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Multiple Solitary Waves For a Non-homogeneous Schrödinger–Maxwell System in $\mathbb{R}^3$

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Abstract

We look for standing waves of nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\Delta\psi+|\psi|^{p-2}\,\psi=g(x)\,e^{i\omega t}$$

coupled with Maxwell's equations. We use the variational formulation introduced by Benci and Fortunato in 1992 for studying an eigenvalue problem for the Schrödinger-Maxwell system in bounded domains. We establish the existence of multiple standing waves both in the homogeneous and the non-homogeneous cases by means of the fibering method introduced by Pohozaev.

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1 Introduction

In this paper we state the existence of multiple radial solutions for the following perturbed system of Schrödinger-Maxwell equations

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + (e\Phi + \hbar\omega)\, u - |u|^{p-2}\, u = g(x), & x \in \mathbb{R}^3, \\ -\Delta\Phi = 4\pi e u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

with $u,\ \Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^3 \to \mathbb{R}$, $p > 2$, $e = \pm 1$ and $\hbar$, m, ω real positive constants. Such a system was first introduced in [2] as a model describing solitary waves for the nonlinear stationary Schrödinger equation in $\mathbb{R}^3$ interacting with the electrostatic field. Here, u is the wave associated with the particle, while $\hbar$ is the Planck's constant, m the mass of the particle, e the electric charge and ω the phase of the wave (for more details, see Section 2).

If $g = 0$, in [2] the existence of infinitely many radial solutions of an eigenvalue problem for the Schrödinger-Maxwell equations in bounded domains was established if $4 < p < 6$. More recently, the existence of infinitely many radial solutions of (1.1) in all $\mathbb{R}^3$ was established in [6] if $4 < p < 6$; while in [7] it was shown that a nontrivial radial solution exists if $4 \leq p < 6$. Moreover, the existence of a non radially symmetric solution was established in [8] if $4 < p < 6$.

If $g \neq 0$, the existence of infinitely many radial solutions of the coupled Schrödinger–Maxwell system in a bounded symmetric domain was proved in [5]. On the other hand, to the author's knowledge, no existence and multiplicity results for (1.1) have been obtained in all of $\mathbb{R}^3$ if $g \neq 0$. Now, using the fibering method introduced by Pohozaev in [9, 10, 11], we will give the following multiplicity results in the homogeneous and non-homogeneous case.

Theorem 1.1 *Let m, ω and $\hbar$ be real positive numbers, $e = \pm 1$ and $4 < p < 6$. Taken $g = 0$, system (1.1) has infinitely many radially symmetric solutions (u_n, Φ_n), $u_n \in H^1(\mathbb{R}^3)$, $\Phi_n \in L^6(\mathbb{R}^3)$ and $|\nabla\Phi_n| \in L^2(\mathbb{R}^3)$, with $u_n \neq 0$ and $\Phi_n \neq 0$, $e\Phi_n \geq 0$. Moreover, there exist two solutions $(u_\pm, \Phi)$, $u_\pm \neq 0$ and $\Phi \neq 0$, with $u_+ \geq 0$, $u_- \leq 0$ and $e\Phi \geq 0$.*

Theorem 1.2 *Let m, e, ω, $\hbar$ and p be as in Theorem 1.1. Taken $g(x) = g(|x|)$, $g \in L^2(\mathbb{R}^3)$ with L^2–norm small enough, system (1.1) has at least three radially symmetric solutions (u_i, Φ_i), $u_i \in H^1(\mathbb{R}^3)$, $\Phi_i \in L^6(\mathbb{R}^3)$ and $|\nabla\Phi_i| \in L^2(\mathbb{R}^3)$, $u_i \neq 0$ and $\Phi_i \neq 0$, $e\Phi_i \geq 0$.*

Remark 1.3 As already observed, if $g = 0$ a multiplicity result like the one stated in Theorem 1.1 has been obtained in [6] by exploiting the symmetry of the problem. On the contrary, a different type of result holds if $g \neq 0$. In fact, by using a suitable perturbative method, infinitely many solutions have been found in [5] for all $g(x) = g(|x|)$, $g \in L^2(\Omega)$, and $4 < p < 6$, but only if the charged particle lies in a bounded symmetric space region Ω.

This paper is organized as follows. In Section 2 we deduce system (1.1), describing a quantum particle interacting with an electrostatic field. In Section 3 we give a variational principle, as stated in [5], which allows us to reduce system (1.1) to an elliptic equation in the only variable u. Moreover, we recall the Pohozaev fibering method, usefull in order to state our multiplicity results. Finally, in Sections 4 and 5 we give the proofs of Theorems 1.1 and 1.2.

2 The Schrödinger-Maxwell equations

In this section we show that system (1.1) arises in the study of solitary waves for non-linear Schrödinger equations coupled with Maxwell equations. To this aim, we adapt the arguments used in [2] (see also [7]) if $g = 0$ to the case $g \neq 0$.

Let us consider the following nonlinear Schrödinger type equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi - |\psi|^{p-2}\,\psi - g(x)\,e^{i\omega t}, \qquad x \in \mathbb{R}^3,$$

where $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ is the wave function. The associated Lagrangian density is

$$\mathcal{L}_S(\psi)(x,t) = \frac{1}{2}\left(i\hbar\frac{\partial\psi}{\partial t}\bar{\psi} - \frac{\hbar^2}{2m}|\nabla\psi|^2\right) + \frac{1}{p}|\psi|^p + \mathrm{Re}\left(g(x)\,e^{i\omega t}\bar{\psi}\right).$$

The interaction of ψ with the electromagnetic field is described by the minimal coupling rule, that is by the formal substitution

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + i\frac{e}{\hbar}\Phi, \qquad \nabla \to \nabla - i\frac{e}{\hbar}\mathbf{A},$$

where $\mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ and $\Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ are the gauge potentials. As in [2], we do not assume that the electromagnetic field (**E**,**H**) is assigned. Then not only the wave function $\psi = \psi(x,t)$ but also the gauge potentials $\mathbf{A}$ and Φ which are related to $\mathbf{E}$, $\mathbf{H}$ by Maxwell's equations

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{H} = \nabla \times \mathbf{A}$$

are unknown. Thus, the Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_{SM}(\psi,\Phi,\mathbf{A})(x,t) \;&=\; \frac{1}{2}\left(i\hbar\frac{\partial\psi}{\partial t}\bar{\psi} - e\Phi\,|\psi|^2 - \frac{\hbar^2}{2m}\left|\nabla\psi - i\frac{e}{\hbar}\mathbf{A}\psi\right|^2\right)\\ &\quad + \frac{1}{p}|\psi|^p + \mathrm{Re}\left(g(x)\,e^{i\omega t}\bar{\psi}\right).\end{aligned}$$

If ψ is written in polar form

$$\psi(x,t) = u(x,t)\,e^{iS(x,t)/\hbar},$$

with $u, S : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, the Lagrangian density becomes

$$\begin{aligned}\mathcal{L}_{SM}(u,S,\Phi,\mathbf{A})(x,t) = &-\frac{1}{2}\left(\frac{\partial S}{\partial t} + e\Phi + \frac{1}{2m}|\nabla S - e\mathbf{A}|^2\right)u^2\\ &+\frac{1}{2}\left(i\hbar u\frac{\partial u}{\partial t} - \frac{\hbar^2}{2m}|\nabla u|^2\right) + \frac{1}{p}|u|^p + g(x)u\,\cos\left(\omega t - \frac{S}{\hbar}\right).\end{aligned}$$

Now, consider the Lagrangian density of the electromagnetic field (**E**,**H**)

$$\mathcal{L}_0(\Phi, \mathbf{A})(x,t) = \frac{|\mathbf{E}|^2 - |\mathbf{H}|^2}{16\pi} = \frac{1}{16\pi}\left(\left|\nabla\Phi + \frac{\partial \mathbf{A}}{\partial t}\right|^2 - |\nabla \times \mathbf{A}|^2\right).$$

Hence, the total action of the system "particle-electromagnetic field" is given by

$$\mathcal{L}(u, S, \Phi, \mathbf{A}) = \int (\mathcal{L}_{SM} + \mathcal{L}_0) dx dt.$$

Making the first variation of $\mathcal{L}$ with respect to u, S, Φ and $\mathbf{A}$ respectively, we obtain the following system of equations:

$$-\frac{\hbar^2}{2m}\Delta u + \left(\frac{\partial S}{\partial t} + e\Phi + \frac{1}{2m}|\nabla S - e\mathbf{A}|^2\right) u - |u|^{p-2} u = g\cos\left(\omega t - \frac{S}{\hbar}\right), \quad (2.1)$$

$$u\frac{\partial u}{\partial t} + \frac{1}{2m} div\left[(\nabla S - e\mathbf{A}) u^2\right] = -\frac{gu}{\hbar}\sin\left(\omega t - \frac{S}{\hbar}\right), \quad (2.2)$$

$$4\pi e u^2 + div\left(\nabla\Phi + \frac{\partial \mathbf{A}}{\partial t}\right) = 0, \quad (2.3)$$

$$\frac{4\pi e}{m}(\nabla S - e\mathbf{A}) u^2 - \frac{\partial}{\partial t}\left(\nabla\Phi + \frac{\partial \mathbf{A}}{\partial t}\right) + \nabla \times (\nabla \times \mathbf{A}) = 0. \quad (2.4)$$

For simplicity, we restrict ourselves to look for standing waves with the same frequency ω of the source in the electrostatic case, or better, we limit ourselves to consider

$$u(x,t) = u(x), \qquad S(x,t) = \hbar\,\omega\, t, \qquad \Phi(x,t) = \Phi(x), \qquad \mathbf{A}(x,t) = 0.$$

Then, equations (2.2) and (2.4) are identically satisfied while (2.1) and (2.3) reduce to system (1.1).

3 The variational principles

Our aim is to find solutions (u, Φ) of (1.1) with

$$u \in H^1, \qquad \Phi \in D^1,$$

where $H^1 = H^1(\mathbb{R}^3)$ is the usual Sobolev space with norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^3}(u^2 + |\nabla u|^2) dx\right)^{\frac{1}{2}}$$

and $D^1 = D^1(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the inner product

$$(v, w)_{D^1} = \int_{\mathbb{R}^3} (\nabla v, \nabla w)\, dx.$$

From now on, all the integrals are taken on $\mathbb{R}^3$ and $\|\cdot\|_s$ is the norm in $L^s(\mathbb{R}^3)$. By Sobolev imbedding theorems,

$$H^1 \hookrightarrow L^s(\mathbb{R}^3) \qquad \text{for all} \quad 2 \le s < 6$$

and

$$D^1 \hookrightarrow L^6(\mathbb{R}^3).$$

Let us point out that system (1.1) has a variational structure; in fact, if we consider the functional

$$\begin{aligned} F_g(u,\Phi) \quad = \quad & \frac{\hbar^2}{4m}\int |\nabla u|^2\,dx - \frac{1}{16\pi}\int |\nabla \Phi|^2\,dx + \frac{1}{2}\int (\hbar\omega + e\Phi)\,u^2 dx \\ & -\frac{1}{p}\int |u|^p\,dx - \int gu dx \end{aligned}$$

defined on $H^1 \times D^1$, standard arguments show that F_g is C^1 on $H^1 \times D^1$ and its critical points are solutions of system (1.1). Since F_g is strongly indefinite, i.e. it is neither bounded from below nor from above even modulo compact perturbations, arguing as in [2] and [3] we reduce the problem to studying the critical points of a new functional of the only variable u. To this aim, we recall the following result.

Lemma 3.1 *There exists a map $\Phi : H^1 \to D^1$ such that for any $u \in H^1$, the function $\Phi(u) \in D^1$ is the unique weak solution of*

$$-\Delta\Phi = 4\pi e u^2, \qquad x \in \mathbb{R}^3. \tag{3.1}$$

Moreover, $\Phi(u)$ satisfies the following properties:

(i) $e\Phi(u) \ge 0$;

(ii) $\Phi(su) = s^2\Phi(u)$ *for all* $s \in \mathbb{R}$, *in particular* $\Phi(u) = \Phi(-u)$;

(iii) if u is radially symmetric, then $\Phi(u)$ is radial.

Proof. The function Φ was introduced in [2] while its properties *(i)–(ii)*, respectively *(iii)*, were proved in [7], respectively [3]. However, for completeness, we give here the proof of this lemma.

For fixed $u \in H^1$, let us consider the linear map $\psi \in D^1 \to \int u^2\psi dx$, which is continuous since by Hölder inequality and Sobolev imbeddings a constant $a_0 > 0$ exists such that

$$\left|\int u^2\psi\,dx\right| \le \|u^2\|_{\frac{6}{5}}\|\psi\|_6 \le a_0\|u\|^2_{\frac{12}{5}}\|\psi\|_{D^1}.$$

By Lax-Milgram's lemma a unique $\Phi \in D^1$ exists such that

$$\int \nabla\Phi\cdot\nabla\psi\,dx = 4\pi e\int u^2\psi\,dx \qquad \text{for all } \psi \in D^1, \tag{3.2}$$

i.e. $\Phi(u) = -4\pi e\Delta^{-1}u^2$ is the unique solution of (3.1). Moreover, Φ is the minimum point of the associated functional, that is

$$\min_{\psi\in D^1}\left\{\frac{1}{2}\int|\nabla\psi|^2\,dx - 4\pi e\int u^2\psi dx\right\} = \frac{1}{2}\int|\nabla\Phi|^2\,dx - 4\pi e\int u^2\Phi dx.$$

If $e = \pm1$, and $\pm|\Phi|$ achieves such a minimum, then by uniqueness we have $\Phi = \pm|\Phi|$ and therefore $e\Phi(u) \geq 0$. Clearly, for all $s \in \mathbb{R}$

$$-\Delta\Phi(su) = 4\pi es^2u^2 = -s^2\Delta\Phi(u) = -\Delta(s^2\Phi(u)),$$

and, again by uniqueness, $\Phi(su) = s^2\Phi(u)$ and, in particular, $\Phi(u) = \Phi(-u)$. Finally, property *(iii)* follows by arguing as in the proof of [3, Lemma 4.2]. ■

Remark 3.2 For fixed $u \in H^1$, by (3.2) it follows that

$$\int|\nabla\Phi(u)|^2\,dx = 4\pi e\int u^2\Phi(u)dx.$$

Hence,

$$\|\Phi(u)\|^2_{D^1} = 4\pi\int u^2\,|\Phi(u)|\,dx \leq 4\pi\|u^2\|_{\frac{6}{5}}\|\Phi(u)\|_6 \leq 4\pi a_0\|u\|^2_{\frac{12}{5}}\|\Phi(u)\|_{D^1}$$

and therefore

$$\|\Phi(u)\|_{D^1} \leq 4\pi a_0\|u\|^2_{\frac{12}{5}}, \qquad \int u^2\,|\Phi(u)|\,dx \leq 4\pi a_0^2\|u\|^4_{\frac{12}{5}}. \tag{3.3}$$

Now, we can state the following proposition (see also [2, Proposition 5]).

Proposition 3.3 *There exists a C^1 functional $J_g : H^1 \to \mathbb{R}$, defined as*

$$J_g(u) = \frac{\hbar^2}{4m}\int|\nabla u|^2dx + \frac{\omega\hbar}{2}\int u^2dx + \frac{e}{4}\int u^2\Phi(u)dx - \frac{1}{p}\int|u|^pdx - \int gudx, \tag{3.4}$$

such that the following statements are equivalent:

(i) $(u,\Phi) \in H^1 \times D^1$ is a critical point of F_g,

(ii) u is a critical point of J_g and $\Phi = \Phi(u)$.

Proof. Standard arguments imply that the map Φ defined in Lemma 3.1 is C^1 on H^1; moreover, by definition, its graph G_Φ is given by

$$G_\Phi = \left\{(u,\Phi) \in H^1 \times D^1 :\ \frac{\partial F_g}{\partial\Phi}(u,\Phi) = 0\right\}. \tag{3.5}$$

Now, let us introduce the functional $J_g : H^1 \to \mathbb{R}$ defined as

$$\begin{aligned} J_g(u) &= F_g(u,\Phi(u)) = \frac{\hbar^2}{4m}\int|\nabla u|^2\,dx - \frac{1}{16\pi}\int|\nabla\Phi(u)|^2\,dx \\ &\quad + \frac{1}{2}\int(\hbar\omega + e\Phi(u))\,u^2dx - \frac{1}{p}\int|u|^p\,dx - \int gudx. \end{aligned}$$

By (3.5), J_g is as in (3.4), it is C^1 and

$$J_g'(u) = \frac{\partial F_g}{\partial u}(u, \Phi(u)) = -\frac{\hbar^2}{2m}\Delta u + (\hbar\omega + e\Phi(u))\, u - |u|^{p-2}\, u - g$$

(as an operator in H^{-1}). Thus, the proof is completed. ■

Hence, looking for solutions of system (1.1) is equivalent to studying critical points of functional J_g depending only on variable u. However, since the embedding of H^1 in $L^s(\mathbb{R}^3)$, $2 \le s < 6$, is not compact, we cannot study directly critical points of the functional J_g, as also this functional is strongly indefinite, i.e. it is neither bounded from above nor from below on H^1 modulo compact perturbation. Consequently, J_g does not satisfy a compactness condition of the Palais-Smale type in an obvious way. Hence, we will restrict J_g to the subspace of the radial functions

$$H_r = \left\{u \in H^1 : u(x) = u(|x|)\right\}.$$

By virtue of Lemma 3.1 *(iii)*, it follows easily that H_r is a natural constraint for J_g, i.e. any critical point of $J_{g|H_r}$ is also a critical point of J_g. Therefore, from now on we will look for critical points of $J_{g|H_r}$, still denoted by J_g. It has been proved (see [4] and [12]) that

$$H_r \hookrightarrow\hookrightarrow L_r^s \qquad 2 < s < 6, \tag{3.6}$$

where $L_r^s = L^s(\mathbb{R}^3) \cap H_r$. Hence, (3.4), (3.6) and Lemma 3.1 *(i)* imply that J_g is bounded from below modulo the compact perturbation

$$u \longmapsto \frac{1}{p}\int |u|^p\, dx.$$

So, it is possible to prove that for $4 \le p < 6$ J_g satisfies the Palais-Smale condition (see [7, Proposition 3.3] if $g = 0$ and [5] if $g \neq 0$). Whence, at least if $g = 0$, the existence of a non trivial critical point of J_0 follows by a direct application of the classical Mountain Pass theorem (see [1] and [7]). Moreover, since in this case J_0 is even, estimate (3.3) on the growth at infinity of the nonlinear term $\int u^2\Phi(u)dx$ allows one to prove that if $4 < p < 6$, J_g satisfies a symmetric version of a linking theorem (see [6]), then infinitely many solutions of system (1.1) can be found. On the contrary, if $g \neq 0$, the problem loses its symmetry and existence and multiplicity results cannot be stated in general. In the next sections we shall prove Theorems 1.1 and 1.2 using the "algebraic" approach based on the fibering method, more precisely on the "spherical fibering" method, introduced by Pohozaev (see [9, 10, 11]). For completeness, here we recall it briefly.

Let Y be a real Banach space with a norm which is differentiable for $w \neq 0$, and let E be a functional on Y of class $C^1(Y \setminus \{0\})$. To functional E we can associate a new functional $\tilde{E}$ defined on $\mathbb{R} \times Y$ by

$$\tilde{E}(t, v) = E(tv).$$

Denoted by S, the unit sphere in Y, the following result holds (see in [10, Theorem 1.2.1]).

Theorem 3.4 *Let Y be a real Banach space with norm differentiable on $Y \setminus \{0\}$, and let $(t, v) \in (\mathbb{R} \setminus \{0\}) \times S$ be a conditionally stationary point of the functional $\tilde{E}$ on $\mathbb{R} \times S$. Then, the vector $u = tv$ is a nonzero "free" stationary point of the functional E, that is, $E'(u) = 0$.*

The previous theorem says that any critical point (t, v) of $\tilde{E}$ restricted on $(\mathbb{R} \setminus \{0\}) \times S$ generates the free nontrivial critical point $u = tv$ of E and vice versa, that is equation

$$E'(u) = 0, \quad u \neq 0,$$

is equivalent to the system

$$\begin{cases} \dfrac{\partial \tilde{E}}{\partial t}(t, v) = 0 \\ \dfrac{\partial \tilde{E}}{\partial v}(t, v) = 0 \end{cases}$$

for $\|v\| = 1$.

In the following, the first scalar equation of the previous system will be called a "bifurcation equation".

4 Proof of Theorem 1.1

Let us assume $g = 0$. Then by Lemma 3.1*(ii)*, the functional J_0 is even. From now on, as $\omega > 0$, we denote by $\|\cdot\|$ the equivalent norm in H^1

$$\|u\| = \left(\frac{\hbar^2}{2m} \int |\nabla u|^2 \, dx + \omega \int u^2 dx \right)^{\frac{1}{2}}.$$

According to the spherical fibering method, we look for critical points $u \in H_r$ of J_0 in the form

$$u = tv \quad \text{where } t \in \mathbb{R}, t \neq 0, \text{ and } v \in S, \text{ with } S = \{v \in H_r : \|v\| = 1\}.$$

Then, by Lemma 3.1*(ii)*, the functional J_0 can be extended to the space $\mathbb{R} \times H_r$ by setting

$$\tilde{J}_0(t, v) = J_0(tv) = \frac{t^2}{2}\|v\|^2 + \frac{t^4 e}{4} \int v^2 \Phi(v) dx - \frac{|t|^p}{p} \int |v|^p dx.$$

Clearly, the restriction of $\tilde{J}_0$ on $\mathbb{R} \times S$, still denoted by $\tilde{J}_0$, becomes

$$\tilde{J}_0(t, v) = \frac{t^2}{2} + \frac{t^4 e}{4} \int v^2 \Phi(v) dx - \frac{|t|^p}{p} \|v\|_p^p.$$

Hence the bifurcation equation $\frac{\partial \tilde{J}_0}{\partial t}(t, v) = 0$ takes the form

$$t + t^3 e \int v^2 \Phi(v) dx - |t|^{p-2} \, t \, \|v\|_p^p = 0$$

or, equivalently, for $t \neq 0$,

$$1 + t^2 e \int v^2 \Phi(v) dx - |t|^{p-2} \|v\|_p^p = 0. \tag{4.1}$$

Let us point out that for any $v \in S$ equation (4.1) has at least two nontrivial solutions $\pm t(v)$. In fact, setting

$$\varphi_v(t) = 1 + t^2 e \int v^2 \Phi(v) dx - |t|^{p-2} \|v\|_p^p,$$

$\lim_{|t| \to +\infty} \varphi_v(t) = -\infty$ (as $p > 4$), $\varphi_v(0) = 1$ and, for $t \neq 0$, $\varphi_v'(t) = 0$ if and only if

$$\bar{t}(v) = \pm \left(\frac{2e \int v^2 \Phi(v) dx}{(p-2) \|v\|_p^p} \right)^{\frac{1}{p-4}}.$$

Hence, the functional $\hat{J}_0(v) = \tilde{J}_0(t(v), v)$ becomes

$$\hat{J}_0(v) = \frac{t^2(v)}{2} + \frac{t^4(v)\, e}{4} \int v^2 \Phi(v) dx - \frac{|t(v)|^p}{p} \|v\|_p^p$$

on the unit sphere S of H_r, or, equivalently, by bifurcation equation (4.1)

$$\hat{J}_0(v) = \left(\frac{1}{2} - \frac{1}{p} \right) t^2(v) + \left(\frac{1}{4} - \frac{1}{p} \right) t^4(v) e \int v^2 \Phi(v) dx. \tag{4.2}$$

Obviously, by (4.2) and $p > 4$ it follows that $\hat{J}_0$ is bounded from below. We claim that $\hat{J}_0$ is weakly continuous on S. Let $(v_n)_n \subset S$ and $v \in H_r$ be such that v_n converges weakly to v in H_r. By (3.6) and $2 < s < 6$ we have

$$\|v_n\|_s \to \|v\|_s \quad \text{as } n \to +\infty. \tag{4.3}$$

Now, we need to prove that

$$\int v_n^2 \Phi(v_n) dx \to \int v^2 \Phi(v) dx. \tag{4.4}$$

To this aim, let us note that

$$\begin{aligned} \left| \int \left(v_n^2 \Phi(v_n) - v^2 \Phi(v) \right) dx \right| \;\leq\; & \int \left| v_n^2 - v^2 \right| |\Phi(v_n)| \, dx \\ & + \int v^2 \left| \Phi(v_n) - \Phi(v) \right| dx. \end{aligned} \tag{4.5}$$

Clearly, by Hölder inequality and (3.3) a positive constant c exists such that

$$\begin{aligned} \int \left| v_n^2 - v^2 \right| |\Phi(v_n)| \, dx &\leq \|v_n - v\|_3 \|v_n + v\|_2 \|\Phi(v_n)\|_6 \\ &\leq a_0 \|v_n - v\|_3 \|v_n + v\|_2 \|\Phi(v_n)\|_{D^1} \\ &\leq 4\pi a_0^2 \|v_n - v\|_3 (1 + \|v\|_2) \|v_n\|_{\frac{12}{5}}^2 \leq c \|v_n - v\|_3. \end{aligned}$$

Hence (4.3) implies

$$\int \left| v_n^2 - v^2 \right| |\Phi(v_n)| \, dx \to 0. \tag{4.6}$$

On the other hand, denoted $L : H_r \to H_r'$ as $Lv = -\Delta v$, we have $\Phi(v) = L^{-1}(-4\pi e v^2)$. Since $(v_n)_n \subset S$, it is bounded in L_r^3. Thus $\left(v_n^2\right)_n$ is bounded in $L_r^{\frac{3}{2}} \hookrightarrow\hookrightarrow H_r'$, where H_r' denotes the dual of H_r. Hence,

$$L^{-1}(v_n^2) \to L^{-1}(v^2) \qquad \text{in } H_r,$$

and therefore

$$\Phi(v_n) \to \Phi(v) \qquad \text{in } H_r \hookrightarrow L_r^{\frac{3}{2}}.$$

Since

$$\int v^2 \, |\Phi(v_n) - \Phi(v)| \, dx \leq \|v^2\|_3 \|\Phi(v_n) - \Phi(v)\|_{\frac{3}{2}}$$

we have that

$$\int v^2 \, |\Phi(v_n) - \Phi(v)| \, dx \to 0. \tag{4.7}$$

Now, (4.4) follows by (4.5) - (4.7). Since (4.3) and (4.4) hold, by the implicit function theorem, the sequence of solutions $(t(v_n))_n$ of equation (4.1) converges to the corresponding solution $t(v)$, and therefore

$$\hat{J}_0(v_n) \to \hat{J}_0(v).$$

Hence, we conclude that $\hat{J}_0$ is weakly continuous on S. Clearly, by the Weierstrass theorem, $\hat{J}_0$ attains its minimum at a point $\bar{v}$ which belongs to the closed unit ball B in H_r. Now, we prove that $\bar{v} \in S$. In fact, taking $\theta > 0$, by the bifurcation equation we obtain

$$\begin{aligned} \frac{d}{d\theta}\hat{J}_0(\theta v) &= \left[t(\theta v)v + t^3(\theta v)v\theta^4 e \int v^2\Phi(v)dx - |t(\theta v)|^{p-2}\, t(\theta v)v\|\theta v\|_p^p \right] \times \\ & \quad \frac{d}{d\theta}t(\theta v) + t^4(\theta v)\theta^3 e \int v^2\Phi(v)dx - |t(\theta v)|^p\, \theta^{p-1}\|v\|_p^p \\ &= -\frac{t^2(\theta v)}{\theta} < 0. \end{aligned}$$

Then, $\hat{J}_0(\theta v)$ decreases with respect to $\theta \in]0,1]$ and attains its minimum for $\theta = 1$, i.e. $\bar{v} \in S$. As, by definition, $\Phi(v) = \Phi(|v|)$, we have $\hat{J}_0(v) = \hat{J}_0(|v|)$. So, according to the fibering method, we conclude that

$$u_+ = t\left(|\bar{v}|\right)|\bar{v}|, \qquad u_- = -t\left(|\bar{v}|\right)|\bar{v}|$$

are a positive and a negative critical point of J_0 respectively. By Proposition 3.3 it follows that $(u_+, \Phi(u_+))$ and $(u_-, \Phi(u_-))$ are two nontrivial solutions of system (1.1) if $g = 0$. Moreover, since $\hat{J}_0$ is even, positive, weakly continuous and of class C^1 on S, by applying the classical Lusternik-Schnirelmann theory, we prove that $\hat{J}_0$ has a sequence of geometrical different constrained critical points $v_1, v_2, \ldots, v_n, \ldots$ on S with $\hat{J}_0(v_n) \to +\infty$ as $n \to +\infty$. Hence, by fibering method, J_0 has a sequence of geometrically different critical points $\pm u_1, \pm u_2, ..., \pm u_n, ...$ with $u_n(x) = t(v_n)v_n$ such that $J_0(v_n) \to +\infty$. So, the conclusion of the proof of Theorem 1.1 follows again by Proposition 3.3. ■

5 Proof of Theorem 1.2

Let us consider the case $g \neq 0$. According to the notations introduced in the previous section, let us denote by $\tilde{J}_g$ both the extension of J_g to space $\mathbb{R} \times H_r$, that is

$$\tilde{J}_g(t,v) = J_g(tv) = \frac{t^2}{2}\|v\|^2 + \frac{t^4 e}{4}\int v^2\Phi(v)dx - \frac{|t|^p}{p}\int |v|^p dx - t\int gvdx,$$

and its restriction to "unit sphere" $\mathbb{R} \times S$, that is

$$\tilde{J}_g(t,v) = \frac{t^2}{2} + \frac{t^4 e}{4}\int v^2\Phi(v)dx - \frac{|t|^p}{p}\int |v|^p dx - t\int gvdx.$$

Now, we will prove that, if g is small enough, for any $v \in S$ the bifurcation equation

$$t + t^3 e\int v^2\Phi(v)dx - |t|^{p-2}\, t\|v\|_p^p - \int gvdx = 0 \tag{5.1}$$

has at least three different roots $t_i(v)$, $i = 1, 2, 3$. To this aim, set

$$\psi_v(t) = t + t^3 e\int v^2\Phi(v)dx - |t|^{p-2}\, t\|v\|_p^p.$$

Obviously, $\psi_v(t)$ is odd, $\lim_{t\to+\infty}\psi_v(t) = -\infty$ (since $p > 4$) and $\psi_v'(t) = 0$ has exactly two solutions since it reduces to an equation like (4.1). Whence, ψ_v has a local maximum M_v and a local minimum $m_v = -M_v$ and equation (5.1) has three distinct roots if

$$\left|\int gvdx\right| < M_v.$$

We are not able to calculate the local maximum M_v. However, by Lemma 3.1*(i)*, for all $v \in S$ we have

$$\psi_v(t) \geq \bar{\psi}_v(t) = t - |t|^{p-2}\, t\|v\|_p^p \qquad \text{for all } t \geq 0$$

and direct calculations show that $\bar{\psi}_v(t)$ has a local maximum $\bar{M}_v$ and a local minimum $\bar{m}_v = -\bar{M}_v$ with

$$\bar{M}_v = \bar{\psi}_v\left(\left((p-1)\|v\|_p^p\right)^{-\frac{1}{p-2}}\right) = (p-2)(p-1)^{-\frac{p-1}{p-2}}\|v\|_p^{-\frac{p}{p-2}}.$$

As $\bar{M}_v \leq M_v$, it follows that the bifurcation equation possesses three isolated smooth branches of solutions $t_i = t_i(v)$, $i = 1, 2, 3$, if we take

$$\sup_{v\in S}\left\{\left|\int gvdx\right|\, \|v\|_p^{\frac{p}{p-2}}\right\} < (p-2)(p-1)^{-\frac{p-1}{p-2}}. \tag{5.2}$$

Hence, we obtain three distinct functionals

$$\begin{aligned}
\hat{J}_{g,i}(v) &= \tilde{J}_g(t_i(v), v) \\
&= \frac{1}{2}t_i^2(v) + \frac{t_i^4(v)}{4}e\int v^2\Phi(v)dx - \frac{1}{p}\,|t_i(v)|^p\,\|v\|_p^p - t_i(v)\int gvdx
\end{aligned}$$

defined on $B \setminus \{0\}$. We will prove that for each $i = 1,2,3$, $\hat{J}_{g,i}$ attains its minimum at a point $\bar{v}_i \in S$ such that $t_i(\bar{v}_i) \neq 0$. Indeed, given a minimizing sequence $(v_{n,i}) \subset S$ for functional $\hat{J}_{g,i}$ on S, there exists $\bar{v}_i \in B$ such that, passing to a subsequence,

$$v_{n,i} \rightharpoonup \bar{v}_i \qquad \text{in } H_r.$$

Arguing as in the case $g = 0$, it is easy to prove that $\hat{J}_{g,i}$ is weakly continuous, and thus

$$\hat{J}_{g,i}(\bar{v}_i) = \inf_{v \in S} \hat{J}_{g,i}(v) < 0,$$

and, therefore, $t_i(\bar{v}_i) \neq 0$. Moreover, for all $\theta > 0$ we have

$$\begin{aligned}\frac{d}{d\theta}\hat{J}_{g,i}(\theta v) \quad = \quad & \left[t_i(\theta v)v + t_i^3(\theta v)v\theta^4 e \int v^2\Phi(v)dx\right] \frac{d}{d\theta}t_i(\theta v) \\ & - \left[|t_i(\theta v)|^{p-2}\, t_i(\theta v)v\|\theta v\|_p^p + v \int g\theta v dx\right] \frac{d}{d\theta}t_i(\theta v) \\ & + t_i^4(\theta v)\theta^3 e \int v^2\Phi(v)dx - |t_i(\theta v)|^p\, \theta^{p-1}\|v\|_p^p - t_i(\theta v) \int gv dx.\end{aligned}$$

Hence, by the bifurcation equation, we deduce

$$\frac{d}{d\theta}\hat{J}_{g,i}(\theta v) = -\frac{1}{\theta}t_i^2(\theta v) < 0 \qquad \text{for any } \theta > 0.$$

Then, $\hat{J}_{g,i}(\theta v)$ decreases with respect to $\theta \in]0,1]$ and attains its minimum for $\theta = 1$, i.e. $\bar{v}_i \in S$. Theorem 3.4 implies that the original action functional J_g has at least three critical points of the form $\bar{u}_i(x) = t_i\bar{v}_i(x)$; hence, the proof of Theorem 1.2 follows again from Proposition 3.3. Finally, as the sign of $t_i(\bar{v}_i)$ depends on the sign of $\int g\bar{v}_i dx$, we have

$$\int g\bar{v}_1 dx \leq 0, \qquad \int g\bar{v}_2 dx \geq 0, \qquad \int g\bar{v}_3 dx \geq 0.$$

■

Remark 5.1 Let us point out that inequality (5.2) is satisfied if the L^2–norm of g is small enough, the estimate for $\|g\|_2$ depending on p and on the embedding constant of H_r into L_r^p.

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