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Asymptotic Behaviour of the Gradient of Large Solutions to Some Nonlinear Elliptic Equations

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Abstract

If h is a nondecreasing real valued function and $0 \leq q \leq 2$, we analyse the boundary behaviour of the gradient of any solution u of $-\Delta u + h(u) + |\nabla u|^q = f$ in a smooth N-dimensional domain Ω with the condition that u tends to infinity when x tends to $\partial\Omega$. We give precise expressions of the blow-up which, in particular, point out the fact that the phenomenon occurs essentially in the normal direction to $\partial\Omega$. Motivated by the blow–up argument in our proof, we also give a symmetry result for some related problems in the half space.

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1 Introduction

Let Ω be a C^2 domain in $\mathbf{R}^N$ ($N \geq 2$), h a continuous nondecreasing function and q a nonnegative real number. The aim of this work is to study the behaviour of solutions of nonlinear equations of the following type

$$-\Delta u + h(u) + |\nabla u|^q = f \qquad \text{in } \Omega \subseteq \mathbf{R}^N, \tag{1.1}$$

satisfying a boundary blow–up condition

$$\lim_{d_\Omega(x)\to 0} u(x) = +\infty \tag{1.2}$$

where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. The interest for solutions of (1.1) satisfying such singular boundary conditions arises from stochastic control problems with state constraints, as explained in [11], where $h(u) = \lambda\, u$. In that situation, u represents the value function of the optimal control problem and $-q\nabla u\, |\nabla u|^{q-2}$ acts as the optimal (feedback) control which forces the process to stay in Ω.

From a purely PDE's point of view, the existence of such solutions depends on the possibility of finding universal interior estimates for (1.1), independently on the behaviour of u at the boundary. In the case $q = 0$ these estimates hold provided the well–known Keller–Osserman condition ([10], [17]) is satisfied, i.e.

$$\int^{+\infty} \frac{ds}{\sqrt{\int_0^s h(t)dt}} < \infty\,. \tag{1.3}$$

A large number of papers has investigated properties of such singular solutions (also called *large*, or *explosive* solutions) when the lower order terms only depend on u (see [3], [4], [5], [14], [15], [16], [20]). In the presence of gradient dependent terms as in (1.1), large solutions in smooth domains have been studied in [2], [8], [7], [11], [18]; roughly speaking, such solutions exist if h satisfies (1.3) or if $1 < q \leq 2$ and h is unbounded at infinity. Indeed, in equation (1.1) both lower order terms may lead to the construction of large solutions, so that existence of solutions to problem (1.1) – (1.2) can be proved even if h is sublinear, provided $q > 1$.

In this paper we consider problem (1.1) – (1.2), mainly referring to the model examples $h(s) = e^{as}$, $a > 0$, and $h(s) = s^\beta$, $\beta > 0$, and we study the asymptotic behaviour of ∇u at the boundary. It turns out, as a quite general rule, that ∇u blows up, in its first approximation, in the normal direction: in the model examples, our results read as follows. We denote by $d_\Omega(x)$ the distance of a point x to $\partial\Omega$, and by ν the outward unit normal vector at $\partial\Omega$.

Theorem 1.1 *Let Ω be a C^2 domain in $\mathbf{R}^N$, ν be the normal outward unit vector to $\partial\Omega$, and assume $f \in L^\infty(\Omega)$.*

A- *Let $a > 0$, and u be a solution of*

$$\begin{cases} -\Delta u + e^{au} + |\nabla u|^q = f & \text{in } \Omega, \\ \lim_{d_\Omega(x)\to 0} u(x) = +\infty\,. \end{cases}$$

Then we have:

(1) If $q = 2$ *and* $a \leq 2$*, then*

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)\nabla u(x) = \nu.$$

(2) if $0 \leq q < 2$*, or if* $q = 2$ *and* $a > 2$*, then*

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)\nabla u(x) = \frac{2}{a}\nu.$$

B- *Let* $\beta > 0$ *and* u *be a solution of*

$$\begin{cases} -\Delta u + |u|^{\beta-1}u + |\nabla u|^q = f & \text{in } \Omega, \\ \lim\limits_{d_\Omega(x)\to 0} u(x) = +\infty\,. \end{cases}$$

Then:

(3) If $q \geq \frac{2\beta}{1+\beta}$*, then*

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)^{\frac{1}{q-1}}\,\nabla u(x) = b\,\nu,$$

in which formula $b = (q-1)^{-\frac{1}{q-1}}$ *if* $q > \frac{2\beta}{1+\beta}$*, and* $b = \left(\frac{1}{a}\right)^{\frac{2-q}{2(q-1)}}\left(\frac{2-q}{q-1}\right)^{\frac{1}{q-1}}$ *if* $q = \frac{2\beta}{1+\beta}$*, where* a *is the solution of* $a - a^{\frac{q}{2}} = 2 - q$*.*

(4) If $q < \frac{2\beta}{1+\beta}$*, then*

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)^{\frac{1+\beta}{\beta-1}}\nabla u(x) = b\,\nu,$$

where $b = \frac{2}{\beta-1}\left[\frac{2(\beta+1))}{(\beta-1)^2)}\right]^{1/(\beta-1)}$.

The previous result generalizes those obtained in [1] and [4] for large solutions of semilinear problems, in case the lower order terms do not depend on ∇u; indeed, our proof follows a similar approach based on a blow–up argument near the boundary and requires some symmetry results on the blown–up functions, which are solutions of a similar problem in the half space. Even in the case $q = 0$, our result extends those previous ones by considering a slightly larger class of nonlinearities $h(s)$. The conclusions of Theorem 1.1 will follow as a particular case of the results which we prove in Section 2.

Moreover, in a third section we will also provide a simple uniqueness result for solutions of (1.1)–(1.2) which is meant to be applied in case h is concave, or the sum of a concave and a convex function. In fact, previous uniqueness results seem to have been proved only if h has a convex type behaviour.

Finally, motivated by the blow–up argument in case $h(s)$ has a power growth at infinity, we prove in the last section some symmetry and uniqueness results for nonnegative solutions of the problem in the half space

$$\begin{cases} -\Delta u + \alpha\, u^p + |\nabla u|^q = 0 \quad \text{in } \mathbf{R}^N_+ := \{\xi = (\xi_1, \xi') \in \mathbf{R}^N \,:\, \xi_1 > 0\}, \\ u(0,\xi') = M \end{cases}$$

where $\alpha \geq 0$, $p > 0$, $1 < q \leq 2$ and M is a nonnegative constant or possibly $M = +\infty$. We give a simple proof, based mainly on comparison with radial or one–dimensional solutions, that any nonnegative solution u is one–dimensional, and uniqueness follows if $\alpha > 0$.

2 Asymptotic behaviour of derivatives

In this section we let $\Omega \subset \mathbf{R}^N$ be a bounded C^2 domain. We denote by $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, and by $\nu(x)$ the outward unit normal vector at any point $x \in \partial\Omega$, or simply ν when meant as a vector field defined on $\partial\Omega$. In the sequel, τ is any unitary tangent vector field defined on $\partial\Omega$ as well, i.e. $\tau \cdot \nu = 0$.

We start by considering the equation

$$\begin{cases} -\Delta u + h(u) + |\nabla u|^2 = f & \text{in } \Omega, \\ \lim\limits_{d_\Omega(x) \to 0} u(x) = +\infty\,, \end{cases} \tag{2.1}$$

where h is an increasing function such that $\lim\limits_{s \to +\infty} h(s) = +\infty$, and $f \in L^\infty(\Omega)$.

It is proved in [18] that problem (2.1) admits a solution, and moreover any solution satisfies the estimate

$$u(x) - F(d_\Omega(x)) \text{ is bounded near } \partial\Omega, \text{ where } F^{-1}(s) = \int_s^{+\infty} \frac{e^{-t}}{[\int_0^t h(\xi)e^{-2\xi}d\xi]^{\frac{1}{2}}}dt. \tag{2.2}$$

Note that the function F has at most a logarithmic blow–up rate. Moreover, if the following limit exists

$$\lim_{\xi \to +\infty} \left(1 + \frac{1}{2}\frac{h(\xi)e^{-2\xi}}{\int_0^\xi h(t)e^{-2t}dt}\right)^{-1};$$

one has, using twice L'Hopital's rule and since both $F^{-1}(\xi)$ and $(F^{-1})'(\xi)$ tend to zero as ξ goes to infinity,

$$\begin{aligned} &\lim_{s \to 0} \frac{F(s)}{|\log s|} = -\lim_{s \to 0} s\,F'(s) = \\ &= -\lim_{\xi \to +\infty} \frac{F^{-1}(\xi)}{(F^{-1})'(\xi)} = -\lim_{\xi \to +\infty} \frac{(F^{-1})'(\xi)}{(F^{-1})''(\xi)} = \lim_{\xi \to +\infty} \left(1 + \frac{1}{2}\frac{h(\xi)e^{-2\xi}}{\int_0^\xi h(t)e^{-2t}dt}\right)^{-1}. \end{aligned} \tag{2.3}$$

Similarly one has

$$\begin{aligned} &\lim_{s \to 0} (F(s) + \log s) = \lim_{\xi \to +\infty} \log(e^\xi F^{-1}(\xi)) = \\ &= \log\left(-\lim_{\xi \to +\infty} \frac{(F^{-1})'(\xi)}{e^{-\xi}}\right) = -\frac{1}{2}\log\left(\lim_{\xi \to +\infty} \int_0^\xi h(t)e^{-2t}dt\right). \end{aligned} \tag{2.4}$$

In particular we deduce that

$$u(x) + \log(d_\Omega(x)) \text{ is bounded near } \partial\Omega \text{ if and only if } \int_0^{+\infty} h(t)e^{-2t}dt < \infty, \tag{2.5}$$

and that

$$\text{if } \lim_{s\to+\infty} \frac{h(s)e^{-2s}}{\int_0^s h(t)e^{-2t}dt} = \lambda \geq 0, \text{ then } \frac{u(x)}{|\log(d_\Omega(x))|} \to \frac{2}{\lambda+2} \text{ as } d_\Omega(x) \to 0. \tag{2.6}$$

In view of these remarks, we will consider three types of situations in our analysis, which are mutually excluding:

(h1) $\int^{+\infty} h(t)e^{-2t}dt < \infty$ and $\lim_{s\to+\infty} h(s)e^{-2s} = 0$.

(h2) $\int^{+\infty} h(t)e^{-2t}dt = \infty$, $\lim_{s\to+\infty} \frac{h(s)e^{-2s}}{\int_0^s h(t)e^{-2t}dt} = 0$, and $\frac{h(s+c)}{h(s)}$ is bounded for large s, and any $c \in \mathbf{R}$.

(h3) $\lim_{s\to+\infty} \frac{h(s)e^{-2s}}{\int_0^s h(t)e^{-2t}dt} = \lambda > 0$, and, for any $t \in \mathbf{R}$, $\exists \lim_{s\to+\infty} \frac{h(s+t)}{h(s)} = e^{(\lambda+2)t}$.

Remark 2.1 Assumption (h1) corresponds to a subcritical case, where the blow–up rate of u only depends on the first order term, whereas (h2) represents the critical case (e.g. $h(s) = e^{2s}$) in which both terms give a contribution and a superposition effect may be observed; in fact, due to (2.5)–(2.6), in both cases we have $\frac{u(x)}{|\log(d_\Omega(x))|} \to 1$, but while under (h1) we have that $u(x) + \log(d_\Omega(x))$ is bounded near $\partial\Omega$, (h2) implies that $u(x) + \log(d_\Omega(x)) \to -\infty$ at the boundary.

As far as (h3) is concerned, it covers exponential–type growths, including the model $h(s) = e^{(2+\lambda)s}s^\beta$ for any $\beta \geq 0$. Let us remark that assuming the existence, for any $t \in \mathbf{R}$, of $\lim_{s\to+\infty} \frac{h(s+t)}{h(s)}$ *automatically* implies that the function $\omega(t) := \lim_{s\to+\infty} \frac{h(s+t)}{h(s)}$ is an exponential. Indeed, since h is increasing, the same is true for ω. Since $\omega(t+t') = \omega(t)\omega(t')$ for every t, $t' \in \mathbf{R}$, the continuity of ω at a point t_0 implies that ω is continuous on $\mathbf{R}$, and then (using also $\omega(0) = 1$) $\omega(t) = e^{a\,t}$ for some $a \in \mathbf{R}$. Moreover, since ω is continuous the above convergence is locally uniform for t in $\mathbf{R}$. Eventually, if

$$\lambda = \lim_{s\to+\infty} \frac{h(s)e^{-2s}}{\int_0^s h(t)e^{-2t}dt}\,, \tag{2.7}$$

we have

$$\frac{\int_0^{s+t} h(\xi)e^{-2\xi}d\xi}{e^{-2s}h(s)} = \frac{\int_0^s h(\xi)e^{-2\xi}d\xi}{e^{-2s}h(s)} + \int_0^t \frac{h(s+\xi)}{h(s)}e^{-2\xi}d\xi \to \frac{1}{\lambda} + \int_0^t e^{(a-2)\xi}d\xi$$

as $s \to \infty$. But L'Hopital's rule also implies

$$\lim_{s\to+\infty} \frac{\int_0^{s+t} h(\xi)e^{-2\xi}d\xi}{\int_0^s h(\xi)e^{-2\xi}d\xi} = e^{(a-2)t},$$

so that we deduce, using also (2.7),

$$\frac{1}{\lambda} + \int_0^t e^{(a-2)\xi}d\xi = \lim_{s\to+\infty} \frac{\int_0^{s+t} h(\xi)e^{-2\xi}d\xi}{e^{-2s}h(s)} = \frac{e^{(a-2)t}}{\lambda},$$

hence $a \neq 2$, and $a = \lambda + 2$.

Theorem 2.1 *Let u be a solution of (2.1). Then we have:*

(1) If $(h1)$ or (h2) hold true,

$$\lim_{\delta\to 0} \delta \frac{\partial u}{\partial \nu(x)}(x - \delta\nu(x)) = 1\,, \qquad \lim_{\delta\to 0} \delta \frac{\partial u}{\partial \tau(x)}(x - \delta\nu(x)) = 0 \tag{2.8}$$

hold uniformly for $x \in \partial\Omega$, hence

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)\nabla u(x) = \nu. \tag{2.9}$$

(2) If (h3) holds true,

$$\lim_{\delta\to 0} \delta \frac{\partial u}{\partial \nu(x)}(x - \delta\nu(x)) = \frac{2}{\lambda+2}\,, \qquad \lim_{\delta\to 0} \delta \frac{\partial u}{\partial \tau(x)}(x - \delta\nu(x)) = 0 \tag{2.10}$$

hold uniformly for $x \in \partial\Omega$, hence

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)\nabla u(x) = \frac{2}{\lambda+2}\nu. \tag{2.11}$$

Proof. Thanks to (2.2), we can fix d_0 and C_0 such that

$$\begin{gathered} |u(x) - F(d_\Omega(x))| \leq C_0 \quad \text{for any } x \in \Omega\text{: } d_\Omega(x) \leq d_0, \\ \text{with } F^{-1}(s) = \int_s^{+\infty} \frac{e^{-t}}{[\int_0^t h(\xi)e^{-2\xi}d\xi]^{\frac{1}{2}}}dt. \end{gathered} \tag{2.12}$$

We use a similar blow–up framework as in [1], [4]. Let $x \in \partial\Omega$ and consider a new system of coordinates $(\eta_1, \ldots, \eta_N)$ centered at x and such that the positive η_1-axis is the direction $-\nu(x)$, where $\nu(x)$ is the outward normal vector at x; thus $x = O$ is the origin and η_1 is the direction of the inner normal vector at x. In the η–space, let us set $P_0 = (d_0, 0, \ldots, 0)$ and define

$$D_\delta = B(O, \delta^{1-\sigma}) \cap B(P_0, d_0)\,, \qquad \text{with } 0 < \sigma < \tfrac{1}{2}.$$

Note that we can assume that Ω satisfies the interior sphere condition with radius d_0 so that $D_\delta \subset \Omega$, and since the operator is invariant under translations and rotations we obtain the same equation for u in the new variable η. Define $\xi = \frac{\eta}{\delta}$ and the function

$$v_\delta(\xi) = u(\eta) - F(\delta) = u(\delta\xi) - F(\delta)\,,$$

where F is defined in (2.12). Then $v_\delta(\xi)$ satisfies the equation

$$-\Delta v_\delta + h(u(\delta\xi))\delta^2 + |\nabla v_\delta|^2 = \delta^2 f(\delta\xi) \qquad \xi \in \frac{1}{\delta}D_\delta.$$

It is readily seen that since $0 < \sigma < \frac{1}{2}$, if $\eta \in \partial B(P_0, d_0) \cap \partial D_\delta$, then $\frac{\eta_1}{\delta} \to 0$ and $\frac{|\eta'|}{\delta} \to +\infty$ as $\delta \to 0$, and that the domain $\frac{1}{\delta}D_\delta$ converges to the half space $\mathbf{R}^N_+ := \{\xi \in \mathbf{R}^N : \xi_1 > 0\}$.

Let us study now the limit of v_δ. First of all, observe that since F^{-1} is a decreasing and convex function (as easily checked), then its inverse function F is also convex. We have then, for any $\lambda < 1$,

$$0 \le F(\lambda s) - F(s) \le -F'(\lambda s)\,\lambda s\,\frac{1-\lambda}{\lambda}\,,$$

and since (see also (2.3)) $0 < -F'(\xi)\xi < C$ for any $\xi \in \mathbf{R}^+$, we deduce that F enjoys the property

$$\exists C > 0 : \quad F(\lambda s) - F(s) \le C\frac{1-\lambda}{\lambda} \qquad \forall \lambda < 1\,, \quad \forall s > 0\,. \tag{2.13}$$

Since $\partial\Omega$ is C^2, we have that for $\eta \in D_\delta$

$$d_\Omega(\eta) = \eta_1 + O(|\eta|^2) = \eta_1 + O(\delta^{2-2\sigma})\,. \tag{2.14}$$

Note in particular that the choice $\sigma < \frac{1}{2}$ implies that

$$\frac{d_\Omega(\delta\xi)}{\delta} \to \xi_1$$

uniformly for $\xi \in \frac{1}{\delta}\,D_\delta$. Thus from (2.12)–(2.13) we deduce that

$$|u(\delta\xi) - F(\delta\,\xi_1 + \delta^{2-2\sigma})| \le C_1 \quad \text{for any } \xi \in \tfrac{1}{\delta}D_\delta, \tag{2.15}$$

so that

$$|v_\delta(\xi)| \le C_1 + |F(\delta\,(\xi_1 + \delta^{1-2\sigma})) - F(\delta)| \quad \text{for any } \xi \in \tfrac{1}{\delta}D_\delta. \tag{2.16}$$

In particular, due to (2.13), (2.16) implies that

$$|v_\delta(\xi)| \le C_1 + C_2\,\max\{\xi_1\,,\,\frac{1}{\xi_1}\}\,,$$

hence v_δ is locally uniformly bounded.

Assume that (h1) holds true: then (see (2.4)) $F(\delta) + \log(\delta)$ is bounded for small δ, so that (2.16) implies that

$$v_\delta(\xi) \geq F(\delta(\xi_1 + \delta^{1-2\sigma})) - F(\delta) - C_1 \geq -\log(\xi_1 + \delta^{1-2\sigma}) - C_2\,,$$

for $\xi \in \frac{1}{\delta} D_\delta$; in particular in the limit (as $\delta \to 0$) we deduce (recall that $\sigma < \frac{1}{2}$)

$$v(\xi) \geq -\log \xi_1 - C_2, \tag{2.17}$$

so that $\lim\limits_{\xi_1 \to 0^+} v(\xi) = +\infty$. Noticing that

$$\begin{aligned}
&\delta^2 h(u(\delta\xi)) = h(v_\delta + F(\delta)) e^{-2(v_\delta + F(\delta))}\, e^{2(v_\delta + F(\delta) + \log\delta)} \\
&\leq C h(v_\delta + F(\delta)) e^{-2(v_\delta + F(\delta))}\, e^{2v_\delta}\,,
\end{aligned}$$

and using that v_δ is locally bounded and $h(s)e^{-2s} \to 0$ as $s \to +\infty$, we deduce

$$\delta^2 h(u(\delta\xi)) \to 0 \quad \text{in } L^\infty_{loc}(\mathbf{R}^N_+)\,. \tag{2.18}$$

Furthermore, standard elliptic estimates for second derivatives imply that $|\nabla v_\delta|$ is also locally uniformly bounded, and, in the end, that v_δ is locally relatively compact in the C^1_{loc}–topology. Let v be the limit of some subsequence v_{δ_k}, as $\delta_k \to 0$. Therefore v is a solution of

$$\begin{cases} -\Delta v + |\nabla v|^2 = 0 & \text{in } \mathbf{R}^N_+, \\ \lim\limits_{\xi_1 \to 0^+} v(\xi) = +\infty\,. \end{cases} \tag{2.19}$$

The function $w = e^{-v}$ is positive and harmonic in $\mathbf{R}^N_+$; it satisfies $w \leq C\xi_1$, from (2.17), hence $w = 0$ on $\{\xi_1 = 0\}$. We deduce (for instance using Kelvin transform, or symmetry results) that there exists $\lambda \in \mathbf{R}_+$ such that $w = \lambda\,\xi_1$, hence $v = -\log\xi_1 - \log\lambda$. In particular, we obtain, locally uniformly in $\mathbf{R}^N_+$:

$$\frac{\partial v_{\delta_k}}{\partial \xi_1} \to -\frac{1}{\xi_1}\,, \qquad \frac{\partial v_{\delta_k}}{\partial \xi_j} \to 0 \quad \forall j = 2, \dots, N,$$

for any convergent subsequence v_{δ_k}. Note that while the limit function v is determined up to the constant $-\log\lambda$, its gradient is uniquely determined. This implies that the whole sequence of derivatives $\frac{\partial v_\delta}{\partial \xi_i}$ will be converging to this limit. We have proved then that it holds:

$$\delta \frac{\partial u(\delta\xi)}{\partial \xi_1} \to -\frac{1}{\xi_1}\,, \qquad \delta \frac{\partial u(\delta\xi)}{\partial \xi_j} \to 0 \quad \forall j = 2, \dots, N.$$

Recalling that ξ_1 is the direction of the inner normal vector and the point $\eta = (\delta, 0, \dots, 0)$ coincides with $x - \delta\nu(x)$, we fix $\xi_1 = 1$ and obtain (2.8).

Let us now assume (h2). In this case $F(\delta) + \log(\delta)$ is unbounded, but we still have (see (2.3))

$$F'(\delta)\delta \to -1 \qquad \text{as } \delta \to 0.$$

In particular, for any $\alpha < 1$ there exists an interval $(0, s_\alpha)$ such that the function $F(s) + \alpha\log s$ is decreasing in $(0, s_\alpha)$; therefore, for $\xi_1 < 1$ and δ small enough, we have

$$F(\delta(\xi_1 + \delta^{1-2\sigma})) - F(\delta) \geq -\alpha\log(\xi_1 + \delta^{1-2\sigma})\,.$$

Together with (2.16) it leads to

$$v_\delta(\xi) \geq F(\delta(\xi_1 + \delta^{1-2\sigma})) - F(\delta) - C_1 \geq -\alpha \log(\xi_1 + \delta^{1-2\sigma}) - C_1.$$

Hence, for any possible limit function v, we deduce that $v \geq -\alpha \log \xi_1 - C_1$ for ξ_1 near zero. This implies in particular that v blows–up uniformly on $\{\xi_1 = 0\}$. Writing again

$$\delta^2 h(u(\delta\xi)) = \frac{h(v_\delta + F(\delta))}{h(F(\delta))} \, \frac{h(F(\delta))e^{-2F(\delta)}}{\int_0^{F(\delta)} h(s)e^{-2s}ds} \, e^{2\log(\delta e^{F(\delta)}[\int_0^{F(\delta)} h(s)e^{-2s}ds]^{\frac{1}{2}})}, \quad (2.20)$$

and using (h2) and (see (2.3))

$$\lim_{t\to+\infty} F^{-1}(t)e^t[\int_0^t h(s)e^{-2s}ds]^{\frac{1}{2}} = \lim_{t\to+\infty} -\frac{F^{-1}(t)}{(F^{-1})'(t)} = 1,$$

we conclude that (2.18) still holds true. Then, passing to the limit in δ, any limit function v will satisfy (2.19). Again, we have that $w = e^{-v}$ is harmonic in $\mathbf{R}^N_+$ and $w \leq C\xi_1^\alpha$ in a neighborhood of $\{\xi_1 = 0\}$, so that $w = 0$ on $\partial\mathbf{R}^N_+$. We conclude as above that $w = \lambda\xi_1$ for some $\lambda \in \mathbf{R}_+$, and then $v = -\log \xi_1 - \log \lambda$. As before, the convergence of ∇v_δ to ∇v then implies (2.8) and (2.9).

Finally, let us assume (h3), and let again v be such that (a subsequence of) v_δ converges to v locally uniformly. Due to the monotonicity of h, we have (see Remark 2.1):

$$\lim_{s\to+\infty} \frac{h(s+t)}{h(s)} = e^{(\lambda+2)t} \quad \text{locally uniformly in } t$$

so that

$$\lim_{\delta\to 0} \frac{h(v_\delta + F(\delta))}{h(F(\delta))} = e^{(\lambda+2)v} \quad \text{in } L^\infty_{loc}(\mathbf{R}^N_+).$$

Since under (h3) we also have (see (2.3))

$$\lim_{t\to+\infty} F^{-1}(t)e^t[\int_0^t h(s)e^{-2s}ds]^{\frac{1}{2}} = \lim_{t\to+\infty} -\frac{F^{-1}(t)}{(F^{-1})'(t)} = \lim_{s\to 0} -F'(s)s = \frac{2}{\lambda+2}, \quad (2.21)$$

then (2.20) now implies

$$\lim_{\delta\to 0} \delta^2 h(u(\delta\xi)) = e^{(\lambda+2)v} \, \lambda \, e^{2\log(\frac{2}{\lambda+2})} = c_\lambda e^{(\lambda+2)v} \quad (2.22)$$

where $c_\lambda = \frac{4\lambda}{(\lambda+2)^2}$. Moreover we also deduce from (2.21) that there exist an interval $(0, \sigma_0)$ and constants $\alpha_0 < \frac{2}{\lambda+2}$ and $\alpha_1 > \frac{2}{\lambda+2}$ such that $F(t) + \alpha_0 \log t$ is decreasing and $F(t) + \alpha_1 \log t$ is increasing in $(0, \sigma_0)$. In particular we have

$$F(\delta(\xi_1 + \delta^{1-2\sigma})) - F(\delta) \geq -\alpha_0 \log(\xi_1 + \delta^{1-2\sigma}) \qquad \text{if } \xi_1 \leq 1 - \delta^{1-2\sigma},$$

and

$$F(\delta(\xi_1 + \delta^{1-2\sigma})) - F(\delta) \geq -\alpha_1 \log(\xi_1 + \delta^{1-2\sigma}) \qquad \text{if } 1 < \xi_1 < \tfrac{\sigma_0}{\delta} - \delta^{1-2\sigma},$$

which together with (2.16) imply

$$v_\delta(\xi) \geq -\alpha_0 \log(\xi_1 + \delta^{1-2\sigma}) - c_0 \qquad \text{if } \xi_1 \leq 1 - \delta^{1-2\sigma}, \tag{2.23}$$

and

$$v_\delta(\xi) \geq -\alpha_1 \log(\xi_1 + \delta^{1-2\sigma}) - c_1 \qquad \text{if } 1 < \xi_1 < \tfrac{\sigma_0}{\delta} - \delta^{1-2\sigma}. \tag{2.24}$$

From (2.22) and (2.23)–(2.24) we deduce, passing to the limit in δ, that v satisfies

$$\begin{cases} -\Delta v + c_\lambda\, e^{(\lambda+2)v} + |\nabla v|^2 = 0 \quad \text{in } \mathbf{R}^N_+, \\ \lim\limits_{\xi_1 \to 0^+} v(\xi) = +\infty\,, \end{cases} \tag{2.25}$$

and the further estimate

$$v(\xi) \geq -\alpha_1 \log \xi_1 - c_1 \qquad \text{if } 1 < \xi_1. \tag{2.26}$$

We proved in [19] (Corollary 2.6) that any solution of (2.25) only depends on the ξ_1 variable, moreover condition (2.26) implies that we have exactly

$$v = \frac{2}{\lambda+2}\log\Big(\frac{1}{\xi_1}\Big) + \frac{1}{\lambda+2}\log\Big(\frac{2\lambda}{c_\lambda(\lambda+2)^2}\Big) = \frac{2}{\lambda+2}\log\Big(\frac{1}{\xi_1}\Big) - \frac{\log 2}{\lambda+2}\,.$$

We obtain that

$$\frac{\partial v_\delta}{\partial \xi_1} \to -\frac{2}{(\lambda+2)\xi_1}\,, \qquad \frac{\partial v_\delta}{\partial \xi_j} \to 0 \quad \forall j = 2, \ldots, N,$$

which, as before, gives (2.10) and (2.11).

Remark 2.2 The same proof applies if one only requires on the right hand side that $\lim\limits_{d_\Omega(x)\to 0} d_\Omega^2(x) f(x) = 0$, which implies that $\lim\limits_{\delta\to 0} \delta^2 f(\delta\xi) = 0$ locally uniformly for $\xi \in \mathbf{R}^N_+$.

Remark 2.3 Under assumption (h3), the previous proof gives that the rescaled sequence v_δ converges towards $v = \frac{2}{\lambda+2}\log(\frac{1}{\xi_1}) - \frac{\log 2}{\lambda+2}$. Setting $\xi_1 = 1$ we deduce that

$$u(x) - F(d_\Omega(x)) \to -\frac{\log 2}{\lambda+2}$$

which improves estimate (2.2). As a consequence, this also implies that $u_1(x) - u_2(x) \to 0$ for any two large solutions u_1, u_2, hence in this case uniqueness of solutions of (2.1) follows immediately by the maximum principle.

We consider now the problem

$$\begin{cases} -\Delta u + h(u) + |\nabla u|^q = f \quad \text{in } \Omega, \\ \lim\limits_{d_\Omega(x)\to 0} u(x) = +\infty\,, \end{cases} \tag{2.27}$$

with $0 \leq q < 2$. In this case if h has an exponential growth at infinity, the gradient term does not affect the behaviour of solutions near the boundary, so that the asymptotic behaviour of this problem turns out to be the same as for the semilinear equation with $q = 0$. In order to adapt the above proof we will need the following uniqueness result for solutions in the half space.

Lemma 2.1 *Let $a > 0$ and v be a solution of*

$$\begin{cases} -\Delta v + e^{av} = 0 & \text{in } \mathbf{R}^N_+, \\ \lim\limits_{\xi_1 \to 0^+} v(\xi) = +\infty & \text{locally uniformly with respect to } \xi' \in \mathbf{R}^{N-1}. \end{cases}$$

Assume that v satisfies the following assumption:

$$\exists \alpha\,,\, m\,,\, S_0 > 0\,:\quad v(\xi) \geq -\alpha \log S - m \quad \forall \xi \in \mathbf{R}^N\,:\, \xi_1 \leq S\,,\quad \forall S > S_0\,. \tag{2.28}$$

Then $v = -\frac{2}{a}\log \xi_1 + \frac{1}{a}\log\frac{2}{a}$.

Proof. We can assume $a = 1$, up to replacing v by $\frac{1}{a}v - \frac{1}{a}\log a$. We follow the approach used in [19] (see Proposition 4.1); for any $R > 0$, $S > S_0$, define ω_R as the solution of the problem

$$\begin{cases} -\Delta \omega_R + e^{\omega_R} = 0 & \text{in } B_R(0), \\ \lim\limits_{\rho \uparrow R} \omega_R(\rho) = +\infty\,, \end{cases}$$

and define $\underline{\omega}_{R,S}$ as the solution of the problem

$$\begin{cases} -\Delta \underline{\omega}_{R,S} + e^{\underline{\omega}_{R,S}} = 0 & \text{in } B_{R+S}(0) \setminus B_R(0), \\ \lim\limits_{\rho \downarrow R} \underline{\omega}_{R,S}(\rho) = +\infty\,, & \underline{\omega}_{R,S}(R+S) = -\alpha \log S - m. \end{cases}$$

Now fix $\xi' \in \mathbf{R}^{N-1}$, and consider the points $\xi_R = (R, \xi')$, $\eta_R = (-R, \xi')$ and the functions $\omega_R(\cdot - \xi_R)$ and $\underline{\omega}_{R,S}(\cdot - \eta_R)$. By comparison, and using (2.28), we have

$$v \leq \omega_R(\cdot - \xi_R) \quad \text{in } B_R(\xi_R), \qquad v \geq \underline{\omega}_{R,S}(\cdot - \eta_R) \quad \text{in } B_{R+S}(\eta_R) \cap \mathbf{R}^N_+. \tag{2.29}$$

It is readily seen that the sequence $\{\omega_R(\cdot - \xi_R)\}$ is decreasing and converges, as $R \to +\infty$, to a function ω_∞ which only depends on the ξ_1–variable and is the maximal solution of

$$-z'' + e^z = 0\,, \qquad \lim_{t \to 0^+} z(t) = +\infty\,. \tag{2.30}$$

In particular, from a straightforward computation of solutions of (2.30), we obtain $\omega_\infty(\xi_1) = -2\log \xi_1 + \log 2$.

Let $S > S_0$; without loss of generality we can replace the constants α and m in (2.28) with possibly larger values. In particular, we can assume that $\alpha > 2$ and $e^{-m} < 2S_0^{\alpha-2}$: let then $w(\rho) = -2\log(\rho - R) - (\alpha - 2)\log S - m$, computing we have, for $\rho \in (R, R+S)$:

$$\begin{aligned} -\Delta w + e^w &= \frac{2(N-1)(\rho - R)S^{\alpha-2} - (2S^{\alpha-2} - e^{-m})\rho}{(\rho - R)^2 S^{\alpha-2}\rho} \\ &\leq \frac{2(N-1)S^{\alpha-1} - (2S^{\alpha-2} - e^{-m})R}{(\rho - R)^2 S^{\alpha-2}\rho}\,, \end{aligned}$$

so that there exists a value $R_0(S)$ such that

$$-\Delta w + e^w \le 0 \qquad \text{in } B_{R+S}(0) \setminus B_R(0) \text{ for any } R \ge R_0(S).$$

Since $w(R+S) = -\alpha \log S - m$ we deduce that

$$\underline{\omega}_{R,S} \ge w \ge -\alpha \log S - m \qquad \text{for any } R \ge R_0(S).$$

In particular, for any $R > R' > R_0(S)$, comparing $\underline{\omega}_{R,S}(\cdot - \eta_R)$ and $\underline{\omega}_{R',S}(\cdot - \eta_{R'})$ (on their common domain $B_{R'+S}(\eta_{R'}) \setminus B_R(\eta_R)$) we deduce that

$$\underline{\omega}_{R,S}(\cdot - \eta_R) \ge \omega_{R',S}(\cdot - \eta_{R'}).$$

Hence for any fixed S the sequence $\{\underline{\omega}_{R,S}(\cdot - \eta_R)\}_R$ is definitively increasing and converges to a function $\underline{\omega}_S$ which only depends on the ξ_1–variable and solves

$$-\underline{\omega}_S'' + e^{\underline{\omega}_S} = 0\,, \quad \lim_{t\to 0^+} \underline{\omega}_S(t) = +\infty\,, \quad \underline{\omega}_S(S) = -\alpha \log S - m\,. \tag{2.31}$$

Thus from (2.29), passing to the limit in R, we derive

$$\underline{\omega}_S(\xi_1) \le v(\xi) \le -2\log \xi_1 + \log 2 \qquad \forall \xi \in \mathbf{R}^N_+ \,:\, \xi_1 \le S\,, \quad \forall S > S_0\,. \tag{2.32}$$

Next, letting $e^{-m} \le 2$, we observe that the function z defined by $z(t) = -2\log t - (\alpha - 2)\log(t+1) - m$ satisfies

$$-z'' + e^z = -\frac{2}{t^2} - \frac{\alpha-2}{(t+1)^2} + \frac{e^{-m}}{t^2(t+1)^{\alpha-2}} \le \frac{-2(t+1)^{\alpha-2} + e^{-m}}{t^2(t+1)^{\alpha-2}} \le 0\,,$$

and since $z(S) < -\alpha \log S - m$ we have that it is a subsolution for the problem (2.31), hence

$$-2\log t - (\alpha-2)\log(t+1) - m \le \underline{\omega}_S(t) \le -2\log t + \log 2\,. \tag{2.33}$$

The sequence $\{\underline{\omega}_S(t)\}_{S \ge S_0}$ is then locally bounded and, up to subsequences, converges (locally in the C^2–topology) to a solution $\underline{\omega}_\infty$ of (2.30); but estimate (2.33) implies (due to the classification of all solutions of (2.30), see e.g. [19]) that the only possible limit is $\underline{\omega}_\infty = -2\log t + \log 2$. Letting S go to infinity, we conclude from (2.32) that $v = -2\log \xi_1 + \log 2$.

We are ready now to deal with the case that $q < 2$ and h has an exponential scaling at infinity. Our next result extends the one in [1], where $q = 0$ and $h(t) \equiv e^{\lambda t}$.

Theorem 2.2 *Let $f \in L^\infty(\Omega)$, and let u be a solution of (2.27), with $0 \le q < 2$. Assume that*

$$\lim_{s\to+\infty} \frac{h(s)}{\int_0^s h(t)dt} = \lambda > 0\,, \quad \textit{for every } t \in \mathbf{R}, \quad \exists \lim_{s\to+\infty} \frac{h(s+t)}{h(s)} := e^{\lambda t}. \tag{2.34}$$

Then we have:

$$\lim_{\delta\to 0}\delta\,\frac{\partial u}{\partial\nu(x)}(x-\delta\nu(x))=\frac{2}{\lambda}\,,\qquad \lim_{\delta\to 0}\delta\,\frac{\partial u}{\partial\tau(x)}(x-\delta\nu(x))=0 \tag{2.35}$$

uniformly for $x\in\partial\Omega$, *and then*

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)\nabla u(x)=\frac{2}{\lambda}\nu. \tag{2.36}$$

Proof. We use the same framework of the proof of Theorem 2.1, setting

$$v_\delta=u(\delta\xi)-\tilde F(\delta),$$

where the function $\tilde F$ is defined by

$$\tilde F^{-1}(s)=\int_s^{+\infty}\frac{1}{[2\int_0^t h(\xi)d\xi]^{\frac12}}dt. \tag{2.37}$$

Indeed, as a consequence of Keller-Osserman estimate and due to (2.34), there holds

$$|u(x)-\tilde F(d_\Omega(x))|\le C_0\quad \text{for any } x\in\Omega\text{: } d_\Omega(x)\le d_0,\,. \tag{2.38}$$

Observe that, since $\lim\limits_{s\to+\infty}\frac{h(s)}{\int_0^s h(t)dt}=\lambda>0$, one can prove (as in (2.3)) that $\tilde F'(t)t$ is bounded on $\mathbf{R}^+$ and

$$\tilde F'(\delta)\delta\to-\frac{2}{\lambda}\qquad \text{as } \delta\to 0. \tag{2.39}$$

Moreover the function $\tilde F$ is convex, so that we still have (2.13), and then again

$$|u(\delta\xi)-\tilde F(\delta\,(\xi_1+\delta^{1-2\sigma}))|\le C_1\quad\text{for any } \xi\in\tfrac{1}{\delta}D_\delta. \tag{2.40}$$

Reasoning as in the proof of Theorem 2.1 we deduce that there exist positive constants α_0, α_1, σ_0 such that

$$\tilde F(\delta(\xi_1+\delta^{1-2\sigma}))-\tilde F(\delta)\ge-\alpha_0\log(\xi_1+\delta^{1-2\sigma})\qquad \text{if } \xi_1\le 1-\delta^{1-2\sigma},$$

and

$$\tilde F(\delta(\xi_1+\delta^{1-2\sigma}))-\tilde F(\delta)\ge-\alpha_1\log(\xi_1+\delta^{1-2\sigma})\qquad \text{if } 1<\xi_1<\tfrac{\sigma_0}{\delta}-\delta^{1-2\sigma},$$

which together with (2.40) imply

$$v_\delta(\xi)\ge-\alpha_0\log(\xi_1+\delta^{1-2\sigma})-c_0\qquad \text{if } \xi_1\le 1-\delta^{1-2\sigma}, \tag{2.41}$$

and

$$v_\delta(\xi)\ge-\alpha_1\log(\xi_1+\delta^{1-2\sigma})-c_1\qquad \text{if } 1<\xi_1<\tfrac{\sigma_0}{\delta}-\delta^{1-2\sigma}. \tag{2.42}$$

Now the function v_δ satisfies the equation

$$-\Delta v_\delta+h(u(\delta\xi))\delta^2+|\nabla v_\delta|^q\,\delta^{2-q}=\delta^2 f(\delta\xi)\qquad \xi\in\frac{1}{\delta}D_\delta$$

and v_δ is locally uniformly bounded. Since

$$\delta^2 h(u(\delta\xi)) = \frac{h(v_\delta + \tilde{F}(\delta))}{h(\tilde{F}(\delta))}\,\frac{h(\tilde{F}(\delta))}{\int_0^{\tilde{F}(\delta)} h(s)ds}\, e^{2\log(\delta[\int_0^{\tilde{F}(\delta)} h(s)ds]^{\frac{1}{2}})}\,,$$

as in the proof of Theorem 2.1 we obtain, using (2.34) and (2.39), that $\delta^2 h(u(\delta\xi))$ is locally uniformly bounded. Furthermore

$$\lim_{\delta\to 0} \delta^2 h(u(\delta\xi)) = e^{\lambda v}\frac{2}{\lambda}$$

locally uniformly, where v is the limit of a subsequence (not relabeled) of v_δ. When $q > 1$, local estimates of Bernstein's type (see e.g. [11], [13] and the remark therein of the regularity of f), imply that any solution of (2.27) satisfies, for a constant $C > 0$,

$$|\nabla u(x)| \leq C d_\Omega(x)^{-\frac{1}{q-1}}\,.$$

In particular v_δ verifies an equation of type

$$-\Delta v_\delta + F_\delta \cdot \nabla v_\delta = g_\delta\,, \tag{2.43}$$

where g_δ, F_δ are a function, and a field respectively, which are locally uniformly bounded. By elliptic estimates we get that ∇v_δ is also locally uniformly bounded, and v_δ is relatively compact in the C^1_{loc}-topology. We have therefore

$$\lim_{\delta\to 0} |\nabla v_\delta|^q\, \delta^{2-q} = 0\,.$$

When $0 \leq q \leq 1$, since v_δ bounded in $L^\infty_{loc}(\Omega)\cap H^1_{loc}(\Omega)$ implies Δv_δ bounded in $L^{2/q}_{loc}(\Omega)$ and $(\frac{2}{q})^* > 2$, by elliptic equations regularity theory and a standard bootstraping argument, it follows again that ∇v_δ remains locally bounded and the above limit holds true. Thus, by replacing g_δ by its expression and using also (2.41)–(2.42), it turns out that v is a solution of

$$\begin{cases} -\Delta v + \frac{2}{\lambda}\, e^{\lambda v} = 0 & \text{in } \mathbf{R}^N_+, \\ \lim\limits_{\xi_1\to 0^+} v(\xi) = +\infty\,, \end{cases} \tag{2.44}$$

satisfying in addition that there exists α, $C > 0$ such that for any $S > 1$ we have

$$v(\xi) \geq -\alpha \log S - C \qquad \text{for any } \xi\text{: } \xi_1 \leq S. \tag{2.45}$$

By Lemma 2.1 we conclude that $v = -\frac{2}{\lambda}\log \xi_1$, and this uniqueness result implies also that the whole sequence v_δ is converging in $C^1_{loc}(\mathbf{R}^N_+)$. The convergence of ∇v_δ to ∇v then yields (2.35) and (2.36).

Remark 2.4 As a byproduct of the scaling argument, from the convergence of $v_\delta = u(\delta\xi) - \tilde{F}(\delta)$ to $-\frac{2}{\lambda}\log \xi_1$, we obtained, setting $\xi = 1$, that

$$u(x) - \tilde{F}(d_\Omega(x)) \to 0 \qquad \text{as } d_\Omega(x) \to 0,$$

where $\tilde{F}$ is defined in (2.37). In case $q = 0$ we recover a result of [12].

Finally, we consider the case that h has a power–type asymptotic rescaling at infinity: we extend then some results proved in [4] for the case $q = 0$.

Theorem 2.3 *Let $f \in L^\infty(\Omega)$ and u be a solution of (2.27), with $0 \le q < 2$.*

(i) Assume that

$$\lim_{s\to+\infty} \frac{h(s)^{\frac{2}{q}}}{\int_0^s h(t)dt} = +\infty\,, \tag{2.46}$$

and

$$\text{for every } t \in \mathbf{R}^+ \quad \exists \lim_{s\to+\infty} \tfrac{h(st)}{h(s)} := t^\alpha, \text{ with } \alpha > 1. \tag{2.47}$$

Then we have, uniformly for $x \in \partial\Omega$,

$$\lim_{\delta\to 0} \frac{1}{\tilde{F}'(\delta)} \frac{\partial u}{\partial \nu(x)}(x - \delta\nu(x)) = 1\,, \qquad \lim_{\delta\to 0} \frac{1}{\tilde{F}'(\delta)} \frac{\partial u}{\partial \tau(x)}(x - \delta\nu(x)) = 0 \tag{2.48}$$

where $\tilde{F}^{-1}(s)$ is defined in (2.37), and in particular

$$\lim_{d_\Omega(x)\to 0} \frac{\nabla u(x)}{\tilde{F}'(d_\Omega(x))} = \nu. \tag{2.49}$$

(ii) Assume that $q > 1$ and

$$\lim_{s\to+\infty} \frac{h(s)^{\frac{2}{q}}}{\int_0^s h(t)dt} = l\,, \tag{2.50}$$

for some $l > 0$, and let a be such that $\frac{a}{2-q} - a^{\frac{q}{2}} = (\frac{2-q}{2} l)^{\frac{q}{2-q}}$.
Then we have, uniformly for $x \in \partial\Omega$,

$$\lim_{\delta\to 0} \delta^{\frac{1}{q-1}} \frac{\partial u}{\partial \nu(x)}(x - \delta\nu(x)) = b_q\,, \qquad \lim_{\delta\to 0} \delta^{\frac{1}{q-1}} \frac{\partial u}{\partial \tau(x)}(x - \delta\nu(x)) = 0 \tag{2.51}$$

where $b_q = \left(\frac{1}{a}\right)^{\frac{2-q}{2(q-1)}} \left(\frac{2-q}{q-1}\right)^{\frac{1}{q-1}}$, and then

$$\lim_{d_\Omega(x)\to 0} d_\Omega(x)^{\frac{1}{q-1}} \nabla u(x) = b_q \nu. \tag{2.52}$$

Remark 2.5 As pointed out in Remark 2.1, the existence of the limit in (2.47) automatically implies that this limit is a power function.

Proof. (i) Under assumption (2.46), we can apply the results in [2] and use that

$$\lim_{d_\Omega(x)\to 0} \frac{u(x)}{\tilde{F}(d_\Omega(x))} = 1. \tag{2.53}$$

In other words, the behaviour of u is determined by the Keller–Osserman estimate in this case. Let us now use the framework of Theorem 2.1, introducing the system of coordinates

$(\eta_1, \ldots, \eta_N)$ whose η_1–axis is the inner normal direction. Define $O_\delta = (\delta, \ldots, 0)$ and the domain

$$\tilde{D}_\delta = B(O_\delta, \delta^{1-\sigma}) \cap B(P_0, d_0 - \delta), \qquad \sigma \in (0, \frac{1}{2}).$$

Again we have that $\tilde{D}_\delta$ converges to the half space $\{\xi \,:\, \xi_1 > 0\}$. Now we set $\xi = \frac{\eta - O_\delta}{\delta}$ and we introduce the blown-up function

$$v_\delta = \frac{u(\delta\xi + O_\delta)}{\tilde{F}(\delta)}.$$

This time let us choose d_0 such that $d_\Omega(x) < d_0$ implies $|\frac{u(x)}{\tilde{F}(d_\Omega(x))} - 1| \le \varepsilon_0$; thanks to (2.14) it follows

$$(1-\varepsilon_0)\tilde{F}(\delta(\xi_1+1) + O(\delta^{2-2\sigma})) \le u(\delta\xi + O_\delta) \le (1+\varepsilon_0)\tilde{F}(\delta(\xi_1+1) + O(\delta^{2-2\sigma}))\,.$$

In particular we deduce that $0 \le v_\delta \le (1+\varepsilon_0)$, i.e. v_δ is uniformly bounded and satisfies

$$-\Delta v_\delta + \frac{h(u(\delta\xi + O_\delta))\delta^2}{\tilde{F}(\delta)} + \tilde{F}(\delta)^{q-1}\,|\nabla v_\delta|^q \delta^{2-q} = f(\delta\xi + O_\delta)\frac{\delta^2}{\tilde{F}(\delta)}\,.$$

Note that (2.47) implies

$$\lim_{t\to+\infty} \tilde{F}^{-1}(t)\sqrt{\frac{h(t)}{t}} = \lim_{t\to+\infty} \int_1^{+\infty} \frac{1}{\sqrt{2\int_0^s \frac{h(t\xi)}{h(t)}d\xi}} ds = \frac{\sqrt{2(\alpha+1)}}{\alpha - 1}, \tag{2.54}$$

so that

$$\lim_{\delta\to 0} \frac{h(\tilde{F}(\delta))\delta^2}{\tilde{F}(\delta)} = \frac{2(\alpha+1)}{(\alpha-1)^2}.$$

Set $c_\alpha = \frac{2(\alpha+1)}{(\alpha-1)^2}$; then we have, using that v_δ (up to subsequences) converges, locally uniformly, to a function v, and $\frac{h(st)}{h(s)}$ converges to t^α locally uniformly in $\mathbf{R}$,

$$\frac{h(u(\delta\xi + O_\delta))\delta^2}{\tilde{F}(\delta)} = \frac{h(v_\delta \tilde{F}(\delta))}{h(\tilde{F}(\delta))}\ \frac{h(\tilde{F}(\delta))\delta^2}{\tilde{F}(\delta)} \to c_\alpha\, v^\alpha\,.$$

As in the previous theorem, we can use the local estimates on ∇u for solutions of (2.27), in order to get

$$|\nabla u(x)| \le C d_\Omega(x)^{-\frac{1}{q-1}}$$

when $q > 1$ for some constant $C > 0$. This implies that $\tilde{F}(\delta)^{q-1}\,|\nabla v_\delta|^{q-1}\delta^{2-q}$ is locally uniformly bounded. Hence v_δ satisfies an equation like (2.43) with g_δ and F_δ locally bounded. We deduce with a simple bootstrap argument and elliptic regularity that v_δ is relatively compact in the C^1_{loc}-topology. Moreover assumption (2.46) implies that

$$\lim_{t\to+\infty} \frac{1}{t^{\frac{2}{2-q}}} \int_0^t h(s)ds = +\infty\,,$$

which in turn, by definition of $\tilde{F}$ and by De L'Hopital's rule, gives that

$$\lim_{\delta\to 0} \tilde{F}(\delta)^{q-1}\,\delta^{2-q} = 0\,.$$

Therefore we conclude that

$$\lim_{\delta\to 0} \tilde{F}(\delta)^{q-1}|\nabla v_\delta|^q \delta^{2-q} = 0\,.$$

When $0 \le q \le 1$, the bootstrap argument applies directly to the equation of v_δ and the same conclusion holds. Note that in both cases we have that the function v satisfies, in the limit, the equation

$$-\Delta v + c_\alpha v^\alpha = 0 \qquad \text{in } \mathbf{R}^N_+, \tag{2.55}$$

and it is uniformly bounded.

By (2.47) and the dominated converge theorem,

$$\lim_{\xi\to+\infty} \frac{\int_0^\xi h(s)ds}{\xi\, h(\xi)} = \lim_{\xi\to+\infty} \frac{\int_0^1 h(\xi s)ds}{h(\xi)} = \int_0^1 \lim_{\xi\to+\infty} \frac{h(\xi s)}{h(\xi)} ds = \frac{1}{\alpha+1}, \tag{2.56}$$

then, using (2.54), there holds

$$\begin{aligned} \lim_{t\to 0} \frac{t\tilde{F}'(t)}{\tilde{F}(t)} &= \lim_{\xi\to+\infty} -\frac{\tilde{F}^{-1}(\xi)\sqrt{2\int_0^\xi h(s)ds}}{\xi} = \\ &= \lim_{\xi\to+\infty} -\tilde{F}^{-1}(\xi)\sqrt{\frac{h(\xi)}{\xi}}\sqrt{\frac{2\int_0^\xi h(s)ds}{\xi\, h(\xi)}} = -\frac{\sqrt{2(\alpha+1)}}{\alpha-1}\sqrt{\frac{2}{\alpha+1}} = -\frac{2}{\alpha-1}\,. \end{aligned} \tag{2.57}$$

Moreover, the function $\frac{\tilde{F}'(\xi)}{\tilde{F}(\xi)}$ is increasing, so that for any $\lambda > 1$,

$$1 \ge \frac{\tilde{F}(\lambda\delta)}{\tilde{F}(\delta)} = \exp\left[\log(\tilde{F}(\lambda\delta)) - \log(\tilde{F}(\delta))\right] \ge \exp\left[\frac{\tilde{F}'(\delta)\delta}{\tilde{F}(\delta)}(\lambda-1)\right] \ge \varepsilon_0 > 0\,.$$

Thus, for any $\lambda > 1$ the sequence $\frac{\tilde{F}(\lambda\,\delta)}{\tilde{F}(\delta)}$ is bounded, strictly positive, and satisfies, in view of (2.57) and (2.56), and by definition of $\tilde{F}^{-1}$,

$$\begin{aligned} \frac{\tilde{F}(\lambda\,\delta)}{\tilde{F}(\delta)} &= \frac{\tilde{F}'(\lambda\,\delta)}{\tilde{F}'(\delta)}\,\lambda(1+o(1)) = \sqrt{\frac{\int_0^{\tilde{F}(\lambda\delta)} h(s)ds}{\int_0^{\tilde{F}(\delta)} h(s)ds}}\,\lambda(1+o(1)) \\ &= \sqrt{\frac{\tilde{F}(\lambda\delta)}{\tilde{F}(\delta)}\,\frac{h(\tilde{F}(\lambda\,\delta))}{h(\tilde{F}(\delta))}}\,\lambda(1+o(1))\,. \end{aligned}$$

Using (2.47) we deduce the existence and the value of the limit below,

$$\lim_{\delta\to 0} \frac{\tilde{F}(\lambda\,\delta)}{\tilde{F}(\delta)} = \lambda^{-\frac{2}{\alpha-1}}\,. \tag{2.58}$$

Then, since we have

$$v_\delta(\xi) = \frac{u(\delta\xi + O_\delta)}{\tilde{F}(\delta(\xi_1+1) + O(\delta^{2-2\sigma}))} \frac{\tilde{F}(\delta(\xi_1+1) + O(\delta^{2-2\sigma}))}{\tilde{F}(\delta)} \tag{2.59}$$

from (2.53) (recall that $\eta = \delta\xi + O_\delta$ and dist $(\eta, \partial\Omega)$ is estimated in (2.14)) and (2.59), we conclude that

$$v_\delta(\xi) \to (1+\xi_1)^{-\frac{2}{\alpha-1}}.$$

Hence $v = (1+\xi_1)^{-\frac{2}{\alpha-1}}$. The C^1_{loc} convergence of v_δ gives then

$$\nabla v_\delta(\xi) \to -\frac{2}{\alpha-1}(1+\xi_1)^{-\frac{\alpha+1}{\alpha-1}}(1,0,\ldots,0)\,.$$

Now recall that $\nabla_\xi u(\delta\xi + 0_\delta) = \frac{\tilde{F}(\delta)}{\delta}\nabla v_\delta(\xi)$, hence using (2.57)–(2.58) we get

$$\frac{\nabla_\xi u(\delta\xi + 0_\delta)}{\tilde{F}'(\delta(1+\xi_1))} \to (1,0,\ldots,0)\,,$$

which gives (2.48) and (2.49).

(ii) Using (2.50), we have from [2] and [8]:

$$\lim_{d_\Omega(x)\to 0} \frac{u(x)}{c_q d_\Omega(x)^{-\frac{2-q}{q-1}}} = 1\,, \tag{2.60}$$

where $c_q = \left(\frac{2-q}{(q-1)\sqrt{a}}\right)^{\frac{2-q}{q-1}}$. With the same notations as above we set

$$v_\delta(\xi) = \frac{u(\delta\xi + O_\delta)}{c_q \delta^{-\frac{2-q}{q-1}}}.$$

As before, we deduce that v_δ is uniformly bounded, and satisfies

$$-\Delta v_\delta + \frac{h(u(\delta\xi + O_\delta))\delta^{\frac{q}{q-1}}}{c_q} + c_q^{q-1}\left|\nabla v_\delta\right|^q = f(\delta\xi + O_\delta)\frac{\delta^{\frac{q}{q-1}}}{c_q}\,.$$

Now assumption (2.50) implies

$$\lim_{t\to+\infty} \frac{1}{t^{\frac{2}{2-q}}} \int_0^t h(s)ds = \left(\frac{2-q}{2}\, l^{\frac{q}{2}}\right)^{\frac{2}{2-q}}\,. \tag{2.61}$$

Noticing that

$$h(u(\delta\xi + O_\delta))\delta^{\frac{q}{q-1}} = \frac{h(u(\delta\xi + O_\delta))}{\left(\int_0^{u(\delta\xi+O_\delta)} h(s)ds\right)^{\frac{q}{2}}} \frac{\left(\int_0^{u(\delta\xi+O_\delta)} h(s)ds\right)^{\frac{q}{2}}}{u(\delta\xi + O_\delta)^{\frac{q}{2-q}}}(c_q\, v_\delta)^{\frac{q}{2-q}}\,,$$

and using (2.61) and assumption (2.50), we get

$$\lim_{\delta\to 0} h(u(\delta\xi + O_\delta))\delta^{\frac{q}{q-1}} = \left(\frac{2-q}{2}\, l\right)^{\frac{q}{2-q}} (c_q\, v)^{\frac{q}{2-q}} \, .$$

Reasoning as before, we get then that v_δ is relatively compact in the C^1_{loc}-topology; observe that it will converge, up to subsequences, to a solution v of

$$-\Delta v + (\frac{2-q}{2}\, l)^{\frac{q}{2-q}}\, c_q^{\frac{q}{2-q}-1}\, v^{\frac{q}{2-q}} + c_q^{q-1}\, |\nabla v|^q = 0 \qquad \text{in } \mathbf{R}^N_+ . \tag{2.62}$$

Then, similarly as for (i), using (2.60) we can conclude that $v = (\xi_1 + 1)^{-\frac{2-q}{q-1}}$, and for the whole sequence v_δ there holds

$$\lim_{\delta\to 0} v_\delta = (\xi_1 + 1)^{-\frac{2-q}{q-1}}.$$

Moreover, let us remark that v is the unique solution of (2.62) which also satisfies

$$\lim_{\xi_1\to 0^+} v(\xi) = 1\, . \tag{2.63}$$

For the interested reader, we give a proof of such a type of results in the half space, in Theorem 4.1 below.

To conclude our proof, we deduce that $\nabla v_\delta(\xi)$ converges to $\nabla(\xi_1 + 1)^{-\frac{2-q}{q-1}}$ locally uniformly, and setting $\xi = (1, 0, \ldots, 0)$, we obtain relations (2.51)–(2.52).

Remark 2.6 The result of Theorem 2.2 still holds if one relax the assumptions on the right hand side: for the case (i), it is enough to require that $\lim\limits_{d_\Omega(x)\to 0} \frac{f(x)}{h(\tilde F(d_\Omega(x)))} = 0$, where $\tilde F$ is defined through (2.37). Note that if $h(s) = |s|^{\beta-1}s$ $(\beta > 1)$, this means $\lim\limits_{d_\Omega(x)\to 0} d_\Omega(x)^{\frac{2\beta}{\beta-1}} f(x) = 0$.

In case (ii), it would be enough to have $\lim\limits_{d_\Omega(x)\to 0} d_\Omega(x)^{\frac{q}{q-1}} f(x) = 0$; in fact, this corresponds to the case $h(s) = s^\beta$, with $\beta \le \frac{q}{2-q}$.

Remark 2.7 In case $h(u) = \lambda\, u$, the (unique) solution of (1.1)–(1.2) is the value function of an associated suitable stochastic control problem with state constraint, which is described in [11]. In that context, the field $-q|\nabla u|^{q-2}\nabla u$ is exactly the optimal feedback control, whose role is to keep the process to stay inside Ω (minimizing a certain cost functional). Our results (Theorem 2.1 and Theorem 2.3) prove the first order asymptotics for the control, i.e. $-q|\nabla u(x)|^{q-2}\nabla u(x) \sim -\frac{q'}{d_\Omega(x)}\, \nu(x)$ as $d_\Omega(x) \to 0$.

3 On the uniqueness of explosive solutions in case of concavity

In this section we give a uniqueness result for solutions of

$$\begin{cases} -\Delta u + h(u) + |\nabla u|^q = f & \text{in } \Omega, \\ \lim\limits_{d_\Omega(x)\to 0} u(x) = +\infty\,, \end{cases} \tag{3.1}$$

which applies to the case when $h(s)$ is concave. We restrict ourselves to $q > 1$, which is the significant case. Our basic criterion for uniqueness is the following.

Theorem 3.1 *Let Ω be a bounded domain and $f \in L^\infty(\Omega)$. Assume $1 < q \le 2$, and that h is a continuous increasing function satisfying the following assumption:*

$$\begin{aligned} &\exists \text{ a positive, continuous function } m(s), \text{ and constants } c_0,\, \varepsilon_0 > 0 \text{ such that} \\ &h((1+\varepsilon)a + \varepsilon b) - (1+\varepsilon)h(a) \ge \varepsilon\, b\, m(a) - c_0\, \varepsilon(1 + a^+)\,, \\ &\quad \forall a \in \mathbf{R},\ \varepsilon \in (0, \varepsilon_0),\ 0 \le b \le \tfrac{1}{\varepsilon_0}. \end{aligned} \tag{3.2}$$

If u_1, u_2 are two solutions of (3.1) such that

$$\lim_{d_\Omega(x)\to 0} \frac{u_1}{u_2} = 1\,, \qquad \lim_{d_\Omega(x)\to 0} \frac{|\nabla u_i|^q}{u_i} = +\infty \quad \forall i = 1, 2, \tag{3.3}$$

then $u_1 = u_2$.

Proof. We set $A(v) = -\Delta v + h(v) + |\nabla v|^q$. Define $u_2^\varepsilon = (1+\varepsilon)u_2 + \varepsilon T$, where T is a positive constant to be chosen later. Then

$$A(u_2^\varepsilon) = \ h((1+\varepsilon)u_2 + \varepsilon T) - (1+\varepsilon)h(u_2) + (1+\varepsilon)((1+\varepsilon)^{q-1} - 1)|\nabla u_2|^q + (1+\varepsilon)f\,,$$

and using (3.2) and that $f \in L^\infty(\Omega)$

$$A(u_2^\varepsilon) \ge f + \varepsilon[m(u_2)T + (q-1)|\nabla u_2|^q - c_1(1 + u_2^+)]\,. \tag{3.4}$$

By assumption (3.3), there exists a positive, bounded, compactly supported function $\psi(x)$ such that

$$(q-1)|\nabla u_2|^q - c_1(1 + u_2) \ge -\psi(x)\,.$$

If $K \subset \Omega$ is a compact set containing the support of ψ, we have that u_2 is bounded on K and since $m(s)$ is positive we have $\inf\limits_K m(u_2) > 0$. Setting $T = \frac{\|\psi\|_\infty}{\inf\limits_K m(u_2)}$ then it implies

$$A(u_2^\varepsilon) \ge f = A(u_1) \qquad \text{in } \Omega.$$

Moreover since $\frac{u_1}{u_2} \to 1$ as $d_\Omega(x) \to 0$, we have that $u_1 - u_2^\varepsilon \le 0$ near $\partial\Omega$. Inside Ω, we use that h is increasing to deduce that $u_1 - u_2^\varepsilon \le 0$ on any maximum point, so that we can conclude that

$$u_1 \le (1+\varepsilon)u_2 + \varepsilon T \qquad \text{in } \Omega.$$

Letting $\varepsilon \to 0$ we get $u_1 \leq u_2$. Interchanging the roles of u_1, u_2, we conclude that $u_1 = u_2$.

Let us make some comments and remarks about the previous result:

1) Assumption (3.2) is satisfied if $h(s) = h_1(s) + h_2(s)$, where h_1 is a nondecreasing convex function and h_2 is an increasing concave function. Indeed, one has, taking into account the sublinear behaviour of the concave part,

$$\begin{aligned} h((1+\varepsilon)a+\varepsilon b)-(1+\varepsilon)h(a) &\geq -\varepsilon h_1(0)+h_2((1+\varepsilon)a+\varepsilon b)-h_2((1+\varepsilon)a) \\ &\quad + h_2((1+\varepsilon)a)-h_2(a)-\varepsilon h_2(a) \\ &\geq \varepsilon\, b\, m(a) - c_0\,\varepsilon(1+a^+)\,, \end{aligned}$$

with, for instance, $m(a) = h_2'(2a^+ + 1)$ if h is differentiable, or $m(a) = h_2(2a^+ + 1) - h_2(2a^+)$ otherwise.

2) As remarked above, the previous result is meant to apply to the case that h is the sum of a convex function and an *increasing* concave function. On the other hand, we recall that in case h is purely convex the uniqueness of solutions has been proved in previous papers (see e.g. [11]), essentially using the following standard argument: if $\frac{u_1}{u_2} \to 1$ as $d_\Omega(x) \to 0$, then it is enough to take $T > -h^{-1}(m)$ where $m = \inf_\Omega f$, in order to have

$$\begin{aligned} A((1+\varepsilon)u_2+\varepsilon T) &\geq h((1+\varepsilon)u_2+\varepsilon T)-(1+\varepsilon)h(u_2) \\ &\qquad + (1+\varepsilon)((1+\varepsilon)^{q-1}-1)|\nabla u_2|^q + (1+\varepsilon)f \\ &\geq f+\varepsilon[f-h(-T)] \\ &\geq f = A(u_1), \end{aligned}$$

which yields $u_1 \leq u_2$ for any u_1, u_2 large solutions such that $\frac{u_1}{u_2} \to 1$ as $d_\Omega(x) \to 0$. Note that in this case one does not need to have any information with respect to the gradients.

3) Assumption (3.3) is not really restrictive, and is certainly satisfied in smooth domains Ω and in almost all significant situations. Indeed, this is a consequence of the results on the asymptotic behaviour of u and ∇u which are given in Section 2, so that in particular (3.3) is verified for all the situations considered in Theorem 2.1, Theorem 2.2 and Theorem 2.3, which deal with possibly power or exponential growths of h at infinity.

   In particular, this applies to the case that h is concave (which implies assumption (h1) in Theorem 2.1 and assumption (2.50) in Theorem 2.3), hence condition (3.3) follows from Section 2 and (3.2) also holds true. We get then the following corollary.

Corollary 3.1 *Let Ω be a smooth domain and $f \in L^\infty(\Omega)$. If h is increasing and concave, for any $q > 1$ problem (3.1) has a unique solution.*

On the other hand, note that for possibly larger growths of h than considered in Section 2, more precisely when

$$\text{either } q = 2 \text{ and } \lim_{s\to+\infty} \frac{h(s)e^{-2s}}{\int_0^s h(t)e^{-2t}dt} = +\infty,$$
$$\text{or } q < 2 \text{ and } \lim_{s\to+\infty} \frac{h(s)}{\int_0^s h(t)dt} = +\infty,$$

uniqueness of large solutions follows easily since one can prove directly that $u_1(x) - u_2(x) \to 0$ as $d_\Omega(x) \to 0$ for any two solutions u_1, u_2. Therefore the problem of uniqueness is really significant when h satisfies growth conditions of a similar to the ones of Section 2.

4 On some symmetry results in the half space

In the proof of Theorem 2.3 we mentioned a uniqueness result for solutions of (2.62)–(2.63). Here we give a self–contained proof of a more general result on the uniqueness, or symmetry, of nonnegative solutions of such type of problems in the half space: note that no conditions at infinity are required. More precisely, consider the problem

$$\begin{cases} -\Delta z + \alpha z^p + \beta\,|\nabla z|^q = 0 & \text{in } \mathbf{R}^N_+ := \{\xi = (\xi_1, \xi') \in \mathbf{R}^N \,:\, \xi_1 > 0\}, \\ z \geq 0 & \text{in } \mathbf{R}^N_+, \\ \lim\limits_{\xi_1\to 0^+} z(\xi_1, \xi') = M & \text{locally uniformly with respect to } \xi', \end{cases} \tag{2.1}$$

where $0 \leq M \leq \infty$, $\beta, p > 0$, and $\alpha \geq 0$.

Next we prove that the solutions of (2.1) are one–dimensional, and in particular unique if $\alpha > 0$.

Theorem 4.1 *Let $1 < q \leq 2$, $\alpha \geq 0$, $p > 0$ and $\beta > 0$. Let also $M \in [0, +\infty]$. Then*

(i) if $\alpha > 0$ problem (2.1) admits a unique solution z, and $z = z(\xi_1)$

(ii) if $\alpha = 0$ any solution of (2.1) is a function of the only variable ξ_1. In particular,

(a) if $q = 2$ then (necessarily) $M < \infty$ and $z \equiv M$.

(b) if $q < 2$ and $M < \infty$ then either $z \equiv M$ or there exists $l \in [0, M)$ such that

$$z(\xi) = l + \beta^{-\frac{1}{q-1}} \int_{\xi_1}^{+\infty} [(q-1)s + c_{M,l}]^{-\frac{1}{q-1}} ds\,,$$

where $c_{M,l}$ is uniquely determined by the implicit relation

$$\beta^{-\frac{1}{q-1}} \int_0^{+\infty} [(q-1)s + c_{M,l}]^{-\frac{1}{q-1}} ds = M - l\,,$$

while if $M = +\infty$ then there exists $l \in [0, +\infty)$ such that

$$z(\xi) = l + \beta^{-\frac{1}{q-1}} \int_{\xi_1}^{+\infty} [(q-1)s]^{-\frac{1}{q-1}} ds.$$

Proof. (i) Let $\alpha > 0$. First of all, as in Lemma 2.1, consider the radial solutions ω_R of

$$\begin{cases} -\Delta\omega_R + \alpha\omega_R^p + \beta\,|\nabla\omega_R|^q = 0 & \text{in } B_R(0), \\ \lim\limits_{\rho\uparrow R} \omega_R(\rho) = +\infty\,, \end{cases}$$

and the sequence $\{\omega_R(\xi - \xi_R)\}_{R>0}$, where $\xi_R = (R, 0)$. Note that this sequence exists since $\alpha > 0$ and $q > 1$. By local estimates we have that $\omega_R(\cdot - \xi_R)$ is locally bounded and moreover it is a decreasing sequence converging towards a function $\omega_\infty(\xi_1)$ which is the unique solution of

$$\begin{cases} -\omega_\infty'' + \alpha\omega_\infty^p + \beta\,|\omega_\infty'|^q = 0 & \text{in } (0, +\infty) \\ \omega_\infty(0) = +\infty\,. \end{cases} \tag{2.2}$$

Indeed, ω_∞ is a positive, decreasing convex function and converges to zero as ξ_1 tends to infinity. Since any solution z of (2.1) is below $\omega_R(\xi - \xi_R)$ on $B_R(R, 0)$, we deduce in the limit that

$$z \le \omega_\infty(\xi_1)\,. \tag{2.3}$$

In particular, z tends to zero as ξ_1 tends to infinity. Now, for R, $S > 0$, consider the radial solutions $\underline{\omega}_{R,S}(\rho)$ of

$$\begin{cases} -\Delta\underline{\omega}_{R,S} + \alpha\underline{\omega}_{R,S}^p + \beta\,|\nabla\underline{\omega}_{R,S}|^q = 0 & \text{in } B_{R+S}(0) \setminus B_R(0), \\ \lim\limits_{\rho\downarrow R} \underline{\omega}_{R,S} = M\,, \qquad \underline{\omega}_{R,S}(R+S) = 0 \end{cases}$$

and the sequence $\{\underline{\omega}_{R,S}(\xi - \eta_R)\}_{R,S}$, where $\eta_R = (-R, 0)$. It can be easily checked that, since $\underline{\omega}_{R,S}$ is positive and decreasing with respect to ρ, the sequence $\{\underline{\omega}_{R,S}(\cdot - \eta_R)\}_{R,S}$ is increasing respect to R and S. Letting successively $R \to \infty$ and $S \to \infty$, its limit ω_M is a one–dimensional solution of (2.1). By comparison we have that $\{\underline{\omega}_{R,S}(\xi - \eta_R)\}_R \le z(x)$, for any solution z of (2.1), hence we get in the limit

$$\omega_M(\xi_1) \le z(x) \quad \forall \xi_1 > 0. \tag{2.4}$$

Frther, since $\alpha > 0$ the one dimensional solution of (2.1) is unique; thus if $M = +\infty$, we have obtained that $z \equiv \omega_\infty(\xi_1)$.

If $M < \infty$, we need a sharper upper bound for $z(x)$. To this purpose, let $t \in (0, 1)$; we write $\xi = (\xi_1, \xi')$ and denote $B_R^{N-1} = \{|\xi'| < R\} \subset \mathbf{R^{N-1}}$. Next we construct a supersolution in the cylinder $(0, L) \times B_R^{N-1}$:

Let $\varphi_{t,L}(\xi_1)$ be the solution of the one-dimensional problem

$$\begin{cases} -\varphi_{t,L}'' + t^{p-1}\alpha\varphi_{t,L}^p + t\beta|\varphi_{t,L}'|^q = 0 & \text{in } (0, L), \\ \varphi_{t,L}(0) = \frac{M}{t} \quad \varphi_{t,L}(L) = \frac{1}{t}\omega_\infty(L)\,, \end{cases}$$

where ω_∞ is the solution defined in (2.2). We also set

$$f_t(s) = \begin{cases} (1 - t^2)^{\frac{p-1}{2}} s^p & \text{if } p \ge 1, \\ \frac{\left(\omega_\infty(L) + \sqrt{1-t^2}\, s\right)^p - \omega_\infty^p(L)}{\sqrt{1-t^2}} & \text{if } 0 < p < 1. \end{cases}$$

Now consider the function $\psi_{t,R}(\xi')$ solution of

$$\begin{cases} -\Delta\psi_{t,R} + \alpha f_t(\psi_{t,R}) + \beta\sqrt{1-t^2}\,|\nabla\psi_{t,R}|^q = 0 & \text{in } B_R^{N-1} \subset \mathbf{R^{N-1}}, \\ \lim\limits_{|\xi'|\uparrow R} \psi_{t,R} = +\infty\,. \end{cases}$$

Note that such a function exists since $q > 1$ and $f_t(s)$ is an increasing unbounded function (in fact, $f_t(s)$ behaves like $(1-t^2)^{\frac{p-1}{2}} s^p$ for s large).

Define now $\bar z(\xi_1, \xi') = t\varphi_{t,L}(\xi_1) + \sqrt{1-t^2}\,\psi_{t,R}(\xi')$, we claim that $\bar z$ is a supersolution. Indeed, using that $t\varphi_{t,L} \geq \omega_\infty(L)$ (L is meant to be large enough so that $\omega_\infty(L) < M$), we have

$$\bar z^p \geq t^p \varphi_{t,L}^p + \sqrt{1-t^2}\, f_t(\psi_{t,R}).$$

Moreover

$$|\nabla \bar z|^q = \left(t^2|\varphi'_{t,L}|^2 + (1-t^2)|\nabla\psi_{t,R}|^2\right)^{\frac{q}{2}} \geq t^2|\varphi'_{t,L}|^q + (1-t^2)|\nabla\psi_{t,R}|^q$$

by concavity since $q \leq 2$, so that

$$\begin{aligned} -\Delta\bar z + \alpha\bar z^p + \beta\,|\nabla\bar z|^q &\geq t[-\varphi''_{t,R} + t^{p-1}\alpha\varphi^p_{t,L} + t\beta|\varphi'_{t,L}|^q] \\ &+ \sqrt{1-t^2}[-\Delta\psi_{t,R} + \alpha f_t(\psi_{t,R}) + \beta\sqrt{1-t^2}\,|\nabla\psi_{t,R}|^q] = 0. \end{aligned}$$

Thus $\bar z$ is a supersolution of the equation in the cylinder $(0, L) \times B_R^{N-1}$. Moreover, since $\psi_{t,R}$ blows up at the boundary and is positive, and using (2.3), we have that $z(x) \leq \bar z(x)$ on the boundary of the cylinder. By the comparison principle we deduce that

$$z(\xi) \leq t\varphi_{t,L}(\xi_1) + \sqrt{1-t^2}\,\psi_{t,R}(\xi') \qquad \text{in } (0, L) \times B_R^{N-1}.$$

Now let R go to infinity, and use that $\psi_{t,R}$ converges to zero (as a consequence of the local estimates which depend on the distance to the boundary); we obtain that

$$z(\xi) \leq t\varphi_{t,L}(\xi_1)\,,$$

and then, letting L go to infinity,

$$z(\xi) \leq t\varphi_t(\xi_1)$$

where φ_t solves the problem

$$-\varphi''_t + t^{p-1}\alpha\varphi_t^p + t\beta|\varphi'_t|^q = 0 \quad \text{in } (0, +\infty), \quad \varphi_t(0) = \frac{M}{t}.$$

As t tends to 1, clearly φ_t converges to the unique one–dimensional solution of (2.1), which we called $\omega_M(\xi_1)$. Therefore $z \leq \omega_M(\xi_1)$, which together with (2.4) gives the claimed result.

(ii) Let now $\alpha = 0$. Up to multiplying z by a constant, we can assume that $\beta = 1$. We consider first the case $q < 2$.

First observe that, since z is a solution in $B_{\xi_1}(\xi_1,\xi')$, by the local estimates on ∇z (see e.g. [11], [13]), we have

$$|\nabla z(\xi_1,\xi')| \le C\xi_1^{-\frac{1}{q-1}} \qquad \forall (\xi_1,\xi') \in \mathbf{R}^N_+. \tag{2.5}$$

In particular,

$$|z(\eta_1,\xi') - z(\xi_1,\xi')| \le C\int_{\xi_1}^{\eta_1} t^{-\frac{1}{q-1}}\,dt, \tag{2.6}$$

and since $\frac{1}{q-1} > 1$ we deduce that $z(\xi_1,\xi')$ has a finite limit as ξ_1 goes to infinity. Due to (2.5) this limit does not depend on ξ', thus we set

$$l := \lim_{\xi_1\to+\infty} u(\xi_1,\xi')\,.$$

Using again (2.6) we also deduce the estimate:

$$l - C\,\xi_1^{-\frac{2-q}{q-1}} \le z(\xi_1,\xi') \le l + C\,\xi_1^{-\frac{2-q}{q-1}} \qquad \forall (\xi_1,\xi') \in \mathbf{R}^N_+. \tag{2.7}$$

Our goal is now to prove that $z(\xi) = \omega_l(\xi_1)$, which is the unique solution of

$$\omega_l'' = |\omega_l'|^q \quad \text{in } (0,+\infty), \quad \omega_l(0) = M\,, \qquad \lim_{\xi_1\to+\infty}\omega_l(\xi_1) = l.$$

In order to prove that $z \le \omega_l$, let $t\in(0,1)$, $C\in\mathbf{R}$, and consider the problem on $\mathbf{R^{N-1}}$:

$$\begin{cases} -\Delta\psi_{t,R} + \sqrt{1-t^2}\,|\nabla\psi_{t,R}|^q + C_R = 0 & \text{in } B_R^{N-1}\subset \mathbf{R^{N-1}}, \\ \psi_{t,R}(0) = 0\,, \quad \lim\limits_{|\xi'|\uparrow R}\psi_{t,R}(\xi') = +\infty\,. \end{cases} \tag{2.8}$$

It can be proved (see e.g. [11] for a more general result in the context of ergodic problems) that there exists a unique constant C_R such that problem (2.8) admits a solution $\psi_{t,R}$, which is also unique. Note that $C_R > 0$; moreover, by a simple scaling argument, we have

$$C_R = R^{-\frac{q}{q-1}}C_1\,, \quad \psi_{t,R} = R^{-\frac{2-q}{q-1}}\psi_{t,1}\left(\frac{|\xi'|}{R}\right)\,, \tag{2.9}$$

where C_1, $\psi_{t,1}$ are the solutions of the same problem in the unit ball B_1^{N-1}. Clearly, we also have that $\psi_{t,R}$ achieves its minimum in zero, hence $\psi_{t,R}\ge 0$. Consider also $\varphi_{t,L,R}$ solution of

$$\begin{cases} -\varphi'' + t|\varphi'|^q = \frac{\sqrt{1-t^2}}{t}C_R & \text{in } (0,L), \\ \varphi(0) = \frac{M}{t} \quad \varphi(L) = \frac{1}{t}(l + CL^{-\frac{2-q}{q-1}})\,. \end{cases}$$

As in the above case (i), using the concavity of the function $s^{\frac{q}{2}}$, one can check that the function $\bar z = t\varphi_{t,L,R}(\xi_1) + \sqrt{1-t^2}\,\psi_{t,R}(\xi')$ is a supersolution of (2.1) in the cylinder $(0,L)\times B_R^{N-1}$. Moreover, due to (2.7) and to the properties of $\psi_{t,R}$, we have $\bar z \ge z$ on the boundary, so that we deduce

$$z(\xi) \le t\varphi_{t,L,R}(\xi_1) + \sqrt{1-t^2}\,\psi_{t,R}(\xi') \quad \forall (\xi_1,\xi')\in(0,L)\times B_R^{N-1}.$$

In particular for $\xi' = 0$ we have $z(\xi_1, 0) \le t\varphi_{t,L,R}(\xi_1)$. Of course we can translate the origin in the ξ'–axis, so that we have in fact

$$z(\xi) \le t\varphi_{t,L,R}(\xi_1) \qquad \forall \xi \in \mathbf{R}^N_+.$$

Now let R go to infinity; using (2.9) we see that C_R tends to zero, hence we get

$$z(\xi) \le t\varphi_{t,L}(\xi_1), \tag{2.10}$$

where $\varphi_{t,L}$ solves

$$\begin{cases} -\varphi''_{t,L} + t|\varphi'_{t,L}|^q = 0 \quad \text{in } (0,L), \\ \varphi_{t,L}(0) = \frac{M}{t} \quad \varphi_{t,L}(L) = \frac{1}{t}\,(l + CL^{-\frac{2-q}{q-1}})\,. \end{cases}$$

As L goes to infinity, $\varphi_{t,L}$ converges to the solution of

$$-\varphi''_t + t|\varphi'_t|^q = 0 \quad \text{in } (0,\infty), \quad \varphi_t(0) = \frac{M}{t} \quad \lim_{\xi_1 \to +\infty} \varphi_t(\xi_1) = \frac{1}{t}\,\min\{l, M\}\,.$$

Then, inequality (2.10) implies, after taking the limit in L, that $z(\xi) \le t\varphi_t(\xi_1)$ for any $t \in (0,1)$. Note that, in particular, this gives $z \le M$ on the whole half space $\mathbf{R}^N_+$; by definition of l, this implies that $l \le M$. Now, as t tends to 1, clearly φ_t converges to the function $\omega_l(\xi_1)$ defined above. We conclude that

$$z(\xi) \le \omega_l(\xi_1). \tag{2.11}$$

In order to establish the reverse inequality, let $a \ge 0$, and consider the radial solutions $\omega = \omega_{a,R,S}$ of the problems

$$\begin{cases} -\Delta\omega + |\nabla\omega|^q = 0 \quad \text{in } B_{R+S}(0) \setminus B_R(0), \\ \lim\limits_{\rho \downarrow R} \omega = M\,, \qquad \omega(R+S) = a\,. \end{cases} \tag{2.12}$$

Let as before $\eta_R = (-R, \xi')$. We have that the sequence $\{\omega_{a,R,S}(\xi - \eta_R)\}_R$ is increasing and converges to a one–dimensional function $\omega_{a,S}(\xi_1)$ which is the unique solution of $\omega''_{a,S} = |\omega'_{a,S}|^q$ satisfying $\omega_{a,S}(0) = M$ and $\omega_{a,S}(S) = a$. As S goes to infinity, we have that $\omega_{a,S}$ converges to $\omega_a(\xi_1)$, which is the unique solution of

$$\omega''_a = |\omega'_a|^q \quad \text{in } (0,+\infty), \quad \omega_a(0) = M\,, \qquad \lim_{\xi_1 \to +\infty} \omega_a(\xi_1) = a.$$

In particular, if we know that $z(\xi) \ge a$ for every $\xi \in \mathbf{R}^N_+$, by comparison we deduce that $z(\xi) \ge \omega_{a,R}(\xi - \eta_R)$, and then, after letting R and S go to infinity, that $z(\xi) \ge \omega_a(\xi_1)$. Thus we have the implication

$$z(\xi) \ge a \text{ for every } \xi \in \mathbf{R}^N_+ \text{ implies } z(\xi) \ge \omega_a(\xi_1). \tag{2.13}$$

As a first step, since $z \ge 0$, this implies that $z \ge \omega_0(\xi_1)$, which together with (2.7) implies

$$z(\xi) \ge a_1 \;:= \min\left[\max\{\omega_0(\xi_1)\,,\, l - C\xi_1^{-\frac{2-q}{q-1}}\}\right]$$

Note that $0 < a_1 < l$; applying (2.13) we deduce that $z(\xi) \geq \omega_{a_1}(\xi_1)$ and in particular

$$z(\xi) \geq a_2 \ := \min \left[\max\{\omega_{a_1}(\xi_1)\,,\, l - C\xi_1^{-\frac{2-q}{q-1}}\} \right]$$

Iterating this process we define a sequence of positive real numbers $\{a_n\}$ and a sequence of functions $\{\omega_{a_n}(\xi_1)\}$ such that

$$z \geq \omega_{a_n}(\xi_1)\,, \quad a_n = \min \left[\max\{\omega_{a_{n-1}}(\xi_1)\,,\, l - C\xi_1^{-\frac{2-q}{q-1}}\} \right] .$$

As n goes to infinity, clearly we have that $a_n \uparrow l$ and $\omega_{a_n}(\xi_1)$ converges to $\omega_l(\xi_1)$, which allows to conclude that

$$z \geq \omega_l(\xi_1)\,.$$

Together with (2.11) this concludes the proof.

The case $q = 2$ is much simpler. Indeed, if $M < \infty$ it should be noted that the only nonnegative solution of $\omega'' = |\omega'|^2$ is the constant $\omega \equiv M$. In particular, one can define $\varphi_{t,L,R}$ as above except for requiring $\varphi_{t,L,R}(L) = +\infty$; in the limit (in R, L, t subsequently) one finds that $z \leq M$, while from below one has that $\omega_{0,S}$ (defined in (2.12) for $a = 0$) also converges to the constant M, so that one gets $z \geq M$, and then $z \equiv M$. If $M = +\infty$, the function $v = e^{-z}$ turns out to be harmonic in $\mathbf{R}^N_+$ with $v = 0$ on $\{\xi_1 = 0\}$; but v is also asked to satisfy $0 < v \leq 1$, and such a function cannot exist.

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