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Multiplicity of Solutions For a Convex-concave Problem With a Nonlinear Boundary Condition

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Abstract

We study the existence of multiple positive solutions for a convex-concave problem, denoted by $(P_{\lambda\mu})$, with a nonlinear boundary condition involving two critical exponents and two positive parameters λ and μ. We obtain a continuous strictly decreasing function f such that $K_1 \equiv \{(f(\mu), \mu) : \mu \in [0, \infty)\}$ divides $[0, \infty) \times [0, \infty) \setminus \{(0, 0)\}$ in two connected sets K_0 and K_2 such that problem $(P_{\lambda\mu})$ has at least two solutions for $(\lambda, \mu) \in K_2$, at least one solution for $(\lambda, \mu) \in K_1$ and no solution for $(\lambda, \mu) \in K_0$.

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1 Introduction

In this paper we study the existence of multiple positive solutions for a convex-concave problem with a nonlinear boundary condition involving two critical exponents and two positive parameters λ and μ of the type

$$\begin{cases} -\Delta u + u = \lambda u^{q_1} + u^{p_1} & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = \mu u^{q_2} + u^{p_2} & \text{on} \quad \partial\Omega, \\ u > 0 & \text{in} \quad \Omega, \end{cases} \qquad (P_{\lambda\mu})$$

where $0 < q_i < 1 < p_i < \infty$ $(i = 1, 2)$, $\Omega \subset I\!\!R^N$ $(N \geq 3)$ is a smooth bounded domain and $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative.

By Pohozaev identity (see [1]), problem $(P_{\lambda\mu})$ does not have any solution when $\lambda, \mu \leq 0$. While for the Dirichlet elliptic problems involving nonlinearities convex-concave, that is, when the nonlinearities are a sum of a sublinear and a superlinear terms, they were studied in [3] (see [4], [13] and [14] for some improvement involving the p-laplacian operator).

Actually in [3] they considered the Dirichlet problem of the form

$$\begin{cases} -\Delta u = \lambda u^{q} + u^{p} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u > 0 & \text{in} \quad \Omega, \end{cases} \qquad (P_{\lambda})$$

where Ω is a smooth bounded domain, $\lambda > 0$, $0 < q < 1$ and $1 < p \leq 2^* - 1 = \frac{N+2}{N-2}$. In the critical case, that is, when $p = 2^* - 1$, they proved that there exists $\Lambda > 0$ such that problem (P_λ) has at least two solutions for all $\lambda \in (0, \Lambda)$, at least one solution for $\lambda = \Lambda$ and no solution for all $\lambda > \Lambda$. After pioneering paper [7], problem (P_λ) corresponding to $0 < q < p = 2^* - 1$ has been studied extensively in recent years (e.g. [21]). For the Neumann case, recently, in [15], a class of elliptic problems involving convex-concave nonlinearities of the form

$$\begin{cases} -\Delta u + u = |u|^{p-1}u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = \mu |u|^{q-1}u & \text{on} \quad \partial\Omega, \end{cases} \qquad (P_{0\mu})$$

was treated, where Ω is a smooth bounded domain, $\mu > 0$, q and p as in (P_λ). They obtained results similar to those obtained in [3]. In [15] is also considered the problem

$$\begin{cases} -\Delta u + u = \lambda |u|^{q-1}u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = |u|^{p-1}u & \text{on} \quad \partial\Omega, \end{cases} \qquad (P_{\lambda 0})$$

where Ω is as (P_λ) with $\lambda > 0$, and $0 < q < 1 < p < 2_* - 1 = \frac{N}{N-2}$. We recall that 2^* and 2_* are the limiting Sobolev exponents for the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{2_*}(\partial\Omega)$ respectively.

For the multiplicity results in the inhomogeneous Neumann problem, one is referred to [11] and the references therein. We would also like to mention the works in ([1, 5, 9, 10, 17, 18], and [20]) for the Neumann problem involving convex nonlinearities. In this paper, by combining argument used in [3] and [17], we study the existence of multiple solutions for the problems related to $(P_{\lambda 0})$ and $(P_{0\mu})$. Notice that our problem involves two

Sobolev critical exponents, bringing two difficulties to our approach because of the lack of compactness of the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{2_*}(\partial\Omega)$. We overcame these difficulties by showing that the critical levels of "approximate solutions " lie below of a number which is a combination of two best constants of the above embedding. As in [7] this information implies that sequence of "approximate solutions " is relatively compact.

By sub and super solution method (see [2]), we obtain a minimal positive solution of $(P_{\lambda\mu})$. We also get some properties of the minimal solution, as well as, for the function $f : [0, M_0] \to [0, \Lambda_0]$ defined by

$$f(\mu) = \sup\{\lambda \in [0, \infty) : \text{problem } \ (P_{\lambda\mu}) \ \text{ has a classical solution}\},$$

where Λ_0 and M_0 are positive constants such that problem $(P_{\lambda 0})$ has at least a solution $\forall\lambda \leq \Lambda_0$ and $\mu = 0$, and problem $(P_{0\mu})$ has at least a solution $\forall\mu \leq M_0$ and $\lambda = 0$, respectively. These constants can be found arguing as in [3] and [15] (see Lemma 2.1).

We will split $[0, \infty) \times [0, \infty) \setminus \{(0, 0)\}$ such that

$$[0, \infty) \times [0, \infty) \setminus \{(0, 0)\} = K_0 \cup K_1 \cup K_2$$

where $K_1 \equiv \{(f(\mu), \mu) : \mu \in [0, M_0]\}$, $K_2 \equiv \{(\lambda, \mu); \quad \mu \in [0, M_0) \ \text{ and } \ 0 \leq \lambda < f(\mu)\} \setminus \{(0, 0)\}$ and $K_0 \equiv (K_1 \cup K_2)^c$.

Finally, we employ a version of the Ambrosetti-Rabinowitz Mountain Pass Theorem due to Ghoussoub and Preiss [16] in order to get the second positive solution. Our main results of this paper are stated as follows.

Theorem 1.1 *There exists a strictly decreasing function $f : [0, M_0] \to [0, \Lambda_0]$ such that:*

i) *For $(f(\mu), \mu) \in K_1$, problem $(P_{f(\mu)\mu})$ has a solution $u_{f(\mu)\mu}$.*

ii) *For all $(\lambda, \mu) \in K_2$, problem $(P_{\lambda\mu})$ has a minimal solution $u_{\lambda\mu}$, such that $I_{\lambda\mu}(u_{\lambda\mu}) < 0$. Moreover, $u_{\lambda\mu}$ are ordered in the sense that $\forall(\lambda_i, \mu_i) \in K_2$, $i = 1, 2$, such that $\lambda_1 \leq \lambda_2$ and $\mu_1 < \mu_2$ or $\lambda_1 < \lambda_2$ and $\mu_1 \leq \mu_2$, we have $u_{\lambda_1\mu_1} < u_{\lambda_2\mu_2}$.*

iii) *For all $(\lambda, \mu) \in K_0$, problem $(P_{\lambda\mu})$ has no solution.*

Remark 1.1 A regularity result due to Brezis and Kato [6], which was proved for Dirichlet problems, still holds for problem $(P_{\lambda\mu})$, since its proof can be adapted for our case (see Appendix). Moreover proceeding as in [8] we can conclude that $u_{\lambda\mu} \in C^{2,\beta}(\bar{\Omega})$, for some $\beta \in (0, 1)$.

Remark 1.2 The solutions of problema $(P_{\lambda\mu})$ are classical solutions if $p_1 \leq 2^* - 1$ and $p_2 \leq 2_* - 1$, while when $p_1 > 2^* - 1$ and $p_2 > 2_* - 1$ the solutions are in $H^1(\Omega) \cap L^{p_1+1}(\Omega) \cap L^{p_2+1}(\partial\Omega)$.

Theorem 1.2 *For all* $(\lambda, \mu) \in K_2$, *problem* $(P_{\lambda\mu})$ *has a second solution* $v_{\lambda\mu} > u_{\lambda\mu}$, *provided that* $p_1 \leq 2^* - 1$ *and* $p_2 \leq 2_* - 1$.

Remark 1.3 Arguing as in [3], we can prove that if $w_{\lambda\mu}$ is a solution of problem $(P_{\lambda\mu})$ such that $w_{\lambda\mu} \neq u_{\lambda\mu}$ and $u_{\lambda\mu}$ is a minimal solution, then $||w_{\lambda\mu}||_\infty \to \infty$ as $|(\lambda, \mu)| \to 0$.

The figure below illustrates the diagram of solution for problem $(P_{\lambda\mu})$.

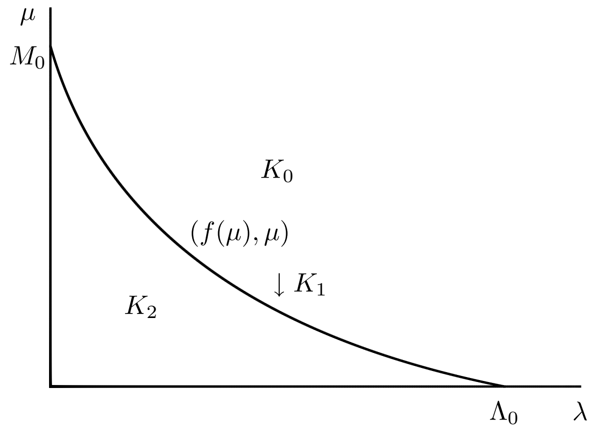

Figure 1: Solution diagram

By solutions to problema $(P_{\lambda\mu})$, we understand as the critical points of the associated energy functional, defined on $H^1(\Omega)$, given by

$$\begin{aligned} I_{\lambda\mu}(u) &= \tfrac{1}{2}\int_\Omega(|\nabla u|^2 + |u|^2)dx - \int_\Omega\left(\frac{\lambda u_+^{q_1+1}}{q_1+1} + \frac{u_+^{p_1+1}}{p_1+1}\right)dx \\ &\quad - \int_{\partial\Omega}\left(\frac{\mu u_+^{q_2+1}}{q_2+1} + \frac{u_+^{p_2+1}}{p_2+1}\right)d\sigma \end{aligned}$$

where $d\sigma$ is the measure on the boundary and $u_+ = max\{0, u\}$. It is standard to see that $I \in C^1(H^1(\Omega), I\!R)$ and

$$\begin{aligned} I'_{\lambda\mu}(u)v &= \int_\Omega(\nabla u \nabla v + uv)dx - \int_\Omega(\lambda u_+^{q_1}v + u_+^{p_1}v)dx \\ &\quad - \int_{\partial\Omega}(\mu u_+^{q_2}v + u_+^{p_2}v)d\sigma. \end{aligned}$$

Notation: In the rest of the paper we will make use of the following notation. $\int_\Omega f(x)dx$ and $\int_{\partial\Omega} g(x)d\sigma$ will be denoted by $\int_\Omega f$ and $\int_{\partial\Omega} g$ respectively; $||.||_p$ means the norm in $L^p(\Omega)$ or $L^p(\partial\Omega)$ spaces; $||u||^2 = \int_\Omega(|\nabla u|^2 + u^2)$ denotes the norm in $H^1(\Omega)$; $C,\ C_a,\ C_b,\ C_1,\ C_2, \ldots,$ mean (possibly different) positive constants.

The organization of this paper is as follows: In Section 2 we discuss the existence and nonexistence of the solution of $(P_{\lambda\mu})$, by using sub and super solution method. In section 3 the existence of second positive solution for $(P_{\lambda\mu})$ is established, by applying a version of the mountain pass Theorem.

2 Proof of Theorem 1.1

Firstly, we shall state the curve limit of the region K_2.

Lemma 2.1 *We have* $0 < f(\mu) < \infty, \ \ \forall \mu \in [0, M_0)$ *and* $f(M_0) = 0$. *Also* $0 < g(\lambda) < \infty, \ \ \forall \lambda \in [0, \Lambda_0)$ *where* $g(\lambda) \equiv sup\{\mu \in (0, \infty) : \ \ problem \ \ (P_{\lambda\mu}) \ \ has \ a \ classical \ solution\}$.

Proof. Let Ψ be the unique positive solution of

$$\begin{cases} -\Delta\Psi + \Psi = 1 & \text{in} \quad \Omega, \\ \frac{\partial\Psi}{\partial\nu} = 1 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $0 < q_i < 1, \quad i = 1, 2$, we can find $\lambda_0 > 0$ and $\mu_0 > 0$ such that for all $(\lambda, \mu) \in (0, \lambda_0) \times (0, \mu_0)$ there exists $M \equiv M(\lambda, \mu) > 0$ satisfying

$$M \geq \lambda M^{q_i} ||\Psi||_\infty^{q_i} + M^{p_i} ||\Psi||_\infty^{p_i}, \quad i = 1, 2.$$

Thus $M\Psi$ is a supersolution of $(P_{\lambda\mu})$. In fact

$$M = -\Delta(M\Psi) + M\Psi \geq \lambda(M\Psi)^{q_1} + (M\Psi)^{p_1} \ \text{ in } \ \Omega$$

$$M = \frac{\partial M\Psi}{\partial\nu} \geq \mu(M\Psi)^{q_2} + (M\Psi)^{p_2} \ \text{ on } \ \partial\Omega$$

On the other hand, taking $\varphi_1 > 0$ the solution of

$$\begin{cases} -\Delta\varphi_1 + \varphi_1 = \lambda_1\varphi_1 & \text{in} \quad \Omega, \\ \frac{\partial\varphi_1}{\partial\nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

we conclude that $\epsilon\varphi_1$ is a subsolution of $(P_{\lambda\mu})$ for all $\epsilon > 0$. Indeed, for all $\lambda, \mu > 0$ we have

$$\epsilon\lambda_1\varphi_1 = -\Delta(\epsilon\varphi_1) + (\epsilon\varphi_1) \leq \lambda(\epsilon\varphi_1)^{q_1} + (\epsilon\varphi_1)^{p_1} \ \text{ in } \ \Omega$$

$$0 \leq \mu(\epsilon\varphi_1)^{q_2} + (\epsilon\varphi_1)^{p_2} \ \text{ on } \ \partial\Omega,$$

for ϵ sufficiently small.

Choose $\epsilon > 0$ small enough such that $\epsilon\varphi_1 < M\Psi$. From sub and super solution method (see [2]) there exists a solution u of $(P_{\lambda\mu})$ such that $\epsilon\varphi_1 \leq u \leq M\Psi, \ \ \forall (\lambda, \mu) \in (0, \lambda_0) \times (0, \mu_0)$. Therefore $f(\mu) \geq \lambda_0$ and $g(\lambda) \geq \mu_0$.
Choose $\bar{\lambda}$ and $\bar{\mu}$ satisfying

$$\bar{\lambda} t^{q_1} + t^{p_1} > \lambda_1 t, \ \ \forall \, t > 0 \tag{2.1}$$

$$\bar{\mu}t^{q_2} + t^{p_2} > \mu_1 t, \ \ \forall\, t > 0, \tag{2.2}$$

where μ_1 verifies (Steklov problem)

$$\left\{ \begin{array}{ccc} -\Delta\phi_1 + \phi_1 = 0 & \text{in} & \Omega, \\ \frac{\partial \phi_1}{\partial \nu} = \mu_1 \phi_1 & \text{on} & \partial\Omega, \\ \phi_1 > 0 & \text{in} & \Omega. \end{array} \right. \tag{2.3}$$

If (λ, μ) is such that $(P_{\lambda\mu})$ has a solution $u = u_{\lambda\mu}$, multiplying $(P_{\lambda\mu})$ by φ_1 and integrating over Ω, we obtain

$$\lambda_1 \int_\Omega u\varphi_1 = \lambda \int_\Omega u^{q_1}\varphi_1 + \int_\Omega u^{p_1}\varphi_1 + \mu \int_{\partial\Omega} u^{q_2}\varphi_1 + \int_{\partial\Omega} u^{p_2}\varphi_1.$$

Combining with (2.1), we conclude that $\lambda < \bar{\lambda}$, that is, $f(\mu) \leq \bar{\lambda}$.

On the other hand, from (2.3) we have

$$\lambda \int_\Omega u^{q_1}\phi_1 + \int_\Omega u^{p_1}\phi_1 + \mu \int_{\partial\Omega} u^{q_2}\phi_1 + \int_{\partial\Omega} u^{p_2}\phi_1 = \mu_1 \int_{\partial\Omega} u\phi_1,$$

by using (2.2), we obtain $\mu < \bar{\mu}$. Thus $g(\lambda) \leq \bar{\mu}$. Therefore $0 < f(\mu) < \infty$, and also $0 < g(\lambda) < \infty$. This is the end of the proof. ■

Now, the following comparison Lemma whose the proof follows arguing as in [3], it will used to prove that problem $(P_{\lambda\mu})$ has a minimal solution.

Lemma 2.2 *Assume that $f(t)$ $g(t)$ are functions such that $t^{-1}f(t)$ and $t^{-1}g(t)$ are decreasing for $t > 0$. Let v and w satisfy*

$$\left\{ \begin{array}{ccc} -\Delta v \leq f(v) & in & \Omega, \\ \frac{\partial v}{\partial \nu} \leq g(v) & on & \partial\Omega, \\ v > 0 & in & \Omega, \end{array} \right.$$

and

$$\left\{ \begin{array}{ccc} -\Delta w \geq f(w) & in & \Omega, \\ \frac{\partial w}{\partial \nu} \geq g(w) & on & \partial\Omega, \\ w > 0 & in & \Omega. \end{array} \right.$$

Then $w \geq v$ in Ω.

We shall prove the following existence results.

Lemma 2.3 *For all $\tilde{\mu} > 0$ such that $\tilde{\mu} \leq \mu$, problem $(P_{\lambda\tilde{\mu}})$, with $0 \leq \lambda < f(\mu)$, has a minimal solution $u_{\lambda\tilde{\mu}}$ such that $I_{\lambda\tilde{\mu}}(u_{\lambda\tilde{\mu}}) < 0$. Moreover, problem $(P_{f(\mu)\mu})$ possesses a solution.*

Lemma 2.4 *For all $\tilde{\lambda} > 0$ such that $\tilde{\lambda} \leq \lambda$, problem $(P_{\tilde{\lambda}\mu})$, with $0 \leq \mu < g(\lambda)$, has a minimal solution $u_{\tilde{\lambda}\mu}$ such that $I_{\tilde{\lambda}\mu}(u_{\tilde{\lambda}\mu}) < 0$. Moreover, problem $(P_{\lambda g(\lambda)})$ possesses a solution.*

Remark 2.1 From Lemma 2.3, problem $(P_{\tilde{\lambda}\tilde{\mu}})$ has a solution, if $\tilde{\mu} < \mu$ and $\tilde{\lambda} < f(\mu)$. Analogously for Lemma 2.4

Proof of Lemma 2.4. The proof of Lemma 2.3 is similar. Given $\tilde{\lambda} \leq \lambda$ and $\mu < g(\lambda)$, let $u_{\lambda_0\mu_0}$ a solution of $(P_{\lambda_0\mu_0})$ with $\tilde{\lambda} \leq \lambda_0 \leq \lambda$ and $\mu < \mu_0 < g(\lambda)$. Notice that $u_{\lambda_0\mu_0}$ is a supersolution for $(P_{\tilde{\lambda}\mu})$. Since $\epsilon\varphi_1 < u_{\lambda_0\mu_0}$, for ϵ sufficiently small, we conclude that $P_{\tilde{\lambda}\mu}$ has a solution, which is positive since $\epsilon\varphi_1 < u_{\tilde{\lambda}\mu}$.

We shall prove that $u_{\tilde{\lambda}\mu}$, with $0 \leq \mu < g(\lambda)$, satisfies the following properties.

Claim 2.1

i) $u_{\tilde{\lambda}\mu}$ is a minimal solution.

ii) $I_{\tilde{\lambda}\mu}(u_{\tilde{\lambda}\mu}) < 0$.

We are going to postpone the proof of this Claim , first completing the proof of Lemma 2.4; more exactly, we prove that problem $(P_{\lambda g(\lambda)})$ possesses a solution.

Letting $\mu_n \in [0, g(\lambda))$, from the Claim we have

$$I_{\lambda\mu_n}(u_{\lambda\mu_n}) \equiv I(u_{\lambda\mu_n}) < 0.$$

This implies that $||u_{\lambda\mu_n}||_{p_i+1}^{p_i+1} \leq C, \quad C > 0, \quad i = 1, 2$. Since $u_{\lambda\mu_n}$ is a solution, we conclude that

$$||u_{\lambda\mu_n}|| \text{ is bounded.}$$

Thus, there exists $u_{\lambda g(\lambda)} \in H^1(\Omega)$ such that $u_{\lambda\mu_n} \rightharpoonup u_{\lambda g(\lambda)}$ weakly in $H^1(\Omega)$, as $n \to \infty$. It is standard to prove that $u_{\lambda g(\lambda)}$ is a positive weak solution of $(P_{\lambda g(\lambda)})$; and this is the end of the proof. ■

Proof of Claim 2.1 i) Let $v_{\tilde{\lambda}\mu}$ be the unique solution of

$$\begin{cases} -\Delta v + v = \tilde{\lambda} v^{q_1} & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} = \mu v^{q_2} & \text{on} \quad \partial\Omega. \end{cases}$$

Let $u > 0$ be a solution of $P_{\tilde{\lambda}\mu}$. Observe that

$$\begin{cases} -\Delta u + u \geq \tilde{\lambda} u^{q_1} & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = \mu u^{q_2} & \text{on} \quad \partial\Omega. \end{cases}$$

From Lemma 2.2, we reach that $u \geq v_{\tilde{\lambda}\mu}$in Ω.

On the other hand, $v_{\tilde{\lambda}\mu}$ is a subsolution of $(P_{\tilde{\lambda}\mu})$. By monotone iteration method we obtain an increasing sequence u_n such that $u_n \to u_{\tilde{\lambda}\mu}$, as $n \to \infty$. But u is a supersolution of $P_{\tilde{\lambda}\mu}$. Therefore $u_n \leq u, \ \forall\, n$. Thus $u_{\tilde{\lambda}\mu} \leq u$, that is, $u_{\tilde{\lambda}\mu}$ is a minimal solution.

To prove Claim 2.1 ii) we shall need the following

Lemma 2.5 *Let $\psi < \bar{\psi}$ be a subsolution and supersolution for $(P_{\lambda\mu})$, respectively. Suppose that ψ is not a solution. Let u be a minimal solution for $(P_{\lambda\mu})$ such that $\psi \le u \le \bar{\psi}$. Then*

$$\lambda_1 \equiv \lambda_1[-\Delta - a(x) - b(y)] \ge 0,$$

where $a(x) = \lambda q_1 u^{q_1-1} + p_1 u^{p_1-1}$, $b(y) = \mu q_2 u^{q_2-1} + p_2 u^{p_2-1}$ and $\lambda_1[-\Delta - a(x) - b(y)]$ denotes the first eigenvalue of the problem

$$\begin{cases} -\Delta\phi + \phi - a(x)\phi = \lambda_1\phi & in \quad \Omega, \\ \frac{\partial\phi}{\partial\nu} - b(y)\phi = \lambda_1\phi & on \quad \partial\Omega. \end{cases}$$

Proof. Notice that there exists

$$\phi \in V = \{v \in H^1(\Omega); \int_\Omega v^2 + \int_{\partial\Omega} v^2 = 1\},$$

$\phi > 0$ verifying

$$J_{\nu_1}(\phi) = inf\{J_{\nu_1}(v); \;\; v \in V\},$$

where

$$J_{\nu_1}(v) = \int_\Omega (|\nabla v|^2 + v^2) - \int_\Omega a v^2 - \int_{\partial\Omega} b v^2, \;\; v \in H^1(\Omega).$$

We claim that $\lambda_1 \ge 0$. Suppose by contradiction that $\lambda_1 < 0$. Arguing as in [3], we conclude that $u - \alpha\phi$ satisfies

$$-\Delta(u - \alpha\phi) + u - \alpha\phi - [\lambda(u - \alpha\phi)^{q_1} + (u - \alpha\phi)^{p_1}] > 0,$$

for $\alpha > 0$ small enough. We also get

$$\frac{\partial(u - \alpha\phi)}{\partial\nu} - [\mu(u - \alpha\phi)^{q_2} + (u - \alpha\phi)^{p_2}] > 0.$$

Thus $(u - \alpha\phi)$ is a supersolution for $(P_{\lambda\mu})$. Since ψ is not a solution, we obtain $u > \psi$. Therefore $u - \alpha\phi \ge \psi$, for α sufficiently small. Hence $(P_{\lambda\mu})$ has a solution $\hat{u}$ satisfying $\psi \le \hat{u} \le u - \alpha\phi$, which is a contradiction because u is a minimal solution. This is the end of the proof. ■

Proof of Claim 2.1 ii). From Lemma 2.5 we have

$$||u_{\lambda\mu}||^2 - \lambda q_1 ||u_{\lambda\mu}||_{q_1+1}^{q_1+1} - p_1 ||u_{\lambda\mu}||_{p_1+1}^{p_1+1} - \mu q_2 ||u_{\lambda\mu}||_{q_2+1}^{q_2+1} - p_2 ||u_{\lambda\mu}||_{p_2+1}^{p_2+1} \ge 0. \quad (2.4)$$

From $I'_{\lambda\mu}(u_{\lambda\mu}) = 0$ and (2.4) we have

$$\lambda(1 - q_1)||u_{\lambda\mu}||_{q_1+1}^{q_1+1} + (1 - p_1)||u_{\lambda\mu}||_{p_1+1}^{p_1+1} + \\ \mu(1 - q_2)||u_{\lambda\mu}||_{q_2+1}^{q_2+1} + (1 - p_2)||u_{\lambda\mu}||_{p_2+1}^{p_2+1} \ge 0. \quad (2.5)$$

Using again that $I'_{\lambda\mu}(u_{\lambda\mu}) = 0$ we have

$$
\begin{aligned}
I_{\lambda\mu}(u_{\lambda\mu}) = & \frac{\lambda}{2}\frac{q_1-1}{q_1+1}||u_{\lambda\mu}||_{q_1+1}^{q_1+1} + \frac{1}{2}\frac{p_1-1}{p_1+1}||u_{\lambda\mu}||_{p_1+1}^{p_1+1} \\
& + \frac{\mu}{2}\frac{q_2-1}{q_2+1}||u_{\lambda\mu}||_{q_2+1}^{q_2+1} + \frac{1}{2}\frac{p_2-1}{p_2+1}||u_{\lambda\mu}||_{p_2+1}^{p_2+1} \\
< & \frac{1}{2}\min\{\frac{1}{q_1+1}, \frac{1}{q_2+1}\}(\lambda(q_1-1)||u_{\lambda\mu}||_{q_1+1}^{q_1+1} + \mu(q_2-1)||u_{\lambda\mu}||_{q_2+1}^{q_2+1}) \\
& + \frac{1}{2}\frac{p_1-1}{p_1+1}||u_{\lambda\mu}||_{p_1+1}^{p_1+1} + \frac{1}{2}\frac{p_2-1}{p_2+1}||u_{\lambda\mu}||_{p_2+1}^{p_2+1} \\
< & \frac{1}{2}\min\{\frac{1}{q_1+1}, \frac{1}{q_2+1}\}[\lambda(q_1-1)||u_{\lambda\mu}||_{q_1+1}^{q_1+1} + \mu(q_2-1)||u_{\lambda\mu}||_{q_2+1}^{q_2+1} \\
& + (p_1-1)||u_{\lambda\mu}||_{p_1+1}^{p_1+1} + (p_2-1)||u_{\lambda\mu}||_{p_2+1}^{p_2+1}].
\end{aligned}
$$

From (2.5) and the above inequality, we conclude that $I_{\lambda\mu}(u_{\lambda\mu}) < 0$. This is the end of the proof of the claim. ■

Now, we will state some properties about the minimal solution.

Lemma 2.6

1) *If* $\lambda_1 < \lambda_2$ *and* $\mu_1 < \mu_2$, *then* $u_{\lambda_1\mu_1} < u_{\lambda_2\mu_2}$.

2) *The function* f *is strictly decreasing.*

3) *The function* f *is invertible and* $f^{-1} = g$.

Proof. **1)** Let $u_{\lambda_i\mu_i}, \quad i = 1, 2$, be solutions of $(P_{\lambda\mu})$. Since $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$, we have

$$
\begin{cases}
-\Delta u_{\lambda_1\mu_1} - u_{\lambda_2\mu_2} \leq 0 & \text{in} \quad \Omega, \\
\frac{\partial(u_{\lambda_1\mu_1} - u_{\lambda_2\mu_2})}{\partial\nu} \leq 0 & \text{on} \quad \partial\Omega.
\end{cases}
$$

By the Hopf maximum principle we conclude $u_{\lambda_1\mu_1} < u_{\lambda_2\mu_2}$.

2) First of all, we will prove that f is non-increasing, that is, if $\mu_1 < \mu_2, \quad \mu_i \in [0, M_0] \quad (i = 1, 2)$ then $f(\mu_1) \geq f(\mu_2)$. Suppose by contradiction that $f(\mu_1) < f(\mu_2)$. Then for all $\lambda \in (f(\mu_1), f(\mu_2))$ there exists a solution $u_{\lambda\mu_2}$ of $(P_{\lambda\mu_2})$. But $u_{\lambda\mu_2}$ is a supersolution for $(P_{\lambda\mu}), \quad \forall\mu < \mu_2$. In particular, $u_{\lambda\mu_1}$ is a solution for $(P_{\lambda\mu_1})$, which is a contradiction since $\lambda > f(\mu_1)$.

Next we conclude that f is strictly decreasing. Suppose by contradiction that $\mu_1 < \mu_2, \quad \mu_i \in [0, M_0] \quad (i = 1, 2)$ and $f(\mu_1) = f(\mu_2) = \lambda_0$.
Let $u_i, \quad i = 1, 2$, be solutions of $(P_{\lambda_0\mu_i}), \quad i = 1, 2$. Since $\mu_1 < \mu_2$, then

$$
\begin{cases}
-\Delta(u_1 - u_2) < 0 & \text{in} \quad \Omega, \\
\frac{\partial(u_1 - u_2)}{\partial\nu} < 0 & \text{on} \quad \partial\Omega,
\end{cases}
$$

which is a contradiction, by the Hopf maximum principle since

$$\frac{\partial(u_1 - u_2)}{\partial\nu}(x_0) \geq 0,$$

for some $x_0 \in \partial\Omega$

3) We shall prove that $g(\lambda) = f^{-1}(\lambda)$. Since $(P_{\lambda f^{-1}(\lambda)})$ has a solution then $g(\lambda) \geq f^{-1}(\lambda)$. Suppose that $g(\lambda) > f^{-1}(\lambda)$, then problem $(P_{\lambda\mu})$ with $f^{-1}(\lambda) < \mu < g(\lambda)$ has a solution, which is a contradiction, because $(\lambda, \mu) \in K_0$. Therefore $f^{-1} = g$. This is the end of the proof. ■

Proof of Theorem 1.1. The proof follows directly by applying the results above.
This is the end of the proof. ■

3 Proof of Theorem 1.2

Again, we will follow some arguments used in [3] (see also [11] and [14, 15]). The main difference here is to use the regularity result due to [6] adapted to the Neumann problems, in order to get a similar results those in [8]. Thus, we will give only the sketch of the proof.

Let u_0 be a minimal positive solution obtained in the first part satisfying $I_{\lambda\mu}(u_0) < 0$. In fact arguing as in [3], u_0 is a local minimum of $I_{\lambda\mu}$ in the C^1 topology. We will find a second positive solution v of the form $u = u_0 + v$. The function v should be a solution of the problem

$$\begin{cases} -\Delta v + v = h_\lambda(x, v) & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} = g_\mu(x, v) & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$h_\lambda(x, s) = \begin{cases} \lambda(u_0 + s)^{q_1} - \lambda u_0^{q_1} + (u_0 + s)^{p_1} - u_0^{p_1} & \text{if} \quad s \geq 0 \\ 0 & \text{if} \quad s < 0 \end{cases}$$

and

$$g_\mu(x, s) = \begin{cases} \mu(u_0 + s)^{q_2} - \mu u_0^{q_2} + (u_0 + s)^{p_2} - u_0^{p_2} & \text{if} \quad s \geq 0 \\ 0 & \text{if} \quad s < 0. \end{cases}$$

Notice that v is a critical point of the C^1 functional

$$J(v) = \frac{1}{2}||v||^2 - \int_\Omega H_\lambda(x, s) - \int_{\partial\Omega} G_\mu(x, s)$$

where

$$H_\lambda(x, s) = \int_0^s h_\lambda(x, s) \quad \text{and} \quad G_\mu(x, s) = \int_0^s g_\mu(x, s).$$

The proof of Theorem 1.2 follows from the results below. We recall that the proofs of these results follow as in [3] (see also and [11] and [14, 15]).

Lemma 3.1 *It folows that $v = 0$ is a local minimum of J in $H^1(\Omega)$.*

To prove Lemma 3.1, we need the following result due to [8].

Lemma 3.2 *If* $v = 0$ *is a local minimum of* J *in* C^1*-topology, then it is a local minimum of* J *in* H^1*-topology.*

Proof. The proof follows as [8], see also [15]. ■

The following compactness result can be obtained by adapting some arguments used in [3], as well as in [15].

Lemma 3.3 *If 0 is the only critical point of* J*, then* J *satisfies* $(PS)_c$ *for all*

$$c < (\frac{1}{2} - \frac{1}{2_*}) max\{S_0^{\frac{2^*}{2^*-2}}, S_0^{\frac{2_*}{2_*-2}}\} \tag{2.6}$$

where

$$S_0 = inf\{\int_{I\!R^N_+} |\nabla u|^2 : \ u \in H^1(I\!R^N_+) \ \ and \ \ \int_{I\!R^{N-1}} u^{2_*} + \int_{I\!R^N_+} u^{2^*} = 1\}.$$

and $I\!R^N_+ = \{(x,t) \in I\!R^N \mid x \in I\!R^{N-1} \ and \ t > 0\}$.

Proof. The proof of $(PS)_c$ condition follows arguing as in [15]. In our situation, since problem $(P_{\lambda\mu})$ involves two critical exponents, the verification of (2.6) needs refinements in some estimates. But its proof is basically made as in [7], more exactly, as it is done in [1], where they used the function

$$w_\epsilon(x,t) = (\frac{\epsilon}{\epsilon^2 + |(x,t) - (x_0,t_0)|^2})^{\frac{N-2}{2}}$$

which is a minimizer of S_0 (see also [9], [12] and [20]).

To obtain (2.6) it suffices to prove that there exists $v_0 \in H^1(\Omega)$, $v_0 \geq 0$ on $\bar{\Omega}$ and $v_0 \neq 0$, such that

$$\sup_{t \geq 0} J(tv_0) < (\frac{1}{2} - \frac{1}{2_*}) max\{S_0^{\frac{2^*}{2^*-2}}, S_0^{\frac{2_*}{2_*-2}}\} \equiv \bar{S}.$$

First of all we will state some estimates. Consider the cut-off function $\varphi \in C^\infty(\bar{\Omega})$ such that $0 \leq \varphi \leq 1$, $(x,t) \in \Omega \subset I\!R^{N-1} \times I\!R$ and $\varphi(x,t) = 1$ on a neighborhood U of (x_0,t_0) such that $U \subset \Omega$. Define

$$u_\epsilon(x,t) = w_\epsilon(x,t)\varphi(x,t)$$

and

$$v_\epsilon(x,t) = \frac{u_\epsilon(x,t)}{(\|u_\epsilon\|^2_{2^*} + \|u_\epsilon\|^2_{2_*})^{\frac{1}{2}}}.$$

The following estimates are proved by combining [17, Lemma 5.2] with the argument used in the proof of [7, Lemma 1.1]:

$$\|\nabla v_\epsilon\|_2^2 = S_0 + O(\epsilon^{N-2}), \tag{2.7}$$

$$\|u_\epsilon\|_{2^*}^{2^*} = \|u_1\|_{2^*, I\!R_+^N}^{2^*} + O(\epsilon^N), \tag{2.8}$$

$$\|u_\epsilon\|_{2_*}^{2_*} = \|u_1\|_{2_*, I\!R^{N-1}}^{2_*} + O(\epsilon^{N-1}), \tag{2.9}$$

and

$$\|v_\epsilon\|_2^2 = \begin{cases} o(\epsilon) & \text{for} \quad N \geq 4 \\ O(\epsilon) & \text{for} \quad N = 3, \end{cases} \tag{2.10}$$

where

$$\|u_1\|_{2^*, I\!R^{N-1}}^{2^*} = \int_{I\!R^{N-1}} u_1^{2^*} \quad \text{and} \quad \|u_1\|_{2^*, I\!R_+^N}^{2^*} = \int_{I\!R_+^N} u_1^{2^*}.$$

As we mentioned before, it is sufficient to show that

$$\sup_{s \geq 0} J(s\tilde{v_\epsilon}) < \bar{S},$$

where $\tilde{v_\epsilon}(x,t) = \alpha v_\epsilon(x,t)$ with $\alpha > 0$ to be chosen later on.

Since $J(s\tilde{v_\epsilon}) \to -\infty$ as $s \to \infty$, there exists $s_\epsilon > 0$ such that

$$\sup_{s \geq 0} J(s\tilde{v_\epsilon}) = J(s_\epsilon \tilde{v_\epsilon}. \tag{2.11}$$

(If $s_\epsilon = 0$ the proof is finished). From (2.11) we obtain

$$X_\epsilon^2 \geq s_\epsilon^{2^*-2} A_\epsilon^{2^*} + s_\epsilon^{2_*-2} B_\epsilon^{2_*},$$

where $\|\nabla \tilde{v_\epsilon}\|_2^2 \equiv X_\epsilon^2$, $\|\tilde{v_\epsilon}\|_{2^*}^{2^*} \equiv A_\epsilon^{2^*}$ and $\|\tilde{v_\epsilon}\|_{2_*}^{2_*} \equiv B_\epsilon^{2_*}$. So,

$$\begin{aligned} \min\{s_\epsilon^{2^*-2}, s_\epsilon^{2_*-2}\} &\leq \frac{X_\epsilon^2}{A_\epsilon^{2^*} + B_\epsilon^{2_*}} \\ &\leq \frac{\alpha^2}{\min\{\alpha^{2^*}, \alpha^{2_*}\}} \left(\frac{\|\nabla v_\epsilon\|_{2,}^2}{\|v_\epsilon\|_{2^*}^{2^*} + \|v_\epsilon\|_{2_*}^{2_*}} \right). \end{aligned} \tag{2.12}$$

Now choosing $\alpha > 0$ such that

$$\frac{1}{\min\{\alpha^{2^*-2}, \alpha^{2_*-2}\}} \left(\frac{1}{\|u_1\|_{2^*, I\!R_+^N}^{2^*} + \|u_1\|_{2_*, I\!R^{N-1}}^{2_*}} \right) \leq 1,$$

from (2.12) results

$$s_\epsilon \leq \max\{S_0^{(2^*-2)^{-1}}, S_0^{(2_*-2)^{-1}}\}. \tag{2.13}$$

Also, by (2.7)-(2.10) we have following estimates

$$X_\epsilon^2 \leq S_0 + O(\epsilon^{N-2}). \tag{2.14}$$

and

$$s_\epsilon^2 X_\epsilon^2 \leq \max\{S_0^{2^*(2^*-2)^{-1}}, S_0^{2_*(2_*-2)^{-1}}\} + O(\epsilon^{N-2}), \text{ for } N \geq 3. \tag{2.15}$$

On the other hand, since the critical level $c > 0$, we can assume that $s_\epsilon \geq c_0 > 0, \ \forall \epsilon > 0$.

From (2.11), we obtain

$$s_\epsilon^{2^*} A_\epsilon^{2^*} + s_\epsilon^{2_*} B_\epsilon^{2_*} \geq s_\epsilon^2 X_\epsilon^2 + O(\epsilon). \tag{2.16}$$

Thus inserting (2.16) into the expression of $J(s_\epsilon \tilde{v_\epsilon})$, from (2.10) and (2.15) we conclude that

$$J(s_\epsilon \tilde{v_\epsilon}) < \bar{S}, \quad \text{for all } \ \epsilon \ \text{ sufficiently small}$$

This proves Lemma 3.3. ■

Proof of Theorem 1.2. From the results above we can conclude that J has a nontrivial positive critical point by applying a version of the Ambrosetti-Rabinowitz Mountain Pass Theorem due to [16]. ■

4 Appendix

Next we shall prove a regularity result for the weak solutions of $(P_{\lambda\mu})$, by closely following [19]. Its proof is based on the classical interactions method due to Moser. Here, we make essentially an adaptation of the proof of a result proved by Brézis-Kato [6] for Dirichlet problem. See also the appendix in Struwe [19]. We consider the following problem

$$\begin{array}{rcll} -\Delta u &=& f(x,u), & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} &=& g(y,u), & \text{on} \quad \partial\Omega, \end{array} \tag{2.17}$$

with Caratheodory functions $f : \Omega \times I\!R \to I\!R$ and $g : \partial\Omega \times I\!R \to I\!R$, that is, $f(x,u)$ is measurable in $x \in \Omega$ and continuous in $u \in I\!R$, and $g(y,u)$ is measurable in $y \in \partial\Omega$ and continuous in $u \in I\!R$. Moreover, we will assume that f and g satisfy the following growth conditions

$$|f(x,u)| \leq a(x)(1+|u|), \ \ x \in \Omega, \ u \in I\!R \tag{2.18}$$

$$|g(y,u)| \leq b(y)(1+|u|), \ \ y \in \partial\Omega, \ u \in I\!R, \tag{2.19}$$

where $a \in L^{N/2}_{loc}(\Omega)$ and $b \in L^{N-1}_{loc}(\partial\Omega)$.

Lemma 4.1 *Let Ω be a domain in $\mathbb{R}^N$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carthéodory functions satisfying (2.18) and (2.19). Also let $u \in H^1_{loc}(\Omega)$ be a weak solution of problem (2.17). Then $u \in L^t_{loc}(\Omega) \cap L^r_{loc}(\partial\Omega)$ for any $1 \le t, r < \infty$. If $u \in H^1(\Omega)$, $a \in L^{N/2}(\Omega)$ and $b \in L^{N-1}(\partial\Omega)$, then $u \in L^t(\Omega) \cap L^r(\partial\Omega)$ for any $1 \le t, r < \infty$.*

Proof. Choose $\eta \in C^\infty(\overline{\Omega})$ and for $s \ge 0$, $L \ge 0$, let $\varphi_{s,L} = u \min\{|u|^{2s}, L^2\}\eta^2 \in H^1(\Omega)$. Testing (2.17) with $\varphi_{s,L}$, we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2 + \frac{s}{2}\int_{\{x\in\Omega\ ;\ |u(x)|^s \le L\}} |\nabla(|u|^2)|^2 |u|^{2s-2}\eta^2 \\
\le\ & -2\int_\Omega \nabla u u \min\{|u|^{2s}, L^2\}\nabla\eta\eta \\
+\ & \int_\Omega a(1+2|u|^2)\min\{|u|^{2s}, L^2\}\eta^2 + \int_{\partial\Omega} b(1+2|u|^2)\min\{|u|^{2s}, L^2\}\eta^2 \\
\le\ & \frac{1}{2}\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2 + C\int_\Omega |u|^2 \min\{|u|^{2s}, L^2\}|\nabla\eta|^2 \\
+\ & C_1\int_\Omega |a||u|^2 \min\{|u|^{2s}, L^2\}\eta^2 + C_2\int_{\partial\Omega} |b||u|^2\min\{|u|^{2s}, L^2\}\eta^2 \\
+\ & \int_\Omega |a|\eta^2 + \int_{\partial\Omega} |b|\eta^2 \\
\le\ & C_3(1+C_a)\int_\Omega |u|^2\min\{|u|^{2s}, L^2\}\eta^2 + C_3C_b\int_{\partial\Omega}|u|^2\min\{|u|^{2s}, L^2\}\eta^2 \\
+\ & C_3\int_{\{x\in\Omega\ ;\ |a(x)|\ge C_a\}} |a||u|^2\min\{|u|^{2s}, L^2\}\eta^2 \\
+\ & C_3\int_{\{y\in\partial\Omega\ ;\ |b(y)|\ge C_b\}} |b||u|^2\min\{|u|^{2s}, L^2\}\eta^2.
\end{aligned}
$$

Suppose $u \in L^{2s+2}(\Omega)$. Then we may conclude that, with constants depending on the L^{2s+2} – norm of u, we have

$$
\begin{aligned}
&\int_\Omega |\nabla(u\min\{|u|^s, L\})|^2 \\
\le\ & C_3(1+C_a+C_b) + C_3\Big(\int_{\{x\in\Omega\ ;\ |a(x)|\ge C_a\}} |a|^{N/2}\Big)^{2/N} \cdot \\
&\cdot\Big(\int_\Omega |u\min\{|u|^s, L\}\eta|^{2N/(N-2)}\Big)^{(N-2)/N} \\
&+C_3\Big(\int_{\{y\in\partial\Omega\ ;\ |b(y)|\ge C_b\}} |b|^{N-1}\Big)^{1/(N-1)} \cdot \\
&\cdot\Big(\int_{\partial\Omega} |u\min\{|u|^s, L\}\eta|^{2(N-1)/(N-2)}\Big)^{(N-2)/(N-1)}
\end{aligned}
$$

$$\leq \quad C_3(1+C_a+C_b)+\Big(\epsilon(C_a)+\epsilon(C_b)\Big)\cdot\int_\Omega |\nabla(u\min\{|u|^s,L\}\eta)|^2,$$

where

$$\epsilon(C_a)=\Big(\int_{\{x\in\Omega\ ;\ |a(x)|\geq C_a\}}|a|^{N/2}\Big)^{2/N}\to 0\quad (C_a\to\infty),$$

$$\epsilon(C_b)=\Big(\int_{\{y\in\partial\Omega\ ;\ |b(y)|\geq C_b\}}|b|^{N-1}\Big)^{1/(N-1)}\to 0\quad (C_b\to\infty).$$

Fix C_a and C_b such that $\epsilon(C_a)+\epsilon(C_b)=1/2$, and note that

$$\int_{\{x\in\Omega\ ;\ |u(x)|^s\leq L\}}|\nabla(|u|^{s+1}\eta)|^2\ \leq C_3(1+C_a+C_b)$$

is uniformly bounded in L. Hence we may let $L\to\infty$ to get that

$$|u|^{s+1}\eta\in H^1(\Omega)\hookrightarrow L^{2N/(N-2)}(\Omega)\cap L^{(N-1)/(N-2)}(\partial\Omega),$$

that is,

$$u\in L^{\frac{(2s+2)N}{N-2}}(\Omega)\cap L^{\frac{(2s+2)(N-1)}{N-2}}(\partial\Omega).$$

Now, let $s_o=0$, $s_i+1=(s_{i-1}+1)\frac{N}{N-2}$, or $s_i+1=(s_{i-1}+1)\frac{N-1}{N-2}$, if $i\geq 1$, to conclude the lemma. If $u\in H^1(\Omega)$, we may let $\eta=1$ to obtain the same conclusion.

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