end{aligned} \tag{3.12}$$

Using (3.10)-(3.12) and $(B1)$, we have

$$\begin{aligned} & \sum_{i\in\Omega} \frac{A_i(f,l)}{l^\beta} \\ \geqslant \quad & \frac{C_0}{l^\beta} \sum_{i\in\Omega} \sum_{F\in\mathfrak{F}} \sup_{\{B^{F_i}_{k,2l}\}} \sum_{k=1}^{N_{2l}(F_i)} \int_{B^{F_i}_{k,2l}\bigcap K_i} |f(x) - f(p_{k,2l}, 2l)|^2 d\mu(x) \\ \geqslant \quad & \frac{1}{l^{\alpha+\beta}} \int_{U(E,l)} \int_{B(x,l)} |f(x)-f(y)|^2 d\mu(y) d\mu(x) + \frac{A(f,l)}{l^\beta}. \end{aligned} \tag{3.13}$$

Combining (3.9) and (3.13), we finally obtain

$$\begin{aligned}
&\sum_{i\in\Omega}\frac{1}{r_i}\overline{I}_0^*(f\circ\psi_i)\\
\geqslant\ &\limsup_{l\to 0}\sum_{i\in\Omega}\left(I_i(f,l)+\frac{A_i(f,l)}{l^\beta}\right)\\
\geqslant\ &\limsup_{l\to 0}\left(\frac{D(f,l)}{l^{\alpha+\beta}}+\frac{A(f,l)}{l^\beta}\right)=\overline{I}_0^*(f).
\end{aligned}$$

The proof of Lemma 3.1 is completed. □

4 Resistence estimates

In this section we show first the Poincaré inequality, and give resistance estimates by the self-similarity of Dirichlet form. Then for the strong recurrent case $\beta > \alpha$, we get a new equivalent characterization of heat kernel estimates.

Let K be an arcwise connected self-similar set in $\mathbb{R}^n$ with OSC and $(B1)$, $(B2)$. Recall notations α, d and μ in the beginning of Subsection 3.2. Let $\mathcal{F} = \Lambda_{2,\infty}^{\beta/2}$ be a nontrivial Besov space on (K, d, μ) satisfying (NE) and (DR), which have been defined in Section 1. Note that we don't require the priori existence of any sub-Gaussian heat kernel here.

According to Theorem 2.2 in [15], there is a strongly local regular Dirichlet form $\mathcal{E}^{(c)}$ on $\mathcal{F}$ such that $\mathcal{E}^{(c)}[f] \simeq N_{\beta/2,\infty}(f)$. Repeating the construction in Subsection 3.2, we obtain also a SSDF $(\mathcal{E}, \mathcal{F})$ such that $\mathcal{E}[f] \simeq N_{\beta/2,\infty}(f)$ and for $r_i = c_i^{\beta-\alpha}$,

$$\mathcal{E}[f] = \sum_{i\in\Omega}\frac{1}{r_i}\,\mathcal{E}[f\circ\psi_i].$$

Let $\Gamma(f, f)$ be the energy measure on K such that

$$\int_K \tilde{g}\,d\Gamma(f,f) = 2\,\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \qquad g \in \mathcal{F}_b, \tag{4.1}$$

where $\tilde{g}$ is the quasi-continuous modification of g and $\mathcal{F}_b$ is the set of functions in $\mathcal{F}$ that are essentially bounded. For details we refer to [8].

Let K_w be of size r. We make two more assumptions on local geometric properties of K as follows:

$(G1)$ if K_v is also of size r intersecting with K_w, then $\psi_w^{-1}(K_w \cap K_v)$ belongs to $\mathfrak{F}$,

$(G2)$ there is an independent constant $c_K > 0$ such that $U(K_w, c_K r)$ is always contained in the union of all K_u of size r intersecting with K_w.

Remark 4.1. $(G2)$ means for any $x \in K_w$ and $y \in B(x, cr)$ there exist certain K_u containing y and $K_w \cap K_u \neq \emptyset$.

Theorem 4.1. *With above assumptions, there are two positive constants C and q such that for any $B = B(x,r)$, $B^* = B(x,qr)$ and $f \in \mathcal{F}$,*

$$\int_B |f(x) - f_B|^2 d\mu(x) \leqslant Cr^\beta \int_{B^*} d\Gamma(f,f), \qquad (PI(\beta))$$

where f_B is the mean integral of f on B.

Proof. Let $I(x,r)$ be composed of $w \in \Omega^*$ such that K_w is of size r and intersects with B. For any $w, v \in I(x,r)$, the Hausdorff distance between K_w and K_v is less than $4r$. Let $w' < w$ and $v' < v$ such that $K_{w'}$ and $K_{v'}$ are of size $4c_K^{-1}r$. Then by $(G2)$, we have $K_{w'} \cap K_{v'} \neq \emptyset$; so there are words $u_0, u_1, \cdots, u_k$ such that $u_0 = w, u_k = v, K_{u_i} \cap K_{u_{i+1}} \neq \emptyset$, and K_{u_i} is of size r for each $0 \leqslant i < k$.

Let $J(x,r)$ be a collection of all above u_i depending on w and v. Then the union $K(x,r) := \bigcup_{u \in J(x,r)} K_u$ is connected. Note that k is uniformly bounded; there is an independent constant q such that $B(x,qr) \supset K(x,r) \supset B$.

Given $w \in J(x,r)$, we have by the Hölder inequality and integral transformation

$$\begin{aligned}
&\int_{K_w} |f(\xi) - f_{K_w}|^2 d\mu(\xi) \\
\leqslant\ & \frac{1}{\mu(K_w)} \int_{K_w} \int_{K_w} |f(\xi) - f(\eta)|^2 d\mu(\eta) d\mu(\xi) \\
=\ & \mu(K_w) \int_K \int_K |f \circ \psi_w(\xi) - f \circ \psi_w(\eta)|^2 d\mu(\eta) d\mu(\xi) \\
\leqslant\ & Cr^\alpha N_{\beta/2,\infty}(f \circ \psi_w) \\
\leqslant\ & Cr^\alpha\, \mathcal{E}[f \circ \psi_w] \leqslant Cr^\beta \int_{K_w} d\Gamma(f,f).
\end{aligned} \tag{4.2}$$

By connectedness, for any $w, v \in J(x,r)$, we can find a team $u_0, u_1 \cdots, u_m \in J(x,r)$ such that $u_0 = w$, $u_m = v$ and K_{u_i} intersects with $K_{u_{i+1}}$ for any $0 \leqslant i < m$. Choose $p_i \in K_{u_i} \cap K_{u_{i+1}}$ with $B(p_i, r) \supset K_{u_i} \cap K_{u_{i+1}}$. Recalling the definition (3.4), we have by $(G1)$ that

$$\begin{aligned}
& r^{-\alpha} \int_{K_w} \int_{K_v} |f(\xi) - f(\eta)|^2 d\mu(\eta) d\mu(\xi) \\
\leqslant\ & Cr^{-\alpha} \int_{K_w} \int_{K_v} |f(\xi) - f(p_0, r)|^2 + |f(p_{k-1}, r) - f(\eta)|^2 \\
& \quad + \sum_{i=0}^{k-2} |f(p_i, r) - f(p_{i+1}, r)|^2 d\mu(\eta) d\mu(\xi) \\
\leqslant\ & C \int_{K_w} |f(\xi) - f_{K_w}|^2 d\mu(\xi) + \int_{K_v} |f(\xi) - f_{K_v}|^2 d\mu(\xi) \\
& \quad + Cr^\alpha \left(\sum_{i=0}^{k-1} |f_{K_{u_i}} - f(p_i, r)|^2 + \sum_{i=1}^{k} |f_{K_{u_i}} - f(p_{i-1}, r)|^2 \right) \\
\leqslant\ & Cr^\beta \int_{B(x,qr)} d\Gamma(f,f),
\end{aligned}$$

the last step being due to (3.6), (3.8) and (4.2). Consequently

$$\begin{aligned}\int_B |f(\xi)-f_B|^2 d\mu(\xi) &\leqslant \frac{1}{\mu(B)}\int_B\int_B |f(\xi)-f(\eta)|^2 d\mu(\eta)d\mu(\xi)\\ &\leqslant Cr^{-\alpha}\sum_{w,v\in J(x,r)}\int_{K_w}\int_{K_v}|f(\xi)-f(\eta)|^2 d\mu(\eta)d\mu(\xi)\\ &\leqslant Cr^{\beta}\int_{B(x,qr)} d\Gamma(f,f).\end{aligned}$$

□

Remark 4.2. Comparing with the proof in [18], we don't consider the capacity of boundary set here.

Define the effective resistance $R_{\text{eff}}(A,B)$ for disjoint subsets A and B as

$$R_{\text{eff}}(A,B)^{-1} = \inf\left\{\mathcal{E}[f] : f = 1 \text{ on } A \text{ and } f = 0 \text{ on } B, f \in \mathcal{F}\right\}.$$

By (DR) and the Markovian property, there is a sequence $f_k \in \mathcal{F}$ with $f_k = 1$ on A, $f_k = 0$ on B and $0 \leqslant f_k \leqslant 1$ on K such that $\mathcal{E}[f_k]$ tends to the infimum. Then there is a subsequence f_{k_l} converging $\mathcal{E}$-weakly to a certain $h \in \mathcal{F}$ by the Banach-Alaoglu's theorem (see for example [5]), which yields a subsequence f_{k_m} of f_{k_l} such that the Cesàro mean $h_n = \sum_{m=1}^{n} f_{k_m}/n$ converges $\mathcal{E}$-strongly to some $h \in \mathcal{F}$. Hence $R_{\text{eff}}(A,B)^{-1} = \mathcal{E}[h]$ with $h = 1$ on A and $h = 0$ on B quasi-everywhere (see [8] for this definition).

Adjusting the definition in [3], we say K fulfills condition $RES(\beta)$ if there exists a constant $C > 0$ such that for any $x \in K$ and $R > 0$,

$$C^{-1}R^{\beta-\alpha} \leqslant R_{\text{eff}}(B(x,R/2), B(x,2R)^c) \leqslant CR^{\beta-\alpha}. \qquad (RES(\beta))$$

Theorem 4.2. *With the above assumptions, $RES(\beta)$ holds for K.*

Proof. Recalling the definition of $I(x_0,r)$, there is a certain $w \in I(x_0,R/2)$ such that K_w contains x_0. Set $A = \bigcup_{u\in I(x_0,R/2)} K_u$ and $B = \bigcup_{v\in I(x_0,R)} K_v$ such that $B(x,R/2) \subset A \subset B(x,R) \subset B \subset B(x,2R)$.

Consider $R_{\text{eff}}(A,B^c)^{-1}$ and h as above. By $(G1)$ we know that the set of the shapes of $\psi_w^{-1}(B)$ is finite, which means $h\circ\psi_w$ is of finite choices on $\psi_w^{-1}(B)$. Note that contraction ratio $c_v \simeq R$, there is a constant $C_1 > 0$ such that

$$\mathcal{E}[h] = \sum_{v\in I(x_0,R)} \frac{1}{c_v^{\beta-\alpha}}\,\mathcal{E}[h\circ\psi_v] \leqslant C_1 R^{\alpha-\beta}.$$

Hence we obtain the lower bound of $R_{\text{eff}}(B(x,R/2), B(x,2R)^c)$ since it is greater than $R_{\text{eff}}(A,B^c)$.

To get the upper bound, we use the same argument as above. There exists a positive constant $\lambda \leqslant 1/2$ such that we can find a word $w' \in I(x_0,\lambda R)$ satisfying

$K_{w'} \subset B(x_0, R/2)$. Set $A' = K_{w'}$ and $B' = \bigcup_{v' \in I(x_0,2R)} K_{v'}$. Let $R_{\text{eff}}(A', B'^c)^{-1} = \mathcal{E}[h']$. Still by $(G1)$, there is a constant $C_2 > 0$ such that

$$\mathcal{E}[h'] = \sum_{v' \in I(x_0,2R)} \frac{1}{c_{v'}^{\beta-\alpha}} \mathcal{E}[h' \circ \psi_{v'}] \geqslant C_2 R^{\alpha-\beta},$$

which yields $R_{\text{eff}}(B(x, R/2), B(x, 2R)^c) \leqslant R_{\text{eff}}(A', B'^c) \leqslant C_2^{-1} R^{\beta-\alpha}$. □

Remark 4.3. The Poincaré inequality implies the upper bound. However, we prefer a direct way to show it here.

If $\beta > \alpha$, it is known that $\mathcal{F}$ is continuously imbedded into the Hölder space $C^{\frac{\beta-\alpha}{2}}(K)$ (see for example Theorem 8.1 in [9]), i.e. there exists a constant $C > 0$ such that for all $f \in \mathcal{F}$ and $x, y \in K$ μ-a.e.

$$|f(x) - f(y)|^2 \leqslant C d(x, y)^{\beta-\alpha} \mathcal{E}[f].$$

So we can define the resistance metric $R(x, y)$ (see [12] and [13]) by

$$R(x, y) = \sup\{\mathcal{E}[f]^{-1} : f(x) = 1, f(y) = 0, f \in \mathcal{F}\}.$$

Let $r = d(x, y)$. Then by the definition, we have a constant C such that

$$C r^{\beta-\alpha} \geqslant R(x, y) \geqslant R_{\text{eff}}(B(x, r/2), B(x, 2r)^c) \geqslant C^{-1} r^{\beta-\alpha}.$$

That is $R(x, y) \simeq d(x, y)^{\beta-\alpha}$, which we label still by $(RES(\beta))$.

Say a metric space (X, ρ) fulfills the midpoint property (see [1]) if for any $x, y \in X$ there exists $z \in X$ such that $\rho(x, z) = \rho(z, y) = \frac{1}{2}\rho(x, y)$. If we impose the midpoint property on K with respect to some appropriate metric equivalent to $d(\cdot, \cdot)$ (for example the geodesic metric on SG), then by Theorem 8.16 in [1] and Theorem 3.1 in [14], we obtain immediately

Corollary 4.1. *With the above assumptions on K, if $\beta > \alpha$, then*

$$HK(\beta) \Longleftrightarrow (NE) + (DR) \Longleftrightarrow RES(\beta).$$

Remark 4.4. Formally speaking, Besov space $\Lambda_{2,\infty}^{\beta/2}$ describes the 'smoothness' of functions through a global integral average. By contrast, heat kernel estimates mainly rely on the local neighborhoods. So the equivalence between them is due to the self-similarity essentially.

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