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Periodic Solutions of Symmetric Elliptic Singular Systems: the Higher Codimension Case

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Abstract

We show the existence of periodic solutions of certain singularly perturbed systems having symmetry properties. Our result applies to some singular systems arising in the study of Hamiltonian systems with a strong restoring force and extends to a higher codimension the result obtained in our previous paper [4].

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1 Introduction

This paper is a continuation of [4] where we studied the problem of bifurcation of periodic orbits of large period in elliptic singular systems such as

$$\begin{aligned} \dot{x} &= f(x, y, \varepsilon) \\ \varepsilon \dot{y} &= J(x)y + \varepsilon g_1(x, y) + \varepsilon^2 g_2(x, y, \varepsilon) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^{2n}$ and $y \in \mathbb{R}^2$ and

$$J(x) = \begin{pmatrix} 0 & 1 \\ -\lambda^2(x) & 0 \end{pmatrix}.$$

It is assumed, there, that the associated degenerate equation $\dot{x} = f(x, 0, 0)$ has a non degenerate homoclinic solution. Then, under some further conditions, it has been shown in [4] that, if the homoclinic orbit is even, then (1.1) has several layers of periodic solutions.

Equations like (1.1) arise, for example, when passing from the D'Alembert equation associated to a second order equation on a certain submanifold $\mathcal{M}$ of $\mathbb{R}^n$ to the so-called *penalized equation* [1, 12]. Here, by D'Alembert equation we mean the equation

$$D_t \dot{z} + P_z F(z) = 0 \tag{1.2}$$

where D_t is the covariant derivative along the tangent bundle $T\mathcal{M}$ of an orientable submanifold $\mathcal{M}$ and $P_z : \mathbb{R}^n \to T_z\mathcal{M}$ is the orthogonal projection on $T_z\mathcal{M}$ along the normal space $N_z\mathcal{M}$. We recall that, given a smooth curve $z(t)$ on a submanifold $\mathcal{M} \subset \mathbb{R}^n$ and a vector field $Y(t) \in T_{z(t)}\mathcal{M}$, the covariant derivative $D_t Y(t)$ is defined as

$$D_t Y(t) = P_{z(t)} \dot{Y}(t)$$

(see [5, p. 305-306]). By *penalized equation*, instead, we mean the equation

$$\ddot{z} + F(z) + \varepsilon^{-2} G(z) = 0 \tag{1.3}$$

where $G(z)$ is a smooth function vanishing on $\mathcal{M}$ and such that, for any $x \in \mathcal{M}$, $G'(x)$ is positive definite on the normal space $N_x\mathcal{M}$. Since in [4] the case $y \in \mathbb{R}^2$ is studied, the result proved therein applies to D'Alembert equations on a hypersurface $\mathcal{M}$. The purpose of this paper is to extend Theorem 5.1 in [4] to the case where $y \in \mathbb{R}^{2m}$, $m \geq 1$. This fact allows us to remove the assumption that $\mathcal{M}$ is a hypersurface, thus extending Theorem 1.1 in [4] to the case where $\mathcal{M}$ is an orientable, codimension $m \geq 1$ submanifold of $\mathbb{R}^n$. As a matter of fact, we will prove the following result.

Theorem 1.1 *Assume that $F(z) \in C^3$, $G(z) \in C^5$ and that $\mathcal{M}$ is an orientable C^5 codimension m submanifold of an open subset $\Omega \subset \mathbb{R}^n$. Let D_t be the covariant derivative along the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and $P_z : \mathbb{R}^n \to T_z\mathcal{M}$ be the orthogonal projection on $T_z\mathcal{M}$ along the normal bundle $N_z\mathcal{M}$. Moreover assume the following conditions hold:*

1) *$G(z) = 0$, for any $z \in \mathcal{M}$ and $P_{z_0} F(z_0) = 0$ for some $z_0 \in \mathcal{M}$;*

2) *$N_z\mathcal{M}$ has an orthonormal basis $\{n_j(z) \mid j = 1, \dots, m\}$ such that $G'(z)n_j(z) = \lambda_j^2(z)n_j(z)$, with $\lambda_j^2(z) \geq \lambda^2 > 0$, for any $z \in \mathcal{M}$;*

3) *the D'Alembert equation*

$$D_t\dot{z} + P_zF(z) = 0 \tag{1.4}$$

has a (nontrivial) symmetric solution $\gamma_0(t) = \gamma_0(-t) \in \mathcal{M}$ homoclinic to the fixed point $z = z_0$ which is hyperbolic for the dynamics of (1.4) restricted on $\mathcal{M}$;

4) *$z(t) = \dot{\gamma}_0(t)$ is the unique bounded solution, up to a multiplicative constant, of the variational equation of (1.4) along $\gamma_0(t)$ such that $z(t) \in T_{\gamma_0(t)}\mathcal{M}$ and*

$$[n_j'(\gamma_0(t))^t z(t)]\dot{\gamma}_0(t) + n_j^t(\gamma_0(t))\dot{z}(t) = 0$$

for any $j = 1, \dots, m$.

Then, there exist $M > 0$, $\delta > 0$, $\mu > 0$, $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a subset $S_\varepsilon \subset (-3\delta^{-1}\ln\varepsilon, \mu\varepsilon^{-1/2})$ such that the Lebesgue measure $m(S_\varepsilon) \geq \frac{1}{2}[\mu\varepsilon^{-1/2} + 3\delta^{-1}\ln\varepsilon]$ and for any $T \in S_\varepsilon$ the penalized equation (1.1) has a $2T$−periodic solution $z(t)$ such that $z(t) = z(-t)$ and the following hold

$$\sup_{-T\leq t\leq T} |z(t) - \gamma_0(t)| \leq M\varepsilon. \tag{1.5}$$

The method of proof of Theorem 1.1 follows that in [4], with the changes that are made necessary by the higher codimension case. In particular we have to face with the problem that, now, we have to study the existence of bounded solutions of such a set of m boundary value problems in $\mathbb{R}^2$ as

$$\begin{cases} \varepsilon\dot{z}_1 = z_2 + \varepsilon h_1(t) \\ \varepsilon\dot{z}_2 = -\lambda_j^2(t)z_1 + \varepsilon h_2(t) \\ z_2(-T) = z_2(T) = 0, \end{cases}$$

where $\lambda_j^2(t)$ are, possibly different, real functions bounded below by a positive number λ. As it has been proved in [4], each of the above systems has bounded solutions (with T-independent bound) provided T is sufficiently large and satisfies a certain non resonance condition. Thus, in Section 2 we show that all the non resonance conditions can be satisfied simultaneously (see Lemma 2.2). Section 3 contains the main result whose proof is only sketched since it is quite similar to Theorem 5.1 in [4]. In Section 4 we give an example of application of our Theorem 1.2 to a second order equation in $\mathbb{R}^3$ whose D'Alembert equation on a non planar curve $\mathcal{M}$ has a homoclinic solution satisfying the assumption of Theorem 1.1.

2 Reduction to a singular equation

Here we assume that $\mathcal{M}$ is a C^r, $r \geq 5$, orientable submanifold of an open and bounded subset $\Omega \subset \mathbb{R}^n$ of codimension m and denote with $N_x\mathcal{M}$ the m-dimensional normal bundle at the point $x \in \mathcal{M}$. Without loss of generality we may (and will) assume that

$z_0 = 0$. Let $N_x\mathcal{M} = \mathrm{span}\{n_1(x), \ldots, n_m(x)\}$ be an orthonormal basis of $N_x\mathcal{M}$. We assume that positive numbers $\lambda_j^2(x) \geq \lambda^2 > 0$ exist such that the following holds:

$$G'(x)n_j(x) = \lambda_j^2(x)n_j(x). \tag{2.1}$$

Then let $U \subset \Omega$ be a neighborhood of $\mathcal{M}$ such that any point $z \in U$ can be written as

$$z = x + \varepsilon \sum_{j=1}^{m} \xi_j n_j(x)$$

with $x \in \mathcal{M}$ and $\xi = (\xi_1, \ldots, \xi_m)^t \in \mathbb{R}^m$ (here and in the rest of this paper v^t and A^t will denote the transpose of the vector v and of the matrix A).

We also extend $n_j(x)$ to the whole U in such a way that $\{n_1(x), \ldots, n_m(x)\}$ is an orthonormal C^{r-1} subset of $\mathbb{R}^n$. Note, that $G'(x)v = 0$ for any $x \in \mathcal{M}$ and any tangent vector v to $\mathcal{M}$ at the point $x \in \mathcal{M}$. We want to derive the differential equation satisfied by (x, ξ). Assume that $z = z(t)$ is a C^2-solution of equation

$$\ddot{z} + F(z) + \varepsilon^{-2}G(z) = 0 \tag{2.2}$$

that belongs to U for any t in a certain interval. Then we can write $z(t) = x(t) + \varepsilon \sum_{j=1}^{m} \xi_j(t)n_j(x(t)) = x(t) + \varepsilon n(x(t))\xi(t)$, where $n(x)$ is the $n \times m$ matrix whose columns are the vectors $n_1(x), \ldots, n_m(x)$, and $t \mapsto x(t) \in \mathcal{M}$, $t \mapsto \xi(t) \in \mathbb{R}^m$ are C^2-functions. Note that according to this notation we have

$$G'(x)n(x) = n(x)\Lambda(x) \tag{2.3}$$

where

$$\Lambda(x) = \begin{pmatrix} \lambda_1^2(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_m^2(x) \end{pmatrix}. \tag{2.4}$$

Now, for any $p \times q$ matrix $A(x) = [a_{ij}(x)]_{\substack{i=1,\ldots,p \\ j=1,\ldots,q}}$ depending smoothly on the parameter $x \in \mathbb{R}^n$, with $A(x)'$ we will denote its derivative, that is for any $v \in \mathbb{R}^n$, $A(x)'v$ is the $p \times q$ matrix whose $(i,j)-$entry is $< \nabla a_{ij}(x), v >$ or else the $i-$th row of $A(x)'v$ is $A_i'(x)v$ where $A_i'(x)$ is the Jacobian matrix of the $i-$th row $A_i(x)$ of $A(x)$.

Then $(x(t), \xi(t))$ satisfies:

$$\begin{aligned} \ddot{x} \ &+ \varepsilon[n(x)'\ddot{x} + n(x)''(\dot{x}, \dot{x})]\xi + 2\varepsilon[n(x)'\dot{x}]\dot{\xi} + \varepsilon n(x)\ddot{\xi} \\ &+ F(x + \varepsilon n(x)\xi) + \varepsilon^{-2}G(x + \varepsilon n(x)\xi) = 0. \end{aligned} \tag{2.5}$$

Now, differentiating the identity $n^t(x)n(x) = \mathbb{I}$, we obtain, for any $v \in \mathbb{R}^n$ and any C^2-function $t \mapsto x(t) \in \mathcal{M}$

$$\begin{aligned} &[n^t(x)'v]\, n(x) + n^t(x)n(x)'v = 0, \\ &[n^t(x)'\ddot{x} + n^t(x)''(\dot{x}, \dot{x})]\, n(x) + 2\,[n^t(x)'\dot{x}]\,[n(x)'\dot{x}] \\ &+ n^t(x)\,[n(x)'\ddot{x} + n(x)''(\dot{x}, \dot{x})] = 0. \end{aligned} \tag{2.6}$$

Moreover, using $n^t(x(t))\dot{x}(t) = 0$ (that follows from $\dot{x}(t) \in T_{x(t)}\mathcal{M}$), we get

$$[n^t(x)'\dot{x}]\dot{x} + n^t(x)\ddot{x} = 0 \tag{2.7}$$

for any C^2-function $x(t)$ whose values are in $\mathcal{M}$. Thus, left multiplying equation (2.5) by $n^t(x)$, we obtain

$$\begin{aligned} \varepsilon\ddot{\xi} \; & -[n^t(x)'\dot{x}]\dot{x} - 2\varepsilon[n^t(x)'\dot{x}]n(x)\dot{\xi} + \varepsilon n^t(x)\,[n(x)'\ddot{x} + n(x)''(\dot{x},\dot{x})]\,\xi \\ & + n^t(x)F(x + \varepsilon n(x)\xi) + \varepsilon^{-2}n^t(x)G(x + \varepsilon n(x)\xi) = 0 \end{aligned} \tag{2.8}$$

and then, plugging into (2.5) the expression for $\varepsilon\ddot{\xi}$ we get from (2.8)

$$\begin{aligned} \ddot{x} \; & + n(x)[n^t(x)'\dot{x}]\dot{x} + \varepsilon[\mathbb{I} - n(x)n^t(x)][n(x)'\ddot{x} + n(x)''(\dot{x},\dot{x})]\xi \\ & + 2\varepsilon[n(x)'\dot{x}]\dot{\xi} + 2\varepsilon n(x)[n^t(x)'\dot{x}][n(x)\dot{\xi}] + [\mathbb{I} - n(x)n^t(x)]F(x + \varepsilon n(x)\xi) \\ & + \varepsilon^{-2}[\mathbb{I} - n(x)n^t(x)]G(x + \varepsilon n(x)\xi) = 0. \end{aligned} \tag{2.9}$$

We remark that, for $x \in \mathcal{M}$ and any vector $v \in \mathbb{R}^n$ one has

$$n(x)n^t(x)v = \sum_{j=1}^{m} \langle v, n_j(x)\rangle n_j(x)$$

and then $\mathbb{I} - n(x)n^t(x)$ is the projection $P_x : \mathbb{R}^n \to T_x\mathcal{M}$ along the normal space $N_x\mathcal{M}$. Moreover

$$\begin{aligned} [\mathbb{I} - n(x)n^t(x)]G(x + \varepsilon n(x)\xi) & = [\mathbb{I} - n(x)n^t(x)][\varepsilon G'(x)n(x)\xi + O(\varepsilon^2\|\xi\|^2)] \\ & = [\mathbb{I} - n(x)n^t(x)][\varepsilon n(x)\Lambda(x)\xi + O(\varepsilon^2\|\xi\|^2)] \\ & = O(\varepsilon^2\|\xi\|^2). \end{aligned} \tag{2.10}$$

Thus we multiply (2.8) by ε and get:

$$\begin{aligned} \varepsilon^2\ddot{\xi} \; & + \Lambda(x)\xi - \varepsilon[n^t(x)'\dot{x}]\dot{x} - 2\varepsilon^2[n^t(x)'\dot{x}]n(x)\dot{\xi} + \varepsilon^2 n^t(x)\,[n(x)'\ddot{x} + n(x)''(\dot{x},\dot{x})]\,\xi \\ & + \varepsilon n^t(x)F(x + \varepsilon n(x)\xi) + \varepsilon^{-1}n^t(x)[G(x + \varepsilon n(x)\xi) - \varepsilon G'(x)n(x)\xi] = 0. \end{aligned} \tag{2.11}$$

Note that, since $\Lambda(x)$ is an invertible $m \times m$–matrix, equations (2.9), (2.11) for $\varepsilon = 0$ read:

$$\begin{aligned} & \ddot{x} - n(x)n^t(x)\ddot{x} + P_xF(x) = 0 \\ & \xi = 0 \end{aligned} \tag{2.12}$$

that is, since $\mathbb{I} - n(x)n^t(x)$ is the projection onto the tangent space $T_x\mathcal{M}$ along the normal space $N_x\mathcal{M}$, (2.9), (2.11) for $\varepsilon = 0$ is the D'Alembert equation on $\mathcal{M}$. On the other hand, passing to the time $\tau = \frac{t}{\varepsilon}$ and setting $\varepsilon = 0$ we obtain the boundary layer system

$$\xi'' + \Lambda(x)\xi = 0 \tag{2.13}$$

with $x \in \mathcal{M}$. Thus the manifold $\mathcal{M}$ is a manifold of fixed points for (2.12) which is not normally hyperbolic for (2.13). As in [4] we see that the standard Geometric Theory of singular equations (see [8]) cannot be applied.

We now write equations (2.9)–(2.11) as a first order system, setting $x_1 = x$, $x_2 = \dot{x}$, $y_1 = \xi$, $y_2 = \varepsilon\dot{\xi}$. Using also the equality $[n(x)'\widetilde{x}]\xi = [n(x)\xi]'\widetilde{x}$, we get, after some algebra:

$$\begin{aligned}
\dot{x}_1 =\; & x_2\\
\dot{x}_2 =\; & \varphi(x_1,x_2,y_1,y_2,\varepsilon) := -\{\mathbb{I}+\varepsilon[\mathbb{I}-n(x_1)n^t(x_1)][n(x_1)y_1]'\}^{-1}\\
& \{[n(x_1)n^t(x_1)'x_2]x_2+\varepsilon[\mathbb{I}-n(x_1)n^t(x_1)][n''(x_1)(x_2,x_2)]y_1\\
& +2[n'(x_1)x_2]y_2+2n(x_1)[n^t(x_1)'x_2][n(x_1)y_2]\\
& +[\mathbb{I}-n(x_1)n^t(x_1)]F(x_1+\varepsilon n(x_1)y_1)\\
& +\varepsilon^{-2}[\mathbb{I}-n(x_1)n^t(x_1)]G(x_1+\varepsilon n(x_1)y_1)\} \qquad (2.14)\\
\varepsilon\dot{y}_1 =\; & y_2\\
\varepsilon\dot{y}_2 =\; & -\Lambda(x_1)y_1+\varepsilon[n^t(x_1)'x_2]x_2+2\varepsilon[n^t(x_1)'x_2]n(x_1)y_2\\
& -\varepsilon^2 n^t(x_1)\left[n(x_1)'\varphi(x_1,x_2,y_1,y_2,\varepsilon)+n(x_1)''(x_2,x_2)\right]y_1\\
& -\varepsilon n^t(x_1)F(x_1+\varepsilon n(x_1)y_1)\\
& -\varepsilon^{-1}n^t(x_1)[G(x_1+\varepsilon n(x_1)y_1)-\varepsilon G'(x_1)n(x_1)y_1].
\end{aligned}$$

Now, since (2.14) is just equation (2.2) in the new coordinates (x_1,x_2,y_1,y_2), (2.14) should be considered as a dynamical system on $\mathcal{S}\times\mathbb{R}^{2m}$ where

$$\mathcal{S} = \left\{ \begin{pmatrix} x_1\\ x_2 \end{pmatrix} \mid x_1\in\mathcal{M}, x_2\in T_{x_1}\mathcal{M} \right\}. \qquad (2.15)$$

Here we show that, in fact, $\mathcal{S}\times\mathbb{R}^{2m}$ is invariant for equations (2.14). To prove this we only need to prove that if $(x_1,x_2)\in\mathcal{S}$, then

$$\begin{pmatrix} x_2\\ \varphi(x_1,x_2,y_1,y_2,\varepsilon)\end{pmatrix} \in T_{(x_1,x_2)}\mathcal{S}.$$

It is easy to see that

$$T_{(x_1,x_2)}\mathcal{S} = \left\{ \begin{pmatrix} u_1\\ u_2 \end{pmatrix} \in\mathbb{R}^{2n} \mid\; n^t(x_1)u_1 = 0 \;\text{ and }\; [n^t(x_1)'u_1]x_2 + n^t(x_1)u_2 = 0 \right\}.$$

We recall that the $j-$th row of $n^t(x)'u$ is given by the product of the Jacobian matrix of the $j-$th row of $n^t(x)$ with the vector u. That is $n_j'(x)u$, $j = 1,\ldots,m$. In other words $\binom{u_1}{u_2}$ belongs to $T_{(x_1,x_2)}\mathcal{S}$ if and only if, for any $j = 1,\ldots,m$, one has

$$< n_j(x_1), u_1 >= 0 \;\text{ and }\; < n_j'(x_1)u_1, x_2 > + < n_j(x_1), u_2 >= 0.$$

Hence we have to show that for any $(x_1,x_2)\in\mathcal{S}$ we have

$$n^t(x_1)x_2 = 0 \quad\text{and}\quad [n^t(x_1)'x_2]x_2 + n^t(x_1)\varphi(x_1,x_2,y_1,y_2,\varepsilon) = 0.$$

The first of the above equalities follows directly from the definition of $\mathcal{S}$. As for the second we have, using $n^t(x_1)n(x_1) = \mathbb{I}$:

$$\begin{aligned} n^t(x_1)\varphi(x_1,x_2,y_1,y_2,\varepsilon) = &\ n^t(x_1)\{\mathbb{I}+\varepsilon[\mathbb{I}-n(x_1)n^t(x_1)][n(x_1)y_1]'\}\varphi(x_1,x_2,y_1,y_2,\varepsilon) \\ = &\ -n^t(x_1)\{[n(x_1)n^t(x_1)'x_2]x_2+ \\ &\ \varepsilon[\mathbb{I}-n(x_1)n^t(x_1)][n(x_1)''(x_2,x_2)]y_1 \\ &\ +2[n(x_1)'x_2]y_2+2n(x_1)[n^t(x_1)'x_2][n(x_1)y_2] \\ &\ +[\mathbb{I}-n(x_1)n^t(x_1)]F(x_1+\varepsilon n(x_1)y_1) \\ &\ +\varepsilon^{-2}[\mathbb{I}-n(x_1)n^t(x_1)]G(x_1+\varepsilon n(x_1)y_1)\} \\ = &\ -[n^t(x_1)'x_2]x_2-2n^t(x_1)[n(x_1)'x_2]y_2 \\ &\ -2[n^t(x_1)'x_2][n(x_1)y_2] \\ = &\ -[n^t(x_1)'x_2]x_2. \end{aligned}$$

Note that to show the last equality we made use of the first equation in (2.6) with $v = x_2$. This concludes the proof of the invariance of $\mathcal{S}\times\mathbb{R}^{2m}$.

To simplify notation we set

$$J(x) = \begin{pmatrix} 0 & \mathbb{I} \\ -\Lambda(x_1) & 0 \end{pmatrix}, \tag{2.16}$$

$$f_0(x_1,x_2,y_1,y_2) = \begin{pmatrix} x_2 \\ \\ -n(x_1)[n^t(x_1)'x_2]x_2 - 2n(x_1)'x_2y_2 \\ -2n(x_1)[n^t(x_1)'x_2][n(x_1)y_2] - [\mathbb{I}-n(x_1)n^t(x_1)]\,F(x_1) \\ -[\mathbb{I}-n(x_1)n^t(x_1)]G''(x_1)(n(x_1)y_1,n(x_1)y_1) \end{pmatrix}$$

$$g_1(x_1,x_2,y_1,y_2) = \begin{pmatrix} 0 \\ \\ [n^t(x_1)'x_2]\,x_2 + 2[n^t(x_1)'x_2][n(x_1)y_2] \\ -n^t(x_1)F(x_1) - n^t(x_1)G''(x_1)(n(x_1)y_1,n(x_1)y_1) \end{pmatrix}$$

and write (2.14) as

$$\begin{aligned} \dot{x} &= f_0(x,y)+\varepsilon f_1(x,y,\varepsilon) \\ \varepsilon\dot{y} &= q(x,y,\varepsilon) := J(x)y+\varepsilon g_1(x,y)+\varepsilon^2 g_2(x,y,\varepsilon) \end{aligned} \tag{2.17}$$

where $x = (x_1,x_2)\in\mathcal{S}$, $y = (y_1,y_2)\in\mathbb{R}^{2m}$ and $f_1(x,y,\varepsilon)$, $g_2(x,y,\varepsilon)$, $q(x,y,\varepsilon)$ are defined by the equalities. Sometimes we will also write $f(x,y,\varepsilon)$ and $g(x,y,\varepsilon)$ for $f_0(x,y)+\varepsilon f_1(x,y,\varepsilon)$ and $g_1(x,y)+\varepsilon g_2(x,y,\varepsilon)$ respectively.

Remark 2.1 Note that, when $f(x,y,\varepsilon)$ and $q(x,y,\varepsilon)$ are taken as in (2.14), then f and q are C^3-functions because of the smoothness properties of F, G and $\mathcal{M}$. Moreover $\mathcal{S}\times\mathbb{R}^{2m}$ is an invariant manifold for (2.14) and the following symmetry properties hold.

Let $U_1:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{2n}$ and $U_2:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^{2m}$ be the involutions defined as:

$$U_1(x) = U_1(x_1,x_2) = (x_1,-x_2), \qquad U_2(y) = U_2(y_1,y_2) = (y_1,-y_2), \tag{2.18}$$

then:

$$U_1(f(x,y,\varepsilon)) = -f(U_1(x),U_2(y),\varepsilon), \quad U_2(q(x,y,\varepsilon)) = -q(U_1(x),U_2(y),\varepsilon). \quad (2.19)$$

Moreover, $f(x,y,\varepsilon)$ and $g(x,y,\varepsilon)$ satisfy

2-i) $[\mathbb{I}+U_2]g(x,0,0)=0$ and $g_y(x,0,0)=0$,

2-ii) $f_y(x,0,0)[\mathbb{I}+U_2]=0$,

2-iii) $f(0,0,\varepsilon)=0$.

Finally, for any $x=(x_1,x_2)\in\mathcal{S}$, $N_x\mathcal{S}$, has the C^3-smooth and U_1-invariant basis: $\{v_1(x),\ldots,v_{2m}(x)\}$ where

$$v_j(x) = w_j(x)/\|w_j(x)\|$$

and

$$w_{2j-1}(x) = \begin{pmatrix} n_j'(x_1)^t x_2 - n_j(x_1) \\ n_j(x_1) \end{pmatrix}$$
$$w_{2j}(x) = \begin{pmatrix} n_j'(x_1)^t x_2 + n_j(x_1) \\ n_j(x_1) \end{pmatrix}.$$

We recall the following definition [4]:

Definition 2.1 *A basis $\{v_1(x),\ldots,v_{2m}(x)\}$ of $N_x\mathcal{S}$ is said to be U_1-invariant if for any $x\in\mathcal{S}$ and any $j=1,\ldots,m$ the following holds:*

$$U_1 v_{2j-1}(x) = -v_{2j}(U_1(x)).$$

Thus, we also have $U_1v_{2j}(x)=U_1v_{2j}(U_1^2(x))=-U_1^2v_{2j-1}(U_1(x))=-v_{2j-1}(U_1(x))$ that is

$$U_1 v_{2j}(x) = -v_{2j-1}(U_1(x)).$$

Now, let $C^k(I,\mathbb{R}^n)$ be the space of functions on the interval $I\subset\mathbb{R}$ that are bounded together with their first k derivatives, on I. For $h(t)\in C^0([-T,0],\mathbb{R}^n)\cap C^0([0,T],\mathbb{R}^n)$ we consider the following system in $\mathbb{R}^{2m}$:

$$\varepsilon\dot{y} = J(x(t))y + \varepsilon h(t) \quad (2.20)$$

where $x(t)\in C^1(\mathbb{R},\mathbb{R})$ is an even (i.e. $x(t)=x(-t)$) function taking values on $\mathcal{M}$, together with the boundary conditions

$$(\mathbb{I}-U_2)y(-T) = (\mathbb{I}-U_2)y(T) = 0 \quad (2.21)$$

where U_2 is defined as in (2.18). Due to the form of the matrix $J(x)$ (see (2.4) and (2.16)) system (2.20) splits in m equations in $\mathbb{R}^2$ that look like:

$$\begin{cases} \varepsilon\dot{y}_j = y_{m+j} + \varepsilon h_1^{(j)}(t) \\ \varepsilon\dot{y}_{m+j} = -\lambda_j^2(t)y_j + \varepsilon h_2^{(j)}(t) \\ y_{m+j}(-T) = y_{m+j}(T) = 0 \end{cases} \quad (2.22)$$

to which we may apply the result of Lemma 4.2 in [4]. However, for our purposes, we need to look closely at the proof of the same Lemma 4.2. Such a closer look shows that the following result holds:

Lemma 2.1 *Assume that $\lambda_j(t) \in C^1(\mathbb{R}, \mathbb{R})$ is an even function on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} \frac{|\dot{\lambda}_j(t)|}{\lambda_j(t)} dt < \infty.$$

Let

$$\omega_j(t) = \int_0^t \lambda_j(\tau) d\tau.$$

Then there are two points $\theta_{1,j}, \theta_{2,j} \in [0, \pi]$ such that for any $\delta > 0$ and for any $j = 1, .., m$, there exist $T_j(\delta) > 0$ such that if $T > T_j(\delta)$ and $\varepsilon > 0$ satisfy

$$\left|\varepsilon^{-1}\omega_j(T) - \theta_{i,j} - k\pi\right| > \delta$$

for any $i = 1, 2$ and $k \in \mathbb{Z}$, then (2.22) has a unique solution $Y_j(t) := (y_j(t), y_{m+j}(t)) \in C^0([-T, T]) \cap C^1([-T, 0]) \cap C^1([0, T])$. Moreover there exist positive constants K, K_1, independent of T and ε (but possibly not of δ) such that

$$\|Y_j\| \leq KT\|h\|$$

if $h(t) \in C^0([-T, 0]) \cap C^0([0, T])$ and

$$\|Y_j\| \leq \varepsilon K_1 T[\|h\| + \|\dot{h}\|]$$

when $h(t) \in C^1([-T, 0]) \cap C^1([0, T])$. Here, the norms are those in $C^0([-T, 0]) \cap C^0([0, T])$.

Since we want to solve all the above systems for $j = 1, \ldots, m$, we need to show that it is not empty the set of $T > \max\{T_j(\delta) \mid j = 1, \ldots, m\}$ for which the inequalities

$$\left|\varepsilon^{-1}\omega_j(T) - \theta_{i,j} - k\pi\right| > \delta \tag{2.23}$$

are satisfied for any $j = 1, \ldots, m$, $i = 1, 2$ and $k \in \mathbb{Z}$. To this end we show the following

Lemma 2.2 *Let $A, B : (0, \infty) \to [0, \infty)$ be two functions such that $A(\varepsilon) < B(\varepsilon)$, $\lim\limits_{\varepsilon \to 0^+} B(\varepsilon) = \infty$ and $\lim\limits_{\varepsilon \to 0^+} \dfrac{A(\varepsilon)}{B(\varepsilon)} = 0$. Let $\phi_{i,j} \in [0, \pi]$ for $i = 1, 2$ and $j = 1, 2, \cdots, m$. Then there exists a constant $K_0 \geq 1$ such that for any $\delta < \frac{\pi}{4mK_0^2}$ there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the set*

$$S_{\varepsilon,\delta} := \left\{T \in [A(\varepsilon), B(\varepsilon)] \mid \left|\varepsilon^{-1}\omega_j(T) - \phi_{i,j} - k\pi\right| > \delta, \forall i = 1, 2; j = 1, \ldots, m; k \in \mathbb{Z}\right\}$$

is nonempty. More precisely,

$$\liminf_{\varepsilon \to 0} \frac{m(S_{\varepsilon,\delta})}{B(\varepsilon) - A(\varepsilon)} \geq 1 - \frac{4mK_0^2\delta}{\pi}.$$

Here $m(S)$ is the Lebesgue measure of the set S. As a consequence, if $\varepsilon > 0$ is sufficiently small, the set of $T \in [A(\varepsilon), B(\varepsilon)]$, for which equation (2.20) with the boundary condition (2.21) has a solution, has a positive measure.

Proof. The complement $S'_{\varepsilon,\delta}$ of $S_{\varepsilon,\delta}$ in $[A(\varepsilon), B(\varepsilon)]$ is the set of those $T \in [A(\varepsilon), B(\varepsilon)]$ for which (i, j, k) exist such that

$$\left|\varepsilon^{-1}\omega_j(T) - \phi_{i,j} - k\pi\right| \le \delta \tag{2.24}$$

or

$$\varepsilon(k\pi + \phi_{i,j} - \delta) \le \omega_j(T) \le \varepsilon(k\pi + \phi_{i,j} + \delta).$$

Let

$$\begin{array}{l} \lambda := \inf\{\lambda_j(t) \mid t \in \mathbb{R}, \quad j = 1, \dots, m\} \\ \Lambda := \sup\{\lambda_j(t) \mid t \in \mathbb{R}, \quad j = 1, \dots, m\} \end{array}$$

Then, for $t_1, t_2 \in \mathbb{R}$, one has

$$\lambda|t_2 - t_1| \le |\omega_j(t_2) - \omega_j(t_1)| \le \Lambda|t_2 - t_1|.$$

Hence, since $\omega_j(t)$ is monotone increasing, it easily follows that

$$\begin{array}{l} |\omega_j(t_2) - \omega_j(t_1)| \le \Lambda|t_2 - t_1|, \\ |\omega_j^{-1}(t_2) - \omega_j^{-1}(t_1)| \le \lambda^{-1}|t_2 - t_1|. \end{array}$$

We set

$$K_0 = \max\{\Lambda, \lambda^{-1}\} \ge 1.$$

Then, the length of the interval

$$\omega_j^{-1}\Big(\varepsilon(k\pi + \phi_{i,j} - \delta)\Big) \le T \le \omega_j^{-1}\Big(\varepsilon(k\pi + \phi_{i,j} + \delta)\Big) \tag{2.25}$$

is less than $2K_0\varepsilon\delta$. Since $0 \le T \le B(\varepsilon)$, and $0 \le 4mK_0^2\delta < \pi$, (2.25) implies that

$$-1 - \frac{1}{4mK_0^2} \le k \le \frac{1}{4mK_0^2} + \frac{K_0 B(\varepsilon)}{\pi\varepsilon}.$$

As a consequence, the measure of $S'_{\varepsilon,\delta}$ satisfies

$$m(S'_{\varepsilon,\delta}) \le 4mK_0\delta\left(\frac{K_0 B(\varepsilon)}{\pi} + \frac{4mK_0^2 + 1}{2mK_0^2}\varepsilon\right)$$

or

$$m(S_{\varepsilon,\delta}) \ge B(\varepsilon) - A(\varepsilon) - 4mK_0\delta\left(\frac{K_0 B(\varepsilon)}{\pi} + \frac{4mK_0^2 + 1}{2mK_0^2}\varepsilon\right)$$

which gives

$$\frac{m(S_{\varepsilon,\delta})}{B(\varepsilon) - A(\varepsilon)} \ge 1 - 4mK_0\delta\left(\frac{K_0 B(\varepsilon)}{\pi(B(\varepsilon) - A(\varepsilon))} + \frac{[4mK_0^2 + 1]\varepsilon}{2mK_0^2(B(\varepsilon) - A(\varepsilon))}\right) \to 1 - \frac{4mK_0^2\delta}{\pi}.$$

as $\varepsilon \to 0^+$. The proof is finished.

Remark 2.2 i) The opportunity being given we observe that the symmetry of $\lambda(t)$ has an interesting consequence that has not been observed in [4]. Let $Y(t)$ be the fundamental matrix of the system in $\mathbb{R}^2$

$$\dot{y} = Jy, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then it has been proved in [4] that, if $B(t)$ is a 2×2 matrix such that $\|B(t)\| \in L^1(\mathbb{R})$, the fundamental matrix $Y_B(t)$ of the linear system

$$\varepsilon \dot{y} = [J + \varepsilon B(t)]y \tag{2.26}$$

is bounded and there exist invertible 2×2 matrices $Y_\pm$ such that the following holds:

$$\lim_{t \to \pm\infty} Y(\varepsilon^{-1}t)^{-1} Y_B(t) = Y_\pm = \begin{pmatrix} \phi_{11}^\pm & \phi_{21}^\pm \\ \phi_{12}^\pm & \phi_{22}^\pm \end{pmatrix}.$$

Now, assume that $(y_j(t), y_{j+m}(t))$ is a solution of (2.22) with $h_1(t) = h_2(t) = 0$. Then setting, as in [4],

$$z_j(t) = y_j(\omega_j^{-1}(t)), \quad z_{j+m}(t) = \frac{y_{j+m}(\omega_j^{-1}(t))}{\lambda_j(\omega_j(t))}$$

with

$$\omega_j(t) = \int_0^t \lambda_j(\tau) d\tau,$$

we see that $(z_j(t), z_{j+m}(t))$ satisfies equation (2.26) with

$$B(t) = \begin{pmatrix} 0 & 0 \\ 0 & -\Gamma_j(t) \end{pmatrix}$$

and $\Gamma_j(t) = \dfrac{\dot{\lambda}_j(\omega_j^{-1}(\tau))}{\lambda_j(\omega_j^{-1}(\tau))}$. It is easy to see that, when $\lambda(t)$ is even then $\omega_j(t)$ and $\Gamma_j(t)$ are odd. Moreover, if

$$U_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we have

$$U_2 J = -J U_2 \quad \text{and} \quad U_2 B(-t) = -B(t) U_2.$$

Hence

$$U_2 Y_B(-t) = Y_B(t) U_2 \tag{2.27}$$

since both matrices satisfy the Cauchy problem

$$\begin{cases} \varepsilon \dot{W} = [J + \varepsilon B(t)]W \\ W(0) = U_2. \end{cases}$$

Note that, when $B(t)=0$, (2.27) gives $U_2Y(-t)=Y(t)U_2$. As a consequence we obtain:

$$U_2Y_- = \lim_{t\to\infty} U_2Y(-\varepsilon^{-1}t)^{-1}Y_B(-t) = \lim_{t\to\infty} Y(\varepsilon^{-1}t)Y_B(t)U_2 = Y_+U_2$$

or,

$$\phi_{11}^- = \phi_{11}^+, \quad \phi_{12}^- = -\phi_{21}^+, \quad \phi_{21}^- = -\phi_{12}^+, \quad \phi_{22}^- = \phi_{22}^+.$$

Hence the quantities A_∞, B_∞, C_∞ that have been defined in [4, p. 181] have the following simpler form:

$$\begin{aligned} A_\infty &= -2\phi_{12}^+\phi_{22}^+ \\ B_\infty &= -2[\phi_{11}^+\phi_{22}^+ + \phi_{12}^+\phi_{21}^+] \\ C_\infty &= -2\phi_{11}^+\phi_{21}^+. \end{aligned}$$

This implies that $B_\infty^2 - 4A_\infty C_\infty = (\det Z_+)^2 > 0$, which is a much simpler expression of $B_\infty^2 - 4A_\infty C_\infty$ than the one derived in [4]. We recall that it is this condition that guarantees that a certain equation has only two solutions in $[0,\pi]$, and eventually implies the conclusion of Lemma 4.2 in [4].
ii) Assume that $U_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $U_2 : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ are involutions on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$ such that

$$U_2 q(x,y,\varepsilon) = -q(U_1x, U_2y, \varepsilon), \tag{2.28}$$

where $q(x,y,\varepsilon)$ is as in (2.17). We claim that if for some $x_0 \in \mathbb{R}^{2n}$ we have $J(x_0) = J(U_1x_0)$ (we recall that $J(x)$ is defined in (2.16)) and all the $\lambda_j(x_0)$ are different from each other, then U_2 has the form given in (2.18). In fact, setting $\varepsilon = 0$ in (2.28) we get

$$U_2J(x) = -J(U_1x)U_2. \tag{2.29}$$

Let $\{e_j\}$ be the canonical basis of $\mathbb{R}^m$ and set

$$e_j^\pm := \begin{pmatrix} e_j \\ \pm i\lambda_j(x_0)e_j \end{pmatrix}, \quad j = 1,2,\ldots,m$$

where $i^2=-1$. We extend both $J(x_0)$ and U_2 linearly to $\mathbb{C}^{2m}$ and note that (2.29) with $x = x_0$ continues to hold because of uniqueness of analytic continuation. Then

$$J(x_0)e_j^\pm = \pm i\lambda_j(x_0)e_j^\pm$$

or, using (2.29) with $x=x_0$, and $J(U_1x_0) = J(x_0)$

$$J(x_0)U_2e_j^\pm = \mp i\lambda_j(x_0)U_2e_j^\pm\,. \tag{2.30}$$

Thus $U_2e_j^\pm$ is an eigenvector of $J(x_0)$ with $\mp i\lambda_j(x_0)$ as eigenvalue. Since we assumed that all $\lambda_j(x_0)$ are different from each other, it follows that

$$U_2e_j^+ = e_j^-\,, \quad \text{and} \quad U_2e_j^- = e_j^+.$$

In fact, the equality $J(x_0)v = -i\lambda_j(x_0)v$ writes:

$$\begin{pmatrix} v_2 \\ -\Lambda(x_0)v_1 \end{pmatrix} = \begin{pmatrix} -i\lambda_j(x_0)v_1 \\ -i\lambda_j(x_0)v_2 \end{pmatrix} \quad \text{with} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Hence $v_2 = -i\lambda_j(x_0)v_1$ and $\Lambda(x_0)v_1 = \lambda_j^2(x_0)v_1$. Using the fact that all the $\lambda_j(x_0)$ are different we get then, $v_1 = e_j$ and $v_2 = -i\lambda_j(x_0)e_j$ that is $v = e_j^-$. A similar argument shows that $U_2 e_j^- = e_j^+$ (note that we may also obtain this equality by applying U_2 to $U_2 e_j^+ = e_j^-$ and using $U_2^2 = \mathbb{I}$).

Let $\{b_j\}_{j=1,\dots,2m}$ be the standard basis on $\mathbb{R}^{2m}$. Since for $j = 1, \dots, m$ we have

$$b_j = \frac{e_j^+ + e_j^-}{2}$$
$$b_{j+m} = \frac{e_j^+ - e_j^-}{2i\lambda_j(x_0)}$$

we get, for $j = 1, \dots, m$:

$$U_2 b_j = \frac{1}{2}(e_j^- + e_j^+) = b_j$$
$$U_2 b_{j+m} = \frac{1}{2i\lambda_j(x_0)}(e_j^- - e_j^+) = -b_{j+m}.$$

Hence, generically, U_2 has the form given in (2.18).

Now we state the following result which is a slight variation of Proposition 3.1 in [4]. Consider equation (2.17) and let U_1, U_2 be involutions on $\mathbb{R}^{2n}$, $\mathbb{R}^{2m}$ respectively that satisfy (2.19). Then the following result holds.

Proposition 2.1 *Suppose that system (2.17) has an invariant, C^3-smooth, orientable manifold $\mathcal{S} \times \mathbb{R}^{2m}$, with $\mathcal{S} \subset \mathbb{R}^{2n}$, which is U_1-invariant and such that the normal vector bundle $N\mathcal{S}$ has an U_1-invariant basis. Let $\widetilde{\mathcal{S}} \subset \mathcal{S}$ be a U_1-invariant, open (in the relative topology) and bounded subset of $\mathcal{S}$. Then, there exist an open, U_1-invariant neighborhood $\widetilde{\Omega}$ of $\widetilde{\mathcal{S}}$ in $\mathbb{R}^{2n}$, with $\widetilde{\Omega} \cap \mathcal{S} = \widetilde{\mathcal{S}}$, and functions $\widetilde{x}(w)$, $\widetilde{f}(w, y, \varepsilon)$, $\widetilde{g}(w, y, \varepsilon)$, $w \in \widetilde{\Omega}$, $y \in \mathbb{R}^{2m}$ such that the following hold:*

a) *$\widetilde{x}(w) = w$, $\widetilde{f}(w, y, \varepsilon) = f(w, y, \varepsilon)$ and $\widetilde{g}(w, y, \varepsilon) = g(w, y, \varepsilon)$ for any $(w, y) \in \widetilde{\mathcal{S}} \times \mathbb{R}^{2m}$;*

b) *$\widetilde{\mathcal{S}} \times \mathbb{R}^{2m}$ is a normally hyperbolic invariant manifold of the system*

$$\begin{aligned} \dot{w} &= \widetilde{f}(w, y, \varepsilon) \\ \varepsilon \dot{y} &= J(\widetilde{x}(w))y + \varepsilon \widetilde{g}(w, y, \varepsilon). \end{aligned} \tag{2.31}$$

c) *System (2.31) is U-reversible.*

d) *If* $(x_0(t), y_0(t)) \in \widetilde{\mathcal{S}} \times \mathbb{R}^{2m}$ *is a solution of* (2.31) *which is bounded on* $\mathbb{R}$ *then the space of bounded solution on* $\mathbb{R}$ *of the linear system*

$$\begin{aligned}&\dot{x} = \widetilde{f}_x(x_0(t), y_0(t), \varepsilon)x + \widetilde{f}_y(x_0(t), y_0(t), \varepsilon)y\\ &\varepsilon\dot{y} = J'(x_0(t))xy_0(t) + J(x_0(t))y + \varepsilon\widetilde{g}_x(x_0(t), y_0(t), \varepsilon)x + \varepsilon\widetilde{g}_y(x_0(t), y_0(t), \varepsilon)y\end{aligned}$$

has the same dimension as the space of the bounded solutions on $\mathbb{R}$ *of the linear system*

$$\begin{aligned}\dot{x} &= f_x(x_0(t), y_0(t), \varepsilon)x + f_y(x_0(t), y_0(t), \varepsilon)y\\ \varepsilon\dot{y} &= q_x(x_0(t), y_0(t), \varepsilon)x + q_y(x_0(t), y_0(t), \varepsilon)y\end{aligned}$$

such that $x(t) \in T_{x_0(t)}\mathcal{S}$*. Similarly, the space of bounded solutions on* $\mathbb{R}$ *of the linear system*

$$\dot{x} = \widetilde{f}_x(x_0(t), 0, 0)x$$

has the same dimension as the space of the bounded solutions on $\mathbb{R}$ *of the linear system*

$$\dot{x} = f_x(x_0(t), 0, 0)x$$

such that $x(t) \in T_{x_0(t)}\mathcal{S}$*.*

Finally, $\widetilde{f}(w, y, \varepsilon)$ *and* $\widetilde{g}(w, y, \varepsilon)$ *satisfy* 2-i)*,* 2-ii) *and* 2-iii)*.*

Here we note that the slight difference we mentioned is that Proposition 3.1 in [4] has been proved only for $m = 1$. However the same proof works even when $m > 1$. Thus, from now on we will consider (2.17) as a system in $\mathbb{R}^{2(n+m)}$ having the invariant manifold $\mathcal{S} \times \mathbb{R}^{2m}$ with the property that at any point $x \in \mathcal{S}$, $N_x\mathcal{S}$ has the U_1-invariant basis $\{v_1(x), \dots, v_{2m}(x)\}$ and the properties 2-i), 2-ii), 2-iii) hold.

3 Existence of periodic solutions

In this Section we consider a system in the general form (2.17) and assume that the vector field $(f(x, y, \varepsilon), q(x, y, \varepsilon))$ is $(U_1, U_2)-$invariant, where $U_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $U_2 : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ are involutions on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. According to Remark 2.2 ii) we assume that U_2 has the form given in (2.18). Moreover, from [13] it follows that we may choose a scalar product in $\mathbb{R}^{2n}$, denoted by $< \cdot, \cdot >$, so that $U_1 = U_1^*$. We assume that this choice has been made and write A^* for the linear map adjoint to A. The purpose of this Section is to prove the following result.

Theorem 3.1 *Assume that system* (2.17) *has an invariant,* C^3*-smooth, orientable manifold* $\mathcal{S} \times \mathbb{R}^{2m}$*, with* $\mathcal{S} \subset \mathbb{R}^{2n}$*, which is* U_1*-invariant and such that the normal vector bundle* $N\mathcal{S}$ *has an* U_1*-invariant basis. Assume, also, that* $\gamma(t) \in \mathcal{S}$ *is a solution of the equation* $\dot{x} = f_0(x, 0)$*,* $x \in \mathbb{R}^{2n}$*, which is homoclinic to a hyperbolic fixed point* $x = x_0 \in \mathcal{S} \cap \operatorname{Fix} U_1$*. Here* $\operatorname{Fix} U_1 = \{x \in \mathbb{R}^{2n} \mid U_1 x = x\}$ *is the set of fixed points of* U_1 *and the the hyperbolicity of* x_0 *is considered with respect to the dynamics restricted to* $\mathcal{S}$*. Moreover, assume that* $\dot{\gamma}(t)$ *is the unique, up to a multiplicative constant, bounded solution of the variational system* $\dot{x} = f_{0,x}(\gamma(t), 0)x$ *along the tangent bundle* $T\mathcal{S}$*, and that the following conditions hold:*

i) $J(U_1x) = J(x)$, $(\mathbb{I} + U_2)g_1(x, 0) = 0$ *and* $g_{1,y}(x, 0) = 0$;

ii) $f_{0,y}(x, 0)(\mathbb{I} + U_2) = 0$ *and* $f_1(x_0, 0, 0) = 0$;

iii) *system* (2.17) *is* (U_1, U_2)*-reversible, that is*

$$U_1 f(x, y, \varepsilon) = -f(U_1x, U_2y, \varepsilon), \qquad U_2 q(x, y, \varepsilon) = -q(U_1x, U_2y, \varepsilon);$$

iv) $\gamma(t)$ *is* U_1-*symmetric that is* $U_1\gamma(t) = \gamma(-t)$.

Then, there exist $M > 0$, $\delta > 0$, $\mu > 0$, $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $T > 0$ *that satisfies condition* (2.23) *together with* $-3\delta^{-1} \ln \varepsilon < T < \frac{\mu}{\sqrt{\varepsilon}}$, *system* (2.17) *has a* $2T-$*periodic solution* $(x(t), y(t))$ *such that the following hold*

$$\begin{aligned} &\sup_{-T \le t \le T} |x(t) - \gamma(t)| \le M\varepsilon \\ &\sup_{-T \le t \le T} |y(t)| \le M\varepsilon \\ &U_1x(t) = x(-t), \quad U_2y(t) = y(-t). \end{aligned} \tag{3.1}$$

Proof. Since the proof is very similar to that given in [4], we mainly emphasize the differences, due to the different notation we use in this paper, and refer to [4] for more details.

First we note that, changing x with $x + x_0$ and $f(x, y, \varepsilon)$, $g(x, y, \varepsilon)$ and $J(x)$ with $f(x + x_0, y, \varepsilon)$, $g(x + x_0, y, \varepsilon)$ and $J(x + x_0)$, assumptions i), ii), iii) are still satisfied because of $U_1x_0 = x_0$. Thus we may assume, without loss of generality, that $x_0 = 0$.

As in [4], we note that, according to Proposition 2.1, $x = 0$ is a hyperbolic fixed point for the expanded system $\dot{x} = \widetilde{f}_0(x, 0)$ and that $\dot{\gamma}(t)$ is the unique, up to a multiplicative constant, bounded solution on $\mathbb{R}$ of the linear system $\dot{x} = \widetilde{f}_{0x}(\gamma(t), 0)x$. In other words, passing to the expanded system (2.31), we may assume that the conditions of this theorem hold for the dynamics in the whole $\mathbb{R}^{2(n+m)}$. Next, we note that it is enough to prove the result for the expanded system (2.31). In fact, since $\mathcal{S} \times \mathbb{R}^{2m}$ is a normally hyperbolic invariant manifold for (2.31) any periodic solution of (2.31) in a small neighborhood of $\mathcal{S} \times \mathbb{R}^{2m}$ actually belongs to $\mathcal{S} \times \mathbb{R}^{2m}$. But then such a periodic solution solves also (2.17) since on $\mathcal{S} \times \mathbb{R}^{2m}$ (2.31) is the same as (2.17). Thus we may and will assume that the assumptions of this theorem hold for the dynamics in the whole $\mathbb{R}^{2(n+m)}$. For simplicity we will continue to write f and g and q instead of $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{q}$. Finally taking $A(\varepsilon) = -3\delta^{-1} \ln \varepsilon$ and $B(\varepsilon) = \frac{\mu}{\sqrt{\varepsilon}}$ we see that the conditions of Lemma 2.2 are satisfied and hence, for any $\varepsilon > 0$ sufficiently small, the set of T that satisfy the assumptions of this theorem has positive measure.

Now, we observe that, from iii), i) we get

$$\begin{aligned} &U_2J(x) = -J(U_1x)U_2 = -J(x)U_2 \\ &U_1f_0(x, y) = -f_0(U_1x, U_2y), \\ &U_1f_1(x, y, 0) = -f_1(U_1x, U_2y, 0). \end{aligned} \tag{3.2}$$

Let

$$y_0(t) := -J(\gamma(t))^{-1}g_1(\gamma(t), 0).$$

From (3.2) we get immediately: $(\mathbb{I}-U_2)J(x)^{-1}=J(x)^{-1}(\mathbb{I}+U_2)$. Hence

$$(\mathbb{I}-U_2)y_0(t)=0. \tag{3.3}$$

Moreover, it is easy to see that

$$U_2y_0(t)=y_0(-t). \tag{3.4}$$

In fact from iii) we get

$$U_2g_1(x,y)=-g_1(U_1x,U_2y)$$

and then, using (3.2),

$$\begin{aligned} U_2y_0(t) &= -U_2J(\gamma(t))^{-1}g_1(\gamma(t),0)=J(U_1\gamma(t))^{-1}U_2g_1(\gamma(t),0)\\ &= -J(\gamma(-t))^{-1}g_1(U_1\gamma(t),0)=-J(\gamma(-t))^{-1}g_1(\gamma(-t),0)=y_0(-t). \end{aligned}$$

Next, differentiating the second equation in (3.2) with respect to x, at $(x,0,0)$ we obtain

$$U_1f_{0,x}(x,0)=-f_{0,x}(U_1x,0)U_1. \tag{3.5}$$

Let $\psi(t)$ be the bounded solution of the adjoint system $\dot{x}=-f^*_{0,x}(\gamma(t),0)x$ with $|\psi(0)|=1$. It is known (see for example [4, 13]) that $U_1\psi(t)=\psi(-t)$. As in [4] we see that the system

$$\begin{cases} \dot{x}-f_{0,x}(\gamma(t),0)x=f_1(\gamma(t),0,0)\,,\\ \langle\dot{\gamma}(0),x(0)\rangle=0 \end{cases} \tag{3.6}$$

has a unique bounded solution $x_0(t)\in C^1_b(\mathbb{R})$ (see, also [10]) such that

$$|x_0(t)|,|\dot{x}_0(t)|\le c_0e^{-\delta|t|/2}$$

and, because of uniqueness:

$$U_1x_0(t)=x_0(-t). \tag{3.7}$$

Now, as in [4], we replace $(x(t),y(t))$ in equation (2.17) with $(\gamma(t)+\varepsilon x_0(t)+\varepsilon x(t),\varepsilon y_0(t)+\varepsilon y(t))$. From ii) and (3.3) we obtain first:

$$f_{0,y}(x,0)y_0(t)=-f_{0,y}(x,0)U_2y_0(t)=-f_{0,y}(x,0)y_0(t)$$

and then we obtain the following system ($x\in\mathbb{R}^{2n}$, $y\in\mathbb{R}^{2m}$)

$$\begin{aligned} \varepsilon\dot{y}\ &-[J(\gamma(t))+\varepsilon J'(\gamma(t))x_0(t)]y=g_1(\gamma(t)+\varepsilon x_0(t)+\varepsilon x,\varepsilon y_0(t)+\varepsilon y)\\ &-g_1(\gamma(t),0)-\varepsilon\dot{y}_0(t)+\varepsilon g_2(\gamma(t)+\varepsilon x_0(t)+\varepsilon x,\varepsilon y_0(t)+\varepsilon y,\varepsilon)\\ &+[J(\gamma(t)+\varepsilon x_0(t)+\varepsilon x)-J(\gamma(t))]y_0(t)\\ &+[J(\gamma(t)+\varepsilon x_0(t)+\varepsilon x)-J(\gamma(t))-\varepsilon J'(\gamma(t))x_0(t)]y\\ \dot{x}\ &-f_{0,x}(\gamma(t),0)x=\varepsilon^{-1}[f_0(\gamma(t)+\varepsilon x_0(t)+\varepsilon x,\varepsilon y_0(t)+\varepsilon y)\\ &-f_0(\gamma(t),0)-f_{0,x}(\gamma(t),0)(\varepsilon x_0(t)+\varepsilon x)-f_{0,y}(\gamma(t),0)\varepsilon y_0(t)]\\ &+f_1(\gamma(t)+\varepsilon x_0(t)+\varepsilon x,\varepsilon y_0(t)+\varepsilon y,\varepsilon)-f_1(\gamma(t),0,0) \end{aligned} \tag{3.8}$$

with the boundary condition

$$x(-T)-x(T)=x_0(T)-x_0(-T)+\varepsilon^{-1}[\gamma(T)-\gamma(-T)]. \tag{3.9}$$

Arguing as in [4] we see that the matrix $J(\gamma(t))+\varepsilon J'(\gamma(t))x_0(t)$ satisfies the assumptions of Lemma 4.2 and that taking $\rho_1 > 0$, $\rho_2 > 0$ and then $\mu > 0$ so that

$$4\hat{C}_2 C_6 \rho_2 < \rho_1, \quad \widetilde{C}\mu(2+\rho_1\rho_2) \le \rho_2 \tag{3.10}$$

then there exists $\varepsilon_0 = \varepsilon_0(\rho_1, \rho_2, \mu)$ such that if $0 < \varepsilon \le \varepsilon_0$, $-3\delta^{-1}\ln\varepsilon < T < \mu/\sqrt{\varepsilon}$, and T satisfies (2.23), system (3.8) with the boundary condition (3.9) has a unique bounded solution $(\widetilde{x}(t), \widetilde{y}(t))$ such that

$$\begin{array}{l} \|\widetilde{x}\|_1 \le \sqrt{\varepsilon}\rho_1, \quad \|\widetilde{y}\|_0 \le \sqrt{\varepsilon}\rho_2\,, \\ (\mathbb{I}-U_2)\widetilde{y}(-T) = (\mathbb{I}-U_2)\widetilde{y}(T) = 0\,, \\ \widetilde{x}(0^+) - \widetilde{x}(0^-) = \langle \widetilde{x}(0^+) - \widetilde{x}(0^-), \psi(0)\rangle\psi(0)\,. \end{array} \tag{3.11}$$

Then

$$(x(t), y(t)) = (\gamma(t) + \varepsilon x_0(t) + \varepsilon\widetilde{x}(t), \varepsilon y_0(t) + \varepsilon\widetilde{y}(t))$$

is a solution of the expanded system (2.31), with a possible jump at $t = 0$, that satisfies

$$x(-T) = x(T) \quad \text{and} \quad (\mathbb{I}-U_2)y(\pm T) = 0.$$

To conclude the proof of the theorem we only need to show that:

$$x(0^+) = x(0^-) \quad \text{and} \quad U_2 y(-t) = y(t).$$

In fact, since $(\mathbb{I}-U_2)y(\pm T) = 0$, the above conditions imply that $(x(t), y(t))$ is a C^1, periodic solution of the expanded system (2.31). Thus it belongs to the invariant manifold $\mathcal{S}\times\mathbb{R}^{2m}$, as we have observed at the beginning of this proof and is in fact a periodic solution of the original system (2.17).

Let

$$\bar{x}(t) = U_1\widetilde{x}(-t), \quad \bar{y}(t) = U_2\widetilde{y}(-t).$$

It is easily seen, from (3.4), (3.7), iv) and v), that $(\bar{x}(t), \bar{y}(t))$ is another solution of system (3.8) such that $\|\bar{x}\|_1 \le \sqrt{\varepsilon}\rho_1$, $\|\bar{y}\|_0 \le \sqrt{\varepsilon}\rho_2$ (here we used also $\|U_1\| = \|U_2\| = 1$). Moreover

$$(\mathbb{I}-U_2)\bar{y}(t) = (\mathbb{I}-U_2)U_2\widetilde{y}(-t) = -(\mathbb{I}-U_2)\widetilde{y}(-t)$$

and hence $(\mathbb{I}-U_2)\bar{y}(-T) = (\mathbb{I}-U_2)\bar{y}(T) = 0$. Next, as in [4] we see that

$$< \bar{x}(0^+), \dot{\gamma}(0) >= 0$$

and

$$\bar{x}(0^+) - \bar{x}(0^-) =< \psi(0), \bar{x}(0^+) - \bar{x}(0^-) > \psi(0).$$

Thus, owing to the uniqueness of the fixed point that satisfies (3.11),

$$U_1\widetilde{x}(t) = \widetilde{x}(-t), \quad U_2\widetilde{y}(t) = \widetilde{y}(-t).$$

As a consequence

$$\begin{aligned} \widetilde{x}(0^+) - \widetilde{x}(0^-) &=< \widetilde{x}(0^+) - \widetilde{x}(0^-), \psi(0) > \psi(0) = \\ &=< U_1(\widetilde{x}(0^-) - \widetilde{x}(0^+)), \psi(0) > \psi(0) \\ &= - < \widetilde{x}(0^+) - \widetilde{x}(0^-), \psi(0) > \psi(0) \\ &= \widetilde{x}(0^-) - \widetilde{x}(0^+) = 0. \end{aligned}$$

So $\widetilde{x}(0^+) = \widetilde{x}(0^-)$ and then $x(0^+) = x(0^-)$. Finally, using also (3.4):

$$\begin{aligned} U_1 x(t) &= U_1\gamma(t) + \varepsilon U_1 x_0(t) + \varepsilon U_1 \widetilde{x}(t) = x(-t)\,, \\ U_2 y(t) &= \varepsilon U_2 y_0(t) + \varepsilon U_2 \widetilde{y}(t) = y(-t)\,. \end{aligned}$$

This concludes the proof of the Theorem.

We are now able to give the

Proof of Theorem 1.1. We apply Theorem 3.1 to equation (2.14). Thus we show that the assumptions of Theorem 3.1 follow from those of Theorem 1.1. Of course we take $\mathcal{S}$ as in (2.15), $J(x)$ as in (2.16) and (U_1, U_2) as in (2.18). From Remark 2.1 and the fact that equation $\dot{x} = f_0(x, 0)$ is the D'Alembert equation (2.12), it follows that we only need to show that

$$\gamma(t) = \begin{pmatrix} \gamma_0(t) \\ \dot{\gamma}_0(t) \end{pmatrix}$$

satisfies the assumptions of Theorem 3.1. Now, the U_1-symmetry of $\gamma(t)$ directly follows from $\gamma_0(t) = \gamma_0(-t)$. Thus, in order to apply Theorem 3.1 we only have to prove that the space of bounded solutions $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ of the variational system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -[n(\gamma_0(t))' x_1][n^t(\gamma_0(t))'\dot{\gamma}_0(t)]\dot{\gamma}_0(t) - n(\gamma_0(t))[n^t(\gamma_0(t))'' x_1 \dot{\gamma}_0(t)]\dot{\gamma}_0(t) \\ \qquad -n(\gamma_0(t))[n^t(\gamma_0(t))'\dot{\gamma}_0(t)]x_2 - n(\gamma_0(t))[n^t(\gamma_0(t))' x_2]\dot{\gamma}_0(t) \\ \qquad -[\mathbb{I} - n(\gamma_0(t))n^t(\gamma_0(t))]F'(\gamma_0(t))x_1 \\ \qquad +[n(\gamma_0(t))' x_1 n^t(\gamma_0(t)) + n(\gamma_0(t))n^t(\gamma_0(t))' x_1]F(\gamma_0(t)) \end{cases}$$

such that $x(t) \in T_{\gamma(t)}\mathcal{S}$ is one dimensional (and then spanned by $\dot{\gamma}(t)$). Now, setting $z = x_1$, $\dot{z} = x_2$ and using $[n^t(\gamma_0(t))'\dot{\gamma}_0(t)]\dot{\gamma}_0(t) = -n^t(\gamma_0(t))\ddot{\gamma}_0(t)$, (see (2.7)) we see that proving this is equivalent to prove that the space of solutions of

$$\begin{aligned} \ddot{z} - \; & [n(\gamma_0(t))' z][n^t(\gamma_0(t))\ddot{\gamma}_0(t)] + n(\gamma_0(t))[n^t(\gamma_0(t))'' z \dot{\gamma}_0(t)]\dot{\gamma}_0(t) \\ & + n(\gamma_0(t))[n^t(\gamma_0(t))'\dot{\gamma}_0(t)]\dot{z}(t) + n(\gamma_0(t))[n^t(\gamma_0(t))'\dot{z}(t)]\dot{\gamma}_0(t) \\ & + [\mathbb{I} - n(\gamma_0(t))n^t(\gamma_0(t))]F'(\gamma_0(t))z \\ & - [n(\gamma_0(t))' z n^t(\gamma_0(t)) + n(\gamma_0(t))n^t(\gamma_0(t))' z]F(\gamma_0(t)) = 0 \end{aligned} \tag{3.12}$$

that are bounded on $\mathbb{R}$ and satisfy:

$$n^t(\gamma_0(t))z(t) = 0 \quad \text{and} \quad [n^t(\gamma_0(t))' z(t)]\dot{\gamma}_0(t) + n^t(\gamma_0(t))\dot{z}(t) = 0 \tag{3.13}$$

is one dimensional and spanned by $\dot{\gamma}_0(t)$. Now, the linearization of the D'Alembert equation (1.4) along $\gamma_0(t)$ is:

$$\begin{aligned} [\mathbb{I} - \; & n(\gamma_0(t))n^t(\gamma_0(t))][\ddot{z} + F'(\gamma_0(t))z] \\ = \; & [n(\gamma_0(t))' z n^t(\gamma_0(t)) + n(\gamma_0(t))n^t(\gamma_0(t))' z][\ddot{\gamma}_0(t) + F(\gamma_0(t))]. \end{aligned} \tag{3.14}$$

Hence we see that if $z(t)$ satisfies the following condition

$$\begin{aligned} & [n^t(\gamma_0(t))'' z(t)\dot{\gamma}_0(t)]\dot{\gamma}_0(t) + [n^t(\gamma_0(t))'\dot{\gamma}_0(t)]\dot{z}(t) \\ & + [n^t(\gamma_0(t))'\dot{z}(t)]\dot{\gamma}_0(t) + n^t(\gamma_0(t))\ddot{z}(t) + [n^t(\gamma_0(t))' z(t)]\ddot{\gamma}_0(t) = 0 \end{aligned} \tag{3.15}$$

then $z(t)$ satisfies (3.12)–(3.13) if and only if $z(t)$ solves (3.14) with the condition in Theorem 1.1 point 4) (which is just the second condition in (3.13)). But then from the assumption 4) in Theorem 1.1 the space of bounded solutions of (3.12) that satisfy (3.13) is one dimensional, this concluding the proof. So, we prove that (3.13) implies (3.15). Differentiating the second equality in (3.13) with respect to t we see that (3.15) is equivalent to

$$[n^t(\gamma_0(t))''z(t)\dot{\gamma}_0(t) - n^t(\gamma_0(t))''\dot{\gamma}_0(t)z(t)]\dot{\gamma}_0(t) = 0,$$

which of course is valid since $n(x)$ is at least C^2-smooth. Thus the assumptions of Theorem 3.1 are satisfied and the conclusion follows.

4 Example

Here we give an example of application of our main theorem. As $\mathcal{M}$ we take the non planar curve in the torus of radii $R = 1$ and $r = \frac{1}{2}$:

$$z(\theta) = \begin{pmatrix} z_1(\theta) \\ z_2(\theta) \\ z_3(\theta) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (2+\cos\theta)\cos\theta \\ (2+\cos\theta)\sin\theta \\ \sin\theta \end{pmatrix}$$

$-\pi \le \theta \le \pi$. We have

$$z'(\theta) = \frac{1}{2}\begin{pmatrix} -2\sin\theta - \sin 2\theta \\ 2\cos\theta + \cos 2\theta \\ \cos\theta \end{pmatrix}$$

and

$$z''(\theta) = -\begin{pmatrix} \cos\theta + \cos 2\theta \\ \sin\theta + \sin 2\theta \\ \frac{1}{2}\sin\theta \end{pmatrix}.$$

So, the tangent vector at the point $z(\theta) = (z_1(\theta), z_2(\theta), z_3(\theta))$ is

$$T(\theta) = \frac{z'(\theta)}{\|z'(\theta)\|} = \frac{1}{\sqrt{(2+\cos\theta)^2+1}}\begin{pmatrix} -2\sin\theta - \sin 2\theta \\ 2\cos\theta + \cos 2\theta \\ \cos\theta \end{pmatrix}$$

and the Binormal vector is $\frac{B(\theta)}{\|B(\theta)\|}$ where

$$B(\theta) = \begin{pmatrix} \sin 3\theta + 3\sin\theta \\ -\cos 3\theta - 3\cos\theta - 4 \\ 12(1+\cos\theta) \end{pmatrix}$$

hence $B(\theta) \neq 0$, for any $\theta \in [-\pi, \pi]$, and, since $B(0) \neq B(\pi)$, the curve is not planar. Another normal vector to the curve is found taking the cross product between the unitary tangent vector and the binormal vector. We obtain $\frac{N(\theta)}{\|N(\theta)\|}$, where

$$N(\theta) = \begin{pmatrix} \cos 4\theta + 12\cos 3\theta + 52\cos 2\theta + 68\cos\theta + 27 \\ \sin 4\theta + 12\sin 3\theta + 52\sin 2\theta + 60\sin\theta \\ 4(\sin 2\theta + 5\sin\theta) \end{pmatrix}.$$

Then, the covariant derivative is

$$D_t\left[\frac{d}{dt}z(\theta(t))\right] =< z'(\theta(t))\ddot\theta(t) + z''(\theta(t))\dot\theta^2(t), T(\theta(t)) > T(\theta(t)) =$$

and hence, inserting the equations for $z(\theta)$, $z'(\theta)$, $z''(\theta)$, $T(\theta)$ we obtain, after some algebra:

$$D_t\dot z = \frac{1}{2}\left[\sqrt{(2+\cos\theta)^2+1}\ddot\theta - \frac{(2+\cos\theta)\sin\theta}{\sqrt{(2+\cos\theta)^2+1}}\dot\theta^2\right]T(\theta)$$

whereas the scalar product $\langle F(z(\theta)), T(\theta)\rangle$ is

$$\frac{-(2\sin\theta+\sin 2\theta)f_1(\theta) + (2\cos\theta+\cos 2\theta)f_2(\theta) + f_3(\theta)\cos\theta}{\sqrt{(2+\cos\theta)^2+1}},$$

(here $f_j(\theta) := F_j(z(\theta))$, $j = 1, 2, 3$). As a consequence the D'Alembert equation of the equation

$$\ddot z + F(z) = 0$$

on the manifold $\mathcal{M}$ is equivalent to the second order equation in $\mathbb{R}$:

$$\ddot\theta - \frac{\sin\theta(2+\cos\theta)}{(2+\cos\theta)^2+1}\dot\theta^2 = 2\frac{(2\sin\theta+\sin 2\theta)f_1(\theta) - (2\cos\theta+\cos 2\theta)f_2(\theta) - f_3(\theta)\cos\theta}{(2+\cos\theta)^2+1}$$

Now, we take

$$F(z) = \begin{pmatrix} -4z_1 \\ -4z_2 - z_3 \\ 5z_2 - 2z_3 \end{pmatrix}.$$

After some algebra, we see that the above equation reads:

$$((2+\cos\theta)^2+1)\ddot\theta - \sin\theta(2+\cos\theta)\dot\theta^2 = \Phi(\theta) \tag{4.1}$$

where $\Phi(\theta) = -\frac{39}{4}\sin\theta - 5\sin 2\theta - \frac{3}{4}\sin 3\theta$. Setting:

$$\begin{aligned} u &= \theta \\ v &= [(2+\cos\theta)^2+1]\dot\theta \end{aligned}$$

equation (4.1) is equivalent to the following system:

$$\begin{cases} \dot u = \dfrac{1}{[(2+\cos u)^2+1]}v \\ \dot v = \Phi(u) - \dfrac{(2+\cos u)\sin u}{[(2+\cos u)^2+1]^2}v^2. \end{cases} \tag{4.2}$$

It is straightforward to verify that equation (4.1) has the solution

$$\theta_0(t) = 2\arctan(\sinh t),$$

or equivalently, that system (4.2) has the solution

$$\gamma_0(t) = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} = \begin{pmatrix} 2\arctan(\sinh t) \\ \dfrac{4}{\cosh t}\left[1 + \dfrac{2}{\cosh^2 t} + \dfrac{2}{\cosh^4 t}\right] \end{pmatrix}$$

that corresponds to the homoclinic orbit of the D'Alembert equation:

$$z_0(t) = \begin{pmatrix} \dfrac{4-\cosh^4 t}{2\cosh^4 t} \\[2ex] \dfrac{(\cosh^2 t + 2)\sinh t}{\cosh^4 t} \\[2ex] \dfrac{\sinh t}{\cosh^2 t} \end{pmatrix} \to \hat{z}_0 := \begin{pmatrix} -\frac{1}{2} \\[2ex] 0 \\[2ex] 0 \end{pmatrix}$$

as $t \to \pm\infty$. Note that $\gamma_0(t) \to \binom{\pm\pi}{0}$, as $t \to \pm\infty$ and that the equilibrium point $\hat{z}_0$, corresponding to both $\theta = -\pi$ and $\theta = \pi$, is hyperbolic since the Jacobian matrix of (4.2) at $(\pm\pi, 0)$ is

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}.$$

Now, we define the function $G(z)$ so that the assumptions of our main theorem are satisfied. To this end we observe that the manifold $\mathcal{M}$ is the intersection of the two surfaces in $\mathbb{R}^3$:

$$\begin{aligned} \mathcal{T}_1 &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid g_1(z_1, z_2, z_3) := z_1^2 + z_2^2 + z_3^2 - 2\sqrt{z_1^2 + z_2^2} + \tfrac{3}{4} = 0\} \\ \mathcal{T}_2 &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 \mid g_2(z_1, z_2, z_3) := z_2[z_1^2 + z_2^2 + z_3^2 - \tfrac{5}{4}] - 2z_1 z_3 = 0\}. \end{aligned}$$

in a tubular neighborhood of $\mathcal{M}$ itself. In fact it is obvious that any point of $\mathcal{M}$ belongs to the torus $\mathcal{T}_1$. Then, since $\mathcal{T}_1$ has the parametric equation:

$$\begin{cases} z_1 = z_1(\theta, \phi) := (1 + \frac{1}{2}\cos\theta)\cos\phi \\ z_2 = z_2(\theta, \phi) := (1 + \frac{1}{2}\cos\theta)\sin\phi \\ z_3 = z_3(\theta) := \frac{1}{2}\sin\theta, \end{cases}$$

we see that a point in $\mathcal{T}_1$ belongs to $\mathcal{T}_2$ if and only if $g_2(z_1(\theta, \phi), z_2(\theta, \phi), z_3(\theta)) = 0$. After some algebra we see that the above equation is equivalent to:

$$\sin(\phi - \theta) = 0.$$

This mean that besides the manifold $\mathcal{M}$, $\mathcal{T}_1 \cap \mathcal{T}_2$ also contains the points we obtain taking $\phi = \theta + \pi$ that is

$$\begin{cases} z_1 = -(1 + \frac{1}{2}\cos\theta)\cos\theta \\ z_2 = -(1 + \frac{1}{2}\cos\theta)\sin\theta \\ z_3 = \frac{1}{2}\sin\theta. \end{cases}$$

The above points describe another manifold $\mathcal{M}_1$ which is in some sense parallel to $\mathcal{M}$. Hence $\mathcal{M}$ is locally isolated. Then, for $z = (z_1, z_2, z_3) \in \mathbb{R}^3$, we set

$$U(z) = g_1^2(z) + g_2^2(z)$$

and $G(z) = U'(z) = 2g_1(z)\nabla g_1(z) + 2g_2(z)\nabla g_2(z)$. Of course $G(z) = 0$ on $\mathcal{M}$, and, for $z \in \mathcal{M}$, we have

$$\frac{1}{2}G'(z) = \nabla g_1(z) \otimes \nabla g_1(z) + \nabla g_2(z) \otimes \nabla g_2(z)$$

where $\nabla g(z) \otimes \nabla g(z)$ stands for the rank one, $n \times n$ matrix

$$\nabla g(z)[\nabla g(z)]^t = [g_{z_i}(z) g_{z_j}(z)]_{i,j}.$$

Now, it is not difficult to see that $\|\nabla g_1(z)\| = 1$ on $\mathcal{M}$ and $\|\nabla g_1(z) \times \nabla g_2(z)\| = \cos^2\theta + 4\cos\theta + 5 \geq 2$ for $z \in \mathcal{M}$, so that $\nabla g_1(z)$, $\nabla g_2(z)$ are independent on $\mathcal{M}$. Hence $N_z\mathcal{M}$ is spanned by the vectors $\nabla g_1(z)$, $\nabla g_2(z)$, and then any vector $v \in N_z\mathcal{M}$ can be written as

$$v = c_1\nabla g_1(z) + c_2\frac{\nabla g_2(z)}{\|\nabla g_2(z)\|}.$$

Then for any $z \in \mathcal{M}$ we have

$$\begin{aligned}\tfrac{1}{2}G'(z)v \ &= c_1\nabla g_1(z) + c_2\tfrac{\nabla g_1(z)\cdot\nabla g_2(z)}{\|\nabla g_2(z)\|}\nabla g_1(z)\\ &\quad + c_1[\nabla g_1(z)\cdot\nabla g_2(z)]\nabla g_2(z) + c_2\|\nabla g_2(z)\|\nabla g_2(z)\end{aligned}$$

from which it follows that $N_z\mathcal{M}$ is invariant under $G'(z)$. Although it is clear from the above equality that the directions of $\nabla g_1(z)$ and $\nabla g_2(z)$ are not invariant with respect to $G'(z)_{|N_z\mathcal{M}} : N_z\mathcal{M} \to N_z\mathcal{M}$, the matrix of $G'(z)_{|N_z\mathcal{M}}$ with respect to the basis $\{\nabla g_1(z), \nabla g_2(z)\}$ of $N_z\mathcal{M}$, $z \in \mathcal{M}$, is

$$H(z) = \begin{pmatrix} 1 & \nabla g_1(z)\cdot\nabla g_2(z) \\ \nabla g_1(z)\cdot\nabla g_2(z) & \|\nabla g_2(z)\|^2 \end{pmatrix}.$$

So, $\mathrm{tr}H(z) = 1 + \|\nabla g_2(z)\|^2 \geq 1 > 0$ and $\det H(z) = \|\nabla g_2(z)\|^2 - (\nabla g_1(z)\cdot\nabla g_2(z))^2 \geq \kappa > 0$, (by the Cauchy-Schwartz inequality) uniformly with respect to $z \in \mathcal{M}$. As a consequence $G'(z)_{|N_z\mathcal{M}}$ is symmetric and positive definite uniformly with respect to $z \in \mathcal{M}$. On the other hand, passing to the functions

$$\widetilde{g}_1(z) = g_1(z), \quad \widetilde{g}_2(z) = g_2(z) - [\nabla g_1(z)\cdot\nabla g_2(z)]g_1(z)$$

in place of $g_1(z)$ and $g_2(z)$, we see that $\nabla\widetilde{g}_1(z)\cdot\nabla\widetilde{g}_2(z) = 0$ for $z \in \mathcal{M}$. Hence the corresponding matrix $H(z)$ is diagonal in the basis $\{\nabla\widetilde{g}_1(z), \nabla\widetilde{g}_2(z)\}$. Consequently, we can take

$$n_1(z) = \nabla\widetilde{g}_1(z), \quad n_2(z) = \nabla\widetilde{g}_2(z)/\|\nabla\widetilde{g}_2(z)\|$$

in the notation at the beginning of Section 2. Summarizing, the assumptions of our main Theorem are satisfied. Hence there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ the equation

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_3 \end{pmatrix} = \begin{pmatrix} 4z_1 \\ 4z_2 + z_3 \\ -5z_2 + 2z_3 \end{pmatrix} - 2\varepsilon^{-2}[g_1(z_1, z_2, z_3)\nabla g_1(z_1, z_2, z_3) + g_2(z_1, z_2, z_3)\nabla g_2(z_1, z_2, z_3)]$$

has several layers of periodic solutions accumulating on the homoclinic solution $z_0(t)$.

Remark 4.1 The previous example can be extended to the case where $\mathcal{M}$ is a complete intersection, codimension m submanifold of $\mathbb{R}^m$, that is when there exists a neighborhood Ω of $\mathcal{M}$ and smooth functions $g_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, m$ such that

$$\mathcal{M} \cap \Omega = \{z \in \Omega \mid g_1(z) = \ldots = g_m(z) = 0\}$$

and

$$\operatorname{rank} \left(\frac{\partial g_i}{\partial z_j}(z) \right)_{i,j} = m$$

for any $z \in \mathcal{M}$. Then if we take $U(z) = \sum_{i=1}^m g_i^2(z)$ and $G(z) = \frac{1}{2}\nabla U(z) = \sum_{i=i}^m g_i(z)\nabla g_i(z)$ it is easy to see, by the same arguments as above, that a basis of $N_z\mathcal{M}$, at $z \in \mathcal{M}$, is $\{\nabla g_1(z), \ldots, \nabla g_m(z)\}$, and that $G(z)$ satisfies the assumptions of our main theorem. Thus, in order to apply our result, we only need to find an example of a second order equation in $\mathbb{R}^n$:

$$\ddot{z} + F(z) = 0$$

whose D'Alembert equation on $\mathcal{M}$ has a non degenerate (with respect to the dynamics on $\mathcal{M}$) homoclinic orbit.

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