Advanced Nonlinear Studies **6** (2006), 69 – 108

The p-Laplace Heat Equation With a Source Term: Self-similar Solutions Revisited

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Received 2 November 2005
Communicated by Laurent Veron

Abstract

We study the self-similar solutions of any sign of the equation

$$u_t - div(|\nabla u|^{p-2} \nabla u) = |u|^{q-1} u,$$

in $\mathbb{R}^N$, where $p, q > 1$. We extend the results of Haraux-Weissler obtained for $p = 2$ to the case $q > p-1 > 0$. In particular we study the existence of slow or fast decaying solutions. For given $t > 0$, the fast solutions $u(t, .)$ have a compact support in $\mathbb{R}^N$ when $p > 2$, and $|x|^{p/(2-p)} u(t, x)$ is bounded at infinity when $p < 2$. We describe the behaviour for large $|x|$ of all the solutions. According to the position of q with respect to the first critical exponent $p - 1 + p/N$ and the critical Sobolev exponent q^*, we study the existence of positive solutions, or the number of the zeros of $u(t, .)$. We prove that any solution $u(t, .)$ is oscillatory when $p < 2$ and q is closed to 1.

2000 Mathematics Subject Classification. 35K55, 35B05, 34C35, 34C10.
Key words. Degenerate heat equation, self-similar, ordinary differential equations, non autonomous systems.

1 Introduction and main results

In this paper we study the existence of self-similar solutions of degenerate parabolic equations with a source term, involving the p-Laplace operator in $\mathbb{R}^N \times (0, \infty)$, $N \geq 1$,

$$u_t - div(|\nabla u|^{p-2} \nabla u) = |u|^{q-1} u, \tag{1.1}$$

where $p > 1, q > 1$. The semilinear problem, relative to the case $p = 2$,

$$u_t - \Delta u = |u|^{q-1} u, \tag{1.2}$$

has been treated by [18], and [26], [27], [20]. In particular, for any $a > 0$, there exists a self-similar solution of the form

$$u = t^{-1/(q-1)} \omega(t^{-1/2} |x|)$$

of (1.2), unique, such that $\omega \in C^2([0,\infty))$, $\omega(0) = a$ and $\omega'(0) = 0$. Any solution of this form satisfies $\lim_{|\xi|\to\infty} |\xi|^{2/(q-1)} \omega(\xi) = L \in \mathbb{R}$. It is called slowly decaying if $L \neq 0$ and fast decaying if $L = 0$. Let us recall the main results:

• *If $(N+2)/N < q$, there exist positive solutions.*

• *If $(N+2)/N < q < (N+2)/(N-2)$, there exist positive solutions of each type; in particular there exists a fast decaying one with an exponential decay:*

$$\lim_{|z|\to\infty} e^{|z|^2/4} |z|^{N-2/(q-1)} \omega(z) = A \in \mathbb{R},$$

thus for the solution u of (1.2) , $u(.,t) \in L^s(R^N)$ for any $s \geq 1$, $\lim_{t\to 0} \|u(.,t)\|_s = 0$ whenever $s < N(q-1)/2$, and $\lim_{t\to 0} \sup_{|x|\geq\varepsilon} |u(x,t)| = 0$ for any $\varepsilon > 0$. Moreover, for any integer $m \geq 1$, there exists a fast decaying solution ω with precisely m zeros.

• *If $(N+2)/(N-2) \leq q$, all the solutions $\omega \not\equiv 0$ have a constant sign and a slow decay.*

• *If $q \leq (N+2)/N$, then all the solutions $\omega \not\equiv 0$ have a finite positive number of zeros, and there exists an infinity of solutions of each type.*

The uniqueness of the positive fast decaying solution was proved later in [28] and [11], and more results about the solutions can be found in [16], [15] and [17].

Next we assume $p \neq 2$. If u is a solution of (1.1), then for any $\alpha_0, \beta_0 \in \mathbb{R}$, $u_\lambda(x,t) = \lambda^{\alpha_0} u(\lambda x, \lambda^{\beta_0} t)$ is a solution if and only if

$$\alpha_0 = p/(q+1-p), \qquad \beta_0 = (q-1)\alpha_0.$$

This leads to search self-similar solutions of the form

$$u(x,t) = (\beta_0 t)^{-1/(q-1)} w(r), \qquad r = (\beta_0 t)^{-1/\beta_0} |x|, \tag{1.3}$$

and the equation reduces to

$$\left(|w'|^{p-2} w'\right)' + \frac{N-1}{r} |w'|^{p-2} w' + r w' + \alpha_0 w + |w|^{q-1} w = 0 \qquad \text{in } (0,\infty). \tag{1.4}$$

In the sequel, some critical exponents are involved:

$$p_1 = \frac{2N}{N+1}, \qquad p_2 = \frac{2N}{N+2},$$

$$q_1 = p - 1 + \frac{p}{N}, \qquad q^* = \frac{N(p-1)+p}{N-p};$$

with the convention $q^* = \infty$ if $N \leq p$. Observe that $p - 1 < q_1 < q^*$; moreover $p_1 < p \Leftrightarrow 1 < q_1$, and $p_2 < p \Leftrightarrow 1 < q^*$. We also set

$$\delta = \frac{p}{2-p}, \quad \text{and} \quad \eta = \frac{N-p}{p-1}. \tag{1.5}$$

Thus $\delta > 0 \Longleftrightarrow p < 2$. Notice that

$$p_1 < p < 2 \Longleftrightarrow N < \delta \Longleftrightarrow \eta < N, \tag{1.6}$$

$$p_2 < p < 2 \Longleftrightarrow N < 2\delta. \tag{1.7}$$

Problem (1.1) was studied before in [22]. In the range $q_1 < q < q^\star$ and $p_1 < p$, the existence of a nonnegative solution u was claimed, such that w has a compact support when $p > 2$, or $w > 0$ when $p < 2$, with $w(z) = o(\,|z|^{(-p+\varepsilon)/(2-p)})$ at infinity, for any small $\varepsilon > 0$. However some parts of the proofs are not clear. The equation was studied independently for $p > 2$ in [3], but the existence of a nonnegative solution with compact support was not established, and some proofs are incomplete. Here we clarify and improve the former assertions, treat the case $p \leq p_1$, and give new informations on the existence of changing sign solutions. In particular, a new phenomenon appears, namely the possible existence of an infinity of zeros of w. Also all the solutions have a constant sign when $p \leq p_2$.

Theorem 1.1 *Let $q > \max(1, p-1)$.*
(i) For any $a > 0$, there exists a self-similar solution of the form

$$u(t,x) = (\beta_0 t)^{-1/(q-1)} w((\beta_0 t)^{-1/\beta_0} |x|) \tag{1.8}$$

of (1.4), unique, such that $w \in C^2((0,\infty)) \cap C^1([0,\infty))$, $w(0) = a$ and $w'(0) = 0$. Any solution of this form satisfies $\lim_{|z|\to\infty} |z|^{\alpha_0} w(z) = L \in \mathbb{R}$.

(ii) If $q_1 < q$, there exists positive solutions with $L > 0$, also called slow decaying.

(iii) If $q_1 < q < q^\star$, there exists a nonnegative solution $w \not\equiv 0$ such that $L = 0$, called fast decaying, and

$$u(t) \in L^s(\mathbb{R}^N) \quad \text{for any } s \geq 1, \qquad \lim_{t\to 0} \|u(t)\|_s = 0 \quad \text{whenever } s < N/\alpha_0,$$

$$\lim_{t\to 0} \sup_{|x|\geq\varepsilon} |u(x,t)| = 0 \quad \text{for any } \varepsilon > 0.$$

More precisely, when $p > 2$, w has a compact support in $(0,\infty)$; when $p < 2$, w is positive and

$$\begin{aligned} &\lim_{|z|\to\infty} |z|^{p/(2-p)} w(z) = \ell(N,p,q) > 0 && \text{if} \quad p_1 < p < 2, \\ &\lim_{|z|\to\infty} |z|^{(N-p)/(p-1)} w(z) = c > 0 && \text{if} \quad 1 < p < p_1, \\ &\lim_{r\to\infty} r^N (\ln r)^{(N+1)/2} w = \varrho(N,p,q) > 0 && \text{if} \quad p = p_1. \end{aligned} \tag{1.9}$$

(iv) If $q_1 < q < q^\star$, for any integer $m \geq 1$, there exists a fast decaying solution $w \not\equiv 0$ with at least m isolated zeros and a compact support when $p > 2$; there exists a fast decaying solution w precisely m zeros, and $|w|$ has the behaviour (1.1) when $p < 2$.

(v) If $p \leq p_2$, or if $p > p_2$ and $q \geq q^\star$, all the solutions $w \not\equiv 0$ have a constant sign and are slowly decaying.

(vi) If $q \leq q_1$, (hence $p_1 < p$), all the solutions $w \not\equiv 0$, assume both positive and negative values. There exists an infinity of fast decaying solutions such that w has a compact support when $p > 2$, and $|z|^{p/(2-p)} w(z)$ is bounded near ∞ when $p < 2$. Moreover, if $p < 2$, q is close to q_1, and p close to 2, then all the solutions $w \not\equiv 0$ have a finite number of zeros. If $p < 2$ and q is close to 1, all of them are oscillatory.

In the sequel we study more generally the equation

$$\left(|w'|^{p-2} w'\right)' + \frac{N-1}{r} |w'|^{p-2} w' + rw' + \alpha w + |w|^{q-1} w = 0 \text{ in } (0,\infty), \quad (1.10)$$

where $\alpha > 0$ is a parameter, and we only assume $q > 1$. The problem without source

$$u_t - div(|\nabla u|^{p-2} \nabla u) = 0 \quad (1.11)$$

was treated in [23] when $p < 2$ for positive solutions. In [5] we make a complete description of the solutions of any sign of (1.11) for $p < 2$, and study the equation

$$\left(|w'|^{p-2} w'\right)' + \frac{N-1}{r} |w'|^{p-2} w' + rw' + \alpha w = 0 \qquad \text{in } (0,\infty), \quad (1.12)$$

for arbitrary $\alpha \in \mathbb{R}$. A main point is that equation (1.10) appears as a perturbation of (1.12) when w is small enough. When $\alpha > 0$ and $(\delta - N)(\delta - \alpha) > 0$, observe that (1.12) has a particular solution of the form $w(r) = \ell r^{-\delta}$, where

$$\ell = \left(\delta^{p-1} \frac{\delta - N}{\delta - \alpha}\right)^{1/(2-p)}. \quad (1.13)$$

A critical value of α appears in studying (1.12) when $p_2 < p$:

$$\alpha^* = \delta + \frac{\delta(N-\delta)}{(p-1)(2\delta - N)}, \quad (1.14)$$

In the case $p > 2$, eqution (1.12) is treated in [13] and [6].

Our paper is organized as follows:

In Section 2, we give general properties about equation (1.10). Among the solutions defined on $(0,\infty)$, we show the existence and uniqueness of global solutions $w = w(.,a) \in C^2((0,\infty)) \cap C^1([0,\infty))$ of problem (1.10) such that for some $a \in \mathbb{R}$

$$w(0) = a, \qquad w'(0) = 0. \quad (1.15)$$

By symmetry, we restrict to the case $a \geq 0$. We give the first informations on the number of zeros of the solutions, and upper estimates near ∞ of any solution of any sign.

In Section 3, we study the case $(2-p)\alpha < p$. We first show that any solution w satisfies $\lim_{r\to\infty} r^{\alpha} w = L \in \mathbb{R}$. Moreover, we prove that the function $a \longmapsto L(a) = \lim_{r\to\infty} r^{\alpha} w(r,a)$ is continuous on $\mathbb{R}$. When $L = 0$, then any solution w has a compact support if $p > 2$, and $r^{\delta} w$ is bounded if $p < 2$ and we give a complete description of the behaviour of w near infinity. Then we study the existence of fast decaying solutions of equation 1.10, positive or changing sign, according to the value of α, see theorems 3.9 and 3.6. We give sufficient conditions on p, q, α, in order that all the functions $w(.,a)$ are positive and slowly decaying, see Theorem 3.11; some of them are new, even in the case $p = 2$. Finally we prove that all the solutions w are oscillatory when $p_1 < p < 2$ and α is close to δ, see Theorem 3.15; this type of behaviour never occurs in the case $p = 2$.

In Section 4 we study the case $p \leq (2-p)\alpha$, for which equation (1.10) has no more link with problem (1.1), but is interesting in itself. Here $r^{\delta} w$ is bounded at ∞, except in the case $p = (2-p)\alpha < p_1$ where a logarithm appears. Moreover, if $p_1 < p$, or $p_1 = p < (2-p)\alpha$, then all the solutions are oscillatory. As in section 3 we study the existence of positive solutions, see Theorems 4.9 and 4.11. In Theorem 4.6 we prove a difficult result of convergence in the range $\alpha < \eta$,w here the solutions are nonoscillatory.

Section 5 is devoted to the proof of Theorem 1.1, by taking $\alpha = \alpha_0$ and applying the results of Section 3, since $(2-p)\alpha_0 < p$.

2 General properties

2.1 Equivalent formulations, and energy functions

Equation (1.10) can be written under equivalent forms

$$\left(r^{N-1} |w'|^{p-2} w'\right)' + r^{N-1}(rw' + \alpha w + |w|^{q-1} w) = 0 \qquad \text{in } (0,\infty), \tag{2.1}$$

$$\left(r^{N}(w + r^{-1} |w'|^{p-2} w')\right)' + r^{N-1}\left((\alpha - N)w + |w|^{q-1} w\right) = 0 \qquad \text{in } (0,\infty). \tag{2.2}$$

Defining

$$J_N(r) = r^N \left(w + r^{-1} |w'|^{p-2} w'\right), \tag{2.3}$$

then (2.2) is equivalent to

$$J_N'(r) = r^{N-1}(N - \alpha - |w|^{q-1})w. \tag{2.4}$$

We also use the function

$$J_\alpha(r) = r^\alpha \left(w + r^{-1} |w'|^{p-2} w'\right) = r^{\alpha - N} J_N(r), \tag{2.5}$$

which satisfies

$$J_\alpha'(r) = r^{\alpha-1}\left((\alpha - N)r^{-1} |w'|^{p-2} w' - |w|^{q-1} w\right). \tag{2.6}$$

The simplest energy function,

$$E(r) = \frac{1}{p'} |w'|^p + \frac{\alpha}{2} w^2 + \frac{|w|^{q+1}}{q+1}, \tag{2.7}$$

obtained by multiplying (1.10) by w', is nonincreasing, since

$$E'(r) = -(N-1)r^{-1} |w'|^p - rw'^2. \tag{2.8}$$

More generally, we introduce a Pohozaev-Pucci-Serrin type function with parameters $\lambda > 0, \sigma, e \in \mathbb{R}$:

$$V_{\lambda,\sigma,e}(r) = r^\lambda \left(\frac{|w'|^p}{p'} + \frac{|w|^{q+1}}{q+1} + e\frac{w^2}{2} + \sigma r^{-1} w |w'|^{p-2} w' \right). \tag{2.9}$$

Such functions have been used intensively in [21]. After computation we find

$$\begin{aligned} r^{1-\lambda} V'_{\lambda,\sigma,e}(r) = {} & -(N-1-\sigma-\frac{\lambda}{p'}) |w'|^p - \left(\sigma - \frac{\lambda}{q+1}\right) |w|^{q+1} \\ & + \sigma(\lambda - N) r^{-1} w |w'|^{p-2} w' \\ & - \left(rw' + \frac{\sigma - e + \alpha}{2} w \right)^2 - (\sigma\alpha - \frac{e\lambda}{2} - \frac{(\sigma+\alpha-e)^2}{4}) w^2. \end{aligned} \tag{2.10}$$

Notice that $E = V_{0,0,\alpha}$.

In all the sequel we use a logarithmic substitution; for given $d \in \mathbb{R}$,

$$w(r) = r^{-d} y_d(\tau), \qquad \tau = \ln r. \tag{2.11}$$

We get the equation, at each point τ such that $w'(r) \neq 0$,

$$\begin{aligned} & y_d'' + (\eta - 2d) y_d' - d(\eta - d) y_d \\ & \quad + \frac{1}{p-1} e^{((p-2)d+p)\tau} |d y_d - y_d'|^{2-p} \left(y_d' - (d-\alpha) y_d + e^{-d(q-1)\tau} |y_d|^{q-1} y_d \right) = 0. \end{aligned} \tag{2.12}$$

Setting

$$Y_d(\tau) = -r^{(d+1)(p-1)} |w'|^{p-2} w', \tag{2.13}$$

we can write (2.12) as a system:

$$\left\{ \begin{array}{c} y_d' = d y_d - |Y_d|^{(2-p)/(p-1)} Y_d, \\ Y_d' = (p-1)(d-\eta) Y_d \\ + e^{(p+(p-2)d)\tau} (\alpha y_d + e^{-\delta(q-1)\tau} |y_d|^{q-1} y_d - |Y_d|^{(2-p)/(p-1)} Y_d). \end{array} \right. \tag{2.14}$$

In particular, the case $d = \delta$ plays a great role: setting

$$w(r) = r^{-\delta} y(\tau), \quad Y(\tau) = -r^{(\delta+1)(p-1)} |w'|^{p-2} w', \qquad \tau = \ln r, \tag{2.15}$$

equation (2.12) takes the form

$$\begin{aligned}(p-1)y'' + (N-\delta p)y' + (\delta - N)\delta y \\ + |\delta y - y'|^{2-p}\left(y' - (\delta-\alpha)y + e^{-\delta(q-1)\tau}|y|^{q-1}y\right) = 0,\end{aligned} \tag{2.16}$$

and system (2.14) becomes

$$\begin{cases} y' = \delta y - |Y|^{(2-p)/(p-1)}Y \\ Y' = (\delta - N)Y - |Y|^{(2-p)/(p-1)}Y + \alpha y + e^{-\delta(q-1)\tau}|y|^{q-1}y. \end{cases} \tag{2.17}$$

As $\tau \to \infty$, this system appears as a perturbation of an autonomous system

$$\begin{cases} y' = \delta y - |Y|^{(2-p)/(p-1)}Y \\ Y' = (\delta - N)Y - |Y|^{(2-p)/(p-1)}Y + \alpha y \end{cases} \tag{2.18}$$

corresponding to the problem (1.12). The existence of such a system is one of the key points of the new results in [5]. If $\delta(\delta-N)(\delta-\alpha) \le 0$, it has only one stationnary point $(0,0)$. If $\delta(\delta-N)(\delta-\alpha) > 0$, which implies $p < 2$, it has three stationary points:

$$(0,0), \quad M_\ell = (\ell, (\delta\ell)^{p-1}), \text{ and } \quad M'_\ell = -M_\ell, \tag{2.19}$$

where ℓ is defined at (1.13). The critical value α^* of α, defined at (1.14) corresponds to the case where the eigenvalues of the linearized problem at M_ℓ are imaginary. Observe the relation

$$J_N(r) = e^{(N-\delta)\tau}(y(\tau) - Y(\tau)). \tag{2.20}$$

As in [4] and [5], we construct a new energy function, adapted to system (2.17), by using the Anderson and Leighton formula for autonomous systems, see [1]. Let

$$\mathcal{W}(y,Y) = \frac{(2\delta - N)\delta^{p-1}}{p}|y|^p + \frac{|Y|^{p'}}{p'} - \delta yY + \frac{\alpha-\delta}{2}y^2, \tag{2.21}$$

$$W(\tau) = \mathcal{W}(y(\tau), Y(\tau)) + \frac{1}{q+1}e^{-\delta(q-1)\tau}|y(\tau)|^{q+1}. \tag{2.22}$$

Then

$$W'(\tau) = \mathcal{U}(y(\tau), Y(\tau)) - \frac{\delta(q-1)}{q+1}e^{-\delta(q-1)\tau}|y(\tau)|^{q+1} \tag{2.23}$$

with

$$\mathcal{U}(y,Y) = \left(\delta y - |Y|^{(2-p)/(p-1)}Y\right)\left((|\delta y|)^{p-2}\delta y - Y\right)(2\delta - N - \mathcal{H}(y,Y)), \tag{2.24}$$

$$\mathcal{H}(y,Y) = \begin{cases} \left(\delta y - |Y|^{(2-p)/(p-1)}Y\right) / \left(|\delta y|^{p-2}\delta y - Y\right) & \text{if} \quad |\delta y|)^{p-2}\delta y \neq Y, \\ |\delta y|^{2-p}/(p-1) & \text{if} \quad |\delta y|)^{p-2}\delta y = Y. \end{cases} \tag{2.25}$$

If $2\delta \leq N$, then $\mathcal{U}(y,Y) \leq 0$ on $\mathbb{R}^2$; thus W is nonincreasing. If $2\delta \geq N$, the set

$$\mathcal{L} = \left\{(y,Y) \in \mathbb{R}^2 : \mathcal{H}(y,Y) = 2\delta - N\right\} \tag{2.26}$$

is a closed curve surrounding $(0,0)$, symmetric with respect to $(0,0)$, and bounded, since for any $(y,Y) \in \mathbb{R}^2$,

$$\mathcal{H}(y,Y) \geq \frac{1}{2}((\delta y)^{2-p} + |Y|^{(2-p)/(p-1)}). \tag{2.27}$$

Introducing the domain $\mathcal{S}$ of $\mathbb{R}^2$ with boundary $\mathcal{L}$ and containing $(0,0)$,

$$\mathcal{S} = \left\{(y,Y) \in \mathbb{R}^2 : \mathcal{H}(y,Y) < 2\delta - N\right\}, \tag{2.28}$$

then $W'(\tau) \leq 0$ for any τ such that $(y(\tau), Y(\tau)) \notin \mathcal{S}$, from (2.23).

2.2 Existence of global solutions

The first question concerning problem (1.10), (1.15) is the local existence and uniqueness near 0. It is not straightforward in the case $p > 2$, and the regularity of the solution differs according to the value of p. It is shown in [3] when $p > 2$ and $\alpha = \alpha_0$, by following the arguments of [14]. We recall and extend the proof to the general case.

Theorem 2.1 *For any $a \neq 0$, problem (1.10), (1.15) admits a unique solution $w = w(.,a) \in C^1([0,\infty))$ such that $|w'|^{p-2}w' \in C^1([0,\infty))$, and*

$$\lim_{r\to 0} |w'|^{p-2} w'/rw = -(\alpha/N + a^{q+1}); \tag{2.29}$$

thus $w \in C^2([0,\infty))$ if $p < 2$. And $|w(r)| \leq a$ on $[0,\infty)$.

Proof. **Step** 1 : **Local existence and uniqueness.** We can suppose $a > 0$. Let $\rho > 0$. From (2.2), any $w \in C^1([0,\rho])$, such that $|w'|^{p-2}w' \in C^1([0,\rho])$, solution of the problem satisfies $w = T(w)$, where

$$\begin{aligned} T(w)(r) &= a - \int_0^r |H(w)|^{(2-p)/(p-1)} H(w) ds, \\ H(w(r)) &= rw - r^{1-N} J_N(r) = rw - r^{1-N}\int_0^r s^{N-1} j(w(s)) ds, \end{aligned} \tag{2.30}$$

and $j(r) = (N-\alpha)r - |r|^{q-1}r$. Reciprocally, the mapping T is well defined from $C^0([0,\rho])$ into itself. If $w \in C^0([0,\rho])$ and $w = T(w)$, then $w \in C^1((0,\rho])$ and $|w'|^{p-2}w' = H(w)$; hence $|w'|^{p-2}w' \in C^1((0,\rho])$ and w satisfies (1.10) in $(0,\rho]$. Moreover, $\lim_{r\to 0} j(w(r)) = a^q - (N-\alpha)a$, hence $|w'|^{p-2}w'(r) = -r((\alpha/N + a^{q-1}) + o(1))$; in particular, $\lim_{r\to 0} w'(r) = 0$, and $|w'|^{p-2}w' \in C^1([0,\rho])$, and w satisfies (1.10) and (1.15), and (2.29) holds. We consider the ball

$$\mathcal{B}_{R,M} = \left\{w \in C^0([0,\rho]) : \|w - a\|_{C^0([0,R])} \leq M\right\},$$

where M is a parameter such that $0 < M < a/2$. Notice that j is locally Lipschitz continuous, since $q > 1$. In case $p < 2$, then the function $r \mapsto |r|^{(2-p)/(p-1)} r$ has the same property; hence T is a strict contraction from $\mathcal{B}_{\rho,M}$ into itself for ρ and M small enough. Now suppose $p > 2$. Let $K = K(a, M)$ be the best Lipschitz constant of j on $[a - M, a + M]$. For any $w \in \mathcal{B}_{R,M}$, and any $r \in [0, \rho]$, from (2.30)

$$\left(a - M - \frac{j(a) + MK_M}{N}\right) r \leq H(w(r)) \leq \left(a + M + \frac{-j(a) + MK}{N}\right) r \tag{2.31}$$

hence, setting $\mu(a) = a - j(a)/N = (a^q + \alpha a)/N > 0$,

$$\mu(a)r/2 < H(w(r)) < 2\mu(a)r$$

as long as $M \leq M(a)$ is small enough. Then from (2.30),

$$\|T(w) - a\|_{C^0([0,R])} \leq (2\mu(a))^{1/(p-1)} R^{p/(p-1)}$$

and hence $T(w) \in \mathcal{B}_{\rho,M}$ for $\rho = \rho(a)$ small enough. Now for any $w_1, w_2 \in \mathcal{B}_{\rho,M}$, and any $r \in [0, \rho]$,

$$\begin{aligned}&|T(w_1)(r) - T(w_2)(r)| \\ &\leq \int_0^r \left| |H(w)|^{(2-p)/(p-1)} H(w_1) - |H(w_2)|^{(2-p)/(p-1)} H(w) \right| (s) ds\end{aligned}$$

and for any $s \in [0, r]$, from [14, p.185], and

$$\begin{aligned}&\left| |H(w)|^{(2-p)/(p-1)} H(w_1) - |H(w_2)|^{(2-p)/(p-1)} H(w) \right| (s) \\ &\leq H(w_2)^{(2-p)/(p-1)} |H(w_1) - H(w_2)| (s) \\ &\leq (2\mu(a))^{(2-p)/(p-1)} s^{1/(p-1)} \left(|w_1 - w_2| + Ks^{-N} \int_0^s \sigma^{N-1} |w_1 - w_2| d\sigma \right) \\ &\leq C(a)s^{1/(p-1)} \|w_1 - w_2\|_{C^0([0,R])}\end{aligned} \tag{2.32}$$

with $C(a) = (2\mu(a))^{(2-p)/(p-1)} (1 + K/N)$

$$\|T(w_1) - T(w_2)\|_{C^0([0,R])} \leq C(a)\rho^{p'} \|w_1 - w_2\|_{C^0([0,R])} \leq \frac{1}{2} \|w_1 - w_2\|_{C^0([0,R])}$$

if $\rho(a)$ is small enough. Then T is a strict contraction from $\mathcal{B}_{\rho,M}$ into itself. Moreover if $\rho(a)$ and $M(a)$ are small enough, then for any $b \in [a/2, 3a/2]$,

$$\|w(., b) - w(., a)\|_{C^0([0,\rho])} \leq |b - a| + \frac{1}{2} \|w(., a) - w(., b)\|_{C^0([0,R]);}$$

that means $w(a, .)$ is Lipschitz dependent on a in $[0, \rho(a)]$. The same happens for $w'(., a)$, as in (2.32), since

$$\begin{aligned}&|w'(., b) - w'(., a)| \\ &= \left| |H(w(., b))|^{(2-p)/(p-1)} H(w(., b)) - |H(w(., a))|^{(2-p)/(p-1)} H(w(., a)) \right|.\end{aligned}$$

Step 2 : **Global existence and uniqueness.** The function w on $[0, \rho(a)]$ can be extended to $[0, \infty)$. Indeed, in the domain of definition,

$$E(r) = \frac{1}{p'} |w'|^p + \frac{\alpha}{2} w^2 + \frac{1}{q+1} |w|^{q+1} \leq E(0) = \frac{\alpha}{2} a^2 + a^{q+1}, \tag{2.33}$$

and hence w and w' stay bounded, and $|w(r)| \leq a$ on $[0, \infty)$. The extended function is unique. Indeed existence and uniqueness hold near any point $r_1 > 0$ such that $w'(r_1) \neq 0$ or $p \leq 2$ from the Cauchy-Lipschitz theorem; if $w'(r_1) = 0, w(r_1) \neq 0$ and $p > 2$, it follows from fixed point theorem as above; finally if $w(r_1) = w'(r_1) = 0$, then $w \equiv 0$ on $[r_1, \infty)$ since E is nonincreasing. ∎

Remark 2.2 For any $r_1 \geq 0$, we have a local continuous dependence of w and w' on function of $c_1 = w(r_1)$ and $c_2 = w'(r_1)$. Indeed the only delicate case is $c_1 = c_2 = 0$. Since E is nonincreasing, then for any $\varepsilon > 0$,, if $|w(r_1)| + |w'(r_1)| \leq \varepsilon$, then $\sup_{[r_1,\infty)} |w(r)| + |w'(r)| \leq C(\varepsilon)$, where C is continuous; thus the dependence holds on all of $[r_1, \infty)$. In particular, for any $a \in \mathbb{R}$, $w(., a)$ and $w'(., a)$ depend continuously on a on any segment $[0, R]$. If for some a_0, $w(., a_0)$ has a compact support, the dependance is continuous on $\mathbb{R}$. As a consequence, $w(., .)$ and $w'(., .) \in C^0 ([0, \infty) \times \mathbb{R})$.

Remark 2.3 Any local solution w of problem (1.10) near a point $r_1 > 0$ is defined on a maximal interval (R_w, ∞) with $0 \leq R_w < r_1$.

2.3 First oscillatory properties

Let us begin by simple remarks on the behaviour of the solutions.

Proposition 2.4 *Let w be any solution of problem (1.10). Then*

$$\lim_{r\to\infty} w(r) = 0, \qquad \lim_{r\to\infty} w'(r) = 0. \tag{2.34}$$

If $w > 0$ for large r, then $w' < 0$ for large r.

Proof. Let w be any solution on $[r_0, \infty)$, $r_0 > 0$. Since the function E is nonincreasing, w and w' are bounded, and E has a finite limit $\xi \geq 0$. Consider the function $V = V_{\lambda,d,e}$ defined at (2.9) with $\lambda = 0$, $\sigma = (N-1)/2$, $e = \alpha + \sigma$. It is bounded near ∞ and satisfies

$$\begin{aligned} -rV'(r) &= \frac{N-1}{2}(|w'|^p + |w|^{q+1} + \alpha w^2 + \frac{N}{2} r^{-1} w |w'|^{p-2} w' + r^2 w'^2) \\ &\geq \frac{N-1}{2} E(r) + o(1) \geq \frac{N-1}{2} \xi + o(1). \end{aligned}$$

If $\xi > 0$, then V is not integrable, which is contradictory. Thus $\xi = 0$, and (2.34) holds. Moreover, at each extremal point r such that $w(r) > 0$, from

$$(|w'|^{p-2} w')'(r) = -(\alpha + w(r)^{q+1}) w(r), \tag{2.35}$$

r is unique and it is a maximum. If $w(r) > 0$ for large r, then from (2.34) necessarily $w' < 0$ for large r. ■

Now we give some first results concerning the possible zeros of the solutions. If $p < 2$ then any solution $w \not\equiv 0$ of (1.10) has only isolated zeros, from the Cauchy-Lipschitz Theorem. On the contrary, if $p > 2$, there can exist $r_1 > 0$ such that $w(r_1) = w'(r_1) = 0$. Then, by uniqueness, $w \equiv 0$ on $[r_1, \infty)$.

Proposition 2.5 *(i) Assume $\alpha < N$. Let $\underline{a} = (N - \alpha)^{1/(q-1)}$. Then for any $a \in (0, \underline{a}]$, $w(r, a) > 0$ on $[0, \infty)$.*
(ii) Assume $p_1 < p$ and $N \leq \alpha$. Then for any $a \neq 0$, $w(r, a)$ has at least one isolated zero.
(iii) Assume $p < 2$. Then for any $0 < m < M < \infty$, any solution w of (1.10) has a finite number of zeros in $[m, M]$, or $w \equiv 0$ in $[m, M]$.
(iv) Assume $p > 2$ or $\alpha < \max(N, \eta)$. Then for any $m > 0$, any solution w of problem (1.10) w has a finite number of isolated zeros in $[m, \infty)$, or $w \equiv 0$ in $[m, \infty)$.

Proof. (i) Let $a \in (0, \underline{a}]$. Assume that there exists a first $r_1 > 0$ such that $w(r_1, a) = 0$, hence $w'(r_1, a) \leq 0$. Let us consider J_N defined by (2.3). Then $J'_N(r) \geq 0$ on $[0, r_1)$, as $0 \leq w(r) \leq a$; $J_N(0) = 0$, and $J_N(r_1) = r_1^{N-1} |w'(r_1)|^{p-2} w'(r_1) \leq 0$. Thus $J'_N \equiv 0$ on $[0, r_1]$, and $w \equiv \underline{a}$, which contradicts (1.10).

(ii) Suppose that for some $a > 0$, $w(r) = w(r, a) > 0$ on $[0, \infty)$. Since $N \leq \alpha$, there holds $J'_N(r) < 0$ on $[0, \infty)$, and $J_N(0) = 0$, and hence $J_N(r) \leq 0$. Then $r \longmapsto r^{p'} - \delta w^{-\delta}$ is nonincreasing.

• If $p > 2$, it is impossible, thus w has a first zero r_1, and $J'_N(r) < 0$ on $[0, r_1)$, and hence $J_N(r_1) < 0$. Then $w'(r_1) < 0$ and r_1 is isolated.

• If $p < 2$, there exists $c > 0$ such that for large r, $J_N(r) \leq -c$, hence $w(r) + cr^{-N} \leq |w'(r)|^{p-1}/r$. Then there exists another $c > 0$ such that $w' + cr^{(1-N)/(p-1)} \leq 0$. If $N = 1$ it contradicts Proposition 2.4. If $2 \leq N$, then $p < N$, and $w - cr^{-\eta}/\eta$ decreases to 0, thus $\delta \leq \eta$, which contradicts $N < \delta$, which means $p_1 < p$, from (1.6).

(iii) Suppose that w has an infinity of isolated zeros in $[m, M]$. Then there exists a sequence of zeros converging to some $\overline{r} \in [m, M]$. We can extract an increasing (or a decreasing) subsequence of zeros (r_n) such that $w > 0$ on (r_{2n}, r_{2n+1}) and $w < 0$ on (r_{2n-1}, r_{2n}). There exists $s_n \in (r_n, r_{n+1})$ such that $w'(s_n) = 0$; since $w \in C^1[0, \infty)$, it implies $w(\overline{r}) = w'(\overline{r}) = 0$. It is impossible because $p < 2$.

(iv) Suppose that $w \not\equiv 0$ in $[m, \infty)$. Let Z be the set of its isolated zeros in $[m, \infty)$. Notice that m is not an accumulation point of Z, since $(w(m), w'(m)) \neq (0, 0)$. Let $\rho_1 < \rho_2$, be two consecutive zeros, thus such that ρ_1 is isolated, and $|w| > 0$ on (ρ_1, ρ_2). We make the substitution (2.11), where $d > 0$ will be choosen afterwards. At each point τ such that $y'_d(\tau) = 0$, and $y_d(\tau) \neq 0$, we deduce

$$(p-1)y''_d = y_d \left((p-1)d(\eta - d) + e^{((p-2)d+p)\tau} |dy_d|^{2-p} \left(d - \alpha - e^{-d(q-1)\tau} |y_d|^{q-1} y_d \right) \right); \tag{2.36}$$

if $\tau \in (e^{\rho_1}, e^{\rho_2})$ is an maximal point of $|y_d|$, it follows that

$$e^{((p-2)d+p)\tau} \left|dy_d(\tau)\right|^{2-p} \left(d - \alpha - e^{-d(q-1)\tau} \left|y_d(\tau)\right|^{q-1}\right) \leq (p-1)d(d-\eta). \quad (2.37)$$

Setting $\rho = e^{\tau} \in (\rho_1, \rho_2)$, it means

$$\rho^p \left|w(\rho)\right|^{2-p} \left(d - \alpha - \left|w\right|^{q-1}(\rho)\right) \leq (p-1)d^{p-1}(d-\eta). \quad (2.38)$$

If $p > 2$, we fix $d > \alpha$. Since $\lim_{r\to\infty} w(r) = 0$, the coefficient of ρ^p in the left-hand side tends to ∞ as $\rho \to \infty$. Hence ρ is bounded, and also ρ_1, thus Z is bounded. If $\alpha < \eta$, we take $d \in (\alpha, \eta)$. Then the right hand side is negative, and the left hand side is nonnegative for large r, hence again Z is bounded. If $\alpha < N$, we use the function J_N :

$$\begin{aligned}
&J_N(\rho_2) - J_N(\rho_1) \\
&= \rho_2^{N-1} \left|w'\right|^{p-2} w'(\rho_2) - \rho_1^{N-1} \left|w'\right|^{p-2} w'(\rho_1) \\
&= \int_{\rho_1}^{\rho_2} s^{N-1} w(N - \alpha - \left|w\right|^{q-1} w) ds \qquad (2.39)
\end{aligned}$$

and the integral has the sign of w for large ρ, hence a contradiction. In any case Z is bounded. Suppose that Z is infinite; then $p > 2$ from step (iii), and there exists a sequence of zeros (r_n), converging to some $\overline{r} \in (m, \infty)$ such that $w(\overline{r}) = w'(\overline{r}) = 0$. Then there exists a sequence (τ_n) of maximal points of $|y_d|$ converging to $\overline{\tau} = \ln \overline{r}$. Taking $\rho = \rho_n = e^{\tau_n}$ in (2.38) leads to a contradiction, since the left-hand side tends to ∞. ∎

When w has a constant sign for large r, we can give some informations on the behaviour for large τ of the solutions (y, Y) of system (2.17), in particular the convergence to a stationary point of the autonomous system (2.18): We have also a majorization in one case when the solution is changing sign.

Lemma 2.6 *Let w be any solution of (1.10), and (y, Y) be defined by (2.15).*

(i) If $y > 0$ and y is not monotone for large τ, then Y is not monotone for large τ, and either $\max(\alpha, N) < \delta$ and $\lim_{\tau\to\infty} y(\tau) = \ell$, or $\delta < \min(\alpha, N)$ and $\liminf_{\tau\to\infty} y(\tau) \leq \ell \leq \limsup_{\tau\to\infty} y(\tau)$.

(ii) If $y > 0$ and y has a limit l at ∞, then either $l = 0$ and $\lim_{\tau\to\infty} Y(\tau) = 0$, or $(\delta - N)(\delta - \alpha) > 0$ and $l = \ell$ and $\lim_{\tau\to\infty}(y(\tau), Y(\tau)) = M_\ell$, or $\delta = \alpha = N$ and $\lim_{\tau\to\infty} Y(\tau) = (\delta l)^{p-1}$.

(iii) If $y > 0$ and y is nondecreasing for large τ and $\lim_{\tau\to\infty} y(\tau) = \infty$, then $\lim_{\tau\to\infty} Y(\tau) = \infty$.

(iv) If y is changing sign for large τ (which implies $p < 2$) and $\alpha < \delta$, then $N < \delta$ and $|y(\tau)| \leq \ell\,(1 + o(1))$ and $|Y(\tau)| \leq (\delta\ell)^{p-1}(1 + o(1))$ near ∞.

Proof. From Proposition 2.4, $Y(\tau) > 0$ for large τ in cases (i) to (iii).

(i) Suppose that y is not monotone near ∞. Then there exists an increasing sequence (τ_n) such that $\tau_n \to \infty$, $y'(\tau_n) = 0$, $y''(\tau_{2n}) \geq 0$, $y''(\tau_{2n+1}) \leq 0$, $y(\tau_{2n}) \leq y(\tau) \leq y(\tau_{2n+1})$

on (τ_{2n},τ_{2n+1}), $y(\tau_{2n}) \leq y(\tau) \leq y(\tau_{2n-1})$ on (τ_{2n-1},τ_{2n}), and $y(\tau_{2n}) < y(\tau_{2n+1})$. From (2.16),

$$(p-1)y''(\tau_n) = \delta^{2-p}y(\tau_n)\left(y(\tau_n)^{2-p}\left(\delta-\alpha-e^{-\delta(q-1)\tau_n}y(\tau_n)^{q-1}\right)\right)-(\delta-N)\delta^{p-1}\Big) \tag{2.40}$$

From Proposition 2.4, $e^{-\delta\tau}y(\tau)=o(1)$ near ∞ and

$$\begin{aligned}&y(\tau_{2n+1})^{2-p}\left(\alpha-\delta+e^{-\delta(q-1)\tau_{2n+1}}y(\tau_{2n+1})^{q-1})\right)\\&>(N-\delta)\delta^{p-1}\geq y(\tau_{2n})^{2-p}\left(\alpha-\delta+e^{-\delta(q-1)\tau_{2n}}y(\tau_{2n})^{q-1}\right)>y(\tau_{2n})^{2-p}(\alpha-\delta).\end{aligned}$$

Then either $\alpha<\delta, N<\delta$ and $\ell\leq y(\tau_{2n})\leq y(\tau_{2n+1})\leq\ell(1+o(1))$; hence $\lim_{\tau\to\infty}y(\tau)=\ell$. Or $\delta<\alpha$, $\delta<N$, and $y(\tau_{2n})<\ell$, and $\ell\leq y(\tau_{2n+1})(1+o(1))$. If Y is monotone near ∞, then from (2.17), $y''=\delta y'-Y^{(2-p)/(p-1)}Y'$, hence $e^{-\delta t}y'$ is monotone, which contradicts the existence of a sequence (τ_n) as above. Thus Y is not monotone.

(ii) Let $l=\lim_{\tau\to\infty}y\geq 0$. If Y is monotone, either $\lim_{\tau\to\infty}Y=\infty$, which is impossible, since then $y'\to-\infty$; or Y has a finite limit $\lambda\geq 0$. If Y is not monotone, at the extremal points τ of Y, we have

$$|Y(\tau)|^{(2-p)/(p-1)}Y(\tau)-(\delta-N)Y(\tau)=\alpha l+o(1),$$

from (2.17). Thus Y has a limit at these points, hence Y still has a limit λ. From (2.17), y' has a limit, necessarily 0, hence $\lambda=(\delta l)^{p-1}$.Then Y' has a limit, necessarily 0, and $(\delta-N)(\delta l)^{p-1}=(\delta-\alpha)l$; thus $l=0=\lambda$, or $(\delta-N)(\delta-\alpha)>0$ and $l=\ell$, $\lambda=(\delta\ell)^{p-1}$, or $\delta=\alpha=N$.

(iii) Suppose that y is nondecreasing and $\lim_{\tau\to\infty}y(\tau)=\infty$. Then either Y is not monotone, and at minimum points it tends to ∞ from (2.17), then $\lim_{\tau\to\infty}Y(\tau)=\infty$. 0r Y is monotone; if it has a finite limit, then $\lim_{\tau\to\infty}Y'(\tau)=\infty$ from (2.17), which is impossible. Then again $\lim_{\tau\to\infty}Y(\tau)=\infty$.

(iv) Assume that y does not keep a constant sign near ∞; then also w, thus also w', and in turn Y. At any maximal point θ of $|y|$, one finds

$$(p-1)y''(\theta)=\delta^{2-p}y(\theta)\left(|y(\theta)|^{2-p}\left(\delta-\alpha-e^{-\delta(q-1)\theta}|y(\theta)|^{q-1}\right)-(\delta-N)\delta^{p-1}\right),$$

hence

$$|y(\theta)|^{2-p}(\delta-\alpha+o(1))\leq(\delta-N)\delta^{p-1}.$$

Since $\delta-\alpha>0$, it follows that $\delta-N>0$ and $|y(\tau)|\leq\ell(1+o(1))$ near ∞. Similarly at any maximal point ϑ of $|Y|$, one finds

$$\begin{aligned}Y''(\vartheta)&=(\alpha+e^{-\delta(q-1)\vartheta}|y(\vartheta)|^{q-1})y'+\delta(q-1)e^{-\delta(q-1)\vartheta}|y(\vartheta)|^{q-1}y\\0&=(\delta-N)Y(\vartheta)-|Y(\vartheta)|^{(2-p)/(p-1)}Y(\vartheta)+(\alpha+e^{-\delta(q-1)\vartheta}|y(\vartheta)|^{q-1})y(\vartheta)\end{aligned}$$

which implies

$$|Y(\vartheta)|^{(2-p)/(p-1)}(\delta-\alpha+o(1))\leq(\delta-N)\delta;$$

thus $|Y(\tau)|\leq(\delta\ell)^{p-1}(1+o(1))$ near ∞. ■

2.4 Further results by blow up techniques

Next we give two results obtained by rescaling and blow up techniques. The first one consists in a scaling leading to the equation

$$r^{1-N}\left(r^{N-1}\left|v'\right|^{p-2}v'\right)'+\left|v\right|^{q-1}v=0. \tag{2.41}$$

without term in rw', extending the result of ([26, Proposition 3.4]) to the case $p\neq 2$. It gives a result in the subcritical case $q<q^*$, and does not depend on the value of α.

Proposition 2.7 *Assume that* $1<q<q^*$*(thus* $p>p_2$*). Then for any* $m\in\mathbb{N}$*, there exists* $\overline{a_m}$ *such that for any* $a>\overline{a_m}$*,* $w(.,a)$ *admits at least* $m+1$ *isolated zeros. And for fixed* m*, the* m^{th} *zero of* $w(.,a)$ *tends to* 0 *as* a *tends to* ∞*.*

Proof. (i) First we show that there exists $a_*>0$ such that for any $a>a_*$, $w(.,a)$ cannot stay positive on $[0,\infty)$. Suppose that there exists (a_n) tending to ∞, such that $w_n(r)=w(r,a_n)\geq 0$ on $[0,\infty)$, and let

$$v_n(r)=a_n^{-1}w_n(a_n^{-1/\alpha_0}r). \tag{2.42}$$

Then $v_n(0)=1$, $v_n'(0)=0$ and v_n satisfies the equation

$$\left(r^N(a_n^{1-q}v_n+r^{-1}\left|v_n'\right|^{p-2}v_n')\right)'+r^{N-1}\left((\alpha-N)a_n^{1-q}v_n+\left|v_n\right|^{q-1}v_n\right)=0. \tag{2.43}$$

From (2.33) applied to w_n

$$v_n(r)\leq 1,\qquad \left|v_n'(r)\right|^p\leq p'\left(\frac{\alpha}{2}a_n^{1-q}+\frac{1}{q+1}\right)\qquad \text{in }[0,\infty),$$

thus v_n and v_n' are uniformly bounded in $[0,\infty)$. If $p\leq 2$, then v_n'' is uniformly bounded on any compact $\mathcal{K}$ of $(0,\infty)$, from (1.10), and up to a diagonal sequence, v_n converges uniformly in $C^1_{loc}(0,\infty)$ to a function v. If $p>2$, then, from (2.43), the derivatives of $r^N(a_n^{1-q}v_n+\left|v_n'\right|^{p-2}v_n')$ are uniformly bounded on any $\mathcal{K}$, and $a_n^{1-q}v_n$ converges unifomly to 0 in $[0,\infty)$, and up to a diagonal sequence, $\left|v_n'\right|^{p-2}v_n'$ converges uniformly on any $\mathcal{K}$, hence also v_n', thus v_n converges uniformly in $C^1_{loc}(0,\infty)$ to a nonnegative function $v\in C^1(0,\infty)$. For any $r>0$,

$$\left|v_n'\right|^{p-2}v_n'(r)=-a_n^{1-q}rv_n(r)+r^{1-N}\int_0^r s^{N-1}\left(a_n^{1-q}(N-\alpha)v_n-\left|v_n\right|^{q-1}v_n\right)ds,$$

hence

$$\left|v'\right|^{p-2}v'(r)=-r^{1-N}\int_0^r s^{N-1}\left|v\right|^{q-1}vds\qquad \text{in }(0,\infty). \tag{2.44}$$

In particular, $v'(r)\to 0$ as $r\to 0$, and hence v can be extended to a function in $C^1([0,\infty))$ such that $v(0)=1$, and $v'(r)<0$. Using the form (1.10) for the equation in v_n, v_n'' converges uniformly on any $\mathcal{K}$, hence $v\in C^2(0,\infty)\cap C^1([0,\infty))$ and is solution of the equation (2.41) such that $v(0)=1$ and $v'0)=0$. But this equation has no nonnegative solution except 0 since $q < q^*$. Moreover the zeros of function v are all isolated, and form a sequence (r_n) tending to ∞, see [4], [8] and [24]. Then we reach a contradiction.

(ii) Now let $m \geq 0$. As in [26, Proposition 3.4], assume that there exists a sequence (a_n) tending to ∞, such that $w_n(r) = w(r, a_n)$ has at most m isolated zeros, hence also v_n. Up to a subsequence we can suppose that all the $v_n(r)$ have the same number of isolated zeros $\overline{m} : r_{0,n}, r_{1,n}, .., r_{\overline{m},n}$. Let $M > 0$ such that $r_0, r_1, .., r_{\overline{m}} \in (0, M)$. Then for n large enough, $r_{0,n}, r_{1,n}, .., r_{\overline{m},n} \in (0, M+1)$. Either $v_n(r)$ has no zero on $[M+1, \infty)$, or there is a unique zero $r_{\overline{m},n+1}$ such that $v_n(r)$ has a compact support $[0, r_{\overline{m},n+1}]$. Up to a subsequence, all the v_n are nonnegative or nonpositive on $[M+1, \infty)$; then the same holds for v, and we get a contradiction. Thus for a large enough, $w(., a)$ has at least $m+1$ zeros. Moreover, as in [26], the m first zeros stay in a compact set, and from (2.42) the m^{th} zero of $w(., a)$ tends to 0 as $a \to \infty$. ■

Now we make a scaling leading to the problem without source

$$r^{1-N}\left(r^{N-1}\left|v'\right|^{p-2}v'\right)' + rv' + \alpha v = 0. \tag{2.45}$$

It gives informations when the regular solutions of (2.45) are changing sign, in particular $p_2 < p < 2$, and $\delta < \alpha$. It does not depend on the value of q.

Proposition 2.8 *Assume that $p_2 < p < 2$, $\delta < \alpha$. Then there exists an $\alpha_c \in (\eta, \alpha^*)$ such that if $\alpha > \alpha_c$, then for any $m \in \mathbb{N}$, there exists $\overline{a_m}$ such that for any $0 < a < \overline{a_m}$, $w(., a)$ admits at least $m+1$ isolated zeros. And for fixed m, the m^{th} zero of $w(., a)$ tends to 0 as a tends to ∞.*

Proof. Suppose that there exists (a_n) tending to 0, such that $w_n(r) = w(r, a_n) \geq 0$ on $[0, \infty)$, and let

$$v_n(r) = a_n^{-1} w_n(a_n^{-1/\delta} r).$$

Then $v_n(0) = 1$, $v_n'(0) = 0$ and v_n satisfies equation

$$\left(r^N(v_n + r^{-1}\left|v_n'\right|^{p-2}v_n')\right)' + r^{N-1}\left((\alpha - N)v_n + a_n^{q-1}\left|v_n\right|^{q-1}v_n\right) = 0,$$

and estimates

$$v_n(r) \leq 1, \qquad \left|v_n'(r)\right|^p \leq p'\left(\frac{\alpha}{2} + \frac{a_n^{q-1}}{q+1}\right) \qquad \text{in } [0, \infty).$$

As above we construct a solution $v \in C^2(0, \infty) \cap C^1([0, \infty))$ of the equation (2.45). But from [5], there exists $\alpha_c \in (\eta, \alpha^*)$ such that the regular solutions of (2.45) are oscillating for $\alpha > \alpha_c$, hence we conclude as above. ■

Remark 2.9 This scaling does not give any result when the regular solutions of (2.45) have a constant sign: it is the case for example when $\alpha = N$: they are the Barenblatt solutions, they have a compact support when $p > 2$ and a behaviour in $r^{-\delta}$ near ∞ when $p < 2$. Nevertheless if $p > p_1$, all the solutions $w(., a)$ of (1.10) have at least one zero, from Proposition 2.5.

2.5 Upper estimates of the solutions

Here we get the behaviour at infinity for solutions of any sign. We extend the results of [18] obtained for $p = 2$, giving upper estimates with continous dependence, which also improve the results of [22]:

Proposition 2.10 *Let* $d \geq 0$.
(i) Assume that the solution w *of problem (1.10), (1.15) satisfies*

$$|w(r)| \leq C_d(1+r)^{-d}, \tag{2.46}$$

on $[0,\infty)$, *for some* $C_d > 0$. *Then there exists another* $C'_d > 0$, *depending continuously on* C_d *such that*

$$|w'(r)| \leq C'_d(1+r)^{-d-1}. \tag{2.47}$$

(ii) For any solution of (1.10) such that $w(r) = O(r^{-d})$ *near* ∞, *then* $w'(r) = O(r^{-d-1})$ *near* ∞.

Proof. (i) We can assume that $w \not\equiv 0$. Let $r \geq R \geq 0$; we set

$$f_R(r) = \exp\left(\frac{1}{p-1}\int_R^r s\,|w'|^{2-p}\,ds\right). \tag{2.48}$$

The function is well defined when $p < 2$ from (2.29), and $f_R \in C^1([R,\infty))$. When $p > 2$, from Proposition 2.5, (iv), the function w has a finite number of isolated zeros and either there exists a first $\bar{r} > 0$ such that $w(\bar{r}) = w'(\bar{r}) = 0$, or w has no zero for large r, and we set $\bar{r} = \infty$. In the last case case, from Proposition 2.4, the set of zeros of w' is bounded. If $w'(\tilde{r}) = 0$ for some $\tilde{r} \in (0,\bar{r})$, then, from (1.10), $(|w'|^{p-2}\,w')'$ has a nonzero limit λ at $\tilde{r}$, hence $\tilde{r}$ is an isolated zero of w and

$$|w'(s)|^{2-p} = |\lambda|^{(2-p)/(p-1)}\,(s-\tilde{r})^{-1+1/(p-1)}(1+o(1))$$

near $\tilde{r}$. Then $s\,|w'|^{2-p} \in L^1_{loc}\,(R,\infty)$; thus f_R is absolutely continuous on $[R,\bar{r})$ if $\bar{r} = \infty$. Let $k = k(N,p,d) > 0$ be a parameter, such that $K = k-(N-1)/(p-1) > 0$, and $k > 1+d$. By computation, for almost any $r \in (R,\bar{r})$,

$$\left(r^k f_R(w' - Kr^{-1}w)\right)' = -K(k-1)r^{k-2}f_R w - r^{k-1}f'_R w(\alpha + K + |w|^{q-1})$$

and hence for any $r \in [R,\bar{r})$,

$$\begin{aligned} r^k f_R w' &= R^{k-1}(Rw'(R) - Kw(R)) + Kr^{k-1}f_R w \\ &\quad - K(k-1)\int_R^r s^{k-2}f_R w ds - \int_R^r s^{k-1}f'_R w(\alpha + K + |w|^{q-1})ds. \end{aligned} \tag{2.49}$$

Assume (2.46), take $R = 0$, and divide by f_0. From our choice of k, and since $f' \geq 0$, we obtain

$$r^k\,|w'(r)| \leq \tilde{C}_d r^{k-1-d}$$

on $[0,\bar{r})$ and then on $[0,\infty)$, where $\tilde{C}_d = C_d(K+K(k-1)/(k-1-d)+\alpha+K)+C_d^{q-1}$, and $K = K(N,p,d)$; this holds in particular on $[1,\infty)$; on $[0,1]$, from (2.33),

$$|w'(r)| \le p'(\alpha C_d/2 + C_d^{q-1}),$$

and (2.47) holds.

(ii) Let $R \ge 1$ such that w is defined on $[R,\infty)$ and $w(r) \le C_d r^{-d}$ on $[R,\infty)$. Defining $\bar{r}$ as above and dividing (2.49) by f_R and observing that $f_R(r) \ge 1$, and $R^k \le R^{k-1-d} \le r^{k-1-d}$, we deduce

$$r^k |w'(r)| \le R^k |w'(R)| + C_d K R^{k-1-d} + \tilde{C}_d r^{k-1-d} \le (|w'(R)| + C_d K + \tilde{C}_d) r^{k-1-d}$$

on $[R,\bar{r})$ and then on $[R,\infty)$, and we conclude again. ■

Proposition 2.11 *(i) For any* $\gamma \ge 0$, *if* $p > 2$, *any* $\gamma \in [0,\delta)$ *if* $p < 2$, *any solution of (1.10) satifies, near* ∞,

$$w(r) = O(r^{-\gamma}) + O(r^{-\alpha}). \tag{2.50}$$

(ii) The solution $w = w(.,a)$ *of problem (1.10), (1.15) satisfies*

$$|w(r,a)| \le C_\gamma(a)((1+r)^{-\gamma} + (1+r)^{-\alpha}), \tag{2.51}$$

where $C_\gamma(a)$ *is continuous with respect to* a *on* $\mathbb{R}$.

Proof. (i) Here we simplify the proofs of [18] and [22]: using equation (1.10), the function F defined by

$$F(r) = \frac{1}{2}w^2 + r^{-1}|w'|^{p-2} w'w, \tag{2.52}$$

satisfies the relation

$$\begin{aligned}(r^{2\alpha}F)' &= r^{2\alpha-1}(|w'|^p + (2\alpha-N)r^{-1}|w'|^{p-2}w'w - |w|^{q+1}) \\ &\le r^{2\alpha-1}(|w'|^p + (2\alpha-N)r^{-1}|w'|^{p-2}w'w).\end{aligned}$$

Assume that for some $d \ge 0$ and $R > 0$, $|w(r)| \le Cr^{-d}$ on $[R,\infty)$. Then from Proposition 2.10 there exists other constants $C > 0$ such that $\left(r^{2\alpha}F\right)' \le Cr^{2\alpha-1-(d+1)p}$ on $[R,\infty)$. Then $F(r) \le C(r^{-(d+1)p} + r^{-2\alpha})$ on $[R,\infty)$ if $(d+1)p \ne 2\alpha$; and $r^{-1}|w'|^{p-1}|w| \le Cr^{-(d+1)p}$, and thus

$$|w(r)| \le C(r^{-(d+1)p/2} + r^{-\alpha})$$

on $[R,\infty)$. We know that w is bounded on $[R,\infty)$ from Proposition 2.4. Consider the sequence (d_n) defined by $d_0 = 0$, $d_{n+1} = (d_n+1)p/2$. It is increasing and tends to ∞ if $p \ge 2$ and to δ if $p < 2$. After a finite number of steps, we get (2.50) by changing slightly the sequence if it takes the value $2\alpha/p - 1$.

(ii) We have $|w(.,a)| \le a$, from Theorem 2.1. Assuming that for some $d \ge 0$, $|w(r,a)| \le C_d(a)(1+r)^{-d}$ on $[0,\infty)$, and C_d is continuous, then

$$|w(r,a)| \le \tilde{C}_d(a)((1+r)^{-(d+1)p/2} + (1+r)^{-\alpha})$$

from Proposition 2.10, where $\tilde{C}_d$ is also continuous. We deduce (2.51) as above, and C_γ is continuous, since we use is a finite number of steps. Notice in particular that $\lim_{a\to 0} C_\gamma(a) = 0$. ■

As a consequence we can extend a property of zeros given in [26, Proposition 3.1] in case $p = 2$, which improves Proposition 2.5:

Proposition 2.12 *Assume that* $\alpha < N$, *or* $p > 2$, *or* $\alpha < \eta$. *Given* $A > 0$, *there exists* $M(A) > 0$ *such that if* $0 < |a| \leq A$, *then the solution* $w(.,a)$ *of (1.10), (1.15) has at most one isolated zero outside* $[0, M(A)]$.

Proof. From Proposition 2.5, $w(.,a)$ has a finite number of isolated zeros. Let $\rho_1 < \rho_2$ be its two last zeros, where by convention $\rho_2 = \bar{r}$ if $p > 2$ and the function has a compact support $[0, \bar{r}]$. From Proposition 2.11, for any $\mu > 0$, there exists $R = R(A, \mu) > 0$ such that $\max_{|a|\leq A, r\geq R} |w(r,a)| \leq \mu^{1/(q-1)}$. Also $\max_{|a|\leq A, r\geq 0} |w(r,a)| \leq A$, from Theorem 2.1. As in Proposition 2.5, we make the substitution (2.11) for some $d > 0$. If $p > 2$, we choose $d > \alpha$, and fix $\mu = (d-\alpha)/2$. Suppose that $\rho_1 > R$. Then from (2.38), denoting $\mu' = d^{p-1}((p-1)d - N + p)$, there exists $\rho \in (\rho_1, \rho_2)$ such that$\rho^p |w(\rho)|^{2-p} \left(d - \alpha - |w|^{q-1}(\rho)\right) \leq (p-1)d^{p-1}(d-\eta)$,

$$\mu\rho^p \leq \mu' |w(\rho)|^{p-2} \leq \mu' A^{p-2}.$$

Taking $M(A) = \max(R(A,\mu), (\mu'\mu^{-1}A^{p-2})^{1/p})$, we find $\rho_1 \leq M(A)$. If $p < 2$ and $\alpha < \eta$, taking $d \in (\alpha, \eta)$ and the same μ, and $M(A) = R(A,\mu)$, then $\rho_1 \leq M(A)$, from (2.38). If $p < 2$ and $\alpha < N$, we choose $\mu = (N-\alpha)/2$ and $M(A) = R(A,\mu)$ and get $\rho_1 \leq M(A)$ from (2.39) by contradiction. ■

3 The case $(2-p)\alpha < p$

In this paragraph, we suppose that $(2-p)\alpha < p$, or equivalently,

$$p > 2 \qquad \text{or} \qquad (p < 2 \text{ and } \alpha < \delta). \tag{3.1}$$

3.1 Behaviour near infinity

Proposition 3.1 *Assume (3.1) and* $q > 1$. *For any solution* w *of problem (1.10), there exists* $L \in \mathbb{R}$ *such that* $\lim_{r\to\infty} r^\alpha w = L$.

Proof. From Propositions 2.10 and 2.11, $w(r) = O(r^{-\alpha})$ and $w'(r) = O(r^{-\alpha-1})$ near ∞. Indeed it follows from (2.51) by choosing any $\gamma > \alpha$ if $p > 2$ and $\gamma \in (\alpha, \delta)$ if $p < 2$. Consider the function J_α defined in (2.5). Then from (2.6), J'_α is integrable at infinity: indeed $r^{\alpha-2}|w'|^{p-1} = O(r^{(2-p)\alpha-p-1})$ and (3.1) holds, and $r^{\alpha-1}|w|^{q-1}w = O(r^{-1-\alpha(q-1)})$. Then J_α has a limit L as $r \to \infty$. And

$$r^\alpha w = J_\alpha(r) - r^{\alpha-1}|w'|^{p-2}w' = J_\alpha(r) + O(r^{(2-p)\alpha-p}).$$

Thus $\lim_{r\to\infty} r^\alpha w(r) = L$, and

$$L = J_\alpha(r) + \int_r^\infty J_\alpha^{'}(s)ds. \tag{3.2}$$

■

Next we look for precise estimates of fast decaying solutions. It is easy to obtain an approximate estimate. Since $\lim_{r\to\infty} J_\alpha(r) = 0$, we find $J_\alpha(r) = -\int_r^\infty J'_\alpha(s)ds$; thus

$$|w(r)| \leq r^{-1} |w'(r)|^{p-1} + r^{-\alpha} \int_r^\infty s^{\alpha-1} \left(|w|^q + (N+\alpha)s^{-1} |w'|^{p-1} \right) ds. \tag{3.3}$$

Consider any $d \geq \alpha$, with $(2-p)d < p$, such that $w(r) = O(r^{-d})$, hence also $w'(r) = O(r^{-d-1})$ from Proposition 2.10. Then $w(r) = O(r^{-d(p-1)-p}) + O(r^{-qd})$ from (3.3). Setting $d_0 = \alpha$ and $d_{n+1} = \min(d_n(p-1)+p, qd_n)$, the sequence (d_n) is nondecreasing. it tends to ∞ if $p > 2$, and to δ if $p < 2$. Thus

$$w(r) = o(r^{-d}), \quad \text{for any } d \geq 0 \text{ if } p > 2, \quad \text{for any } d < \delta \text{ if } p < 2. \tag{3.4}$$

Next we give better estimates, for any solution of the problem, even changing sign or not everywhere defined.

Proposition 3.2 *Assume that (3.1) holds, and Let w be any solution of (1.10) such that* $\lim_{r\to\infty} r^\alpha w(r) = 0$.
(i)If $p > 2$, then w has a compact support.
(ii) If $p < 2$, then $w(r) = O(r^{-\delta})$ near ∞.

Proof. (i) Case $p > 2$. Assume that w has no compact support. We can suppose that $w > 0$ for large r, from Proposition 2.5. We make the substitution (2.11) for some $d > \alpha$. Since $w(r) = o(r^{-d}), w'(r) = o(r^{-d-1})$ near ∞, we get $y_d(\tau) = o(1)$, $y'_d(\tau) = o(1)$ near ∞. And $\psi = dy_d - y'_d = -r^{d+1}w'$ is positive for large τ from Proposition 2.4. From (2.12),

$$\begin{aligned} &y''_d + (\eta - 2d)y'_d - d(\eta - d)y_d \\ &+ \frac{1}{p-1} e^{((p-2)d+p)\tau} \psi^{2-p} \left(y'_d - (d-\alpha)y_d + e^{-d(q-1)\tau} |y_d|^{q-1} y_d \right) = 0. \end{aligned}$$

As in Proposition 2.5 the maximal points τ of y_d remain in a bounded set, hence y_d is monotone for large τ, $y'_d(\tau) \leq 0$, and $\lim_{\tau\to\infty} e^{((p-2)d+p)\tau} \psi^{2-p} = \lim_{r\to\infty} r^2 |w'|^{2-p} = \infty$. Then

$$(p-1)y''_d = e^{((p-2)d+p)\tau} \psi^{2-p} \left(|y'_d| (1 + o(1) + (d-\alpha)y_d(1+o(1)) \right).$$

Since $d - \alpha > 0$, there exists $C > 0$ such that $y''_d \geq Ce^{((p-2)d+p)\tau} \psi^{3-p}$ for large τ. Then

$$-\psi' = y''_d + d |y'_d| \geq Ce^{((p-2)d+p)\tau} \psi^{3-p},$$

and thus $\psi^{p-2} + Ce^{((p-2)d+p)\tau} / (d + |\delta|)$ is nonincreasing, which is impossible.

(ii) Case $p < 2$. Let us prove that y is bounded near ∞. It holds if y is changing sign, from Lemma 2.6. Next assume that for example $y > 0$ for large τ, thus also Y. If y is not monotone, then $N < \delta$ and $\lim_{\tau\to\infty} y(\tau) = \ell$, from Lemma 2.6. If y is monotone, and unbounded, then is nondecreasing and tending to ∞. Then $Y \leq (\delta y)^{p-1}$ from system (2.17), which implies $Y = o(y)$; then $y - Y > 0$ for large τ. Thus for any $\varepsilon > 0$, for large τ,

$$\begin{aligned}(y-Y)' &= (\delta-\alpha)y + (N-\delta)Y - e^{-\delta(q-1)\tau}\,|y|^{q-1}\,y \\ &= (\delta-\alpha)(y-Y) + (N-\alpha)Y - e^{-\delta(q-1)\tau}\,|y|^{q-1}\,y \geq (\delta-\alpha-\varepsilon)(y-Y)\end{aligned}$$

and $y \geq y - Y \geq Ce^{(\delta-\alpha-\varepsilon)\tau}$, for some $C > 0$, which contradicts (3.4). ■

Next we complete the estimates of Proposition 3.2 when $p < 2$.

Proposition 3.3 *Under the assumptions of Proposition 3.2 with $p < 2$, if w has a finite number of zeros, then*

$$\text{(i) if } p_1 < p, \quad \lim_{r\to\infty} r^{\delta} w = \pm\ell; \tag{3.5}$$

$$\text{(ii) if } p < p_1, \quad \lim_{r\to\infty} r^{\eta} w = c \qquad c \in \mathbb{R}, \quad c \neq 0; \tag{3.6}$$

$$\text{(iii) if } p = p_1, \ \lim_{r\to\infty} r^{N}(\ln r)^{(N+1)/2} w = \pm\varrho, \ \ \varrho = \frac{1}{N}\left(\frac{N(N-1)}{2(N-\alpha)}\right)^{(N+1)/2}. \tag{3.7}$$

Proof. We can assume that $w > 0$ for large r. Then y, Y are positive for large τ, from Proposition 2.4, and y, y' are bounded from Propositions 3.2 and 2.10. If y is not monotone for large τ, then $N < \delta$ from Lemma 2.6; that means $p_1 < p$ from (1.6), and $\lim_{\tau\to\infty} y(\tau) = \ell$, which proves (3.5). So we can assume that y is monotone for large τ. Since it is bounded, then, from Lemma 2.6, either $N < \delta$ and $\lim_{\tau\to\infty} y(\tau) = \ell$ or 0, or $\delta \leq N$ and $\lim_{\tau\to\infty} y(\tau) = 0$. Suppose that $\lim_{\tau\to\infty} y(\tau) = 0$. Then $y'(\tau) \leq 0$ for large τ.

(i) Case $p_1 < p$ $(N < \delta)$. Then $N < \delta p$, and from (2.16),

$$(p-1)y'' + (\delta p - N)\,|y'| + (\delta - N)\delta y = o(|y'|^{3-p}) + o(y^{3-p}). \tag{3.8}$$

Thus y is concave for large τ, which is a contradiction; and (3.5) holds.

(ii) Case $p < p_1$ $(\delta < N)$. We observe that

$$-(p-1)y'' + (\delta p - N)y' + (N-\delta)\delta y \leq 0 \tag{3.9}$$

for $\tau \geq \tau_1$ large enough, since $\alpha < \delta$; and we can suppose $y(\tau) \leq 1$ for $\tau \geq \tau_1$. For any $\varepsilon > 0$, the function $\tau \longmapsto \varepsilon + e^{-\mu(\tau-\tau_1)}$ is a solution of the corresponding equation on $[\tau_1, \infty)$, where

$$\mu = \eta - \delta = (N-\delta)/(p-1) > 0. \tag{3.10}$$

Then $y(\tau) \leq \varepsilon + e^{-\mu(\tau-\tau_1)}$ from the maximum principle; thus $y(\tau) \leq e^{-\mu(\tau-\tau_1)}$ on $[\tau_1, \infty)$. Hence, $w(r) = O(r^{(p-N)/(p-1)})$ near ∞, and $w'(r) = O(r^{(1-N)/(p-1)})$ from Proposition 2.10. Next we make the substitution (2.11), with $d = \eta$. Then functions y_η and y'_η are bounded, and from (2.12)

$$\begin{aligned}&(p-1)(y''_\eta - \eta y'_\eta) \\ &= e^{(p-(2-p)\eta)\tau} \left|\eta y_\eta - y'_\eta\right|^{2-p} \left(-y'_\eta + (\eta-\alpha)y_\eta - e^{-\eta(q-1)\tau} \left|y_\eta\right|^{q-1} y_\eta\right);\end{aligned} \tag{3.11}$$

hence $(e^{-\eta\tau}y'_\eta)' = O(e^{(p-(3-p)\eta)\tau})$. Since $\lim_{\tau\to\infty} e^{-\eta\tau}y'_\eta(\tau) = 0$, and $\delta < \eta$ from (1.6), we find $p < (2-p)\eta < (3-p)\eta$. Hence $e^{-\eta\tau}y'_\eta(\tau) = O(e^{(p-(3-p)\eta)\tau})$, and $y'_\eta(\tau) = O(e^{(p-(2-p)\eta)\tau})$. Then y_η has a limit $c \geq 0$ as $\tau \to \infty$, and

$$\lim_{r\to\infty} r^\eta w = c.$$

Suppose that $c = 0$. Then $y_d(\tau) = O(e^{-\gamma_0\tau})$, with $\gamma_0 = (2-p)d - p > 0$. Assuming that $y_d(\tau) = O(e^{-\gamma_n\tau})$ for some $\gamma_n > 0$, then $y'_d(\tau) = O(e^{-\gamma_n\tau})$ from Proposition 2.10. Hence $(e^{-d\tau}y'_d)' = O(e^{(p-(3-p)d-(3-p)\gamma_n)\tau})$, and in turn $y_d(\tau) = O(e^{-\gamma_{n+1}\tau})$ with $\gamma_{n+1} = (3-p)\gamma_n + (2-p)d - p$. And $\lim_{n\to\infty}\gamma_n = \infty$; thus $w(r) = o(r^{-\gamma})$ for any $\gamma > 0$. Let us again make the substitution (2.11), now with $d > \eta$. The new function y_d satisfies $\lim_{\tau\to\infty} y_d(\tau) = \lim_{\tau\to\infty} y'_d(\tau) = 0$. It is nondecreasing near ∞, since $\alpha \neq d$: indeed at each point τ large enough where $y'_d(\tau) = 0$, $y''_d(\tau)$ has a constant sign from (2.12). Otherwise $\lim_{\tau\to\infty} e^{(p-(2-p)d)\tau} = 0$, since $\delta < d$. Then

$$(p-1)y''_d + (2d - \eta + o(1))\left|y'_d\right| + d(d-\eta+o(1))y_d = 0;$$

thus y''_d is concave for large τ, which is a contradiction. Thus $c > 0$ and (3.6) holds.

(iii) Case $p = p_1$ $(\delta = N)$. Then also $\delta = \eta$. From (2.17),

$$y' - Ny = -Y^{1/(p-1)}, \qquad Y' + Y^{1/(p-1)} = \alpha y + e^{\delta(q-1)\tau}y^q \tag{3.12}$$

hence $Y' + Y^{1/(p-1)} \geq 0$. Thus by integration, $Y(\tau) \geq C_1\tau^{-(p-1)/(2-p)}$ for some $C_1 > 0$ and for large τ. From (3.12), there exists $K_1 > 0$ such that

$$\left(-Ne^{-N\tau}y\right)' \geq K_1 e^{-N\tau}\tau^{-1/(2-p)} \geq -\frac{K_1}{2}\left(e^{-N\tau}\tau^{-1/(2-p)}\right)'$$

for large τ, which implies a lower bound

$$y \geq (K_1/2N)\tau^{-1/(2-p)}.$$

Also $Y' + Y^{1/(p-1)} \leq (\alpha/N + o(1))Y^{1/(p-1)}$, since $y' < 0$. Then for any $\varepsilon > 0$,

$$Y' + (\frac{N-\alpha}{N} - \varepsilon)Y^{1/(p-1)} \leq 0 \tag{3.13}$$

for large τ. Taking ε small enough, we deduce

$$Y(\tau) \leq C_{1,\varepsilon}\tau^{-(p-1)/(2-p)}, \quad \text{with } C_{1,\varepsilon}^{(2-p)/(p-1)} = \frac{p-1}{2-p}(\frac{N-\alpha}{N} - 2\varepsilon)^{-1} \tag{3.14}$$

for large τ. Then

$$\left(-Ne^{-N\tau}y\right)' \leq NC_{1,\varepsilon}^{1/(p-1)}e^{-N\tau}\tau^{-1/(2-p)} \leq -C_{1,\varepsilon}^{1/(p-1)}\left(e^{-N\tau}\tau^{-1/(2-p)}\right)'.$$

Thus we get an upper bound

$$y(\tau) \leq \frac{1}{N}C_{1,\varepsilon}^{1/(p-1)}\tau^{-1/(2-p)}.$$

Moreover from (3.12) and (3.13), $|Y'(\tau)| \leq Y^{1/(p-1)}(\tau)$ for large τ; hence from (3.14), $y'' - Ny' = -Y^{1/(p-1)}Y' = O\left(\tau^{-(3-p)/(2-p)}\right)$. Then

$$\left(e^{-N\tau}y'\right)' = O(e^{-N\tau}\tau^{-(3-p)/(2-p)}),$$

thus $y' = O(\tau^{-(3-p)/(2-p)})$, and $y' = o(y)$ from the lower estimate of y.Then for any $\varepsilon > 0$,

$$Y' + (\frac{N-\alpha}{N} - \varepsilon)Y^{1/(p-1)} \geq 0$$

for large τ; then

$$Y(\tau) \geq C_{2,\varepsilon}\tau^{-(p-1)/(2-p)}, \quad \text{with } C_{2,\varepsilon}^{(2-p)/(p-1)} = \frac{p-1}{2-p}(\frac{N-\alpha}{N} + 2\varepsilon)^{-1}$$

for large τ. Thus

$$\lim_{\tau\to\infty} \tau^{-(p-1)/(2-p)}Y(\tau) = (\frac{p-1}{2-p}\frac{N}{N-\alpha})^{(p-1)/(2-p)} = \lim_{\tau\to\infty}(\tau^{-1/(2-p)}Ny(\tau))^{p-1},$$

so that $\lim_{\tau\to\infty}(\tau^{-1/(2-p)}y(\tau)) = \varrho$, and (3.7) holds. ∎

We can get an asymptotic expansion of the slow decaying solutions, which in fact covers the case $p = 2$, where we find again the results of [26, Theorem 1].

Proposition 3.4 *Assume (3.1). Let w be any solution of (1.10) such that $L = \lim_{r\to\infty} r^\alpha w > 0$. Then*

$$\lim_{r\to\infty} r^{\alpha+1}w' = -\alpha L, \tag{3.15}$$

and

$$w(r) = \begin{cases} r^{-\alpha}\left(L + (K + o(1))\, r^{-k}\right), & \text{if } (q+1-p)\,\alpha > p, \\ r^{-\alpha}\left(L + (K + M + o(1))\, r^{-\alpha(q-1)}\right), & \text{if } (q+1-p)\,\alpha = p, \\ r^{-\alpha}\left(L + (M + o(1))\, r^{-\alpha(q-1)}\right), & \text{if } (q+1-p)\,\alpha < p, \end{cases} \tag{3.16}$$

where

$$k = p - (2-p)\alpha, \qquad K = \frac{(\alpha(p-1) - (N-p))\,(\alpha L)^{1/(p-1)}}{k}, \qquad M = \frac{L^q}{\alpha(q-1)}.$$

Moreover, differentiating term to term gives an expansion of w'.

Proof. We make the substitution (2.11) with $d = \alpha$, thus $w(r) = r^{-\alpha} y_\alpha(\tau)$. For large r, $w'(r) = r^{-(\alpha+1)}(\alpha y_\alpha(\tau) - y'_\alpha(\tau)) < 0$. Thus $\alpha y_\alpha - y'_\alpha > 0$ for large τ. And (2.14) becomes:

$$\begin{cases} y'_\alpha = \alpha y_\alpha - Y_\alpha^{1/(p-1)} \\ Y'_\alpha = (p-1)(\alpha-\eta)Y_\alpha + e^{k\tau}(\alpha y_\alpha - Y_\alpha^{1/(p-1)} + e^{-\alpha(q-1)\tau} y_\alpha^q). \end{cases} \tag{3.17}$$

The function y_α converges to L, and y'_α is bounded near ∞, since $w' = O(r^{-(\alpha+1)})$ near ∞, thus Y_α is bounded. Either Y_α is monotone for large τ, inwhich case it has a finite limit λ; then y'_α converges to $\alpha L - \lambda^{1/(p-1)}$; and hence $\lambda = (\alpha L)^{1/(p-1)}$. Or for large τ, the extremal points of Y_α form an increasing sequence (τ_n) tending to ∞. Then

$$Y_\alpha(\tau_n)^{1/(p-1)} = \alpha y_\alpha(\tau_n) + e^{-\alpha(q-1)\tau_n} y_\alpha^q(\tau_n) + (p-1)(\alpha-\eta)e^{-k\tau_n} Y_\alpha(\tau_n)$$

thus $\lim Y_\alpha(\tau_n) = (\alpha L)^{1/(p-1)}$. In any case $\lim_{\tau\to\infty} Y_\alpha(\tau) = (\alpha L)^{1/(p-1)}$, which is equivalent to (3.15), and implies $\lim_{\tau\to\infty} y'_\alpha(\tau) = 0$. Now consider Y'_α. Either it is monotone for large τ, thus $\lim_{\tau\to\infty} Y'_\alpha(\tau) = 0$; or for large τ, the extremal points of Y'_α form an increasing sequence (s_n) tending to ∞. Then $Y''_\alpha(\tau_n) = 0$, and by computation, at the point $\tau = s_n$,

$$\begin{aligned} &\left(\frac{1}{p-1} Y_\alpha^{(2-p)/(p-1)} - (p-1)(\alpha-\eta)e^{-k\tau}\right) Y'_\alpha \\ &= \left(p + \alpha(p-1) + q e^{-\alpha(q-1)\tau} y_\alpha^{q-1}\right) y'_\alpha + (k - \alpha(q-1))e^{-\alpha(q-1)\tau} y_\alpha^q \end{aligned}$$

thus $\lim Y'_\alpha(s_n) = 0$. In any case, $\lim_{\tau\to\infty} Y'_\alpha(\tau) = 0$. From (3.17), we deduce

$$\begin{aligned} y'_\alpha &= -e^{-\alpha(q-1)\tau} y_\alpha^q - e^{-k\tau}((p-1)(\alpha-\eta)Y_\alpha - Y'_\alpha) \\ &= -(L^q + o(1))e^{-\alpha(q-1)\tau} - k(K + o(1))e^{-k\tau} \end{aligned}$$

thus $y'_\alpha = -k(K + o(1))e^{-k\tau}$ if $\alpha(q-1) > k$, or equivalently $(q+1-p)\alpha > p$; and $y'_\alpha = -(kK + L^q + o(1))e^{-k\tau}$ if $\alpha(q-1) = k$; and $y'_\alpha = -(L^q + o(1))e^{-\alpha(q-1)\tau}$ if $\alpha(q-1) < k$. The estimates (3.16) follow by integration.This gives also an expansion of the derivatives, by computing $w' = -r^{-(\alpha+1)}(\alpha y_\alpha - y'_\alpha)$:

$$w'(r) = \begin{cases} -r^{-(\alpha+1)}\left(\alpha L + (\alpha+k)(K + o(1))r^{-k}\right), & \text{if } (q+1-p)\alpha > p, \\ -r^{-(\alpha+1)}\left(\alpha L + (\alpha+k)(K + M + o(1))r^{-k}\right), & \text{if } (q+1-p)\alpha = p, \\ -r^{-(\alpha+1)}\left(\alpha L + \alpha q(M + o(1))r^{-\alpha(q-1)}\right), & \text{if } (q+1-p)\alpha < p; \end{cases}$$

which corresponds to a derivation term to term. ■

3.2 Continuous dependence and sign properties

Next we extend an important property of continuity with respect to the initial data, given in [18] in the case $p = 2$. The proof is different; it follows from the estimates of Proposition (2.10) and from the expression of $L(a)$ in terms of function J_α.

Theorem 3.5 *Assume (3.1). For any solution $w = w(.,a)$ of problem (1.10), (1.15), setting $L = L(a)$, the function $a \longmapsto L(a)$ is continuous on all of $\mathbb{R}$. Moreover the family of functions $(a \longmapsto (1+r)^\alpha w(r,a))_{r\geq 0}$ is equicontinuous on $\mathbb{R}$.*

Proof. Let $a_0 \in \mathbb{R}$. From Propositions 2.10 and (2.11), there exists a neighborhood V of a_0 and a constant $C = C(V) > 0$ such that for any $a \in V$,

$$|w(r,a)| \leq C(1+r)^{-\alpha}, \qquad |w'(r,a)| \leq C(1+r)^{-(\alpha+1)}. \tag{3.18}$$

From (3.2), we have for any $r \geq 1$,

$$L(a) = J_\alpha(r,a) + \int_r^\infty J_\alpha'(s,a)ds = \int_0^\infty J_\alpha'(s,a)ds \tag{3.19}$$

where $J_\alpha(r,a) = r^\alpha \left(w(r,a) + r^{-1}\left|w'\right|^{p-2} w'(r,a)\right)$, since $J_\alpha(0,a) = 0$. Then with a new constant $C = C(V)$, for any $a \in V$,

$$\int_r^\infty \left|J_\alpha'(s,a)\right| ds \leq C\left(r^{-\alpha(q-1)} + r^{-(p-\alpha(2-p))}\right);$$

hence for any $\varepsilon > 0$, there exists $r_\varepsilon \geq 1$ such that

$$\sup_{a\in V} \int_{r_\varepsilon}^\infty \left|J_\alpha'(s,a)\right| ds \leq \varepsilon.$$

From Remark 2.2, $w(.,a)$ depends continuously on a on any compact set, thus also $J_\alpha'(.,a)$. Then there exists a neighborhood V_ε of a_0 contained in V such that

$$\sup_{a\in V\varepsilon} \int_0^{r_\varepsilon} \left|J_\alpha'(r_\varepsilon,a) - J_\alpha'(r_\varepsilon,a_0)\right| \leq \varepsilon,$$

and consequently $|L(a) - L(a_0)| \leq 3\varepsilon$. This proves that L is continuous at a_0. Moreover

$$\sup_{a\in V\varepsilon} \sup_{r\in[0,\infty)} |J_\alpha(r,a) - J_\alpha(r,a_0)| \leq 2\varepsilon,$$

thus the family of functions $(a \longmapsto J_\alpha(r,a))_{r\geq 0}$ is equicontinuous at a_0. Next for any $r \geq 1$ and any $a \in V$,

$$|r^\alpha w(r,a) - J_\alpha(r,a)| = r^{\alpha-1}\left|w'(r,a)\right|^{p-1} \leq Cr^{(2-p)\alpha-p}.$$

Thus for any $\varepsilon > 0$, there exists $\tilde{r}_\varepsilon \geq r_\varepsilon$ such that

$$\sup_{a\in V, r\geq \tilde{r}_\varepsilon} |r^\alpha w(r,a) - J_\alpha(r,a)| \leq \varepsilon.$$

It implies

$$\sup_{a\in V_\varepsilon, r\geq \tilde{r}_\varepsilon} |(1+r)^\alpha (w(r,a) - w(r,a_0))| \leq (2^\alpha + 2)\varepsilon.$$

And there exists a neighborhood $\tilde{V}_\varepsilon$ of a_0 contained in V_ε such that

$$\sup_{a\in\tilde{V}_\varepsilon, r\leq\tilde{r}_\varepsilon} |(1+r)^\alpha(w(r,a)-w(r,a_0))| \leq \varepsilon.$$

Then

$$\sup_{a\in\tilde{V}_\varepsilon, r\in[0,\infty)} |(1+r)^\alpha(w(r,a)-w(r,a_0))| \leq (2^\alpha+2)\varepsilon,$$

which shows that the family of functions $a \longmapsto (1+r)^\alpha w(r,a)$ $(r \geq 0)$ is equicontinuous at a_0. ∎

As a consequence we obtain some results concerning the number of zeros of the solutions

Theorem 3.6 *Assume (3.1).*
(i) Suppose that for some $a_0 > 0$, $w(.,a_0)$ has a finite number of isolated zeros, denoted by $N(a_0)$. If $L(a_0) \neq 0$, then $N(a) = N(a_0)$ for any a close to a_0.
(ii) Suppose $q < q^*$. Then $\{a > 0 : L(a) = 0\}$ is unbounded from above. Moreover there exists a increasing sequence (a_m) tending to ∞ such that $w(.,a_m)$ has at least $m+1$ isolated zeros and $L(a_m) = 0$.
(iii) Suppose $q < q^*$, $p < 2$ and $\alpha < N$. Then for any $m \in \mathbb{N}$,

$$\bar{a}_m = \inf\{a > 0 : N(a) \geq m+1\} \in (0,\infty),$$

and if $m \geq 1$, then $w(.,\bar{a}_m)$ has precisely m zeros and $L(\bar{a}_m) = 0$.

Proof. (i) Let $r_1 < r_2 < .. < r_{N(a_0)}$ be the isolated zeros of $w(.,a_0)$. Since $L(a_0) \neq 0$, there do not exist other zeros, and there exists $\varepsilon > 0$ such that

$$\inf_{r\geq r_{N(a_0)}+1} r^\alpha |w(r,a_0)| \geq \varepsilon.$$

By Theorem 3.5, there exists a neighborhood V_ε of a_0 such that

$$\inf_{r\geq r_{N(a_0)}+1} r^\alpha |w(r,a)| \geq \varepsilon/2$$

for any $a \in V_\varepsilon$. From Remark 2.2, there exists a neighborhood $\tilde{V}_\varepsilon \subset V_\varepsilon$ such that $w(r,a)$ has exactly $N(a_0)$ zeros on $\left[0, r_{N(a_0)}+1\right]$. Hence $N(a) = N(a_0)$.

(ii) Assume that for some $a^* > 0$, $L(a) \neq 0$ for any $a \in (a^*,\infty)$. By Proposition 2.5, (iii) and (iv), $w(.,a)$ has a finite number of isolated zeros $N(a)$. The set

$$\{a \in (a^*,\infty) : N(a) = N(a^*)+1\}$$

is closed in (a^*,∞) since N is locally constant, and open; then $N(a)$ is constant on (a^*,∞), which contradicts Proposition 2.7. Moreover there exists an increasing sequence $(a^\star_m)$ tending to ∞ such that $w(.,a^\star_m)$ has at least $m+1$ isolated zeros; as above it cannot happen that $L(a) \neq 0$ for any $a \in (a^*_m,\infty)$. Hence there exists $a_m \geq a^*_m$ such that $w(.,a_m)$ has at least $m+1$ isolated zeros and $L(a_m) = 0$.

(iii) Here $w(.,a)$ has only isolated zeros. Following the proof of [26, Propositions 3.5 and 3.7], for any $m \in \mathbb{N}$, the set $B_m = \{a > 0 : \ N(a) \geq m+1\}$ is open and $z_m(a) = m^{\text{th}}$ zero of $w(.,a)$ depends continuously on a. Using Proposition 2.12, one can show that for any $a_0 > 0$, $N(a) = N(a_0)$ or $N(a_0)+1$ for any a in some neighborhood of a_0. Then necessarily $\bar{a}_m \notin B_m$, and $N(\bar{a}_m) = m$, and $L(\bar{a}_m) = 0$ by contradiction in (i). ∎

Remark 3.7 When $q < q^*$ and $p > 2$, for any $a_0 > 0$, we have $N(a) \geq N(a_0)$ for any a in some neighborhood of a_0, but we cannot prove that $N(a) \leq N(a_0) + 2$, thus we have no specific information of the number of zeros of the compact support solutions.

3.3 Existence of nonnegative solutions

Here we study the existence of nonegative solutions of equation (1.10). If such solutions exist, then either $p_1 < p$ and $\alpha < N$, from From Proposition 2.5, or $p < p_1$. Thus $\alpha < \delta \leq N$; in any case $\alpha < N$. Reciprocally, when $\alpha < N$, we first prove the existence of slow decaying solutions for $|a|$ small enough.

Proposition 3.8 *Assume (3.1), and $\alpha < N$. Let $\underline{a} > 0$ be defined at Proposition 2.5. Then for any $a \in (0, \underline{a}]$, $w(r,a) > 0$ on $[0,\infty)$, and $L(a) > 0$.*

Proof. Let $a \in (0, \underline{a}]$. By construction of $\underline{a}$, $w = w(r,a) > 0$, from Proposition 2.5, and the function J_N is nondecreasing, $J_N(0) = 0$; and $J_N(r) \leq r^N w$ near ∞, from Proposition 2.4. Assume that $L(a) = 0$. Then $p < 2$ from Proposition 3.2. From Proposition 3.3, either $N < \delta$, and $r^N w = O(r^{N-\delta})$; or $\delta < N$ and $N < \eta$ from (1.6), and $r^N w = O(r^{N-\eta})$; or $\delta = N$ and $r^N w = O(\ln r)^{-(N+1)/2}$. In any case, $\limsup_{r\to\infty} J_N(r) = 0$; then $J_N \equiv 0$, and hence $J'_N \equiv 0$, which is impossible. ∎

Next we consider the subcritical case $1 < q < q^*$ and prove the existence of fast decaying solutions. Notice that in that range $p > p_2$; if moreover $1 < q < q_1$, then $p > p_1$.

Theorem 3.9 *Assume (3.1) and $\alpha < N$, and $1 < q < q^*$. Then there exists $a > 0$ such that $w(.,a)$ is nonnegative and such that $L(a) = 0$. If $p > 2$, it has a compact support. If $p < 2$, it is positive and satisfies (3.5), (3.6) or (3.7).*

Proof. Let

$$A = \{a > 0 : w(.,a) > 0 \text{ on } (0,\infty) \text{ and } L(a) > 0\}, \tag{3.20}$$

$$B = \{a > 0 : w(.,a) \quad \text{has at least an isolated zero}\}. \tag{3.21}$$

From Proposition 3.8 and 2.7, A and since B is nonempty: $A \supset (0, \underline{a}]$ and $B \supset [\overline{a}, \infty)$. From the local continuous dependence of the solutions on the initial value, B is open. For any $a_0 \in A$,there exists $\varepsilon > 0$ such that $\min_{r\geq 0}(1+r)^\alpha w(r,a_0) \geq \varepsilon$. From Theorem 3.5, there exists a neighborhood V_ε of a_0 such that $\min_{r\geq 0}(1+r)^\alpha w(r,a) \geq \varepsilon/2$ for any $a \in V_\varepsilon$, hence $V_\varepsilon \subset A$, thus A is open. Let $a_{\inf} = \inf B > \underline{a}$ and $a_{\sup} = \sup A < \overline{a}$. Taking $a = a_{\inf}$ or $a_{\sup}$, then $w(.,a)$ is nonnegative, positive if $p < 2$, and $L(a) = 0$, and the conclusion follows from Proposition 3.3. We cannot assert that $a_{\inf} = a_{\sup}$. ∎

Remark 3.10 As it was noticed in [25] for $p = 2$, there exists an infinity of pairs a_1, a_2 such that $0 < a_1 < a_2 < a_{\inf}$; thus $w(., a_1) > 0$, $w(., a_2) > 0$, and $L(a_1) = L(a_2)$. Indeed from the continuity of L proved at Theorem 3.5, L attains at least twice any value in $\left(0, \max_{[0,a_{\inf}]} L\right)$.

In the supercritical case $q \geq q^*$ we give sufficient conditions assuring that all the solutions are positive, and then slowly decaying. Recall that $q^* \leq 1$ whenever $p \leq p_2$.

Theorem 3.11 *Assume (3.1) and one of the following conditions:*
(i) $p_2 < p$, $\alpha \leq N/2$, *and* $q \geq q^*$;
(ii) $p \leq p_2$, *and* $1 < q$;
(iii) $p_2 < p$, $N/2 < \alpha < (N-1)p'/2$, *and* $q \geq q^*_\alpha$, *where* $q^*_\alpha > q^*$ *is given by*

$$\frac{1}{q^*_\alpha + 1} = \frac{N-1}{2\alpha} - \frac{1}{p'}. \tag{3.22}$$

Then for any $a > 0$, $w(r,a) > 0$ *on* $[0, \infty)$, *and* $L(a) > 0$.

Proof. We use the function $V = V_{\lambda,\sigma,e}$ defined in (2.9) , where $\lambda > 0, \sigma, e$ will be chosen after. It is continuous at 0 and $V_{\lambda,\sigma,e}(0) = 0$, from (2.29). Suppose that $w(r_0) = 0$ for some first real $r_0 > 0$. Then $V_{\lambda,\sigma,e}(r_0) = r_0^N |w'(r_0)|^p /p' \geq 0$. Suppose that for some λ, σ, e, the five terms giving V' are nonpositive. Then $V \equiv V' \equiv 0$ on $[0, r_0]$. Hence $rw' + (\sigma - e + \alpha)w/2 \equiv 0$, $r^{(\sigma-e+\alpha)/2}w$ is constant. Thus, $w \equiv 0$ if $\sigma - e + \alpha \neq 0$, or $w \equiv a$ if $\sigma - e + \alpha = 0$. This is impossible since $w(0) \neq w(r_0)$.

Case (i). We take $\lambda = N$, $\sigma = (N-p)/p$ and $e = \sigma + \alpha - N$. Thus

$$V(r) = r^N \left(\frac{|w'|^p}{p'} + \frac{|w|^{q+1}}{q+1} + (\frac{N-p}{p} + \alpha - N)\frac{w^2}{2} + \frac{N-p}{p} r^{-1} w \, |w'|^{p-2} \, w' \right), \tag{3.23}$$

$$r^{1-N}V'(r) = -\left(\frac{N-p}{p} - \frac{N}{q+1}\right)|w|^{q+1} - \frac{N+2}{4p}(p-p_2)(N-2\alpha)w^2 - \left(rw' + \frac{N}{2}w\right)^2 \tag{3.24}$$

and all the terms are nonpositive from our assumptions, thus $w > 0$ on $[0, \infty)$. Moreover suppose that $L(a) = 0$. Then $p < 2$, and from Proposition 3.2, $V(r) = O(r^{N-2\delta})$ as $r \to \infty$. Thus $\lim_{r\to\infty} V(r) = 0$, since $N < 2\delta$ from (1.7). Then $V \equiv 0$ on $[0, \infty)$ which is a contradiction.

Case (ii). We take $\lambda = N = 2\sigma$ and $e = \alpha - N/2$. Thus

$$r^{1-N}V'(r) = -\frac{N+2}{2p}(p_2 - p)|w'|^p - \frac{N(q-1)}{2q+1}|w|^{q+1} - (rw' + Nw)^2, \tag{3.25}$$

and all the terms are nonpositive, and again $w > 0$ on $[0, \infty)$. If $L(a) = 0$, we find $V(r) = O(r^{N-\eta})$ near ∞, from Proposition 3.2, since $p \leq p_2 < p_{1,}$. Then $\lim_{r\to\infty} V(r) = 0$, hence again a contradiction.

Case (iii). We take $\lambda = 2\alpha$, $\sigma = N - 1 - 2\alpha/p'$ and $e = \sigma - \alpha$. Thus

$$r^{1-2\alpha}V'(r) = -\left(\sigma - \frac{2\alpha}{q+1}\right)|w|^{q+1} + \sigma(2\alpha - N)r^{-1}w\,|w'|^{p-2}\,w' - (rw' + \alpha w)^2 .$$

Here the first term is nonpositive from (3.22), and also the second term, since $\sigma > 0$, $N/2 \leq \alpha$ and $w' < 0$ on $(0, r_0)$, from Proposition 2.4. Hence again $w > 0$ on $[0, \infty)$. If $L(a) = 0$, then $p < 2$. From Proposition 3.2, either $p_1 < p$ and $V(r) = O(r^{2(\alpha-\delta)})$ near ∞, where $\alpha < \delta$; or $p < p_1$ and $V(r) = O(r^{2(\alpha-\eta)})$, and $\alpha < \delta < \eta$ from (1.6); or $p = p_1$ and $V(r) = O(\ln r^{-(N+1)/2})$. In any case $\lim_{r\to\infty} V(r) = 0$, hence again a contradiction. ∎

Remark 3.12 With no hypothesis on p, if $w(r_0) = 0$ for some real r_0, then from (3.23), (3.24),

$$\left(\frac{N-p}{p} - \frac{N}{q+1}\right)\int_0^{r_0} r^{N-1}\,|w|^{q+1}\,dr + \frac{(N+2)p-2N}{4p}\,(N-2\alpha)\int_0^{r_0} r^{N-1}w^2 dr$$
$$+\int_0^{r_0} r^{N-1}\left(rw' + \frac{N}{2}w\right)^2 dr = 0.$$

As in [20] such a relation can be extended to the nonradial case and then applied to nonradial solutions w.

Remark 3.13 Property (ii) was proved for equation (1.12) in [23]. It is new in the general case. It can be also obtained by using the energy function W defined at (2.22) instead of V.

The result (iii) is new. It is also true when $p = 2$: if $N/2 < \alpha < N - 1$ and $q \geq q^*_\alpha$, where $q^*_\alpha = (3\alpha - N + 1)/(N - 1 - \alpha) > q^*$; we prove that all the solutions are ground states, with a slow decay. In the case $p = 2$, $q = q^*$ it had been shown by variational methods in [12] that there exist ground states with a fast decay, whenever $N/2 < \alpha < N$ when $N \geq 4$, or if $2 < \alpha < 3$ when $N = 3$; moreover from [2], they do not exist when $1 < \alpha \leq 2$. Apparently nothing was known beyond the critical case.

Remark 3.14 If $1 < p \leq p_1$, then the condition $\alpha < (N-1)p'/2$ is always satisfied, since $\alpha < \delta \leq N \leq (N-1)p'/2$. If $p_1 < p$, our conditions imply $\alpha < N$, which was a necessary condition in order to get positive solutions, from Proposition 2.5.

3.4 Oscillation or nonoscillation criteria

Our next result concerns the case $p < 2$, and $N \leq \alpha$, thus $N \leq \alpha < \delta$ from (3.1), where there exists no positive solutions: all the solutions are changing sign. It is new, and uses the ideas of [5] for the problem without source (1.12). It involves the coefficient α^* defined at (1.14), which here satisfies $\alpha^* < \delta$, and the energy function W defined in (2.23). We use the notations $\mathcal{W}, \mathcal{U}, \mathcal{H}, \mathcal{L}, \mathcal{S}$ of Section 2.1.

Theorem 3.15 *Assume (3.1), $p < 2$, and $N \leq \alpha$.*
(i) If $\alpha < \alpha^*$, then any solution $w(.,a)$ $(a \neq 0)$ has a finite number of zeros.
(ii) There exists $\overline{\alpha} \in (\max(N, \alpha^*), \delta)$ such that for any $\alpha \in (\overline{\alpha}, \delta)$, any solution $w(.,a)$ has a infinity of zeros.

Proof. (i) Suppose $N \leq \alpha < \alpha^*$ (which implies $p > 3/2$). In the phase plane (y, Y) of system (2.17), the stationary point M_ℓ is in the domain $\mathcal{S}$ of boundary $\mathcal{L}$. Indeed denote $P_\mu = (\mu, (\delta\mu)^{p-1})$ for any $\mu > 0$. Setting $\lambda = \delta^{-1}((2\delta - N)(p-1))^{1/(2-p)}$, the point P_λ is on the curve $\mathcal{L}$. Then $(\theta\lambda, (\theta\delta\lambda)^{p-1}) \in \mathcal{S}$ for any $\theta \in [0, 1)$, and $\alpha < \alpha^* \Leftrightarrow \ell < \lambda$. Thus $P_\ell = M_\ell \in \mathcal{S}$, and there exists $\varepsilon \in (0, 1]$ such that $P_{\ell+\varepsilon} \in \mathcal{S}$. Now for any $\mu > 0$ such that $P_\mu \in \mathcal{S}$, the square $\mathcal{K}_\mu = \left\{(y, Y) \in \mathbb{R}^2 : |y| \leq \mu, |Y| \leq (\delta\mu)^{p-1}\right\}$ is contained in $\mathcal{S}$. Indeed $\mathcal{H}(\mu, (\delta\mu)^{p-1}) = (\delta\mu)^{2-p}/(p-1)$, and for any $\xi, \zeta \in [-1, 1]$

$$\mathcal{H}(\xi\mu, \zeta(\delta\mu)^{p-1}) = (\delta\mu)^{2-p} \frac{\xi - |\zeta|^{(2-p)/(p-1)}}{|\xi|^{(2-p)/(p-1)} - \zeta} \leq \mathcal{H}(\mu, (\delta\mu)^{p-1}),$$

since the quotient is majorized by $1/(p-1)$ if $\xi\zeta > 0$, and by 1 if $\xi\zeta < 0$, because $p > 3/2$. From Lemma 2.6, iv, $(y(\tau), Y(\tau)) \in \mathcal{K}_{\ell+\varepsilon}$ for $\tau \geq \tau(\varepsilon)$ large enough, and hence $(y(\tau), Y(\tau)) \in \mathcal{S}$. Thus $\mathcal{U}(y(\tau), Y(\tau)) \geq 0$. Consider the function

$$\tau \mapsto \Psi(\tau) = W(\tau) - \frac{\delta(q-1)}{q+1} \int_\tau^\infty e^{-\delta(q-1)s} |y(s)|^{q+1} ds. \tag{3.26}$$

We find

$$\Psi'(\tau) = W'(\tau) + \frac{\delta(q-1)}{q+1} e^{-\delta(q-1)\tau} |y(\tau)|^{q+1} = \mathcal{U}(y(\tau), Y(\tau)). \tag{3.27}$$

Then Ψ is nondecreasing and bounded near ∞. Thus it has a limit κ, and W has the same limit. And $\mathcal{H}(y, Y) \leq \mathcal{H}(\ell + \varepsilon, (\delta(\ell+\varepsilon))^{p-1}) = 2\delta - N - m$, for some $m = m(\varepsilon) > 0$, and hence

$$\Psi'(\tau) = \mathcal{U}(y(\tau), Y(\tau)) \geq m\left(\delta y - |Y|^{(2-p)/(p-1)} Y\right)\left((|\delta y|)^{p-2}\delta y - Y\right).$$

Now there exists a constant $c = c(p)$ such that for any $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$(a - b)\left(|a|^{p-2} a - |b|^{p-2} b\right) \geq c(|a| + |b|)^{p-2}(a-b)^2,$$

thus from (2.17),

$$\Psi'(\tau) \geq mc\,(2\delta(\ell+1))^{p-2} y'^2(\tau).$$

Then y'^2 is integrable and bounded; hence $\lim_{\tau\to\infty} y'(\tau) = 0$. Suppose that y admits an increasing sequence of zeros (τ_n). Then $W(\tau_n) = |Y(\tau_n)|^{p'}/p' = |y'(\tau_n)|^p/p'$, thus $\lim_{\tau\to\infty} W(\tau) = 0$, and $\lim_{\tau\to\infty} \mathcal{W}(y(\tau), Y(\tau)) = 0$. Also, we have $|Y|^{(2-p)/(p-1)} Y = \delta y - y' = \delta y + o(1)$. Thus

$$\mathcal{W}(y(\tau), Y(\tau)) = \frac{(\delta - N)\delta^{p-1}}{p} |y(\tau)|^p - \frac{\delta - \alpha}{2} y^2(\tau) + o(1),$$

from which it follows that $\lim y(\tau) = 0$ or $\pm\ell$, and necssarily $\lim_{\tau\to\infty} y(\tau) = 0$. And $\lim_{\tau\to\infty} \Psi(\tau) = 0$. Thus $\Psi(\tau) \leq 0$ near ∞, and

$$\frac{(\delta - N)\delta^{p-1}}{p} |y(\tau)|^p - \frac{\delta - \alpha}{2} y^2 \leq \mathcal{W}(y(\tau), Y(\tau)) \leq \frac{\delta(q-1)}{q+1} \int_\tau^\infty e^{-\delta(q-1)s} |y(s)|^{q+1} ds.$$

Then $y(\tau) = O(e^{-k_0\tau})$, with $k_0 = \delta(q-1)/p$. Assuming that $y(\tau) = O(e^{-k_n\tau})$, then we find $y(\tau) = O(e^{-k_{n+1}\tau})$ with $k_{n+1} = k_n(q+1)/p+(q-1)/(2-p)$. Since $q > 1 > p-1$, it follows that $y(\tau) = O(e^{-k\tau})$ for any $k > 0$. Consider the substitution (2.11)for some $d > 0$. Then $y_d(\tau) = O(e^{-k\tau})$ for any $k > 0$. At any maximal point of $|y_d|$ we find from (2.12)

$$(p-1)d(\eta - d) \leq e^{((p-2)d+p)\tau} \left|dy_d\right|^{2-p} \left((\alpha - d) + e^{-d(q-1)\tau} \left|y_d\right|^{q-1}\right).$$

Choosing, for example $d = \eta/2$, we get a contradiction, as the right-hand side tends to 0.

(ii) Suppose $N \leq \alpha$ and $\alpha^* < \alpha$. Assume that there exists a solution w with a finite number of zeros. We can assume that $w(r) > 0$ near ∞. From Propositions 3.1 and 3.3, either $\lim_{r\to\infty} r^\alpha w = L > 0$ or $\lim_{r\to\infty} r^\delta w = \ell$. Now the point M_ℓ is exterior to $\mathcal{S}$, thus $\mathcal{U}(M_\ell) < 0$, and by computation

$$k_\ell := \mathcal{W}M_\ell = \frac{1}{2}(\delta - N)\,\delta^{p-2}\ell^p = \frac{M}{(\delta-\alpha)^\delta} > 0, \tag{3.28}$$

where $M = M(N,p) = (\delta - N)^{\delta+1}\,\delta^{p-2+(p-1)\delta}/2$.

• First case: $\lim_{r\to\infty} r^\delta w = \ell$. Then $\lim_{\tau\to\infty}(y(\tau), Y(\tau)) = M_\ell$. Thus for large τ, $\mathcal{U}(y(\tau), Y(\tau)) < 0$, so that $W'(\tau) < 0$. Then W is decreasing, and $\lim_{\tau\to\infty} W(\tau) = \lim_{\tau\to-\infty} \mathcal{W}(y(\tau), Y(\tau)) = k_\ell$. Moreover, near $-\infty$, we find that $\lim_{\tau\to-\infty} W(\tau) = \lim_{\tau\to-\infty} \mathcal{W}(y(\tau), Y(\tau)) = 0$; indeed near $-\infty$, $y(\tau) = O(e^{\delta\tau})$ and $Y(\tau) = O(e^{\delta\tau})$ from (2.29) and (2.15); hence $e^{-\delta(q-1)\tau}\left|y(\tau)\right|^{q+1} = O(e^{2\delta\tau})$. Then W has at least a maximum point τ_0 such that $W(\tau_0) > k_\ell$. At such a point, $W'(\tau_0) = 0$, and thus $\mathcal{U}(y(\tau_0), Y(\tau_0)) > 0$, and $(y(\tau_0), Y(\tau_0)) \in \mathcal{S}$. Let $C = \max_{(y,Y)\in\overline{\mathcal{S}}}(|y| + |Y|)$. Then $C = C(N,p)$, and from (2.26) and (2.27), $\max_{(y,Y)\in\overline{\mathcal{S}}} \mathcal{W}(y,Y) \leq K = K(N,p)$, since $\alpha - \delta < 0$. Then

$$k_\ell < W(\tau_0) \leq K + \frac{C^{q+1}}{q+1}.$$

From (3.28) it follows that $\delta - \alpha$ is not close to 0. More precisely, there exists $\overline{\alpha} = \overline{\alpha}(N,p) > \max(N, \alpha^*)$ such that $\alpha \leq \overline{\alpha}$.

• Second case: $\lim_{r\to\infty} r^\alpha w = L > 0$. It follows that $\lim_{\tau\to\infty} e^{(\alpha-\delta)\tau} y = L$, and $\lim_{\tau\to\infty} e^{(\alpha-\delta)\tau} Y = (\alpha L)^{1/(p-1)}$, from (3.15). Then $Y(\tau) = O(y^{p-1}(\tau))$ near ∞, and thus

$$\mathcal{W}(y(\tau), Y(\tau)) + \tfrac{\delta-\alpha}{2} y^2(\tau) = O(y^p(\tau)),$$

$$\begin{aligned} &W(\tau) + \frac{\delta-\alpha}{2} y^2(\tau) \\ &= O(y^p(\tau)) + O(e^{-\delta(q-1)\tau} y^{q+1}(\tau)) \quad = O(y^p(\tau)) + O(y^{2-\alpha(q-1)/(\delta-\alpha)}(\tau)); \end{aligned}$$

so $\lim_{\tau\to\infty} \mathcal{W}(y(\tau), Y(\tau)) = \lim_{\tau\to\infty} W(\tau) = -\infty$; again $\lim_{\tau\to-\infty} \mathcal{W}(y(\tau), Y(\tau)) = 0$. From [5, Lemma 4.3] we know the shape of the level curves $\mathcal{C}_k = \{\mathcal{W}(y,Y) = k\}$:

either $k > k_\ell$ and $\mathcal{C}_k$ has two unbounded connected components, or $0 < k < k_\ell$ and $\mathcal{C}_k$ has three connected components and one of them is bounded, or $k = k_\ell$ and $\mathcal{C}_{k_\ell}$ is connected with a double point at M_ℓ, or $k = 0$ and one of the three connected components of $\mathcal{C}_0$ is $\{(0,0)\}$, or $k < 0$ and $\mathcal{C}_k$ has two unbounded connected components. As a consequence there exists τ_1 such that $\mathcal{W}(y(\tau_1), Y(\tau_1)) = k_\ell$; then again $W(\tau_1) > k_\ell$. Thus W has at least a maximum point τ_0 such that $W(\tau_0) > k_\ell$, and the conclusion follows as above. ■

4 The case $p \leq (2-p)\alpha$

In this section we assume that $p \leq (2-p)\alpha$, that means $p < 2$ and $\delta \leq \alpha$.

4.1 Behaviour near infinity

From Proposition 2.11, we deduce approximate estimates near ∞

$$w(r) = o(r^{-\gamma}), \quad \text{for any } \gamma < \delta. \tag{4.1}$$

However it is not straightforward to obtain exact estimates, and they can be false, see Proposition 4.4 below. Here again the key point is the use of enegy function W defined by (2.22).

Proposition 4.1 *Assume $q > 1, p < 2$, and $\delta < \alpha$, or $N \leq \alpha = \delta$. Then any solution w of problem (1.10) satisfies*

$$w(r) = O(r^{-\delta}), \qquad w'(r) = O(r^{-\delta-1}) \qquad \text{near } \infty. \tag{4.2}$$

Proof. **(i) Case** $\delta < \alpha$.

• First assume that $2\delta \leq N$, that means $p \leq p_2$. Then from (2.23), $W'(\tau) \leq 0$ for any τ; hence W is bounded from above near ∞, and in turn y and Y are bounded, because $\delta < \alpha$ and $p < 2$. Thus (4.2) holds.

• Then assume $N < 2\delta$. Let τ_0 be arbitrary. Since $\mathcal{S}$ is bounded, there exists $k > 0$ large enough such that $W(\tau) \leq k$ for any $\tau \geq \tau_0$ such that $(y(\tau), Y(\tau)) \in \mathcal{S}$, and we can choose $k > W(\tau_0)$; and $W'(\tau) \leq 0$ for any $\tau \geq \tau_0$ such that $(y(\tau), Y(\tau)) \notin \mathcal{S}$. Then $W(\tau) \leq k$ for any $\tau \geq \tau_0$; hence again y and Y are bounded for $\tau \geq \tau_0$.

(ii) Case $N \leq \alpha = \delta$. Since $N < 2\delta$, as above, W is bounded from above for large τ. We can write W in the form

$$W(\tau) = \frac{(\delta - N)\delta^{p-1}}{p} |y(\tau)|^p + \Phi(y(\tau), Y(\tau)) + \frac{1}{q+1} e^{-\delta(q-1)\tau} |y(\tau)|^{q+1},$$

where

$$\Phi(y, Y) = \frac{|Y|^{p'}}{p'} - \delta y Y + \frac{|\delta y|^p}{p} \geq 0, \qquad \forall (y, Y) \in \mathbb{R}^2.$$

Thus y is bounded, and so is Y from Hölder inequality. ■

Remark 4.2 Under the assumptions of Proposition 4.1, we can improve the estimate (4.2) for the global solutions: there exists a constant $C = C(N, p)$ independent on a such that all the solutions $w(.a)$ of (1.10), (1.15) satisfy

$$|w(r,a)| \leq Cr^{-\delta}, \qquad \text{for any } r > 0. \tag{4.3}$$

Indeed let w be any solution. Then $\lim_{\tau\to-\infty} y(\tau) = \lim_{\tau\to-\infty} Y(\tau) = 0$, and therefore $\lim_{\tau\to-\infty} W(\tau) = 0$. If $2\delta \leq N$, then $W(\tau) \leq 0$ for any τ, which gives an upper bound for y independent on a. The same happens in case $2\delta > N$: $\mathcal{S}$ is interior to some curve $\mathcal{W}(y, Y) = k$, with k independent on a, and $W(\tau) \leq k$, for any τ. Thus (4.3) holds. As a consequence, then $|w(r,a)| \leq \max(C, a)2^{\delta}(1 + r)^{-\delta}$ for any $r > 0$, from Theorem 2.1.

The case $\alpha = \delta < N$ is not covered by Proposition 4.1. In fact (4.2) is not satisfied, because a logarithm appears:

Proposition 4.3 *Assumeq $> 1, p < 2$, and $\alpha = \delta < N$. Then any solution w of (1.10)satisfies*

$$w = O(r^{-\delta}(\ln r)^{1/(2-p)}) \qquad \textit{near } \infty. \tag{4.4}$$

Proof. From (2.50), we have $w(r) = O(r^{-\delta+\varepsilon})$ for any $\varepsilon > 0$; hence $y(\tau) = O(e^{\varepsilon\tau})$, and w has a finite number of zeros, from Proposition 2.5,(iv), since $\alpha < N$. We can assume that y is positive for large τ. From (2.17),

$$(y - Y)' = (N - \delta)Y - e^{\delta(q-1)\tau}y^q.$$

From Lemma 2.6,(i), y is monotone for large τ. If y is bounded, then (4.4) is trivial. We can assume that $\lim_{\tau\to\infty} y = \infty$. Then also $\lim_{\tau\to\infty} Y = \infty$, from Lemma 2.6, (iii), and $y' \geq 0$ for large τ. Hence $Y^{1/(p-1)} < \delta y$, and $Y = o(y)$ near ∞, since $p < 2$; for any $\varepsilon > 0$, $y \leq (1 + \varepsilon)(y - Y)$ for large τ. Thus

$$(y - Y)' \leq (N - \delta)(\delta y)^{p-1} \leq (N - \delta)\delta^{p-1}(1 + \varepsilon)^{p-1}(y - Y)^{(p-1)}.$$

Hence with a new ε, for large τ, $(y - Y)^{2-p}(\tau) \leq (N - \delta)\delta^{p-1}(2 - p)(1 + \varepsilon)\tau$, which gives the upper bound

$$y^{2-p}(\tau) \leq (N - \delta)\delta^{p-1}(2 - p)(1 + \varepsilon)\tau. \tag{4.5}$$

In particular, (4.4) holds, and the estimate is more precise:

$$\limsup_{r\to\infty} r^{\delta}(\ln r)^{-1/(2-p)}w \leq ((2 - p)\delta^{p-1}(N - \delta))^{1/(2-p)}. \tag{4.6}$$

■

Next we make precise the behaviour of the solutions according to the values of α.

Proposition 4.4 . *Assume $q > 1, p < 2$. Let w be any solution w of problem (1.10) such that w has a finite number of zeros.*

(i) If $\delta < \min(\alpha, N)$, then either

$$\lim_{r\to\infty} r^{\delta} w = \pm\ell, \tag{4.7}$$

or

$$\lim_{r\to\infty} r^{\eta} w = c \neq 0 \tag{4.8}$$

or $r^{\delta}w(r)$ is bounded near ∞ and $r^{\delta}w$ has no limit, and

$$\lim_{r\to\infty} \inf r^{\delta} w \leq \ell \leq \lim \sup_{r\to\infty} r^{\delta} w; \tag{4.9}$$

in the last case $p_2 < p$.
(ii) If $\alpha = \delta < N$, then either

$$\lim_{r\to\infty} r^{\delta}(\ln r)^{-1/(2-p)} w = \pm\eta, \qquad \eta = ((2-p)\delta^{p-1}(N-\delta))^{1/(2-p)}, \tag{4.10}$$

or (4.8) holds.
(iii) If $\alpha = \delta = N$, then

$$\lim_{r\to\infty} r^{N} w = k \neq 0. \tag{4.11}$$

Proof. **(i) Case** $\delta < \min(\alpha, N)$.

• First assume that y is positive and monotone for large τ. Since it is bounded, from Lemma 2.6,(ii) and (iv), either $\lim_{\tau\to\infty}(y, Y) = M_\ell$ and (4.7) holds; or $\lim_{\tau\to\infty}(y, Y) = (0, 0)$, thus y is nonincreasing to 0, and $\lim_{\tau\to\infty} y'(\tau) = 0$. Comparing to the proof of Proposition 3.3, we observe that (3.9) is no longer true because $\delta - \alpha < 0$. Nevertheless, for any small κ and for $\tau \geq \tau_\kappa$ large enough,

$$-(p-1)y'' + (\delta p - N)y' + (N - \delta - \kappa)\delta y \leq 0. \tag{4.12}$$

Let us fix $\kappa < N - \delta$. Since $\lim_{\tau\to\infty} y(\tau) = 0$, we can suppose that $y(\tau) \leq 1$ for $\tau \geq \tau_\kappa$. Then there exists $\mu_\kappa < \mu$, where μ defined at (3.10), with $\mu_\kappa = \mu + O(K)$, such that, for any $\varepsilon > 0$, the function $\tau \longmapsto \varepsilon + e^{-\mu_\kappa(\tau-\tau_\kappa)}$ is a solution of the corresponding equation on $[\tau_\kappa, \infty)$. It follows that $y(\tau) \leq \varepsilon + e^{-\mu_\kappa(\tau-\tau_\kappa)}$, from the maximum principle. Thus $y(\tau) \leq e^{-\mu_\kappa(\tau-\tau_\kappa)}$ on $[\tau_\kappa, \infty)$. We can choose κ small enough such that $\mu_\kappa(3-p) \geq \mu^0 := \mu(4-p)/2 > \mu$. As a consequence, $y(\tau) \leq e^{-\mu^0(\tau-\tau_\kappa)/(3-p)}$; hence $y'(\tau) = O(e^{-\mu^0\tau/(3-p)})$, from Proposition 2.10. From (2.16) there exists $C > 0$ such that for $\tau \geq \tau_C$ large enough, $y(\tau) \leq 1$ and

$$-(p-1)y'' + (\delta p - N)y' + (N-\delta)\delta y \leq Ce^{-\mu^0\tau}.$$

There exists $A > 0$ such that $-Ae^{-\mu^0\tau}$ is a particular solution of the corresponding equation; then $\varepsilon + (1 + A)e^{-\mu(\tau-\tau_C)} - Ae^{-\mu^0(\tau-\tau_C)}$ is also a solution on $[\tau_\kappa, \infty)$. Thus $y(\tau) \leq \varepsilon + (1 + A)e^{-\mu(\tau-\tau_C)}$ on $[\tau_\kappa, \infty)$ from the maximum principle, and thus $y(\tau) \leq (1 + A)e^{-\mu(\tau-\tau_C)}$. Hence $y(\tau) = 0(e^{-\mu\tau})$, which means $w(r) = O(r^{(p-N)/(p-1)})$ near ∞. As in the proof of Proposition 3.3, $r^{\eta}w$ has a limit c at ∞, and that $c \neq 0$.

• Next assume that y is positive, but not monotone for large τ. Then there exists an increasing sequence (τ_n) of extremal points of y, such that $\tau_n \to \infty$, and (4.9) follows from Lemma 2.6. Assume $p \leq p_2$, or equivalently $2\delta \leq N$. The function W is nonincreasing; hence it has a limit $\Lambda \geq -\infty$. Computing at the point τ_n, where $Y(\tau_n) = (\delta y(\tau_n))^{p-1}$, we find

$$W(\tau_n) = (\alpha - \delta)(\frac{y(\tau_n)^2}{2} - \frac{\ell^{2-p} y(\tau_n)^p}{p}) + \frac{1}{q+1} e^{-\delta(q-1)\tau_n} |y(\tau_n)|^{q+1}$$
$$= (\alpha - \delta)(\frac{y(\tau_n)^2 (1 + o(1)}{2} - \frac{\ell^{2-p} y(\tau_n)^p}{p}),$$

thus $y(\tau_n)$ has a finite limit, necessarily equal to ℓ. Then $\lim_{\tau\to\infty} y(\tau) = \ell$.

(ii) Case $\alpha = \delta < N$. From Proposition 2.5 and Lemma 2.6, (i), (ii), w has a finite number of zeros, $\lim_{\tau\to\infty} y = 0$ or $\pm\infty$, and (4.6) holds. If $\lim_{\tau\to\infty} y = \infty$, we write

$$(y - Y)' + e^{\delta(q-1)\tau} |y|^{q-1} y$$
$$= (N - \delta) Y^{1/(p-1)} Y^{-(2-p)/(p-1)} = (N - \delta)(\delta y - y') Y^{-(2-p)/(p-1)}$$

and $Y^{1/(p-1)} < \delta y$, hence for large τ,

$$(y - Y)' + (N - \delta) Y^{-(2-p)/(p-1)} y' \geq y^{p-1}((N - \delta)\delta^{p-1} - y^{2-p} e^{\delta(q-1)\tau} y^{q-1}.$$

Since $y' \geq 0$, and $\lim_{\tau\to\infty} Y = \infty$, for any $\varepsilon > 0$ and for large τ,

$$(y - Y)' + \varepsilon y' \geq y^{p-1}((N - \delta)\delta^{p-1} - e^{\delta(q-1)\tau} y^{q+1-p})$$

and $y(\tau) = O(\tau^{1/(2-p)})$ from (4.5). Thus for any $\varepsilon > 0$ and for large τ,

$$((1 + \varepsilon)y - Y)' \geq (N - \delta)\delta^{p-1}(1 - \varepsilon) y^{p-1}.$$

Setting $\xi = (1 + \varepsilon)y - Y$, we deduce that

$$\xi' \geq (N - \delta)\delta^{p-1}(1 - 2\varepsilon)\xi^{p-1}$$

for large τ, which leads to the lower bound

$$y^{2-p}(\tau) \geq (N - \delta)\delta^{p-1}(2 - p)(1 - 3\varepsilon)\tau, \tag{4.13}$$

and (4.10) follows from (4.6) and (4.13). If $\lim_{\tau\to\infty} y = 0$, (4.8) follows as in case (i).

(iii) Case $\alpha = \delta = N$. From Proposition 4.1, y and Y are bounded. Moreover $Y - y$ has a finite limit K, and $Y - y = K + O(e^{-(q-1)\tau})$. And y has a finite limit l from Lemma 2.6, (i), (ii). Assume that $l = 0$. Then $\lim_{\tau\to\infty} y' = -|K|^{(2-p)/(p-1)} K$, and hence $K = 0$. Thus there exists $C > 0$ such that $y' = Ny - Y^{1/(p-1)} \geq Ny/2 - Ce^{-(q-1)\tau/(p-1)}$ for large τ. This implies $y = O(e^{-\gamma_0 t})$ with $\gamma_0 = e^{-(q-1)\tau/(p-1)}$. Assuming that $y = O(e^{-\gamma_n t})$, then $(Y - y)' = O(e^{-(q-1)\tau} y^q) = O(e^{-(q-1+q\gamma_n)\tau})$, hence $Y = y + O(e^{-(q-1+q\gamma_n)\tau})$. Then there exists another $C > 0$ such that $y' \geq Ny/2 - Ce^{-(q-1+q\gamma_n)\tau/(p-1)}$ for large τ, and $y = O(e^{-\gamma_{n+1} t})$, with $\gamma_{n+1} = (q - 1 + q\gamma_n)/(p - 1)$. Observe that $\lim \gamma_n = \infty$; thus $y = O(e^{-\gamma t}))$ and $w = O(r^{-\gamma})$ for any $\gamma > 0$. We get a contradiction as in Proposition (3.3) by using the substitution (2.11) with $d > N$. ∎

4.2 Oscillation or nonoscillation criteria

As a consequence of Proposition 4.1, we get a first result of existence of oscillating solutions.

Proposition 4.5 *Assume $q > 1, p < 2$, and $N \leq \delta < \alpha$ or $N < \delta = \alpha$. Then for any $m > 0$, any solution $w \not\equiv 0$ of problem (1.10) has a infinite number of zeros in $[m, \infty)$.*

Proof. Suppose that is is not the case. Let $w \not\equiv 0$, with, for example, $w > 0$ and $w' < 0$ near ∞, and hence $y > 0$ and $Y > 0$ for large τ. If $N < \delta = \alpha$, or $N < \delta = \alpha$, then y is bounded from Proposition 4.1. From Lemma 2.6, y is monotone, and $\lim_{\tau\to\infty}(y(\tau), Y(\tau)) = (0,0)$. As in (3.8), if $N < \delta$, then y is concave for large τ, and we reach a contradiction. If $\delta = N < \alpha$, we find

$$(y - Y)' = (N - \alpha)y - e^{-\delta(q-1)\tau} |y|^{q-1} y \leq 0;$$

then $y - Y$ is non increasing to 0, hence $y \geq Y$, $Y' \geq NY - Y^{1/(p-1)} \geq NY/2$ for large τ, which is impossible since $\lim_{\tau\to\infty} Y(\tau) = 0$. ∎

Next we study the case where $\delta < \min(\alpha, N)$. Recall that $\delta < N \Leftrightarrow p < p_1$. This case is difficult because the solutions could be oscillatory, and even if they are not, they have three possible types of behaviour near ∞ : (4.7), (4.8), or (4.9). Here we extend to equation (1.10) a difficult result obtained in ([5]) for equation (1.12). Recall that for system (2.18), if $\alpha < \eta$, there exist no solution satisfying (4.9), and for some $\alpha \in (\eta, \alpha^*)$ there do exist positive solutions satisfying (4.9).

Theorem 4.6 *Assume $p_2 < p < p_1$ and $\delta < \alpha$. If $\alpha < \eta$, (in particular if $\alpha \leq N$), then any solution $w(.,a)$ $(a \neq 0)$ has a finite number of zeros and satisfies (4.7) or (4.8).*

Proof. Assume $\alpha < \eta$. From Proposition 2.5, (iv), any solution $w \not\equiv 0$ has a finite number of zeros. We can assume that $w(.,a)$ and $w'(.,a) < 0$ for large r, from Proposition 2.4. Consider the corresponding trajectory $\mathcal{T}_n$ of the nonautonomous system (2.17) in the phase plane (y, Y). From Proposition (4.1) it is bounded near ∞. Let Γ be the limit set of $\mathcal{T}_n$ at ∞. Then $y \geq 0$ and $Y \geq 0$ for any $(y, Y) \in \Gamma$. From [19], Γ is nonempty, compact and connected, and for any point $P_0 \in \Gamma$, the positive trajectory $\mathcal{T}_a$ of the autonomous system (2.18) issued from P_0 at time 0 is contained in Γ. From [5, Theorem 5.4] we have a complete description of the solutions of system (2.18) when $\alpha < \eta$. Since $\delta < N$, the point $(0,0)$ is a saddle point; since $\alpha < \alpha^*$, the point M_ℓ is a sink. The only possible trajectories of (2.18) ending up in the set $y \geq 0$,$Y \geq 0$ are either the points $0, M_\ell$, or a trajectory $\mathcal{T}_{a,s}$ starting from ∞ and ending up at 0, or trajectories $\mathcal{T}_a$ ending up at M_ℓ. And $\mathcal{T}_{a,s}$ does not meet the curve

$$\mathcal{M} = \left\{(\lambda, (\delta\lambda)^{p-1}) : \lambda > 0\right\}.$$

Then either $\Gamma = \{0\}$, or $\Gamma = \{M_\ell\}$, or Γ contains some point P_0 of $\mathcal{T}_{a,s}$, or $\mathcal{T}_a$, and hence also the part of $\mathcal{T}_{a,s}$ or $\mathcal{T}_a$ issued from P_0. If $\Gamma = \{M_\ell\}$ or $\{0\}$, the trajectory converges to this point. If it is not the case, then y is not monotonous, and there exists a sequence of extremal points of y, such that $(y, Y) \in \mathcal{M}$. Let P_0 be one of these points. Then $P_0 \notin \mathcal{T}_{a,s}$, thus the autonomous trajectory going through P_0 converges to M_ℓ. Then Γ also contains M_ℓ. Hence there exists a sequence (τ_n) tending to ∞ such that $(y(\tau_n), Y(\tau_n))$ converges to M_ℓ. Next we consider again the energy function W defined at (2.21), and still use the notations $\mathcal{W}, \mathcal{U}, \mathcal{H}, \mathcal{L}, \mathcal{S}$ of Section 2.1. Since $\alpha < \alpha^*$, the point M_ℓ is exterior to the set $\mathcal{S}$. Thus

$$\lim W(\tau_n) = \mathcal{W}(M_\ell) = k_\ell < 0,$$

from (3.28). As, here $\delta < N$, $k_\ell = \min_{(y,Y)\in\mathbb{R}^2} \mathcal{W}(y, Y)$, and for large n, $(y(\tau_n), Y(\tau_n))$ is exterior to $\mathcal{S}$, then $\mathcal{U}(y(\tau_n), Y(\tau_n)) < 0$, and $W'(\tau_n) < 0$. Either W is monotone for large τ, in which case $\lim_{\tau\to\infty} W(\tau) = k_\ell$, and thus $\lim_{\tau\to\infty} \mathcal{W}(\tau) = k_\ell$, which implies $\lim_{\tau\to\infty}(y(\tau), Y(\tau)) = M_\ell$, and the trajectory converges to M_ℓ. Or there exists another sequence (s_n) of minimal points of W such that $s_n > \tau_n$ and $W(s_n) < W(\tau_n)$. Then $k_\ell \leq \liminf \mathcal{W}(s_n) \leq \limsup \mathcal{W}(s_n) = \limsup W(s_n) \leq k_\ell$. Thus also $\lim_{\tau\to\infty}(y(s_n), Y(s_n)) = M_\ell$. But

$$0 = W'(s_n) < \mathcal{U}(y(s_n), Y(s_n))$$

thus $(y(s_n), Y(s_n)) \in \mathcal{S}$, which is contradictory. Hence $\Gamma = \{M_\ell\}$ or $\{0\}$, and w satisfies (4.7) or (4.8) from Proposition (4.4). ∎

Remark 4.7 If $\alpha > \alpha^*$, the regular solutions of system (2.18) are oscillatory, see [5, Theorem 5.8]. We cannot prove the same result for equation (1.10), since it is a global problem, and system (2.17) is only a perturbation of (2.18) near infinity; and the use of the energy function W does not allow us to reach the conclusion.

4.3 Existence of positive solutions

From Theorem 4.6, we first prove the existence of positive solutions, and their decay can be qualifieed as slow among the possible behaviours given at Proposition 4.4:

Proposition 4.8 *Assume $\delta \leq \alpha < N$. Let $\underline{a} > 0$ be defined at Proposition 2.5. Then for any $a \in (0, \underline{a}]$, and $w(r, a) > 0$ on $[0, \infty)$ and satisfies (4.7) if $\delta < \alpha$, or (4.10) if $\alpha = \delta$.*

Proof. We still have $w(r, a) > 0$ from Proposition 2.5, and J_N is nondecreasing and $J_N(0) = 0$. If the conclusions were not true, then $w(r) = O(r^{-\eta})$, from Theorem 4.6, then $r^N w = O(r^{N-\eta})$, and $N < \eta$ from (1.6). Then $\limsup_{r\to\infty} J_N(r) \leq 0$, and we reach a contradiction as in Proposition 3.8. ∎

Next we show the existence of positive solutions with a (faster) decay in $r^{-\eta}$ in the subcritical case:

Theorem 4.9 *Assume $p < 2$, $\delta < \alpha < N$, and $1 < q < q^*$. Then there exists $a > 0$ such that $w(., a)$ is positive and satisfies $\lim_{r\to\infty} r^\eta w = c \neq 0$.*

Proof. Let

$$A = \left\{ a > 0 : w(.,a) > 0 \text{ on } (0,\infty) \text{ and } \lim_{r\to\infty} r^\delta w = \ell \right\},$$

$$B = \{a > 0 : w(.,a) \quad \text{has at least an isolated zero}\}.$$

Then A and B are nonempty by Propositions 4.8, 2.7, and $A \supset (0, \underline{a}]$, $B \supset [\overline{a}, \infty)$, and B is open. Now we show that A is open. Let $a_0 \in A$. Then $J_N(., a_0)$ is increasing for large r and tends to ∞. Hence $J_N(r_0, a_0) > 0$ and $J_N'(r_0, a_0) > 0$ for r_0 large enough; and then there exists a neighborhood $\mathcal{V}$ of a_0 such that $w(r,a) > 0$ on $[0, r_0]$ and $J_N(r_0, a) > 0$ and $J_N'(r_0, a) > 0$ for any $a \in \mathcal{V}$. Then $J_N'(r_0, a) > 0$ for any $r \geq r_0$, since $w(.,a)$ is decreasing. Then for any $a \in \mathcal{V}$, from Propositions 4.4 and 2.10, either $\lim_{r\to\infty} r^\eta w = c > 0$, and $\lim_{r\to\infty} r^{\eta+1} w' = -c\eta$, from (2.14) and (2.13) with $d = \eta$; then $\lim_{r\to\infty} J_N(.,a) = -c^{p-1}$, which is impossible. Or necessarily $\lim_{r\to\infty} r^\delta w(.,a) = \ell$; thus $a \in A$. Let $a_{\inf} = \inf B > \underline{a}$ and $a_{\sup} = \sup A < \overline{a}$. Taking $a = a_{\inf}$ or $a_{\sup}$, then $w(.,a)$ is positive and $\lim_{r\to\infty} r^\eta w = c$. ■

Remark 4.10 Under the assumptions of theorem 4.9, any solution $w(.,a)$ $(a \neq 0)$ has a finite number of zeros, and $\lim_{r\to\infty} r^\delta w(.,a) = \Lambda(a)$, with $\Lambda(a) = \pm\ell$ or 0. Here the function Λ is not continuous on $(0,\infty)$. Indeed it would imply that the set $\{a > 0 : \Lambda(a) = \ell\}$ is closed and open in $(0,\infty)$, and non empty, which contradicts the above results.

At last, in the supercritical case, we show the existence of grounds states for any $a > 0$, and they have a (slow) decay:

Theorem 4.11 *Assume $\delta \leq \alpha$. Let $w(r,a)$ be the solution of problem (1.10), (1.15).*
(i) If $p \leq p_2$, then for any $a > 0$, $w(r,a) > 0$ on $[0,\infty)$ and (4.7) or (4.10) holds.
(ii) If $p_2 < p < p_1$, $\alpha < (N-1)p'/2$, and $q \geq q_\alpha^* > q^*$, where q_α^* is given by (1.14), then again $w(r,a) > 0$ on $[0,\infty)$ and (4.7) or (4.10) holds.

Proof. We consider again the function $V = V_{\lambda,\sigma,e}$ defined in (2.9).

(i) Suppose $p \leq p_2$. As in Theorem 3.11, (ii), we take $\lambda = N = 2\sigma$ and $e = \alpha - N/2$. Then $V' \leq 0$ from (3.25) and in the same way $w(r) > 0$ on $[0,\infty)$. From Proposition (4.4), if (4.7) does not hold, then $w = O(r^{-\eta})$, $w' = O(r^{-(\eta+1)})$ near ∞. Then by computation, $V(r) = O(r^{-\eta})$, and thus $\lim_{r\to\infty} V(r) = 0$. Then $V \equiv 0$ on $[0,\infty)$, which is contradictory.

(ii) Suppose $p_2 < p < p_1$, and $\alpha < (N-1)p'/2$. As in Theorem 3.11 (ii) we take $\lambda = 2\alpha$ and $\sigma = N - 1 - 2\alpha/p'$ and $e = \sigma - \alpha$. Observe that $\alpha < \eta$. Thus from Theorem 4.6, if (4.7) does not hold, then again $w = O(r^{-\eta})$, $w' = O(r^{-(\eta+1)})$ near ∞. Then by computation, $V(r) = O(r^{2\alpha-(N-1)p'})$ near ∞, hence $\lim_{r\to\infty} V(r) = 0$ and we reach again a contradiction. ■

5 Back to problem (1.1)

Here we apply to equation (1.4) the results of Section 3, with $\alpha = \alpha_0 = p/(q+1-p)$, and show our main result.

Proof. [*Proof of Theorem 1.1*] One has $\alpha_0 > 0$ since $q > p-1$, and (3.1) holds since $q > 1$.
(i) The existence and behaviour of w follows from Theorem 2.1 and Proposition 3.1.

(ii) Condition $q_1 < q$ is equivalent to $\alpha_0 < N$, and Proposition 3.8 applies.

(iii) If $q_1 < q < q^\star$, then Theorem 3.9 shows the existence of fast nonnegative decaying solutions w. For any $s \geq 1$, there exists $C > 0$ such that for any $t > 0$,

$$\left\|u(t)\right\|_s = Ct^{(N/s\alpha_0 - 1)/(q-1)} \left\|w\right\|_s . \tag{5.1}$$

If $p > 2$, then w has a compact support thus $u(t) \in L^s(\mathbb{R}^N)$. If $p < 2$, then u is positive, and from Proposition (3.3), w satisfies (1.1), with $\ell(N,p,q)$ and $\rho(N,p,q)$ given by (3.5) and (3.7) with $\alpha = \alpha_0$:

$$\ell(N,p,q) = \left(\delta^{p-1}\frac{\delta - N}{\delta - \alpha_0}\right)^{1/(2-p)} \qquad \rho(N,p,q) = \frac{1}{N}\left(\frac{N(N-1)}{2(N-\alpha_0)}\right)^{(N+1)/2};$$

hence again $u(t) \in L^s(\mathbb{R}^N)$. Indeed either $p_1 < p$; and thus $N < \delta$, $w = O(r^{-\delta})$ at ∞, and $\int_1^\infty r^{N-1-\delta s}dr < \infty$; or $p < p_1$, and thus $w = O(r^{-\eta})$, $N < \eta$, and $\int_1^\infty r^{N-1-(N-p)s/(p-1)}dr < \infty$; or $p = p_1$, and $w = O(r^{-N}(\ln r)^{-(N+1)/2})$, and $\int_1^\infty r^{N-1-Ns}(\ln r)^{-(N+1)/2}dr < \infty$. Moreover $\lim_{t\to 0}\left\|u(t)\right\|_s = 0$ whenever $s > N/\alpha_0$, from (5.1). For fixed $\varepsilon > 0$, by Proposition 3.2, either $p > 2$ and $\sup_{|x|\geq\varepsilon}|u(x,t)| = 0$ for $t \leq t(\varepsilon)$ small enough, or $p < 2$ and $\sup_{|x|\geq\varepsilon}|u(x,t)| \leq C(\varepsilon)t^{(\delta/\alpha_0 - 1)/(q-1)}$ for $t \leq t(\varepsilon)$ small enough, and $\alpha_0 < \delta$; hence in any case, $\lim_{t\to 0}\sup_{|x|\geq\varepsilon}|u(x,t)| = 0$.

(iv) The assertions follow from Theorem 3.6 (ii) and (iii), and from Proposition 3.3.

(v) Here we applyTheorem 3.11 (i) and (ii). Indeed if $p > p_2$, and $q \geq q^\star$, then $\alpha_0 \leq (N-p)/p < N/2$.

(vi) If $1 < q \leq q_1$, then $N < \delta$ and $N \leq \alpha_0$. Hence all the solutions w are changing sign, from Proposition 2.5, (ii); and there exists an infinity of fast decaying solutions w, from Theorem 3.6 (ii); the estimates follow from Proposition 3.2. Moreover in the case $p < 2$, from Theorem 3.15, w has a finite number of zeros if α_0 is not too large, in particular if $\alpha_0 < \alpha^*$,where α^* is defined at (1.14) ($\alpha^* < \delta$), which means $1 < p - 1 + p/\alpha^* < q \leq q_1$.This requires $N < \alpha^*$, which means that p is sufficiently close from 2 , more precisely $(2p-3)p > N(2-p)(p-1)$, in particular $p > 3/2$). On the contrary, there exists $\bar{\alpha} \in (\max(N,\alpha^*),\delta)$ such that w is oscillatory if $\alpha_0 > \bar{\alpha}$; hence $1 < q < p - 1 + p/\bar{\alpha}$. ■

Remark 5.1 If $q = q_1$, then $\alpha_0 = N$. Thus for each of these functions w, there exists $C \in \mathbb{R}$ such that the corresponding function u satisfies $\int_{\mathbb{R}^N} u(t)dx = C \int_{\mathbb{R}^N} wdx$, and $\|u(t)\|_1 = |C| \, \|w\|_1$ for any $t > 0$; then there exists a sequence $(t_n) \to 0$ such that $u(t_n)$ converges weakly to a bounded measure μ in $\mathbb{R}^N$. We still have $\lim_{t\to 0} \sup_{|x|\geq\varepsilon} |u(x,t)| = 0$, hence μ has its support at the origin; we cannot assert that μ is a Dirac mass as in the case $p = 2$, see [26], since we have no uniqueness result for equation 1.1, inasmuch as u does not have a constant sign.

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