Advanced Nonlinear Studies **6** (2006), 47 – 56

Multiple Positive Solutions For a Class of Nonlinear Elliptic Equations

Shenghua Weng , Yongqing Li*

Department of Mathematics
Fujian Normal University, Fuzhou 350007, PR China
e-mail: azhen1998@163.com, yqli@fjnu.edu.cn

Received in revised form 5 October 2005
Communicated by Yiming Long

Abstract

This paper deals with a class of nonlinear elliptic Dirichlet boundary value problems where the combined effects of a sublinear and a superlinear term allow us to establish some existence and multiplicity results.

1991 Mathematics Subject Classification. 35J20, 35J25.
Key words. Multiple positive solutions, elliptic equation, Hardy inequality, critical point

1 Introduction

The first purpose of the present paper is to look for positive solutions of

$$\begin{cases} -\triangle u - \dfrac{\mu}{|x|^2} u = u^p + \lambda u^q & \text{in} \quad \Omega \setminus \{0\}, \\ u(x) > 0 & \text{in} \quad \Omega \setminus \{0\}, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \in \Omega$ and $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary,

$$0 \leq \mu < \overline{\mu} = ((N-2)/2)^2;$$

*The author acknowledges the support of NNSF of China (10161010) and Fujian Provincial Natural Science Foundation of China (A0410015)

here $\overline{\mu}$ is the best constant in the Hardy inequality

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

(cf. [8, Lemma 2.1]), $0 < q < 1 < p < 2^* - 1$, where $2^* = 2N/(N-2)$ is the so-called critical Sobolev exponent.

Finally, in Theorem 1.2 we prove, for $\lambda > 0$ and small, the existence of infinitely many solutions of

$$\begin{cases} -\triangle u - \dfrac{\mu}{|x|^2} u = |u|^{p-1} u + \lambda |u|^{q-1} u & \text{in} \quad \Omega \setminus \{0\}, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

by taking advantage of the oddness of the nonlinearity.

In the case $\mu = 0,\ p = 2^* - 1$, problem (1.1) has been studied extensively. For example, when $q = 1$, Capozzi et al [4] has shown that (1.1) has at least one nontrivial solution for $N \geq 5$. When $0 < q < 1$, Ambrosetti et al [1] has proved that there exists Λ^* such that (1.1) has at least two positive solutions for $\lambda \in (0, \Lambda^*)$. All these results are obtained by using critical point theory for the action functional

$$J_0(u) = \frac{1}{2} \int |\nabla u|^2 \, dx - \frac{1}{2^*} \int |u|^{2^*} \, dx - \frac{\lambda}{q+1} \int |u|^{q+1} \, dx, \ u \in H_0^1(\Omega).$$

In the present paper, we use a variational method to deal with problem (1.1) when $\mu > 0$. Since our method is variational in nature, we need to define the following Euler-Lagrange functional of (1.1) on H:

$$\begin{aligned} J_\mu(u) &= \frac{1}{2} \int [|\nabla u|^2 - \frac{\mu}{|x|^2} u^2 - F(u)] dx \\ &= \frac{1}{2} \int (|\nabla u|^2 - \frac{\mu}{|x|^2} u^2) dx - \frac{1}{p+1} \int (u^+)^{p+1} dx - \frac{\lambda}{q+1} \int (u^+)^{q+1} dx, \end{aligned}$$

where $u^+ = \max\{u,\ 0\}$, $f(u) := (u^+)^p + \lambda (u^+)^q$, $F(u) := (u^+)^{p+1}/(p+1) + \lambda (u^+)^{q+1}/(q+1)$ and H denotes the space of $H_0^1(\Omega)$ with the norm:

$$\| u \|_\mu^2 = \int (|\nabla u|^2 - \frac{\mu}{|x|^2} u^2) dx.$$

Due to the Hardy inequality, the norm $\|\cdot\|_\mu$ is equivalent to the usual norm $\|\cdot\|$ of $H_0^1(\Omega)$(cf. [8, Lemma 3.1]). A similar proof as in [12, Appendix B] shows that J_μ is in $C^1(H, \mathbb{R})$ (see also Proposition 5.1 in this paper).

If u is a critical point of $J_\mu(u)$, then for any $\varphi \in C_0^\infty(\Omega)$, there holds $\langle J_\mu'(u), \varphi \rangle = 0$, where

$$\langle J_\mu'(u), \varphi \rangle = \int (\nabla u \nabla \varphi - \frac{\mu}{|x|^2} u \varphi) dx - \int (u^+)^p \varphi dx - \lambda \int (u^+)^q \varphi dx.$$

This is the definition of weak solutions of (1.1). In fact, multiplying the equation $-\triangle u - \frac{\mu}{|x|^2}u = f(u)$ by $u^- = \max\{-u, 0\}$ and integrating over Ω, we find

$$0 = \int (|\nabla u^-|^2 - \frac{\mu}{|x|^2}(u^-)^2)dx = \|u^-\|_\mu^2. \tag{1.3}$$

Hence $u^- = 0$, i.e. $u \geq 0$. A standard elliptic regularity argument [10] implies that $u \in C^2(\Omega \setminus \{0\})$, in which case, by the strong maximum principle u is positive, thus is the solution of problem (1.1). Therefore, the critical points of $J_\mu(u)$ on H are non-negative weak solutions of (1.1).

Our results are

Theorem 1.1 *Suppose that* $0 \leq \mu < \overline{\mu} = ((N-2)/2)^2$*. Then there exists* $\Lambda^* > 0$*, such that (1.1) has at least two solutions in* $H_0^1(\Omega)$ *for any* $\lambda \in (0, \Lambda^*)$*.*

Remark 1.1 We also mention that when $\lambda = 0$, $1 < p < 2^* - 1$, the existence of one solution of (1.1) has been proved in [8]. And when $p = 2^* - 1$, J. Chen [6] has proved that there exists Λ such that (1.1) has at least two positive solutions for $\lambda \in (0, \Lambda)$.

Theorem 1.2 *Suppose that* $0 \leq \mu < \overline{\mu} = ((N-2)/2)^2$*, then there exists* $\Lambda^{**} > 0$*, such that (1.2) has infinitely many solutions for any* $\lambda \in (0, \Lambda^{**})$*.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and Section 4 are devoted to the proof of Theorem 1.1. The proof of Theorem 1.2 is contained in Section 5.

2 Preliminaries

Throughout this paper, the dual space of a Banach space of E will be denoted by E^{-1}, $H_0^1(\Omega)$, $L^t(\Omega)$ are standard Sobolev spaces with the standard norms: $\|\cdot\|$ is induced by the standard inner product and $|\cdot|_t$. $\int$ here means the integral is taken over Ω unless stated otherwise. c, c_i will denote various positive constants, the exact values of which are not important.

Definition 2.1 [13, Definition 1.16] Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E, \mathbb{R})$. We say that I *satisfies the (PS)$_c$ condition* if any sequence $\{u_n\}$ in E such that $I(u_n) \to c$ and $\| I'(u_n) \|_{E^{-1}} \to 0$ has a convergent subsequence. If this holds for every $c \in \mathbb{R}$, we say that I *satisfies* (PS) *condition.*

3 Existence of a local minimizer

In this section, we will prove that there is $\Lambda^* > 0$, such that J_μ can achieve a local minimizer for any $\lambda \in (0, \Lambda^*)$. First we have the following compactness result.

Proposition 3.1 *If $\{u_n\} \subset H$ are such that*

$$J_\mu(u_n) \to c,\ J_\mu^{'}(u_n) \to 0 \quad in \quad H^{-1},$$

then $\{u_n\}$ possesses a convergent subsequence in H.

Proof. Since

$$J_\mu(u_n) = \frac{1}{2}\int(|\nabla u_n|^2 - \frac{\mu}{|x|^2}u_n^2)dx - \frac{1}{p+1}\int(u_n^+)^{p+1}dx - \frac{\lambda}{q+1}\int(u_n^+)^{q+1}dx,$$

$$\langle J_\mu^{'}(u_n), u_n\rangle = \int(|\nabla u_n|^2 - \frac{\mu}{|x|^2}u_n^2)dx - \int(u_n^+)^{p+1}dx - \lambda\int(u_n^+)^{q+1}dx.$$

For n large enough,

$$\begin{aligned}
& (p+1)c + 1 + o(1) \| u_n \|_\mu \\
\geq\ & (p+1)J_\mu(u_n) - \langle J_\mu^{'}(u_n), u_n\rangle \\
=\ & (\frac{p+1}{2} - 1) \| u_n \|_\mu^2 - (\frac{p+1}{q+1} - 1)\lambda\int(u_n^+)^{q+1}dx \\
=\ & \frac{p-1}{2} \| u_n \|_\mu^2 - \frac{\lambda(p-q)}{q+1} \| u_n^+ \|_{q+1}^{q+1} \\
\geq\ & \frac{p-1}{2} \| u_n \|_\mu^2 + \frac{\lambda c(q-p)}{q+1} \| u_n \|_\mu^{q+1}.
\end{aligned}$$

It follows from $0 < q < 1$ that $\{u_n\}$ is bounded in H. Going if necessary to a subsequence, we can assume that

$$\begin{aligned}
& u_n \rightharpoonup u_0 && in \quad H, \\
& u_n \to u_0,\ a.e. && in \quad \Omega, \\
& u_n \to u_0 && in \quad L^r(\Omega),\ 1 < r < 2^*.
\end{aligned}$$

Denoting $w_n := u_n - u_0$, then the Brezis-Lieb Lemma [3] (or see [13, Lemma 1.32]) implies that

$$\begin{aligned}
\langle J_\mu^{'}(u_n), u_n\rangle &= \int(|\nabla u_0|^2 - \frac{\mu}{|x|^2}u_0^2)dx - \int(u_0^+)^{p+1}dx - \lambda\int(u_0^+)^{q+1}dx \\
&+ \int(|\nabla w_n|^2 - \frac{\mu}{|x|^2}w_n^2)dx - \int(w_n^+)^{p+1}dx + o(1) \\
&= \int(|\nabla w_n|^2 - \frac{\mu}{|x|^2}w_n^2)dx - \int(w_n^+)^{p+1}dx + o(1),
\end{aligned}$$

since $u_n \to u_0\ \ in\ L^{p+1}(\Omega)$, we have that $\int(w_n^+)^{p+1}dx \to 0$. Thus

$$\int(|\nabla w_n|^2 - \frac{\mu}{|x|^2}w_n^2)dx \to 0$$

i.e., $\|w_n\|_\mu^2 \to 0$, therefore $u_n \to u_0$ in H. □

Existence of a first positive solution of (1.1)

Let $\phi \in H$ such that $\| \phi \|_\mu = 1$. Then, for $t > 0$, we have

$$J_\mu(t\phi) = \frac{t^2}{2} \| \phi \|_\mu^2 - \frac{t^{p+1}}{p+1} \int (\phi^+)^{p+1} dx - \frac{\lambda t^{q+1}}{q+1} \int (\phi^+)^{q+1} dx,$$

and

$$D_t J_\mu(t\phi) = t\|\phi\|_\mu^2 - t^p \int (\phi^+)^{p+1} dx - \lambda t^q \int (\phi^+)^{q+1} dx.$$

If $\int (\phi^+)^{q+1} dx = 0$, then we have that $\phi^+ = 0, a.e.$ in Ω, thus $D_t J_\mu(t\phi) = t\|\phi\|_\mu^2 > 0$, for any $t > 0$. If $\int (\phi^+)^{q+1} dx \neq 0$, then $J_\mu(t\phi) < 0$ for sufficiently small $t > 0$ and

$$\begin{aligned} D_t J_\mu(t\phi) &= t\|\phi\|_\mu^2 - t^p \int (\phi^+)^{p+1} dx - \lambda t^q \int (\phi^+)^{q+1} dx \\ &\geq t\|\phi\|_\mu^2 - c_1 t^p \|\phi\|_\mu^{p+1} - c_2 \lambda t^q \|\phi\|_\mu^{q+1} \\ &= t - c_1 t^p - c_2 \lambda t^q. \end{aligned}$$

So when λ is small enough, there is $t_\lambda > 0$ such that $t_\lambda - c_1 t_\lambda^p - c_2 \lambda t_\lambda^q = 0$, and it is easy to check that $t_\lambda \to 0$ as $\lambda \to 0$, then there exists $T_\lambda,\ \Lambda^* > 0$ such that $t - c_1 t^p - c_2 \lambda t^q > 0$ for $t_\lambda < t < T_\lambda$ and $\lambda < \Lambda^*$. So we have that $D_t J_\mu(t\phi) > 0$ for $t_\lambda < t < T_\lambda$ and $\lambda \in (0, \Lambda^*)$. Then if we minimize the functional J_μ on the Ball $\overline{B}_\rho \subset H$ for $\rho = t_\lambda + \varepsilon$, for $\varepsilon > 0$ sufficiently small such that $D_t J_\mu(\rho\phi) > 0$, we must have

$$c_\lambda = \inf_{u \in \overline{B}_\rho} J_\mu(u) < 0.$$

Proposition 3.1 implies that J_μ can achieve its minimum c_λ at u_λ, i.e., $c_\lambda = J_\mu(u_\lambda)$, and we know that the minimum can't be achieved on ∂B_ρ. In fact, if $u_\lambda \in \partial B_\rho$, then $J_\mu(t u_\lambda)$ is strictly increasing with respect to t at ρ, this is contradiction. Thus u_λ satisfies (1.1).

4 Existence of a second positive solution of (1.1)

We obtained a minimizer u_λ of functional J_μ in the ball B_ρ in Section 3 which is a positive solution of (1.1). Without losing of generalities, we may assume that u_λ is an isolated minimizer. Since it is a local minimizer of functional J_μ, one can use Mountain Pass Theorem to find another critical point. To show this proof more clearly, we consider a translated functional as in [1, 6] to do this.

For fixed $\lambda \in (0, \Lambda^*)$, we look for a second positive solution of (1.1) of the form $u = u_\lambda + v$, where $v > 0$ in $\Omega \setminus \{0\}$. For $v \in H$, the corresponding equation for v is

$$-\triangle v - \frac{\mu}{| x |^2} v = (u_\lambda + v)^p - u_\lambda^p + \lambda (u_\lambda + v)^q - \lambda u_\lambda^q. \tag{4.1}$$

Let us define

$$g(x,t)=\begin{cases}(u_\lambda+t)^p-u_\lambda^p+\lambda(u_\lambda+t)^q-\lambda u_\lambda^q, & \mathrm{t}\geq 0\\ 0, & \mathrm{t}<0\end{cases} \tag{4.2}$$

$$G(v)=\int_0^v g(x,t)dt,$$

and

$$I_\mu(v)=\frac{1}{2}\int(|\nabla v|^2-\frac{\mu}{|x|^2}v^2)dx-\int G(v)dx.$$

Now we have one-to-one correspondence between critical points of I_μ in H and non-negative weak solutions of (4.1). Moreover, if v satisfies (4.1) in the weak sense, then standard regularity argument shows that v also satisfies (4.1) in the classical sense.

Next we will prove the existence of a second solution of (1.1).

Lemma 4.1 *$v=0$ is a local minimum of I_μ in H.*

Proof. For any $u_\lambda+v\in B_\rho\subset H(B_\rho$ has been chosen in Section 3), write $v=v^+-v^-,\ v^\pm=\max\{\pm v,\ 0\}$. We have

$$\begin{aligned}I_\mu(v) &= \frac{1}{2}\int(|\nabla v|^2-\frac{\mu}{|x|^2}v^2)dx\\ &- \frac{1}{p+1}\int[(u_\lambda+v^+)^{p+1}-u_\lambda^{p+1}-(p+1)u_\lambda^p v^+]dx\\ &- \frac{\lambda}{q+1}\int[(u_\lambda+v^+)^{q+1}-u_\lambda^{q+1}-(q+1)u_\lambda^q v^+]dx.\end{aligned}$$

From the expression of J_μ and direct computation, we obtain that

$$I_\mu(v)=\frac{1}{2}\int(|\nabla v^-|^2-\frac{\mu}{|x|^2}(v^-)^2)dx+J_\mu(u_\lambda+v^+)-J_\mu(u_\lambda).$$

Since u_λ is a isolated local minimizer of J_μ in H, $v=0$ is a local minimum of I_μ in H. Furthermore, since J_μ satisfies (PS) condition, there exists $\delta>0$ and $\alpha>0$ such that

$$I_\mu(v)\geq\delta>0,$$

as $\|v\|_\mu=\alpha$. □

Completion of the proof of Theorem 1.1

It is standard to show that I_μ satisfies $(\mathrm{PS})_c$ for all c, see e.g. [2].

Note that

$$(b+d)^m\geq b^m+d^m+m\,b^{m-1}d,\ \ m>1,\ b,\ d>0.$$

Then for every $a>0$, since $p>1$, from the definition of g (see (4.2)), we have that

$$g(x,a)\geq a^p+p\,u_\lambda^{p-1}a,$$

and so for any $v > 0$,

$$G(tv) \geq \frac{t^{p+1}}{p+1} v^{p+1} + \frac{p\, t^2}{2} u_\lambda^{p-1} v^2.$$

Then we have that $I_\mu(tv) \to -\infty$ as $t \to \infty$ and there exists $v_1 \in H$ such that $I_\mu(v_1) < 0$. We define the following mini-max value

$$c_\lambda^* = \inf_{h \in \Gamma} \max_{t \in [0,1]} I_\mu(h(t)),$$

where $\Gamma = \{h \in C([0,1], H);\ h(0) = 0,\ h(1) = v_1\}$. From Lemma 4.1, $v = 0$ is a local minimizer of I_μ. By using of the Mountain Pass theorem [13, Lemma 1.15], we get a critical point v of I_μ. We note that $c_\lambda^* \geq \delta > 0$. From the definition of g and u_λ is a positive solution of (1.1), we know that $v \neq -u_\lambda$ in Ω. Thus we complete the proof of Theorem 1.1. □

5 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Here we define the following Euler-Lagrange functional of (1.2) on H:

$$\tilde{J}_\mu(u) = \frac{1}{2} \int (| \nabla u |^2 - \frac{\mu}{| x |^2} u^2) dx - \frac{1}{p+1} \int | u |^{p+1} dx - \frac{\lambda}{q+1} \int | u |^{q+1} dx.$$

We make some preparations.

Proposition 5.1 $\tilde{J}_\mu \in C^1(H, \mathbb{R})$.

Proof. Assume that $u_n \to u$ in H. Then for any $\varphi \in C_0^\infty(\Omega)$, we have that

$$\langle \tilde{J}_\mu'(u), \varphi \rangle = \int (\nabla u \nabla \varphi - \frac{\mu}{| x |^2} u \varphi) dx - \int | u |^p \varphi dx - \lambda \int | u |^q \varphi dx.$$

and

$$u_n \to u \quad in \quad L^r(\Omega),\ 1 < r < 2^*.$$

So

$$\begin{aligned}
& \| \tilde{J}_\mu'(u_n) - \tilde{J}_\mu'(u) \|_{H^{-1}} \\
= & \sup_{\|\varphi\| \leq 1} | \langle \tilde{J}_\mu'(u_n) - \tilde{J}_\mu'(u), \varphi \rangle | \\
= & \sup_{\|\varphi\| \leq 1} | \int (\nabla u_n \nabla \varphi - \frac{\mu}{| x |^2} u_n \varphi) dx - \int |u_n|^{p-1} u_n \varphi dx - \lambda \int |u_n|^{q-1} u_n \varphi dx \\
& - \int (\nabla u \nabla \varphi - \frac{\mu}{| x |^2} u \varphi) dx + \int | u |^{p-1} u \varphi dx + \lambda \int | u |^{q-1} u \varphi dx | \\
= & \sup_{\|\varphi\| \leq 1} | \int [\nabla (u_n - u) \nabla \varphi - \frac{\mu}{| x |^2} (u_n - u) \varphi] dx
\end{aligned}$$

$$
\begin{aligned}
& -\int(| u_n |^{p-1} u_n - | u |^{p-1} u)\varphi dx - \lambda\int(| u_n |^{q-1} u_n - | u |^{q-1} u)\varphi dx | \\
\leq & \sup_{\|\varphi\|\leq 1} [|\langle u_n - u, \varphi\rangle| + \int | \, |u_n|^{p-1}u_n - |u|^{p-1}u \, | \; |\varphi| dx \\
& +\lambda\int | \, |u_n|^{q-1}u_n - |u|^{q-1}u \, | \; |\varphi| dx] \\
\leq & \sup_{\|\varphi\|\leq 1} [|\langle u_n - u, \varphi\rangle| + c_1\int(|u_n|^{p-1} + |u|^{p-1})|u_n - u| \, |\varphi| dx \\
& +\lambda c_2\int |u_n - u|^q |\varphi| dx] \\
\leq & \sup_{\|\varphi\|\leq 1} \{|\langle u_n - u, \varphi\rangle| + c_3[(\int |u_n|^{p+1})^{\frac{p-1}{p+1}} \\
& +(\int |u|^{p+1})^{\frac{p-1}{p+1}}](\int |u_n - u|^{p+1})^{\frac{1}{p+1}}(\int |\varphi|^{p+1})^{\frac{1}{p+1}} + \lambda c_2\int |u_n - u|^q |\varphi| dx\} \\
\leq & \| u_n - u \|_\mu + c_4 \| u_n - u \|_{L^{p+1}(\Omega)} + \lambda c_5 \| u_n - u \|^q_{L^{q+1}(\Omega)} \\
\to & \; 0 \quad as \quad n\to\infty.
\end{aligned}
$$

□

For any $u\in H$, we obtain, from Sobolev imbedding theorem, that

$$
\begin{aligned}
\tilde{J}_\mu(u) &= \frac{1}{2}\int(| \nabla u |^2 - \frac{\mu}{| x |^2}u^2)dx - \frac{1}{p+1}\int | u |^{p+1} dx - \frac{\lambda}{q+1}\int | u |^{q+1} dx \\
&\geq \frac{1}{2} \| u \|_\mu^2 - c_1 \| u \|_\mu^{p+1} - \lambda c_2 \| u \|_\mu^{q+1} .
\end{aligned}
$$

From this one readily finds that there exists $\Lambda^{**} > 0$ such that for all $\lambda\in(0,\Lambda^{**})$ there are $\rho,\ \alpha > 0$ such that

$(i)\ \tilde{J}_\mu(u)\geq\alpha\ for\ all\ \| u \|_\mu = \rho;$

$(ii)\ \tilde{J}_\mu\ is\ bounded\ from\ below\ on\ B_\rho\ (B_\rho = \{u\in H : \| u \|_\mu\leq\rho\});$

$(iii)\ \tilde{J}_\mu\ satisfies\ (PS)\ on\ B_\rho.$

Henceforth we fix $\lambda\in(0,\Lambda^{**})$. After these preliminaries, let us give the proof of Theorem 1.2. We use the same method as in Section 5 of [2].

Proof of Theorem 1.2 We set

$$\Sigma = \{A\subset H : 0\notin A,\ u\in A\Rightarrow -u\in A\}.$$

For $A\in\Sigma$, the $\mathbb{Z}_2$-genus of A is denoted by $\gamma(A)$ (see, for example, [2, 12]). We set also

$$\mathcal{A}_{n,\rho} = \{A\in\Sigma : A \text{ compact},\ A\subset B_\rho, \gamma(A)\geq n\}.$$

Clearly, $\mathcal{A}_{n,\rho}\neq\emptyset$ for all $n = 1,\ 2,\ \cdots$, because

$$S_{n,\varepsilon} := \partial(H_n\bigcap B_\varepsilon)\in\mathcal{A}_{n,\rho},$$

here $H_n = \text{span}\{e_1, \cdots, e_n\}$, e_i is the ith eigenfunction of operator $-\triangle - \frac{\mu}{|x|^2}$ [7, Proposition 2.1](see also [11])). Let

$$b_{n,\rho} = \inf_{A \in \mathcal{A}_{n,\rho}} \max_{u \in A} \tilde{J}_\mu.$$

Each $b_{n,\rho}$ is finite because of (ii). Moreover, one has

$$b_{n,\rho} < 0, \quad \forall\, n \in \mathbb{N}. \tag{5.1}$$

Indeed, let $w \in H_n$ be such that $\| u \|_\mu = \varepsilon$. From

$$\tilde{J}_\mu(w) \leq \frac{1}{2}\varepsilon^2 - \lambda\, c_1\, \varepsilon^{q+1},$$

it follows that $\tilde{J}_\mu(w) < 0$ provided $\varepsilon > 0$ small enough, and this suffices to prove (5.1).

Next, let us note that for all $u \in B_\rho \bigcap \{\tilde{J}_\mu \leq 0\}$ the steepest descent flow η_t (defined through the pseudo-gradient vector field, see e.g. the Deformation Lemma in [2]) is well defined for $t \in [0, \infty)$ and

$$\eta_t(u) \in B_\rho \bigcap \{\tilde{J}_\mu \leq 0\} \quad \forall\, t \geq 0,$$

because of (i). Since, by (5.1), $b_{n,\,\rho} < 0$ and (PS) holds in B_ρ, see (iii), we can make use of the Ljusternik-Schnirelman theory to find infinitely many critical points of $\tilde{J}_\mu$ in B_ρ. This proves Theorem 1.2. □

Acknowledgments. The first author would like to express her thanks to Dr. Jianqing Chen and Dr. Shaowei Chen for useful conversations and encouragement. The authors also thank Professor Yiming Long for many valuable comments which improve the manuscript.

References

[1] A. Ambrosetti, H. Brezis, G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519-543.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349-381.

[3] H. Brezis, E. Lieb, *A relation between pointwise convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486-490.

[4] A. Capozzi, D. Fortunato, G. Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **2** (1985), 463-470.

[5] L. Caffarelli, R. Kohn, L. Nirenberg, *First order interpolation inequalities with weighs*, Compositio Math. **53** (1984), 259-275.

[6] J.Q. Chen, *Multiple positive solutions for a class of nonlinear elliptic equations*, J. Math. Anal. Appl. **295** (2004), 341-354.

[7] J.Q. Chen, *Existence of solutions for a nonlinear PDE with an inverse square potential*, J. Differential Equation, **195** (2003), 497-519.

[8] J. Garcia Azorero, I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Differential Equation, **144** (1998), 441-476.

[9] N. Ghoussoub, D. Press, *A general mountain pass principle for locating and classifying critical points*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **6** (1989), 321-330.

[10] D. Gilbarg, NS. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, New York, 1977.

[11] E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equation, **156** (1999), 407-426.

[12] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, Amer. Math. Soc., Providence, RI, 1986.

[13] Michel Willem, Minimax Theorems, Birkhäuser, Boston, 1996.