

# Periodic Solutions of Equations of Emarkov-Pinney Type

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## Abstract

After establishing a relation between the Hill's equations and the Emarkov-Pinney equations, we are able to use degree theory to remove a technical assumption in [14], and give a complete set of non-resonance conditions for differential equations with repulsive singularities.

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## 1 Introduction

The main aim of this paper is to remove a technical assumption in the non-resonance results in [14] and to obtain a complete set of the non-resonance conditions for differential equations with repulsive singularities. In doing so, a nice relation between the Hill's equations and the Emarkov-Pinney equations will be established. This relation itself is useful in studying the stability of periodic solutions of Lagrangian systems of degree of freedom of  $3/2$ . Compared with the proof in [14], here we use quite a different technique in computing the degree. This is accomplished by using the relation established.

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Some historical background on the problem studied in [14] is as follows. It is well-known that the second order ordinary differential equations with singularities model many problems in the applied sciences. The examples are the Brillouin focusing system and the motion of an atom near a charged wire. See also the references in [14]. Mathematically, the equation is

$$\ddot{x} + f(t, x) = 0, \quad (1.1)$$

where the nonlinear forcing  $f = f(t, x) : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$  is continuous and has some singularity, say at 0. That is,  $f(t, x) \rightarrow \infty$  as  $x \rightarrow 0$ . When  $f(t, x)$  is  $T$ -periodic in  $t$ , it is an interesting mathematical problem to study the existence of positive  $T$ -periodic solutions of (1.1). In this decade, we have seen some important progress on this problem. In my opinion, the following results are worth being mentioned.

In two classical papers [5, 3], the authors gave the necessary and sufficient conditions for the existence of (strictly) positive periodic solutions for the case  $f = \pm x^{-\alpha} - h(t)$ ,  $\alpha \geq 1$ .

We say that  $f(t, x)$  has the repulsive singularity at 0 and has semilinear growth at  $+\infty$ , if  $f(t, x)$  satisfies  $\lim_{x \rightarrow 0+} f(t, x) = -\infty$ , and

$$\phi(t) \leq \liminf_{x \rightarrow +\infty} f(t, x)/x \leq \limsup_{x \rightarrow +\infty} f(t, x)/x \leq \Phi(t) \quad (S_\infty)$$

uniformly with respect to  $t$ , for some continuous  $T$ -periodic functions  $\phi(t)$ ,  $\Phi(t)$ .

Under the following so-called *strong force condition* at 0: There exist constants  $c_1, c_2 > 0$  and  $\nu \geq 1$  such that

$$-c_1 x^{-\nu} \leq f(t, x) \leq -c_2 x^{-\nu} \quad \forall t, \forall 0 < x \ll 1, \quad (S_0)$$

some important progresses on the existence conditions for positive  $T$ -periodic solutions of (1.1) are as follows.

- In 1992, del Pino, Manásevich and Montero obtained the following so-called *Uniform Non-resonance Condition* in [2]: There exists  $k \in \mathbb{N}$  such that

$$((k-1)\pi/T)^2 < \phi(t) \leq \Phi(t) < (k\pi/T)^2 \quad \forall t. \quad (U_k)$$

- In 1996 and 1998, Zhang has generalized in [15, 16] the uniform non-resonance condition (with  $k = 1$ ) to the *nonuniform* case. The corresponding condition reads as

$$-\bar{\phi} < 0 \quad \text{and} \quad \underline{\lambda}_1(\Phi) > 0. \quad (N_1)$$

Here  $\bar{\phi}$  is the mean value and  $\{\underline{\lambda}_n(q)\}_{n \geq 1}, \{\bar{\lambda}_n(q)\}_{n \geq 0}$  are the  $2T$ -periodic eigenvalues of

$$\ddot{x} + (\lambda + q(t))x = 0.$$

Note that

$$-\infty < \bar{\lambda}_0(q) < \underline{\lambda}_1(q) \leq \bar{\lambda}_1(q) < \cdots < \underline{\lambda}_n(q) \leq \bar{\lambda}_n(q) < \cdots$$

and  $\underline{\lambda}_n(q), \bar{\lambda}_n(q) \rightarrow +\infty$ . See [7, 17].

• For the higher order uniform non-resonance conditions (with  $k \geq 2$ ) in [2], under a technical assumption (G) described below, Yan and Zhang [14] have obtained in 2003 the following nonuniform non-resonance condition on  $\phi(t)$  and  $\Phi(t)$ : There exists  $k \in \mathbb{N}$  such that

$$\bar{\lambda}_{k-1}(\phi) < 0 \quad \text{and} \quad \underline{\lambda}_k(\Phi) > 0. \quad (N_k)$$

The technical assumption on  $\phi(t)$  and  $\Phi(t)$  in [14] is: There exists a positive constant  $\gamma \in \mathbb{R}$  such that

$$\phi(t) \leq \gamma \leq \Phi(t) \quad \forall t. \quad (G)$$

It is easy to see that if  $(U_k)$  is satisfied, then  $(N_k)$  and the technical assumption (G) are also satisfied. Hence the results in [14] are generalizations of that in [2]. It should be mentioned that some explicit conditions for which  $(N_k)$  holds have been constructed recently by the author [18]. In addition to these works, there have been several papers on this problem in the last couple years. See [4, 9, 10, 11, 12, 13].

In this paper, we finally find that the technical assumption (G) is unnecessary for this problem and we give a complete generalization of the uniform non-resonance conditions  $(U_k)$  in [2] to the nonuniform cases.

The main idea to overcome this is a relation between the Hill's equations

$$\ddot{x} + a(t)x = 0, \quad (1.2)$$

and the corresponding Emarkov-Pinney equations [8]:

$$\ddot{r} + a(t)r = r^{-3}. \quad (1.3)$$

That is, when (1.2) is elliptic, equation (1.3) has a unique positive  $T$ -periodic solution which is also non-degenerate. See Theorems 2.1 and 2.2. Based on this fact, one can find that under condition  $(N_k)$ , (1.1) can be deformed to (1.3) so that all possible  $T$ -periodic solutions of (1.1) are *a priori* bounded. As a consequence of degree theory, (1.1) has at least one positive  $T$ -periodic solution. See Theorems 3.1 and 3.2.

As the Emarkov-Pinney equations satisfy  $(S_0)$  and  $(S_\infty)$ , that is why we refer the singular equations considered in this paper to the Emarkov-Pinney type equations.

## 2 A relation between Hill's equations and Emarkov-Pinney equations

We begin with a connection between the Hill's equation (1.2) and the Emarkov-Pinney equation (1.3), where  $a(t) \in C(\mathbb{R}/T\mathbb{Z})$  (the space of all continuous,  $T$ -periodic functions).

For (1.2), the Poincaré matrix is

$$M_T = \begin{pmatrix} \psi_1(T) & \psi_2(T) \\ \dot{\psi}_1(T) & \dot{\psi}_2(T) \end{pmatrix},$$

where  $\psi_i(t)$  are solutions of (1.2) satisfying  $\psi_1(0) = \dot{\psi}_2(0) = 1$  and  $\dot{\psi}_1(0) = \psi_2(0) = 0$ , respectively. The eigenvalues  $\mu_{1,2}$  of  $M_T$  are then called the *Floquet multipliers* of (1.2).

They satisfy  $\mu_1 \cdot \mu_2 = 1$ . Equation (1.2) is called *elliptic*, *hyperbolic*, or *parabolic*, if  $|\mu_{1,2}| = 1$  but  $\mu_{1,2} \neq \pm 1$ ,  $|\mu_{1,2}| \neq 1$ , or  $\mu_1 = \mu_2 = \pm 1$ , respectively. From the theory of Hill's equations, it is well-known that (1.2) is stable in the sense of Lyapunov only if (1.2) is elliptic, or is parabolic ( $\mu_1 = \mu_2 = \pm 1$ ) with further property that all solutions of (1.2) satisfy  $x(t+T) \equiv x(t)$ , the  $T$ -periodic solutions in case  $\mu_1 = +1$ , or  $x(t+T) \equiv -x(t)$ , the  $T$ -anti-periodic solutions in case  $\mu_1 = -1$ . See, e.g., [7].

It is well-known that there is a correspondence between the solutions of (1.2) and that of (1.3). Since our result for topological degree is obtained from this connection, we provide the details.

Let  $\varphi_1(t)$  and  $\varphi_2(t)$  be linearly independent solutions of (1.2). Set

$$\varphi_1(t) + i\varphi_2(t) = R(t) \exp(i\varphi(t)). \quad (2.1)$$

Then  $R(t)$  and  $\varphi(t)$  are real functions and  $R(t) > 0$  for all  $t$ . It follows from (1.2) that  $R(t)$  and  $\varphi(t)$  satisfy

$$\ddot{R} + a(t)R - R\dot{\varphi}^2 = 0, \quad R\ddot{\varphi} + 2\dot{R}\dot{\varphi} = 0. \quad (2.2)$$

From the second equation, one has

$$\dot{\varphi} = cR^{-2}, \quad (2.3)$$

where  $c = \dot{\varphi}(0)R^2(0)$  which is always nonzero because  $\varphi_1(t)$  and  $\varphi_2(t)$  are linearly independent. In fact, from (2.1) one has

$$R^2(0) = \varphi_1^2(0) + \varphi_2^2(0),$$

and

$$\dot{\varphi}(0) = \frac{\varphi_1(0)\dot{\varphi}_2(0) - \varphi_2(0)\dot{\varphi}_1(0)}{\varphi_1^2(0) + \varphi_2^2(0)}.$$

Hence

$$c = \dot{\varphi}(0)R^2(0) = \varphi_1(0)\dot{\varphi}_2(0) - \varphi_2(0)\dot{\varphi}_1(0) \neq 0. \quad (2.4)$$

Substituting (2.3) into the first equation in (2.2), we know that  $R(t)$  satisfies the following equation

$$\ddot{R} + a(t)R = c^2R^{-3}.$$

Re-scaling  $R(t)$  by

$$r(t) = |c|^{-1/2}R(t), \quad (2.5)$$

we know that  $r(t)$  satisfies the Emarkov-Pinney equation (1.3).

Conversely, suppose that  $r(t)$  is a (positive) solution of (1.3). Define  $\varphi(t)$  by

$$\varphi(t) = \int_0^t \frac{ds}{r^2(s)}.$$

Then all solutions of (1.2) are given by

$$x = c_1 r(t) \cos \varphi(t) + c_2 r(t) \sin \varphi(t) = A r(t) \sin(\varphi(t) + B), \quad c_1, c_2, A, B \in \mathbb{R}. \quad (2.6)$$

Using this connection, we will establish an important relation between the existence of (positive)  $T$ -periodic solution of (1.3) and the fundamental structure of (1.2). A preliminary version of the following fact was found in [6] in studying the twist character and the Lyapunov stability of periodic solutions of Lagrangian systems of degree of freedom of  $3/2$ .

**Theorem 2.1** *The following assertions are equivalent:*

- (i) *The Emarkov-Pinney equation (1.3) has a (positive)  $T$ -periodic solution.*
- (ii) *The Hill's equation (1.2) is either elliptic or parabolic with all solutions being  $T$ -periodic or all being  $T$ -anti-periodic.*
- (iii) *The Hill's equation (1.2) is stable in the sense of Lyapunov.*

*Moreover, the  $T$ -periodic solution of (1.3) is unique when (1.2) is elliptic.*

*Proof.* It is well-known that (ii) is equivalent to (iii). See [7]. Let us prove that (ii) implies (i). First we assume that (1.2) is elliptic. Let  $\mu \in S^1 \setminus \{\pm 1\}$  be a Floquet multiplier, that is,  $\mu$  is an eigenvalue of  $M_T$ . Let  $v = (v_1, v_2)^T \in \mathbb{C}^2$  be an eigenvector associated with  $\mu$ :

$$M_T v = \mu v.$$

Let  $z(t) = \varphi_1(t) + i\varphi_2(t)$  be the (complex-valued) solution of (1.2) with initial data:  $z(0) = v_1, \dot{z}(0) = v_2$ . Then  $z(t) \neq 0$  for all  $t$  and satisfies

$$z(t+T) \equiv \mu z(t). \quad (2.7)$$

(This is the so-called Floquet solution of (1.2) corresponding to  $\mu$ .) Since  $\mu$  is not a real, one can deduce that  $\varphi_1$  and  $\varphi_2$  are linearly independent real solutions of (1.2). Let  $R(t) = |z(t)|$ . Then  $R(t) > 0$  for all  $t$  and, by (2.7),  $R(t+T) \equiv R(t)$ . Now it follows from the constructions (2.1)-(2.4)-(2.5) that  $r(t) = cR(t)$  is a positive  $T$ -periodic solution of (1.3), where

$$c = \frac{1}{\sqrt{|\varphi_1(0)\dot{\varphi}_2(0) - \varphi_2(0)\dot{\varphi}_1(0)|}}. \quad (2.8)$$

Next we assume that (1.2) is parabolic with Floquet multipliers  $\mu = \mu_{1,2} = +1$  or  $-1$  and all solutions of (1.2) satisfying  $x(t+T) \equiv \mu x(t)$ . Then any set of linearly independent solutions  $\varphi_1(t)$  and  $\varphi_2(t)$  of (1.2) can yield a  $T$ -periodic solution of (1.3) which is given by  $r(t) = c(\varphi_1^2(t) + \varphi_2^2(t))^{1/2}$  with a suitable choice of  $c > 0$ . In this case, (1.3) has infinitely many  $T$ -periodic solutions. These prove that (ii)  $\Rightarrow$  (i) holds.

Now we prove that (i)  $\Rightarrow$  (iii) holds. If (1.3) has a  $T$ -periodic solution  $r(t)$ , it follows from (2.6) that each solution of (1.2) is bounded. This means that (1.2) is stable.

Finally, we prove the uniqueness of the  $T$ -periodic solution of (1.3) when (1.2) is elliptic. Let  $r_j(t)$ ,  $j = 1, 2$ , be  $T$ -periodic solutions of (1.3). Define

$$\varphi_j(t) = \int_0^t \frac{ds}{r_j^2(s)}, \quad j = 1, 2.$$

Then

$$\varphi_j(t+T) \equiv \varphi_j(t) + \theta_j, \quad j = 1, 2,$$

where  $\theta_j > 0$ . By (2.6),  $x_j(t) = r_j(t) \exp(i\varphi_j(t))$  are (complex-valued) solutions of (1.2) and satisfy

$$x_j(t+T) \equiv \exp(i\theta_j)x_j(t), \quad j = 1, 2.$$

This means that  $x_j(t)$  are Floquet solutions of (1.2) with the multipliers  $\exp(i\theta_j)$ ,  $j = 1, 2$ , respectively. Therefore, we have either  $\exp(i\theta_2) = \exp(i\theta_1)$  or  $\exp(i\theta_2) = \exp(-i\theta_1)$ . From the construction of Floquet solutions, we know that there exists  $b \in \mathbb{C} \setminus \{0\}$  such that either  $x_2(t) \equiv bx_1(t)$  or  $x_2(t) \equiv \overline{bx_1(t)}$ , where the bar denotes the complex conjugate. Letting  $t = 0$  in these equalities, we get  $r_2(0) = br_1(0)$  in both cases. Thus  $b$  is a positive number. Now we know that (1.3) has  $T$ -periodic solutions  $r_1(t) = |x_1(t)|$  and  $r_2(t) = |x_2(t)| = br_1(t)$ . That is,

$$\ddot{r}_1 + a(t)r_1 = 1/r_1^3,$$

and

$$b\ddot{r}_1 + ba(t)r_1 = 1/(b^3r_1^3).$$

From these, one sees that  $b$  must be 1. Thus  $r_1(t) = r_2(t)$ . This proves the uniqueness.

Now we give a further property of the  $T$ -periodic solution of (1.3) when (1.2) is elliptic.

**Theorem 2.2** *Suppose that the Hill's equation (1.2) is elliptic. Then the unique positive  $T$ -periodic solution  $r(t)$  of the Emarkov-Pinney equation (1.3) is non-degenerate, i.e., the linearization equation of (1.3) along  $r(t)$*

$$\ddot{s} + (a(t) + 3r^{-4}(t))s = 0 \quad (2.9)$$

*has Floquet multipliers not equal to 1.*

*Proof.* We argue by contradiction. Suppose that there exists some elliptic  $a(t)$  such that (1.3) has a  $T$ -periodic solution  $r(t)$  and (2.9) has a nontrivial  $T$ -periodic solution  $s = \Psi(t)$ . Let

$$b(t) = -\Psi(t)/r(t) \in C(\mathbb{R}/T\mathbb{Z}).$$

Consider the following Hill's equation with a small parameter  $\varepsilon$ :

$$\ddot{x} + (a(t) + \varepsilon b(t))x = 0. \quad (2.10)$$

Since (1.2) is elliptic, (2.10) is also elliptic when  $\varepsilon$  is small. Consequently, the following Emarkov-Pinney equation

$$\ddot{r} + (a(t) + \varepsilon b(t))r = r^{-3} \quad (2.11)$$

has a unique  $T$ -periodic solution denoted by  $r = r(t; \varepsilon)$  when  $\varepsilon$  is small. Of course,  $r(t; 0) = r(t)$ .

We assert that  $r(t; \varepsilon)$  depends upon  $\varepsilon$  analytically when  $\varepsilon$  is small. Once this is proved, we let

$$u(t) = \left. \frac{\partial r(t; \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}.$$

Differentiating (2.11) with respect to  $\varepsilon$  and then evaluating at  $\varepsilon = 0$ , we conclude that  $u(t)$  satisfies

$$\ddot{u} + (a(t) + 3r^{-4}(t))u = -b(t)r(t) = \Psi(t). \quad (2.12)$$

Since  $\Psi(t)$  is a non-trivial  $T$ -periodic solution of (2.9), we obtain from (2.12) and from the Fredholm alternative principle that

$$\int_0^T \Psi^2(t)dt = 0,$$

which is impossible.

To complete the proof of the theorem, we need to prove the assertion. In the following we will follow the constructions in the proof of Theorem 2.1. Let  $M_T(\varepsilon)$  be the Poincaré matrix of (2.10). Then  $M_T(\varepsilon)$  is analytic in  $\varepsilon \in \mathbb{R}$ . Let  $\mu_{1,2}(\varepsilon)$  be eigenvalues of  $M_T(\varepsilon)$ . One may assume that  $\mu_i(\varepsilon)$  are continuous in  $\varepsilon$ . Note that  $\mu_i(\varepsilon)$  depend on  $\varepsilon$  only in a continuous way when  $\varepsilon$  is near those  $\varepsilon_0$  such that  $M_T(\varepsilon_0)$  is parabolic. However, as  $M_T(0)$  is elliptic, one knows that  $\mu_i(\varepsilon)$  are analytic in a small neighborhood of 0. One may then choose the eigenvectors  $v(\varepsilon)$  of  $M_T(\varepsilon)$  with the eigenvalues  $\mu_1(\varepsilon)$  such that

$$M_T(\varepsilon)v(\varepsilon) = \mu_1(\varepsilon)v(\varepsilon),$$

and  $v(\varepsilon)$  is analytic in  $\varepsilon$ . Next, the Floquet solutions  $z(t; \varepsilon) = \varphi_1(t; \varepsilon) + i\varphi_2(t; \varepsilon)$  of (2.10) with  $\mu = \mu_1(\varepsilon)$  are then analytical in  $\varepsilon$  in a small neighborhood of 0. Finally, the positive  $T$ -periodic solutions of (2.11) are  $r(t; \varepsilon) = c(\varepsilon)|z(t; \varepsilon)|$ , where

$$c(\varepsilon) = \frac{1}{\sqrt{|\varphi_1(0; \varepsilon)\dot{\varphi}_2(0; \varepsilon) - \varphi_2(0; \varepsilon)\dot{\varphi}_1(0; \varepsilon)|}}.$$

See (2.8). Since  $\varphi_i(t; \varepsilon)$  are analytic in  $\varepsilon$ , then so is  $c(\varepsilon)$ . Hence  $r(t; \varepsilon)$  is analytical in  $\varepsilon$  in a small neighborhood of 0. This finishes the proof of the assertion.

### 3 Non-resonance results

By the result in [14], the non-resonance condition  $(N_k)$  means that for any  $a(t) \in C(\mathbb{R}/T\mathbb{Z})$  with

$$\phi(t) \leq a(t) \leq \Phi(t) \quad \forall t,$$

the corresponding Hill's equation (1.2) is elliptic.

Now we state the main existence result for equation (1.1).

**Theorem 3.1** *Suppose that  $f(t, x)$  satisfies the strong force condition  $(S_0)$  near 0 and the semi-linear growth condition  $(S_\infty)$  near  $+\infty$ . Assume that  $\phi$  in  $(S_0)$  and  $\Phi$  in  $(S_\infty)$  satisfy  $\Phi(t) \geq \phi(t) > 0$  for all  $t$  and the nonuniform non-resonance condition  $(N_k)$  for some  $k \geq 2$ . Then equation (1.1) has at least one positive  $T$ -periodic solution.*

*Proof.* We will use the Leray-Schauder degree to prove the theorem. Since the nonlinearity in equation (1.1) is independent of  $\dot{x}$ , we work in the following space

$$C_T = \{x : [0, T] \rightarrow \mathbb{R} \text{ is continuous and } x(T) = x(0)\}.$$

One can transform the  $T$ -periodic solution problem to a fixed point problem in  $C_T$ . For example,  $x$  is a positive  $T$ -periodic solution if and only if  $x$  is a positive fixed point in  $C_T$  of the following operator

$$Tx(t) := x(0) + \int_0^T G(t, s)f(s, x(s))ds, \quad t \in [0, T], \quad (3.1)$$

where

$$G(t, s) = \begin{cases} \frac{s(T-t)}{T} & \text{if } 0 \leq s \leq t \leq T, \\ \frac{t(T-s)}{T} & \text{if } 0 \leq t \leq s \leq T. \end{cases}$$

Under condition  $(N_k)$ , the *a priori* estimates to possible positive  $T$ -periodic solutions are almost the same as in [2]. See also [14, Lemma 3] for a simpler treatment. Thus we give here only the sketch of the proof.

**Step 1.** Choose any  $\gamma(t) \in C(\mathbb{R}/T\mathbb{Z})$  such that  $\phi(t) \leq \gamma(t) \leq \Phi(t)$  for all  $t$ . Consider the linear homotopy between (1.1) and a simpler Emarkov-Pinney type equation (3.3) below:

$$\ddot{x} + f_\tau(t, x) = 0, \quad \tau \in [0, 1], \quad (3.2)$$

where  $f_\tau(t, x) = (1 - \tau)f(t, x) + \tau(\gamma(t)x - x^{-\nu})$ . It follows from [14, Lemma 3] that the possible positive  $T$ -periodic solutions of (3.2) are *a priori* bounded in the following sense: There exist  $0 < \varepsilon_0 \ll 1 \ll \varepsilon_\infty < \infty$  such that

$$\varepsilon_0 < x(t) < \varepsilon_\infty \quad \forall t$$

for all  $T$ -periodic solutions  $x(t)$  of (3.2). Hence (1.3) is a deformation of the Emarkov-Pinney type equation

$$\ddot{x} + \gamma(t)x = x^{-\nu}. \quad (3.3)$$

**Step 2.** We further deform (3.3) to the Emarkov-Pinney equation

$$\ddot{x} + \gamma(t)x = x^{-3} \quad (3.4)$$

using the homotopy along the singularity order:

$$\ddot{x} + g_\tau(t, x) = 0, \quad \tau \in [0, 1], \quad (3.5)$$

where  $g_\tau(t, x) = \gamma(t)x - x^{-((1-\tau)\nu+3\tau)}$ .

One can check that the estimates in [2, 14] do work also for (3.5) because the strong force condition for the singularity 0 is uniform:

$$1 \leq \min\{3, \nu\} \leq (1 - \tau)\nu + 3\tau \leq \max\{3, \nu\} \quad \forall \tau \in [0, 1].$$

Thus there exist  $0 < \hat{\varepsilon}_0 \leq \varepsilon_0 < 1 < \varepsilon_\infty \leq \hat{\varepsilon}_\infty < \infty$  such that

$$\hat{\varepsilon}_0 < x(t) < \hat{\varepsilon}_\infty \quad \forall t$$



for all possible  $T$ -periodic solutions  $x(t)$  of (3.5).

**Step 3.** Let  $\mathcal{T}_{ep}$  be the operator associated with equation (1.3). Following from (3.1), we know that

$$\mathcal{T}_{ep}x(t) := x(0) + \int_0^T G(t, s) (a(s)x(s) - 1/x^3(s)) ds, \quad t \in [0, T]. \quad (3.6)$$

Let the domain in  $C_T$  be

$$\hat{\Omega} = \{x \in C_T : \hat{\varepsilon}_0 < x(t) < \hat{\varepsilon}_\infty \forall t \in [0, T]\}.$$

By Steps 1 and 2, one has

$$\deg_{LS}(I - \mathcal{T}, \hat{\Omega}, 0) = \deg_{LS}(I - \mathcal{T}_{ep}, \hat{\Omega}, 0).$$

From Theorem 2.1,  $\mathcal{T}_{ep}$  has a unique positive fixed point, say  $x_0$ . It is evident that  $x_0 \in \hat{\Omega}$ .

We are now in a position to compute the Leray-Schauder degree of the operator  $\mathcal{T}_{ep}$ . It is easy to see that  $\mathcal{T}_{ep}$  is differentiable. In fact, by (3.6), the differential of  $\mathcal{T}_{ep}$  at  $x_0(t)$  is

$$D_{x_0}\mathcal{T} \cdot \eta(t) = \eta(0) + \int_0^T G(t, s) (a(s) + 3/x_0^4(s)) \eta(s) ds, \quad t \in [0, T],$$

where  $\eta \in C_T$ . One sees that  $D_{x_0}\mathcal{T}$  is a completely continuous linear operator of  $C_T$ . Moreover, if  $\eta \in C_T$  is a fixed point of  $D_{x_0}\mathcal{T}$ :

$$\eta = D_{x_0}\mathcal{T}\eta,$$

then  $\eta$  is  $T$ -periodic solution of (2.12), where  $r(t) = x_0(t)$ . By Theorem 2.2,  $\eta = 0$ . That is,  $I - D_{x_0}\mathcal{T}$  is non-degenerate in the space  $C_T$ . Hence

$$\deg_{LS}(I - \mathcal{T}_{ep}, \hat{\Omega}, 0) = \text{index}(I - \mathcal{T}_{ep}) = \pm 1.$$

Consequently, equation (1.1) has at least one  $T$ -periodic solution in  $\hat{\Omega}$  which is positive.

**Remark 3.1** (i) The difference between the proof of Theorem 3.1 and that in [2, 14] is as follows. If the technical assumption  $(G)$  is satisfied, one can choose  $\gamma(t) = \gamma$ , where  $\gamma$  is as in  $(G)$ . In this case, (3.4) is an autonomous differential equation. By the results in [1], for such an autonomous equation, the Leray-Schauder degree  $|\deg_{LS}(I - \mathcal{T}_{ep}, \hat{\Omega}, 0)|$  can be reduced to the Brouwer degree

$$|\deg(\gamma x - 1/x^3, (\hat{\varepsilon}_0, \hat{\varepsilon}_\infty), 0)|$$

which is obviously 1. In the present work, since  $\gamma(t)$  is not a constant,  $\deg_{LS}(I - \mathcal{T}_{ep}, \hat{\Omega}, 0)$  cannot be reduced to the Brouwer degree. Our computation for the Leray-Schauder degree is obtained from the relation established in Section 2.

(ii) It is an interesting problem if one can drop the strict positiveness assumption  $\Phi(t) \geq \phi(t) > 0$  for all  $t$  in Theorem 3.1. In our proof, this is used to deduce the *a priori* bounds for possible  $T$ -periodic solutions of (3.2) and (3.5). However, in the original work of Pinney [8], the connection between the Hill's equations and the Emarkov-Pinney equations is independent of the positiveness assumption.

When  $w(t)$  is a positive  $T$ -periodic function, we use

$$0 = \bar{\mu}_0(w) < \underline{\mu}_1(w) \leq \bar{\mu}_1(w) < \cdots < \underline{\mu}_n(w) \leq \bar{\mu}_n(w) < \cdots$$

to denote the  $2T$ -periodic weighted eigenvalues of

$$\ddot{x} + \mu w(t)x = 0.$$

Then the nonuniform non-resonance conditions  $(N_k)$  can also be stated using weighted eigenvalues.

**Theorem 3.2** Suppose that  $f(t, x)$  satisfies  $(S_0)$  and  $(S_\infty)$ . Assume that  $\phi$  in  $(S_0)$  and  $\Phi$  in  $(S_\infty)$  satisfy  $\Phi(t) \geq \phi(t) > 0$  for all  $t$  and the following nonuniform non-resonance condition  $(N'_k)$  for some  $k \geq 2$ :

$$\bar{\mu}_{k-1}(\phi) < 1 \quad \text{and} \quad \underline{\mu}_k(\Phi) > 1. \quad (N'_k)$$

Then equation (1.1) has at least one positive  $T$ -periodic solution.

In conclusion, the higher order non-resonance conditions in Theorems 3.1 and 3.2 and the non-resonance conditions in [15, 16] for the first non-resonance zones give a complete set of non-resonance results for repulsive singular equations with semilinear growth at infinity.

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