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On the Prescribed Scalar and Zero Mean Curvature on 3-Dimensional Manifolds With Umbilic Boundary

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Abstract

In this paper we consider the existence and the compactness of Riemannian metrics of prescribed mean curvature and zero boundary mean curvature on a three dimensional manifold with umbilic boundary (M, g_0). We prove that for three dimensional manifolds with umbilic boundaries, which are not conformally equivalent to the three dimensional standard half sphere, any positive function can be realized as the scalar curvature of a Riemannian metric g conformal to g_0 with respect to which the boundary has zero mean curvature. Moreover, all such metrics stay bounded with respect to the $C^{2,\alpha}$-topology and in the non-degerate case *Morse inequalities* hold.

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1 Introduction

Let (M^n, g) be n-dimensional Riemannian manifold with boundary, $n \geq 3$, and let $\tilde{g} = u^{4/(n-2)}g$, be a conformal metric to g, where u is a smooth positive function. Then the scalar curvatures R_g, $R_{\tilde{g}}$ and the mean curvatures h_g, $h_{\tilde{g}}$, with respect to g and $\tilde{g}$ respectively, are related by the following equations:

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g v + R_g v = R_{\tilde{g}} v^{\frac{n+2}{n-2}}, & \text{in } M; \\ \frac{2}{n-2}\frac{\partial v}{\partial \nu} + h_g v = h_{\tilde{g}} v^{\frac{n}{n-2}}, & \text{on } \partial M, \end{cases} \tag{1.1}$$

see e.g. [6]. In the above equation, ν denotes the outward unit normal to ∂M, with respect to the metric g.

In view of the above equations the following natural question arises:
Given two functions $K : M \to \mathbb{R}$ and $H : \partial M \to \mathbb{R}$, does there exist a metric $\tilde{g}$ conformally equivalent to g such that $R_{\tilde{g}} = K$ and $h_{\tilde{g}} = H$? The answer to this equation is equivalent to finding a smooth positive solution u of the following equation

$$(P_{K,H}) \quad \begin{cases} -c_n \Delta_g u + R_g u & = & K u^{\frac{n+2}{n-2}} & \text{in} & \mathring{M} \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + h_g u & = & H u^{\frac{n}{n-2}} & \text{on} & \partial M \end{cases}$$

We observe that the above problem is a natural generalization of the well known *Scalar Curvature Problems on Closed manifolds*: to find a positive smooth solution to the following equation:

$$(\mathbf{SC}) \; - c_n \, \Delta_g u + R_g \, u = K u^{(n+2)/(n-2)}, \; u > 0 \, \text{in } M$$

and to which a lot of articles have been devoted (see [2], [5], [4], [8], [9], [7], [10], [12], [15], [14], [24], [23], [26], [27], [28], [29], [36], [43], [48], [49], [60], [62] and the references therein).

When K and H are constants, the problem is called *The Yamabe Problem on Manifolds with boundary.* It has also been studied through the works [20], [33], [35], [34], [37], [38], [41], [42] and the references therein. This Problem was first studied by P. Cherrier [30] in 1984, who proved regularity of weak H^1 solutions. For further works on this equation and related ones please see [3] [17], [18], [31], [33], [35] , [41], [42], [44], [47] and the references therein).

To go further into the description of our problem and its main features, we need to set some notations and definitions.

We denote by

$$L_g = -c_n \, \Delta u + R_g \, u, \qquad B_g = \frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u,$$

the conformal operator and its conformal counterpart on the boundary. Let H be the second Fundamental Form of ∂M in (M, g) with respect to the inner normal. We denote its traceless part of the second Fundamenatal Form by U:

$$U(X,Y) := H(X,Y) - h_g g(X,Y).$$

Definition 1.1 A point $q \in \partial M$ is called *umblic* if $U = 0$ at q. ∂M is called *umbilic* if every point of ∂M is an umbilic point.

Observe that if $u > 0$ and $\tilde{g} := u^{\frac{4}{n-2}} \, g$, then

$$h_{\tilde{g}} = \frac{2}{n-2} \, u^{\frac{-n}{n-2}} B_g.$$

Therefore umbilicity is conformally invariant notion.

Now we consider the following eigenvalue problem on (M, g):

$$(P_\lambda) \quad \begin{cases} -L_g \varphi &= \lambda\, \varphi \text{ in } \overset{\circ}{M} \\ B_g \varphi &= 0 \text{ on } \partial M. \end{cases}$$

Let $\lambda_1(M)$ denote the first eigenvalue.

Definition 1.2 We say that a Manifold M is of *positive(negative, zero) type* if $\lambda_1(M) > 0 (< 0, = 0)$.

This notion is conformally invariant, and as it is also the case for the prescribed scalar curvature on closed manifold, the difficult case for our problem $(P_{K,H})$ is when the manifold is of positive type.

The main analytic difficulties of our problem are due to the presence of critical exponent on the right hand side of our equation. Indeed due to the fact that the embeddings $H^1(M) \to L^{\frac{2n}{n-2}}(M)$ and $H^1(M) \to L^{\frac{2(n-1)}{n-2}}(\partial M)$ are not compact, the Euler-Lagrange functional associated to our problem fails to satisfy *the Palais Samale condition*. That is that there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. Therefore it is not possible to apply the standard variational methods to prove existence of solution, although the functional in the positive case has a *mountain pass* structure. From another part, in the familly of problems $(P_{K,H})$ we single out two extreme cases. Namely the one where we prescribe the scalar curvature under minimal boundary conditions, which amounts to solving the following equations:

$$(P_K) \quad \begin{cases} -c_n \Delta_g u + R_g\, u &= K u^{\frac{n+2}{n-2}} \text{ in } \overset{\circ}{M} \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + h_g u &= 0 \text{ on } \partial M. \end{cases}$$

The second one is when we prescribe the mean curvature of scalar flat metric, which corresponds to solving the following equation:

$$(P_H) \quad \begin{cases} -c_n \Delta_g u + R_g u &= 0 \text{ in } \overset{\circ}{M} \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + h_g u &= H u^{\frac{n}{n-2}} \text{ on } \partial M. \end{cases}$$

Although these are particular cases, they summarize somehow all the analytic difficulties of the familly of problems $(P_{K,H})$, in the sense that all intermediate cases are interpolations between these two extreme ones. From another part these two problemes have different analytic features as far as the lack of compactness and existence results are concerned. While we can prove corresponding statements of all the existence results for the prescribed scalar curvature problem on closed manifolds for the problem P_H, this is no longer true in general for the problem P_K and the problems $P_{K,H}$ which behave like P_K. Indeed in this case we have new solutions created by the boundary effect which have no equivalent on a closed manifold from one part and from another part the boundary effect makes the blow up picture more complicated because of the existence under generic conditions on K of *bubbles* having concentration points on the interior of the manifolds as well as on the boundary. Such a situation which makes the blow up analysis much harder cannot occur for the problem P_H.

In this paper, we will be dealing with the problem (P_K) on three dimensional Riemannian manifold where we have a fairly satisfactory answer.

2 Prescribed scalar curvature on 3-Manifolds under minimal boundary conditions

On a 3-dimensional Riemannian Manifold, consider a more general problem than P_K, namely for $1 < q \leq 5$, we consider the family of problems

$$\begin{cases} L_g u = K u^q, \quad u > 0, & \text{in } \mathring{M}, \\ B_g u = 0, & \text{on } \partial M. \end{cases} \tag{$P_{K,q}$}$$

Let $\mathcal{M}_{K,q}$ denote the set of solutions of $P_{K,q}$ in $C^2(M)$. Our first result is an a priori energy estimate, namely we prove the following theorem:

Theorem 2.1 *Let (M, g) be a three dimensional smooth compact Riemannian manifold with umbilic boundary and assume that K is a positive function. Then for all $\varepsilon_0 > 0$*

$$\|u\|_{H^1(M)} \leq C \quad \forall\, u \in \bigcup_{1+\varepsilon_o \leq q \leq 5} \mathcal{M}_{K,q},$$

where C depends only on M, g, ε_0, $\|K\|_{C^2(M)}$ and the positive lower bound of K.

Our next theorem states that for any positive C^2 function K, all such metrics stay bounded with respect to the $C^{2,\alpha}$-norm for some $\alpha < 1$ and the total Leray-Schauder degree of all the solutions of $(P_{K,q})$ is -1.

Theorem 2.2 *Let (M, g) be 3-dimensional smooth compact Riemannian manifold with umbilic boundary and positive type which is not conformally equivalent to the standard three dimensional half sphere. Then, for any $1 < q \leq 5$ and positive function $K \in C^{2,\alpha}(M)$, there exists some constant C, depending only on $M, g, \|K\|_{C^{2,\alpha}}$, the positive lower bound of K and $q-1$, such that*

$$\frac{1}{C} \leq u \leq C \quad \text{and} \quad \|u\|_{C^{2,\alpha}(M)} \leq C$$

for all solutions u of $(P_{K,q})$. Moreover the total degree of all solutions of $(P_{K,q})$ is -1. Consequently, equation $(P_{K,5})$ has at least one solution.

We remark that the hypothesis (M, g) is not conformally equivalent to the standard three dimensional half sphere is necessary since (P_K) may have no solution in this case due to the Kazdan-Warner type obstruction see e.g [19].

To prove Theorem 2.2, we argue as follows: First of all we observe that without loss of generality, we may assume that $h_g \equiv 0$. Indeed, let φ_1 be a positive eigenfunction associated to the first eigenvalue λ_1 of the problem

$$\begin{cases} L_g \varphi = \lambda_1 \varphi, & \text{in } \overset{\circ}{M}, \\ B_g \varphi = 0, & \text{on } \partial M. \end{cases}$$

Setting $\tilde{g} = \varphi_1^4 g$ and $\tilde{u} = \varphi_1^{-1} u$, where u is a solution of $(P_{K,3})$, one can easily check that $R_{\tilde{g}} > 0$, $h_{\tilde{g}} \equiv 0$, and $\tilde{u}$ satisfies

$$\begin{cases} L_{\tilde{g}} \tilde{u} = K \tilde{u}^5, & \text{in } \overset{\circ}{M}, \\ \dfrac{\partial \tilde{u}}{\partial \nu} = 0, & \text{on } \partial M. \end{cases}$$

For the sake of simplicity, we work with $\tilde{g}$ denoting it by g. Since ∂M is umbilic with respect to g and $h_{\tilde{g}} = 0$, it follows that the second fundamental form vanishes at each point of the boundary, that is the boundary is a totally geodesic submanifold. Hence we can take conformal normal coordinates around any point of the boundary [34]. Moreover, due to elliptic estimates and Harnack Inequality, we need only to prove the L^∞ bound.

Suppose the contrary. Then there exists a sequence $q_i \to q \in]1, 5]$ with

$$u_i \in \mathcal{M}_{K,q_i}, \quad \text{and } \max_M u_i \to +\infty.$$

As usual, a sequence of rescaled u_i converges to some function satisfying some limit equation on the whole space $\mathbb{R}^n$ or the half space $\mathbb{R}^n_+$. From some Liouville type theorems for the limit equation [39], [22] or [50], we deduce that q must be 5. It follows from a careful blow up analysis à la Schoen [59] and Y. Y. Li [48], [41] that, after passing to a subsequence, $\{u_i\}_i$ has N ($1 \leq N < \infty$) isolated simple blow-up points denoted by $y^{(1)}, \dots, y^{(N)}$.

Let $y_i^{(\ell)} \to y^{(\ell)}$ denotes a sequence of local maxima . It turns ou that

$$u_i(y_i^{(1)}) u_i \underset{i \to +\infty}{\longrightarrow} h \quad \text{in } C^{2,\alpha}_{\text{loc}}\big(M \setminus \{y^{(1)}, \dots, y^{(N)}\}\big).$$

Since the manifold is of positive type, it follows that:

$$h(y) = \sum_{j=1}^{N} b_j G(y, y^{(j)}),$$

where $b_j > 0$ and $G(., y^{(j)})$ is the Green's function of $-L_g$ with respect to Neumann boundary conditions.

Let $x = (x^1, x^2, x^3)$ be some geodesic normal coordinate system centered at $y_i^{(1)}$. From the Positive Mass Theorem, and the assumption that the manifold is not conformally equivalent to the standard half sphere, we derive that there exists a positive constant A such that

$$h(x) = c|x|^{-1} + A_i + O(|x|^{-\alpha}) \quad \text{for } |x| \text{ close to } 0$$

and $A_i \geq A > 0$. Using the Pohozaev identity argument we derive a contradiction and prove the theorem.

From the compactness result of Theorem 2.2 we deduce that for Problem (P_K) *Morse inequalities* hold in the nondegenerate case. Namely we have the following corollary:

Corollary 2.1 *Under the assumptions of Theorem 2.2 and assuming further that all the solutions of (P_K) are nondegenrate, we have that:*
There are only finitely many solutions to (P_K) and

$$(-1)^\lambda \leq \sum_{\mu=0}^{\lambda} (-1)^{\lambda-\mu} N_\mu, \quad \lambda = 0, 1, ..,$$

where N_μ is the number of solutions with Morse index μ.

The proof uses the corresponding results for $q < 5$, which does satisfy the Palais Smale condition and then Theorem 2.2 to prove that all the critical points of the subcritical approximation converge. The remainder of the paper is organized as follows. In section 3 we recall the main notions of blow-up analysis. In section 4 we prove sharp pointwise estimates to a sequence of solutions near isolated boundary simple blow-up points, then in section 5 we prove that an isolated blow-up is in fact an isolated simple blow up, ruling out the possibility of bubbles on top of bubbles. In section 5 we rule out the possibility of bubble accumulations and establish the compactness results claimed in section 2. In the Appendix, we provide for the convenience of the reader some standard descriptions of singular behaviour of positive solutions to some linear boundary value elliptic equations in punctured half balls and collect some useful results.

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3 Blow up analysis: definitions and preliminary results

In this section we recall the definition of isolated and isolated simple blow up due to R. Schoen, [58] and [59]; we also collect some useful tools and known results.

We may assume, without loss of generality, that $h_g \equiv 0$. Indeed, let φ_1 be a positive eigenfunction associated to the first eigenvalue λ_1 of the problem

$$\begin{cases} L_g \varphi = \lambda_1 \varphi, & \text{in } \overset{\circ}{M}, \\ B_g \varphi = 0, & \text{on } \partial M. \end{cases}$$

Setting $\tilde{g} = \varphi_1^4 g$ and $\tilde{u} = \varphi_1^{-1} u$, where u is a solution of $(P_{K,5})$, one can easily check, that $h_{\tilde{g}} \equiv 0$ and $\tilde{u}$ satisfies

$$\begin{cases} -\Delta_{\tilde{g}} \tilde{u} + R_{\tilde{g}}\, \tilde{u} = K \tilde{u}^5, & \text{in } \overset{\circ}{M}, \\ \dfrac{\partial \tilde{u}}{\partial \nu} = 0, & \text{on } \partial M, \end{cases}$$

where $R_{\tilde{g}}$ is the scalar curvature with respect to the new metric $\tilde{g}$.

For the sake of simplicity, we work with $\tilde{g}$, denoting it by g. Let us first recall the definitions of isolated and isolated simple blow up which were first introduced by R. Schoen [59] and used extensively by Y. Y. Li [48], [49] and Han and Li [41].

Definition 3.1 Let (M, g) be a smooth compact n-dimensional Riemannian manifold with boundary and let $\bar{r} > 0$, $\bar{c} > 0$, $\bar{x} \in M$, $K \in C^0(\overline{B_{\bar{r}}(\bar{x})})$ be some positive function where $B_{\bar{r}}(\bar{x})$ denotes the geodesic ball in (M, g) of radius $\bar{r}$ centered at $\bar{x}$. Suppose that, for some sequences $q_i = 5 - \tau_i$, $\tau_i \to 0$, $K_i \to K$ in $C^2(\overline{B_{\bar{r}}(\bar{x})})$, $\{u_i\}_{i\in\mathbb{N}}$ solves

$$\begin{cases} L_g u_i = K_i u_i^{q_i}, \quad u_i > 0, & \text{in } B_{\bar{r}}(\bar{x}), \\ \dfrac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial M \cap B_{\bar{r}}(\bar{x}). \end{cases} \tag{3.2}$$

We say that $\bar{x}$ is an *isolated blow-up point* of $\{u_i\}_i$ if there exists a sequence of local maxima x_i of u_i such that $x_i \to \bar{x}$ and, for some $C_1 > 0$,

$$\lim_{i\to\infty} u_i(x_i) = +\infty \quad \text{and} \quad u_i(x) \leq C_1 d(x, x_i)^{-\frac{1}{q_i - 1}}, \quad \forall\, x \in B_{\bar{r}}(x_i),\ \forall\, i.$$

In order to describe the behaviour of blowing-up solutions near an isolated blow-up point, we define , following R. Schoen, spherical averages of u_i centered at x_i as follows

$$\bar{u_i}(r) = \fint_{M\cap\partial B_r(\bar{x})} u_i = \frac{1}{\mathrm{Vol}_g(M \cap \partial B_r(\bar{x}))} \int_{M\cap\partial B_r(\bar{x})} u_i.$$

Now we define the notion of isolated simple blow-up point.

Definition 3.2 Let $x_i \to \bar{x}$ be an isolated blow-up point of $\{u_i\}_i$ as in Definition 3.1. We say that $x_i \to \bar{x}$ is an *isolated simple blow-up point of* $\{u_i\}_i$ if, for some positive constants $\tilde{r} \in (0, \bar{r})$ and $C_2 > 1$, the function $\bar{w}_i(r) := r^{\frac{1}{q_i-1}} \bar{u}_i(r)$ satisfies, for large i,

$$\bar{w}_i'(r) < 0 \quad \text{for } r \text{ satisfying } C_2 u_i^{1-q_i}(x_i) \leq r \leq \tilde{r}.$$

For the analysis of interior blow up points, namely $x_i \to x \in \mathring{M}$, we mainly refer to [51] and [48], where the following proposition is proved.

Proposition 3.1 *Assume $\Omega \subseteq \mathring{M}$ to be an open set of a three dimensional manifold and that $\{K_i\}_i$ is uniformly bounded in $C^1(\overline{\Omega})$. Assume that $p_i \leq 5$, and $\{u_i\}_i$ are solutions of*

$$L_g u_i = K_i\, u_i^{p_i}, \qquad u_i > 0 \textit{ in } \Omega.$$

Then, if $\overline{x} \in \Omega$ is a blow up point for u_i, it is then an isolated simple blow up point. Moreover there exists a function $b : B_{\varrho/2}(\overline{x}) \to \mathbb{R}$ satisfying $L_g b = 0$ on $B_{\varrho/2}(\overline{x})$such that, passing to a subsequence, in some geodesic normal coordinate, for some positive constant $a > 0$,

$$u_i(x_i)\, u_i(x) \to a\, G(\overline{x}, x) + b(x), \qquad \textit{in } C^2_{loc}(B_{\varrho/2} \setminus \{\overline{x}\}), \tag{3.3}$$

where $G(\overline{x}, .)$ is the Green's function of L_g on $B_{\varrho/2}(\overline{x})$ under Dirichlet boundary conditions having a pole at $\overline{x}$ and $B_\varrho \subset \Omega$ is some geodesic ball where ϱ is given in Definition 3.2.

The above proposition describes in an accurate way the behavior of interior blow up points. Regarding boundary blow ups, we argue as follows:

For any $\bar{x} \in \partial M$, using the assumption that the boundray is umbilic, we may choose a conformal normal coordinate system centered at $\bar{x}$, and we can assume without loss of generality that

$$\bar{x} = 0, \quad g_{ij}(0) = \delta_{ij}, \quad B_1^+(0) := \{x = (x^1, x^2, x^3) : \ |x| < 1 \text{ and } x^3 > 0\} \subset M,$$
$$\{(x', 0) = (x^1, x^2, 0) : \ |x'| < 1\} \subset \partial M, \quad \Gamma_{ij}^k(0) = 0,$$

where Γ_{ij}^k is the Christofell's symbols. For later use, we denote

$$\mathbb{R}_+^3 = \{(x', x^3) \in \mathbb{R}^2 \times \mathbb{R} : \ x^3 > 0\}, \quad B_r^+(\bar{x}) = \{x = (x', x^3) \in \mathbb{R}_+^3 : \ |x - \bar{x}| < r\},$$
$$B_r^+ = B_r^+(0), \quad \Gamma_1(B_r^+(\bar{x})) = \partial B_r^+(\bar{x}) \cap \partial \mathbb{R}_+^3, \quad \Gamma_2(B_r(\bar{x})) = \partial B_r(\bar{x}) \cap \mathbb{R}_+^3.$$

Let $K_i \to K$ in $C^2(\overline{(B_3^+)})$ be a sequence of positive functions, q_i be a sequence of numbers satisfying $2 \leq q_i \leq 5$ and $\{v_i\}_i \subset C^2(\overline{B_3^+})$ be a sequence of solutions to

$$\begin{cases} L_g v_i = K_i v_i^{q_i}, \quad v_i > 0, & \text{in } B_3^+, \\ \dfrac{\partial v_i}{\partial \nu} = 0, & \text{on } \Gamma_1(B_3^+). \end{cases} \tag{3.4}$$

Before closing this section we state the following lemma which guarantees a Harnack Inequality of the sequence of blowing up solutions near an isolated boundary blow up point. Its proof is contained in [41].

Lemma 3.1 *Let v_i satisfy (Pi) and $y_i \to \bar{y} \in \Gamma_1(B_3^+)$ be an isolated blow-up of $\{v_i\}_i$. Then for any $0 < r < \bar{r}$, we have*

$$\max_{\overline{B_{2r}^+(y_i) \setminus B_{r/2}^+(y_i)}} v_i \leq C_3 \min_{\overline{B_{2r}^+(y_i) \setminus B_{r/2}^+(y_i)}} v_i,$$

where C_3 is some positive constant independent of i and r.

4 Study of isolated simple blow ups

This section is devoted to the study of isolated simple blow ups of equation 3.4. As we have already observed the situation of interior blow up has been treated in [51]; hence we are reduced to considering the case in which the blow up point $\overline{x} \in \partial M$.

Proposition 4.1 *Assume $\{K_i\}_i \subseteq C^1(\overline{B_3^+})$ and satisfies the condition*

$$\frac{1}{A_1} \leq K_i \leq A_1 \quad \text{and} \quad , \|\nabla K_i\|_{C(\overline{B_3^+})} \leq A_2 \tag{4.5}$$

for some $A_1, A_2 > 0$. For every i, let v_i be a positive solution of 3.4, and let $x_i \to \overline{x}' \in \Gamma_1(B_3^+)$ be an isolated blow up point for $\{v_i\}_i$. Then, given $R_i \to +\infty$ and $\varepsilon_i \to 0^+$, after passing to a subsequence of $\{v_i\}_i$ (still denoted by $\{v_i\}_i$) we have

$$\begin{cases} r_i := R_i v_i(x_i)^{-\frac{p_i-1}{2}} \to 0 \quad \text{as } i \to +\infty, \\ \|v_i(x_i)^{-1} v_i(exp_{x_i}(v_i(x_i)^{-\frac{p_i-1}{2}} \cdot)) - (1+k_i|\cdot|^2)^{-1}\|_{C^2(B^+_{2R_i})} \le \varepsilon_i, \\ \|v_i(x_i)^{-1} v_i(exp_{x_i}(v_i(x_i)^{-\frac{p_i-1}{2}} \cdot)) - (1+k_i|\cdot|^2)^{-1}\|_{H^1(B^+_{2R_i})} \le \varepsilon_i, \end{cases} \tag{4.6}$$

and

$$\frac{R_i}{log v_i(x_i)} \to 0 \quad \text{as } i \to \infty, \tag{4.7}$$

where $x = (x^1, x^2, x^3)$ *denotes some geodesic normal coordinates given by the exponential map* exp_{x_i} *with* $\frac{\partial}{\partial x^3}$ *be the unit inner normal of* M *at* $x = 0$.

Proof. Let x be geodesic normal coodinate sytem in a neighborhood of x_i given by $exp_{x_i}^{-1}$. We write $v_i(x)$ for $v_i(exp_{x_i}(x))$. Let $g_i = (g_i)_{\alpha\beta}(x)\, dx^\alpha dx^\beta = g_{\alpha\beta}(u_i^{1-q_i}(y_i)x)\, dx^\alpha dx^\beta$ denote the scaled metric. Consider the functions

$$w_i(x) = v_i(x_i)^{-1}\, v_i\left(v_i(x_i)^{-\frac{p_i-1}{2}}\, x\right), \quad \text{for } x \in B^{-T_i}_{v_i^{q_i-1}(y_i)},.$$

defined on the set

$$B^{-T_i}_{v_i^{q_i-1}(y_i)} := \left\{ z \in \mathbb{R}^3 : \ |z| < v_i^{q_i-1}(y_i) \quad \text{and} \quad z^3 > -T_i \right\}$$

where $T_i = y_i^3 v_i^{q_i-1}(y_i)$. It follows immediately that $w_i(0) = 1$ for all i and that 0 is a local maximum point for w_i. Moreover, from the assumption of isolated blow up we have

$$w_i(x) \le C\, |x|^{-\frac{p_i-1}{2}}, \qquad x \in B^{-T_i}_{v_i^{q_i-1}(y_i)},$$

where $\overline{r}$ is given in Definition 3.1. The function w_i is a solution of the problem

$$\begin{cases} -\Delta_{g_i} w_i(x) \,+\, \frac{1}{8}\, v_i^{2(1-q_i)}(y_i) R_{g_i}(v_i^{1-q_i}(y_i)x) \\ = K_i\left(v_i(x_i)^{\frac{q_i-1}{2}} x\right) w_i(x)^{q_i}, \text{ in } B^{-T_i}_{v_i^{q_i-1}(y_i)}; \\ \frac{\partial w_i}{\partial \nu_{g_i}}(x) = 0 \text{ on } \partial B^{-T_i}_{v_i^{q_i-1}(y_i)}. \end{cases}$$

Regarding the behavior of T_i, two cases may occur, namely:

$$T_i \to +\infty, \qquad \text{or} \qquad T_i \to T \in \mathbb{R}.$$

In the latter one, we can use (4.5) and the elliptic regularity results (see e.g [1]) to prove that the functions w_i converge up to subsequence, and then one can conclude, as in [41], Proposition 1.4. Hence it is sufficient to rule out the first case. In order to do this, define the functions

$$\xi_i(x) = (x_i^3)^{\frac{2}{q_i-1}}\, v_i(x_i + x_i^3\, x).$$

Since we are supposing, by contradiction, that $T_i \to +\infty$, it is clear that ξ_i is defined on the half space $\mathbb{R}^n_1 := \{x \in \mathbb{R}^n \,:\, x_n > -1\}$, and that 0 is an interior blow up point for the functions ξ_i, so from Proposition 3.1 it follows that 0 is an isolated simple blow up point. Using Lemma 3.1 and the inequality

$$\xi_i(x) \leq C\,|x|^{-\frac{p_i-1}{2}},$$

the convergence in (3.3) can be extended to the whole $\mathbb{R}^n_1 \setminus \{0\}$. Namely one has

$$\xi_i(0)\,\xi_i(x) \to h(x) \qquad \text{in } C^2_{loc}(\mathbb{R}^n_1 \setminus \{0\}),$$

where $h(x)$ is a non-negative harmonic function in $\mathbb{R}^n_1 \setminus \{0\}$ singular at 0 and satisfying

$$\frac{\partial h}{\partial x_n} = 0, \qquad \text{on } \partial\mathbb{R}^n_1. \tag{4.8}$$

By equation (4.8) and by the Schwartz's Reflection Principle, the function h possesses an harmonic extension to the set $\mathbb{R}^n \setminus \{0, \tilde{0}\}$, where $\tilde{0}$ is the symmetric point of 0 with respect to the plane $\partial\mathbb{R}^n_1$. By uniqueness of harmonic extensions, this must coincide with the symmetric prolongation of h through $\partial\mathbb{R}^n_1$. Hence the positivity of h implies that $h(x) = a|x|^{2-n} + A + o(|x|)$ for x close to 0 , where $a, A > 0$. Reasoning as in Proposition 3.1 of [48], one can reach a contradiction. □

The following technical Lemma describes the behavior of blowing up subcritcal solutions and will be used in the third section to rule out accumulation of bubbles.

Lemma 4.1 *Suppose that $\{v_i\}_i$ satisfies (3.4) and $\{x_i\}_i \subset \Gamma_1(B^+_1)$ is a sequence of local maximum points of $\{v_i\}_i$ in $\overline{B^+_3}$ such that*

$$\{v_i(x_i)\} \quad \textit{is bounded,}$$

and, for some positive constant C_5,

$$|x - x_i|^{\frac{1}{q_i-1}} v_i(x) \leq C_5, \quad \forall\, x \in B^+_3. \tag{4.9}$$

Then

$$\limsup_{i\to\infty} \max_{\overline{B^+_{1/4}(x_i)}} v_i < \infty. \tag{4.10}$$

Proof. Suppose that, under the assumptions of the Lemma, (4.10) fails; namely that, along a subsequence, for some $\tilde{x}_i \in \overline{B^+_{1/4}(x_i)}$ we have

$$v_i(\tilde{x}_i) = \max_{\overline{B^+_{1/4}(x_i)}} v_i \underset{i\to\infty}{\longrightarrow} +\infty.$$

It follows from (4.9) that $|\tilde{x}_i - x_i| \to 0$. Let us now consider

$$\xi_i(z) = v_i^{-1}(\tilde{x}_i) v_i(\tilde{x}_i + v_i^{1-q_i}(\tilde{x}_i) z)$$

defined on the set

$$B^{-T_i}_{\frac{1}{8}v_i^{q_i-1}(\tilde{x}_i)} := \left\{ z \in \mathbb{R}^n : \ |z| < \frac{1}{8} v_i^{q_i-1}(\tilde{x}_i) \quad \text{and} \quad z^n > -T_i \right\}$$

where $T_i = \tilde{x}_i^n v_i^{q_i-1}(\tilde{x}_i)$. Using equation (3.4), it is easy to see that ξ_i satisfies

$$-\Delta \xi_i = K_i(\tilde{x}_i + v_i^{1-q_i}(\tilde{x}_i)z)\,\xi_i^{q_i}, \quad \xi_i > 0, \qquad z \in B^{-T_i}_{\frac{1}{8}v_i^{q_i-1}(\tilde{x}_i)},$$
$$\frac{\partial \xi_i}{\partial z^n} = 0, \quad z \in \partial B^{-T_i}_{\frac{1}{8}v_i^{q_i-1}(\tilde{x}_i)} \cap \{z = (z', z^n) \in \mathbb{R}^n : \ z^n = -T_i\}, \tag{4.11}$$

and

$$\xi_i(z) \leq \xi_i(0) = 1, \quad \forall\, z \in B^{-T_i}_{\frac{1}{8}v_i^{q_i-1}(\tilde{x}_i)}.$$

It follows from (4.9) that

$$|z|^{\frac{1}{q_i-1}}\xi_i(z) \leq C_1, \quad \forall\, z \in B^{-T_i}_{\frac{1}{8}v_i^{q_i-1}(\tilde{x}_i)}.$$

Since $\{\xi_i\}_i$ is locally bounded, applying L^p-estimates, Schauder estimates and Lemma 3.1, we have that, up to a subsequence, there exists some positive function ξ such that

$$\lim_{i\to\infty} \|\xi_i - \xi\|_{C^2(\mathbb{R}^n_{-T_i} \cap \overline{B_R})} = 0, \quad \forall\, R > 1,$$

where $\mathbb{R}^n_{-T_i} = \{z = (z', z^n) \in \mathbb{R}^n : \ z^n > -T_i\}$ and, for $T = \lim_{i\to\infty} T_i \in [0, +\infty]$, ξ satisfies

$$\begin{cases} -\Delta \xi = K(x_0)\xi^{\frac{n+2}{n-2}}, \quad \xi > 0, \quad \text{in } \mathbb{R}^n_{-T}, \\ \frac{\partial \xi}{\partial z^n} = 0, \qquad\qquad \text{on} \quad \partial\mathbb{R}^n_{-T}, \end{cases} \tag{4.12}$$

where $x_0 := \lim x_i$.

It follows that, for all $R > 1$

$$\min_{\overline{B}^{-T_i}_{Rv_i^{-(q_i-1)}(\tilde{x}_i)}(\tilde{x}_i)} v_i = v_i(\tilde{x}_i) \min_{\overline{B}^{-T_i}_R(0)} \xi_i \underset{i\to\infty}{\longrightarrow} \infty.$$

Since $\{v_i(x_i)\}_i$ is bounded, we have that, for any $R > 1$, $x_i \notin \bar{B}^{-T_i}_{Rv_i^{q_i-1}(\tilde{x}_i)}(\tilde{x}_i)$ for large i, namely

$$R < v_i^{q_i-1}(\tilde{x}_i)|\tilde{x}_i - x_i|.$$

Hence we have that

$$|\tilde{x}_i - x_i|^{\frac{1}{q_i-1}} v_i(\tilde{x}_i) > R^{\frac{1}{q_i-1}}$$

which contradicts (4.9). □

Next, we establish the counterpart of Proposition 3.1 for boundary blow up points . Now we state our main estimate on isolated simple blow-up points.

Proposition 4.2 *Suppose $\{K_i\}_i \subseteq C^1(\overline{B_1^+})$, satisfying condition (4.5) for some $A_1, A_2 > 0$. Suppose that for every i, v_i satisfies (3.4) and that $y_i \to 0$ is an isolated simple blow up. Then for some positive constant C depending only on C_1, $\tilde{r}$, $\|K_i\|_{C^2(\Gamma_1(B_3^+))}$, and $\inf_{y\in\Gamma_1(B_1^+)} K_i(y)$, we have*

$$v_i(y) \leq C v_i^{-1}(y_i) d(y, y_i)^{-1}, \quad \textit{for} \quad d(y, y_i) \leq \frac{\tilde{r}}{2} \tag{4.13}$$

where C_1 and $\tilde{r}$ are given in Definitions 3.1 and 3.2. Furthermore, after passing to some subsequence, for some positive constant b,

$$v_i(y_i)v_i \underset{i\to+\infty}{\longrightarrow} b\,G(\cdot,\bar{y})+E \quad in\ C^2_{\mathrm{loc}}(\overline{B^+_{\tilde{\rho}}(\bar{y})}\setminus\{\bar{y}\})$$

where $\tilde{\rho}=\min(\delta_0,\tilde{r}/2)$ and $E\in C^2(B^+_{\tilde{\rho}}(\bar{y}))$ satisfy

$$\begin{cases}-\Delta_g E+\dfrac{1}{8}R_g E=0, & in\ B^+_{\tilde{\rho}},\\ \dfrac{\partial E}{\partial \nu}=0, & on\ \Gamma_1(B^+_{\tilde{\rho}}).\end{cases}$$

Proposition 4.2 will be established through a series of lemmas.

Lemma 4.2 *Let v_i satisfy (3.4) and $y_i\to\bar{y}\in\Gamma_1(B^+_1)$ be an isolated simple blow-up. Assume $R_i\to+\infty$ and $0<\varepsilon_i<e^{-R_i}$ are sequences for which (4.6) and (4.7) hold. Then for any given $0<\delta<1/100$, there exists $\rho_1\in(0,\tilde{r})$ which is independent of i (but depending on δ), such that*

$$v_i(y_i)\le C_4 v_i^{-\lambda_i}(y_i)d(y,y_i)^{-1+\delta}, \qquad \forall\, r_i\le d(y,y_i)\le\rho_1, \tag{4.14}$$

$$\nabla_g v_i(y_i)\le C_4 v_i^{-\lambda_i}(y_i)d(y,y_i)^{-2+\delta}, \qquad \forall\, r_i\le d(y,y_i)\le\rho_1, \tag{4.15}$$

$$\nabla^2_g v_i(y_i)\le C_4 v_i^{-\lambda_i}(y_i)d(y,y_i)^{-3+\delta}, \qquad \forall\, r_i\le d(y,y_i)\le\rho_1, \tag{4.16}$$

where $r_i=R_i v_i^{1-q_i}(y_i)$, $\lambda_i=(1-\delta)(q_i-1)-1$, and C_4 is some positive constant independent of i.

Proof. We assume, for simplicity, that g is the flat metric. The general case can be derived essentially in the same way. Let $r_i=R_i v_i^{1-q_i}(y_i)$. It follows from Proposition 4.1 that

$$v_i(y)\le c v_i(y_i)R_i^{-1}, \quad \text{for } d(y,y_i)=r_i. \tag{4.17}$$

We then derive from Lemma 3.1, (4.17), and the definition of isolated simple blow-up that, for $r_i\le d(y,y_i)\le\tilde{r}$, we have

$$v_i^{q_i-1}(y)\le cR_i^{-1+o(1)}d(y,y_i)^{-1}. \tag{4.18}$$

Set $T_i=y_i^3 v_i^{q_i-1}(y_i)$. From the proof of Proposition 4.1 we know that , without loss of generality, we may take $\lim_i T_i=0$. It is not restrictive to take $y_i=(0,0,y_i^3)$. Thus we have $d(0,y_i^3)=o(r_i)$. So

$$B^+_1(0)\setminus B^+_{2r_i}(0)\subset\left\{\frac{3}{2}r_i\le d(y,y_i)\le\frac{3}{2}\right\}.$$

Now following [41], we construct comparision functions and apply the maximum priciple as it is stated in Theorem 1.1 in the Appendix. To this aim, set

$$\varphi_i(y)=M_i\left(|y|^{-\delta}-\varepsilon|y|^{\delta-1}y^3\right)+Av_i^{-\lambda_i}(y_i)\left(|y|^{-1+\delta}\right)\,-v_i(y)$$

with M_i and A to be chosen later, and let $\mathcal{L}_i := \Delta + K_i\, v_i^{q_i-1}$ and Φ_i be the boundary operator defined by

$$\Phi_i(v) = \frac{\partial v}{\partial \nu}.$$

A direct computation yields

$$\Delta\varphi_i(y) = M_i|y|^{-\delta}[-\delta(1-\delta)+O(\varepsilon)] + |y|^{-(3-\delta)}Av_i^{-\lambda_i}(y_i)[-\delta(1-\delta)+O(\varepsilon)].$$

So one can choose $\varepsilon = O(\delta)$ such that $\mathcal{L}_i\varphi_i \leq 0$.

Another straightforward computation, taking into account (4.18), shows that for $\delta > 0$ there exists $\rho_1(\delta) > 0$ such that

$$\Phi_i\varphi_i > 0 \quad \text{on } \Gamma_1(B^+_{\rho_1}).$$

Taking

$$\begin{aligned} &\Omega = D_i = B^+_{\rho_1} \setminus B^+_{2r_i}(0) && \\ &\Sigma = \Gamma_1(D_i), && \Gamma = \Gamma_2(D_i), \\ &V = K_i v_i^{q_i-1}, && h \equiv 0, \\ &\psi = v_i, && v = \varphi_i - v_i, \end{aligned}$$

and choosing $A = O(\delta)$ such that $\varphi_i \geq 0$ on $\Gamma_2(D_i)$ and $M_i = \max_{\Gamma_1(B^+_{\rho_1})} v_i$, we deduce from Theorem 1.1 of the Appendix that

$$v_i(x) \leq \varphi_i(x). \tag{4.19}$$

By the Harnack inequality and the assumption that the blow-up is isolated simple, we derive that

$$M_i \leq c v_i^{-\lambda_i}(y_i). \tag{4.20}$$

The estimate (4.14) of the lemma follows from (4.19) and (4.20).

To derive (4.15) from (4.14), we argue as follows. For $r_i \leq |\tilde{y}| \leq \rho_1/2$, we consider

$$w_i(z) = |\tilde{y}|^{1-\delta} v_i^{\lambda_i}(y_i) v_i(|\tilde{y}|z), \quad \text{for } \frac{1}{2} \leq |z| \leq 2,\ z^3 \geq 0.$$

It follows from (3.4) that w_i satisfies

$$\begin{cases} -\Delta w_i = K_i(|\tilde{y}|z)|\tilde{y}|^{-\lambda_i} v_i^{\lambda_i(1-q_i)}(y_i) w_i^{q_i}, & \text{in } \left\{\frac{1}{2} < |z| < 2 :\ z^3 > 0\right\}, \\ \dfrac{\partial w_i}{\partial \nu} = 0, & \text{on } \left\{\frac{1}{2} < |z| < 2 :\ z^3 = 0\right\}. \end{cases} \tag{4.21}$$

In view of (4.14), we have $w_i(z) \leq c$ for any $\frac{1}{2} \leq |z| \leq 2$, $z^3 \geq 0$. We then derive from (4.21) and gradient elliptic estimates that

$$|\nabla w_i(z)| \leq c, \qquad z \in \Gamma_2(B^+_1)$$

which implies that

$$|\nabla v_i(\tilde{y})| \leq c|\tilde{y}|^{-2+\delta} v_i^{-\lambda_i}(y_i).$$

This establishes (4.15). Estimate (4.16) can be derived in a similar way. We omit the details. Lemma 4.2 is thus established. □

Lemma 4.3 *Let $K \in C^2(\overline{B_1^+})$ and $u \in C^2(\overline{B_1^+})$ satisfy, for $q > 0$,*

$$\begin{cases} -\Delta_g u + \frac{1}{8} R_g u = K u^q, \quad u > 0, & \text{in } B_1^+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1(B_1^+). \end{cases}$$

Then we have, for any r such that $0 < r \leq 1$,

$$\begin{aligned} &\frac{1}{q+1} \int_{B_r^+} (x \cdot \nabla_x K) u^{q+1} \, dx + \left(\frac{3}{q+1} - \frac{1}{2} \right) \int_{B_r^+} K u^{q+1} \, dx \\ &- \frac{1}{16} \int_{B_r^+} (x \cdot \nabla R_g) u^2 \, dx - \frac{3}{16} \int_{B_r^+} R_g \, u^2 \, dx - \frac{r}{16} \int_{\Gamma_2(B_r^+)} R_g \, u^2 \, ds \\ &- \frac{r}{q+1} \int_{\Gamma_1(B_r^+)} K u^{q+1} \, ds = \int_{\Gamma_2(B_r^+)} B(r, x, u, \nabla u) \, dv + A(g, u) \end{aligned}$$

where

$$B(r, x, u, \nabla u) = \frac{1}{2} \frac{\partial u}{\partial \nu} u + \frac{1}{2} r \left(\frac{\partial u}{\partial \nu} \right)^2 - \frac{1}{2} r |\nabla_T u|^2, \tag{4.22}$$

$\nabla_T u$ denotes the component of ∇u which is tangent to $\Gamma_2(B_r^+)$,

$$\begin{aligned} A(g, u) = &\int_{B_r^+} (x^k \partial_k u)(g_{ij} - \delta_{ij}) \partial_{ij} u \, dx - \int_{B_r^+} (x^l \partial_l u)(g_{ij} - \Gamma_{ij}^k \partial_k u) \, dx \\ &+ \frac{1}{2} \int_{B_r^+} u (g^{ij} - \delta^{ij}) \partial_{ij} u \, dx - \frac{1}{2} \int_{B_r^+} u \, g^{ij} \Gamma_{ij}^k \partial_k u \, dx \\ &- \int_{\Gamma_1(B_r^+)} x^i \frac{\partial u}{\partial x_i} (g^{ij} - \delta^{ij}) \frac{\partial u}{\partial x_i} \nu_j - \frac{1}{2} \int_{\Gamma_1(B_r^+)} (g^{ij} - \delta^{ij}) \frac{\partial u}{\partial x_i} \nu_j u, \end{aligned} \tag{4.23}$$

and Γ_{ij}^k denotes the Christoffel symbol.

Regarding the term $A(g, u_i)$, where u_i is a solution of (3.4), we have the following estimate, the proof of which is a direct consequence of Lemma 4.1 and Lemma 4.2.

Lemma 4.4 *Let $\{u_i\}_i$ satisfy (3.4), $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated simple blow-up point. Assume $R_i \to +\infty$ and $0 < \varepsilon_i < e^{-R_i}$ are sequences for which (4.6) and (4.7) hold. Then, for $0 < r < \rho_1$, we have*

$$|A(g, u_i)| \leq C_5 r u_i^{-2\lambda_i}(y_i)$$

where C_5 is some constant independent of i and r.

Using Proposition 4.1, Proposition 4.2, Lemma 4.3, Lemma 4.4, and standard elliptic estimates, we derive the following estimate about the rate of blow-up of the solutions of (3.4).

Lemma 4.5 *Let u_i satisfy (3.4) and $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated simple blow-up point. Assume $R_i \to +\infty$ and $0 < \varepsilon_i < e^{-R_i}$ are sequences for which (4.6) and (4.7) hold. Then*

$$\tau_i = O\left(u_i^{-2\lambda_i}(y_i)\right).$$

Consequently $u_i^{\tau_i}(y_i) \to 1$ as $i \to \infty$.

Lemma 4.6 *Let u_i satisfy (3.4) and $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated simple blow-up point. Then, for $0 < r < \tilde{r}/2$, we have*

$$\limsup_{i \to +\infty} \max_{y \in \Gamma_2(B_r^+(y_i))} u_i(y_i) u_i(y) \le C(r).$$

Proof. Due to Lemma 3.1, it is enough to establish the lemma for $r > 0$ sufficiently small. Without loss of generality we may take $\bar{r} = 1$. If we pick any $y_r \in \Gamma_2(B_r^+)$ and set

$$\xi_i(y) = u_i^{-1}(y_r) u_i(y),$$

then ξ_i satisfies

$$\begin{cases} -\Delta_g \xi_i + \dfrac{1}{8} R_g \xi_i = K_i u_i^{q_i - 1}(y_r) \xi_i^{q_i},, & \text{in } B_{1/2}^+(\bar{y}), \\ \dfrac{\partial \xi_i}{\partial \nu} = 0, & \text{on } \Gamma_1(B_{1/2}^+(\bar{y})). \end{cases}$$

It follows from Lemma 3.1 that for any compact set $K \subset B_{1/2}^+(\bar{y}) \setminus \{\bar{y}\}$, there exists some constant $c(K)$ such that

$$c(K)^{-1} \le \xi_i \le c(K), \quad \text{on } K.$$

We also know from (4.14) that $u_i(y_r) \to 0$ as $i \to +\infty$. Then by elliptic theories, we have, after passing to a subsequence, that $\xi_i \to \xi$ in $C^2_{\text{loc}}(B_{1/2}^+(\bar{y}) \setminus \{\bar{y}\})$, where ξ satisfies

$$\begin{cases} -\Delta_g \xi + \dfrac{1}{8} R_g \xi = 0, & \text{in } B_{1/2}^+(\bar{y}), \\ \dfrac{\partial \xi}{\partial \nu} = 0, & \text{on } \Gamma_1(B_{1/2}^+) \setminus \{\bar{y}\}. \end{cases}$$

From the assumption that $y_i \to \bar{y}$ is an isolated simple blow-up point of $\{u_i\}_i$, we know that the function $r^{1/2}\bar{\xi}(r)$ is nonincreasing in the interval $(0, \tilde{r})$ and so we deduce that ξ is singular at $\bar{y}$. So it follows from Corollary 1.1 in the Appendix that for r small enough there exists some positive constant $m > 0$ independent of i such that for i large we have

$$-\int_{B_r^+} \Delta_g \xi_i = -\int_{\Gamma_2(B_r^+)} \nabla_g \xi_i \cdot \nu = -\int_{\Gamma_2(B_r^+)} \nabla_g \xi \cdot \nu + o(1) > m$$

which implies that

$$-\int_{B_r^+} \Delta_g \xi_i > m. \tag{4.24}$$

On the other hand, since $R_g \geq 0$,

$$-\int_{B_r^+} \Delta_g \xi_i = \int_{B_r^+} \left(K_i u_i^{-1}(y_r)\, v_i^{q_i} \; - \frac{1}{8}\, R_g \xi_i\right) \leq u_i^{-1}(y_r) \int_{B_r^+)} K_i u_i^{q_i}. \tag{4.25}$$

Using Lemma 4.1 and Lemma 4.2, we derive that

$$\int_{B_r^+} K_i u_i^{q_i} \leq c u_i^{-1}(y_i). \tag{4.26}$$

Hence our lemma follows from (4.24), (4.25), and (4.26). □

Now we are able to give the proof of Proposition 4.2.

Proof of Proposition 4.2. We first establish (4.13) arguing by contradiction. Suppose the contrary. Then, possibly passing to a subsequence still denoted as v_i, there exists a sequence $\{\tilde{y_i}\}_i$ such that $d(\tilde{y_i}, y_i) \leq \tilde{r}/2$ and

$$v_i(\tilde{y_i}) v_i(y_i) d(\tilde{y_i}, y_i) \underset{i \to +\infty}{\longrightarrow} +\infty. \tag{4.27}$$

Set $\tilde{r}_i = d(\tilde{y_i}, y_i)$. From Lemma 4.1 it is clear that $\tilde{r}_i \geq r_i = R_i v_i^{1-q_i}(y_i)$. Set

$$\tilde{v_i}(x) = \tilde{r}_i^{\frac{1}{q_i - 1}} v_i(y_i + \tilde{r}_i x) \quad \text{in } B_2^{-T_i}, \quad T_i = \tilde{r}_i^{-1} y_i^3.$$

Clearly $\tilde{v_i}$ satisfies

$$\begin{cases} -\Delta_{g_i} \tilde{v_i} + \dfrac{1}{8}\, \widetilde{R}_{g_i} \tilde{v_i} = \widetilde{K}_i(x) \tilde{v_i}^{q_i}(x), \quad v_i > 0, \quad \text{in } B_2^{-T_i}, \\ \dfrac{\partial \tilde{v_i}}{\partial \nu} = 0, \text{on } \partial B_2^{-T_i} \cap \{x^3 = -T_i\}, \end{cases}$$

where

$$\begin{aligned} (g_i)_{\alpha\beta} &= g_{\alpha\beta}(\tilde{r}_i x)\, dx^\alpha dx^\beta, \\ \widetilde{R}_{g_i}(x) &= \tilde{r}_i^2 R_{g_i}(y_i + \tilde{r}_i x), \end{aligned}$$

and

$$\widetilde{K}_i(x) = K_i(y_i + \tilde{r}_i x).$$

Lemma 4.6 yields that

$$\max_{x \in \Gamma_2(B_{1/2}^+)} \tilde{v_i}(0) \tilde{v_i}(x) \leq c$$

for some positive constant c, and so

$$v_i(\tilde{y_i}) v_i(y_i) d(y_i, y_i) \leq c.$$

This contradicts (4.27). Therefore (4.13) is established. Now take

$$w_i(x) = v_i(y_i) v_i(x).$$

From (3.4) it is clear that w_i satisfies

$$\begin{cases} -\Delta_g w_i + \frac{1}{8} R_g w_i = K_i(x) v_i^{1-q_i}(y_i) w_i^{q_i}, & \text{in } B_3^+, \\ \frac{\partial w_i}{\partial \nu} = 0 & \text{on } \Gamma_1(B_3^+). \end{cases}$$

Estimate (4.13) implies that $w_i(x) \leq c\, d(x, y_i)^{-1}$. Since $y_i \to \bar{y}$, w_i is locally bounded in any compact set not containing $\bar{y}$. Then, up to a subsequence, $w_i \to w$ in $C^2_{\text{loc}}(B_{\tilde{\rho}}(\bar{y}) \setminus \{\bar{y}\})$ for some $w > 0$, satisfying

$$\begin{cases} -\Delta_g w + \frac{1}{8} R_g w = 0, & \text{in } B_{\tilde{\rho}}^+(\bar{y}), \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \Gamma_1(B_{\tilde{\rho}}^+) \setminus \{\bar{y}\}. \end{cases}$$

From Proposition 1.2 of the Appendix, we have that

$$w = b\, G(\cdot, \bar{y}) + E, \quad \text{in } B_{\tilde{\rho}}^+ \setminus \{0\},$$

where $b \geq 0$, E is a regular function satisfying

$$\begin{cases} -\Delta_g E + \frac{1}{8} R_g E = 0, & \text{in } B_{\tilde{\rho}}^+, \\ \frac{\partial E}{\partial \nu} = 0, & \text{on } \Gamma_1(B_{\tilde{\rho}}^+), \end{cases}$$

and $G \in C^2(B_{\tilde{\rho}}^+ \setminus \{\bar{y}\})$ satisfies

$$\begin{cases} -L_g G(\cdot, \bar{y}) = 0, & \text{in } B_{\tilde{\rho}}^+, \\ \frac{\partial G_a}{\partial \nu} = 0, & \text{on } \Gamma_1(B_{\tilde{\rho}}^+) \setminus \{\bar{y}\}, \end{cases}$$

and $\lim_{y \to \bar{y}} d(y, \bar{y}) G(y, \bar{y})$ is a constant. Moreover, w is singular at $\bar{y}$. Indeed from the definition of isolated simple blow-up we know that the function $r^{1/2}\bar{w}(r)$ is nonincreasing in the interval $(0, \tilde{r})$, which implies that w is singular at the origin and hence $b > 0$. The proof of Proposition 4.2 is thereby complete. □

Lemma 4.7 *Suppose that the hypotheses of Proposition 4.2 hold true. Then we have the following estimates:*

$$\int_{(B_{r_i}(x_i))_+} |x - x_i|^s\, v_i(x)^{p_i+1} = O\left(v_i(x_i)^{-\frac{2s}{n-2}}\right), \qquad 0 \leq s < n;$$

$$\int_{(B_1(x_i))_+ \setminus (B_{r_i}(x_i))_+} |x - x_i|^s\, v_i(x)^{p_i+1} = o\left(v_i(x_i)^{-\frac{2s}{n-2}}\right), \qquad 0 \leq s < n;$$

$$\int_{\partial_1 B_{r_i}(x_i)} |x' - x_i'|^s\, v_i(x')^{\frac{p_i+3}{2}} = O\left(v_i(x_i)^{-\frac{2s}{n-2}}\right), \qquad 0 \leq s < n-1;$$

$$\int_{\partial_1 B_1(x_i)\setminus \partial_1 B_{r_i}(x_i)} |x'-x_i'|^s\, v_i(x')^{\frac{p_i+3}{2}} = o\left(v_i(x_i')^{-\frac{2s}{n-2}}\right), \qquad 0 \le s < n-1;$$

$$\int_{\partial_1 B_1(x_i)} |x'-x_i'|^s\, v_i(x')^{p_i+1} = O\left(v_i(x_i)^{-2\frac{n-1}{n-2}}\log v_i(x_i)\right), \qquad s = n-1.$$

Proof. The proof is a simple consequence of Proposition 4.1 and Proposition 4.2. □

Using Proposition 4.2, one can strengthen the results of Lemmas 4.2 and 4.4 by just using (4.13) instead of (4.14), thus obtaining the following corollary.

Corollary 4.1 *Let $\{v_i\}_i$ satisfy (3.4), $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated simple blow-up point. Assume $R_i \to +\infty$ and $0 < \varepsilon_i < e^{-R_i}$ are sequences for which (4.6) and (4.7) hold. Then there exists $\rho_1 \in (0,\tilde{r})$ such that*

$$|\nabla_g v_i(y)| \le C_4 v_i^{-1}(y_i) d(y,y_i)^{-2}, \quad \textit{for all } r_i \le d(y,y_i) \le \rho_1, \tag{4.28}$$

and

$$|\nabla_g^2 v_i(y)| \le C_4 v_i^{-1}(y_i) d(y,y_i)^{-3}, \quad \textit{for all } r_i \le d(y,y_i) \le \rho_1, \tag{4.29}$$

where $r_i = R_i v_i^{1-q_i}(y_i)$ and C_4 is some positive constant independent of i. Moreover

$$|A(g,v_i)| \le C_5 r v_i^{-2}(y_i),$$

for some positive constant C_5 independent of i.

Let us obtain an upper bound estimate for $\nabla_g K_i(y_i)$.

Lemma 4.8 *Let v_i satisfy (3.4) and $y_i \to \bar{y} \in \Gamma_1(B_1^+)$ be an isolated simple blow-up point. Then*

$$\nabla_T K_i(y_i) = O(v_i^{-2}(y_i)),$$

where ∇_T denotes the tangential part of the gradient.

Proof. Let $x = (x^1, x^2, x^3)$ be some geodesic normal coordinates given by the exponential map exp_{y_i} with $\frac{\partial}{\partial y^3}$ being the unit inner normal to $\Gamma(B:1^+)$. Choose a test function $\eta \in C^\infty(B_1)$ which satisfies

$$\eta(x) = 1, \quad x \in B_{1/4}^+; \qquad\qquad \eta(x) = 0, \quad x \in B_1^+ \setminus B_{1/2}^+.$$

Multiplying equation 3.4 by $\eta\,\frac{\partial v_i}{\partial x_j}$, $j = 1, 2$, we obtain

$$\int_{(B_1)_+} (-\Delta v_i)\eta\,\frac{\partial v_i}{\partial x_j} = \frac{1}{8}\int_{(B_1)_+} K_i\, v_i^{q_i}\eta\,\frac{\partial v_i}{\partial x_j}.$$

Integrating by parts, we deduce

$$\int_{B_1^+} K_i\, v_i^{q_i}\eta\,\frac{\partial v_i}{\partial x_j} = -\frac{1}{q_i+1}\int_{(B_1)_+} v_i^{q_i+1}\left(\eta\,\frac{\partial K_i}{\partial x_j} + K_i\,\frac{\partial \eta}{\partial x_j}\right).$$

and also

$$\int_{(B_1)_+} (-\Delta v_i)\, \eta \, \frac{\partial v_i}{\partial x_j} = -\frac{1}{2} \int_{(B_1)_+} |\nabla v_i|^2 \, \frac{\partial \eta}{\partial x_j} + \int_{(B_1)_+} \nabla v_i \, \frac{\partial v_i}{\partial x_j} \, \nabla \eta.$$

From the above equations, Proposition 4.2, and the fact that $\nabla \eta$ has support in

$$\overline{(B_{1/2})_+} \setminus (B_{1/4})_+,$$

we obtain

$$\frac{1}{q_i+1} \int_{(B_1)_+} v_i^{q_i+1} \, \eta \, \frac{\partial K_i}{\partial x_j} \; = O(v_i(x_i)^{-2}).$$

Using the uniform bounds on the second derivatives of K_i and taking into account Lemma 4.5, we deduce

$$\frac{1}{6} \, \frac{\partial K_i}{\partial x_j}(x_i') \int_{(B_1)_+} v_i^{q_i+1} \, \eta \; = O(v_i(x_i)^{-2}). \tag{4.30}$$

□

Corollary 4.2 *Under the same assumptions of Lemma 4.8, one has that*

$$\int_{\Gamma_1(B_r^+)} x' \cdot \nabla_{x'} K_i v_i^{q_i+1} \, d\sigma = O(v_i^{-4}(y_i)).$$

Proof. We have that

$$\int_{\Gamma_1(B_r^+)} x' \cdot \nabla_{x'} K_i v_i^{q_i+1} \, d\sigma = \int_{\Gamma_1(B_r^+)} \nabla_{x'} K_i(y_i) \cdot (x' - y_i) v_i^{q_i+1} \, d\sigma$$
$$+ O\bigg(\int_{\Gamma_1(B_r^+)} |x'|^2 v_i^{q_i+1} \, d\sigma \bigg).$$

Since, using Proposition 4.2 and Lemma 4.1, $\int_{\Gamma_1(B_r^+)} (x' - y_i) v_i^{q_i+1} \, d\sigma = O(v_i^{-2}(y_i))$, from the previous lemma, Corollary 4.1, and (4.6), we reach the conclusion. □

Proposition 4.3 *Let v_i satisfy (3.4), $y_i \to \bar{y}$ be an isolated simple blow-up point and, for some $\tilde{\rho} > 0$,*

$$v_i(y_i) v_i \underset{i \to +\infty}{\longrightarrow} h, \quad \text{in } C^2_{\text{loc}}(B^+_{\tilde{\rho}}(\bar{y}) \setminus \{\bar{y}\}).$$

Assume, for some $\beta > 0$, that in some geodesic normal coordinate system $x = (x^1, x^2, x^3)$,

$$h(x) = \frac{\beta}{|x|} + A + o(1), \quad \text{as } |x| \to 0.$$

Then $A \leq 0$.

Proof. For $r > 0$ small, the Pohozaev type identity of Lemma 4.3 yields

$$\begin{aligned}&\frac{1}{q_i+1}\int_{\Gamma_1(B_r^+)}(x'\cdot\nabla_{x'}K_i)v_i^{q_i+1}\,ds+\Big(\frac{2}{q_i+1}-\frac{1}{2}\Big)\int_{\Gamma_1(B_r^+)}K_i v_i^{q_i+1}\,ds\\&-\frac{1}{16}\int_{B_r^+}(x\cdot\nabla R_g)v_i^2\,dx-\frac{1}{8}\int_{B_r^+}R_g\,v_i^2\,dx-\frac{r}{16}\int_{\Gamma_2(B_r^+)}R_g\,v_i^2\,ds\\&-\frac{r}{q_i+1}\int_{\partial\Gamma_1(B_r^+)}K_i v_i^{q_i+1}=\int_{\Gamma_2(B_r^+)}B(r,x,v_i,\nabla v_i)\,ds+A(g,v_i)\end{aligned}\tag{4.31}$$

where B and $A(g, v_i)$ are defined in (4.22) and (4.23) respectively. Multiply (4.31) by $v_i^2(y_i)$ and let $i \to \infty$. Using Corollary 4.1, Lemma 4.1, and Corollary 4.2, one has that

$$\lim_{r\to 0^+}\int_{\Gamma_2(B_r^+)}B(r,x,h,\nabla h)=\lim_{r\to 0^+}\limsup_{i\to\infty}v_i^2(y_i)\int_{\Gamma_2(B_r^+)}B(r,x,v_i,\nabla v_i)\geq 0.\tag{4.32}$$

On the other hand, a direct calculation yields

$$\lim_{r\to 0^+}\int_{\Gamma_2(B_r^+)}B(r,x,h,\nabla h)=-c\,A\tag{4.33}$$

for some $c > 0$. The conclusion follows from (4.32) and (4.33). □

Now we ready prove that an isolated blow-up point is in fact an isolated simple blow-up point.

Proposition 4.4 *Let v_i satisfy (3.4) and $y_i \to \bar{y}$ be an isolated blow-up point. Then $\bar{y}$ must be an isolated simple blow-up point.*

Proof. From Lemma 4.1, it follows that

$$\bar{w_i}'(r) < 0 \quad \text{for every } C_2 v_i^{1-q_i}(y_i) \leq r \leq r_i.\tag{4.34}$$

Suppose that the blow-up is not simple. Then there exist some sequences

$$\tilde{r}_i \to 0^+,\ \ \tilde{c}_i \to\ +\infty \text{ such that } \tilde{c}_i v_i^{1-q_i}(y_i) \leq \tilde{r}_i$$

and, after passing to a subsequence,

$$\bar{w}_i'(\tilde{r}_i) \geq 0.\tag{4.35}$$

From (4.34) and (4.35) it is clear that $\tilde{r}_i \geq r_i$ and $\bar{w}_i$ has at least one critical point in the interval $[r_i, \tilde{r}_i]$. Let μ_i be the smallest critical point of $\bar{w}_i$ in this interval. We have that

$$\tilde{r}_i \geq \mu_i \geq r_i \quad \text{and} \quad \lim_{i\to\infty}\mu_i = 0.$$

Let $g_i = (g_i)_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta}(\mu_i x)\, dx^\alpha dx^\beta$ be the scaled metric and

$$\xi_i(x) = \mu_i^{\frac{1}{q_i-1}} v_i(y_i + \mu_i x).$$

Then ξ_i satisfies

$$\begin{cases} -\Delta_{g_i}\xi_i + \dfrac{1}{8} R_{g_i}\xi_i = \widetilde{K}_i(x)\,\xi_i^{q_i}, & \text{in } B_{1/\mu_i}^{-T_i}, \\ \dfrac{\partial \xi_i}{\partial \nu} = 0, & \text{on } \partial B_{1/\mu_i}^{-T_i} \cap \{x^3 = -T_i\}, \\ \lim_{i\to\infty} \xi_i(0) = \infty & \text{and 0 is a local maximum point of } \xi_i, \\ r^{\frac{1}{q_i-1}}\bar{\xi}_i(r) & \text{has negative derivative in } \quad c\,\xi_i(0)^{1-q_i} < r < 1, \\ \frac{d}{dr}\left(r^{\frac{1}{q_i-1}}\bar{\xi}_i(r)\right)\Big|_{r=1} = 0, & \end{cases} \tag{4.36}$$

where $T_i = \mu_i^{-1} y_i^3$, $\tilde{a}_i(x) = \mu_i a_i(y_i + \mu_i x)$, and $\widetilde{K}_i(x) = K_i(y_i + \mu v_i x)$. Arguing as in the proof of Lemma 4.1, we can easily prove that $T_i \to 0$. Since 0 is an isolated simple blow-up point, by Proposition 4.2 and Lemma 3.1, we have that, for some $\beta > 0$,

$$\xi_i(0)\xi_i \underset{i\to+\infty}{\longrightarrow} h = \beta\,|x|^{-1} + E \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^3_+ \setminus \{0\}) \tag{4.37}$$

with E satisfying

$$\begin{cases} -\Delta E = 0, & \text{in } \mathbb{R}^3_+, \\ \dfrac{\partial E}{\partial \nu} = 0, & \text{on } \partial\mathbb{R}^3_+. \end{cases}$$

By the Maximum Principle we have that $E \geq 0$. Reflecting E to be defined on all $\mathbb{R}^3$ and thus using the Liouville Theorem, we deduce that E is a constant. Using the last equality in (4.36) and (4.37), we deduce that $E \equiv b$. Therefore, $h(x) = b(G_a(x, \bar{y}) + 1)$ and this fact contradicts Proposition 4.3. □

5 Ruling out bubble accumulations

In this section we use the blow up analysis performed in the previous sections and following the original arguments of R. Schoen [58] and Z.C. Han and Y.Y. Li see[41], to rule out the possible accumulations of bubbles , and this implies that only isolated blow-up points may occur to blowing-up sequences of solutions of the approximated problem. As we have already recalled in Section 3, accumulation of interior blow ups is ruled out through the work of Y.Y. Li [48] and Y.Y. Li and M. Zhu [51]. So we have only to deal with boundary blow ups.

Proposition 5.1 *Let (M, g) be a smooth, compact, three dimensional Riemannian manifold with umbilic boundary. For any $R \geq 1$, $0 < \varepsilon < 1$, there exist positive constants δ_0, c_0, and c_1 depending only on M, g, $\|K\|_{C^2(\partial M)}$, $\inf_{y\in\partial M} K(y)$, R, and ε, such that for all u in*

$$\bigcup_{5-\delta_0 \leq q \leq 5} \mathcal{M}_{K,q}$$

with $\max_M u \geq c_0$, there exists $\mathcal{S} = \{p_1, \ldots, p_N\} \subset \partial M$ with $N \geq 1$ such that

(i) *each* p_i *is a local maximum point of* u *in* M *and*

$$\overline{B_{\bar r_i}(p_i)} \cap \overline{B_{\bar r_j}(p_j)} = \emptyset, \quad \text{for } i \neq j,$$

where $\bar r_i = Ru^{1-q}(p_i)$ *and* $\overline{B_{\bar r_i}(p_i)}$ *denotes the geodesic ball in* (M,g) *of radius* $\bar r_i$ *and centered at* p_i*;*

(ii)

$$\left\| u^{-1}(p_i)u(\exp_{p_i}(yu^{1-q}(p_i)) - \left(\frac{1}{1+h^2|x'|^2}\right)^{1/2} \right\|_{C^2(B^M_{2R}(0))} < \varepsilon$$

where

$$B^M_{2R}(0) = \{y \in T_{p_i}M : \ |y| \leq 2R, \ u^{1-q}(p_i)y \in \exp^{-1}_{p_i}(B_\delta(p_i))\},$$

and $h > 0$*;*

(iii) $d^{\frac{1}{q-1}}(p_j,p_i)u(p_j) \geq c_0$*, for* $j > i$*, while* $d(p,\mathcal{S})^{\frac{1}{q-1}}u(p) \leq c_1$*,* $\forall\, p \in M$*, where* $d(\cdot,\cdot)$ *denotes the distance function in metric* g*.*

For the proof of Proposition 5.1 we need the following Lemma.

Lemma 5.1 *Let* (M,g) *be a smooth, compact, 3-dimensional Riemannian manifold. Given* $R \leq 1$ *and* $\varepsilon < 1$*, there exist positive constants* $\delta_0 = \delta_0(M,g,R,\varepsilon)$ *and* $C_0 = C_0(M,g,R,\varepsilon)$ *such that, for any compact* $C \subset M$ *and any* $u \in \bigcup_{5-\delta_0 \leq q \leq 5} \mathcal{M}_q$ *with* $\max_{p \in \overline{M \setminus C}} d^{\frac{1}{q-1}}(p,C)u(p) \leq C_0$*, we have that there exists* $p_0 \in M \setminus C$ *which is a local maximum point of* u *in* M *such that* $p_0 \in \partial M$ *and*

$$\left\| u^{-1}(p_0)u(\exp_{p_0}(yu^{1-q}(p_0)) - \left(\frac{1}{(1+\kappa|x'|^2}\right)^{\frac{1}{2}} \right\|_{C^2(B^M_{2R}(0))} < \varepsilon$$

where $B^M_{2R}(0)$ *is as in Proposition 5.1,* κ *is a positive constant with depends only on* K*,* $d(p,C)$ *denotes the distance of* p *to* C*, with* $d(p,C) = 1$ *if* $C = \emptyset$*.*

Proof. Suppose the contrary. Then there exist compacta $C_i \subset M$, $5 - \frac{1}{i} \leq q_i \leq 5$, and solutions $u_i \in \mathcal{M}_{q_i}$ such that

$$\max_{p \in \overline{M \setminus C_i}} d^{\frac{1}{q_i-1}}(p,C_i)u_i(p) \geq i.$$

It follows from the Hopf Lemma that $u_i > 0$ in M. Let $\hat p_i \in \overline{M \setminus C}$ be such that

$$d^{\frac{1}{q_i-1}}(\hat p_i,C_i)u_i(\hat p_i) = \max_{p \in \overline{M \setminus C_i}} d^{\frac{1}{q_i-1}}(p,C_i)u_i(p).$$

Let x be a geodesic normal coordinate system in a neighbourhood of $\hat p_i$ given by $\exp^{-1}_{\hat p_i}$. We write $u_i(x)$ for $u_i(\exp_{\hat p_i}(x))$ and denote $\lambda_i = u_i^{q_i-1}(p_i)$. We rescale x by $y = \lambda_i x$

and define $\hat{v}_i(y) = \lambda_i^{-\frac{1}{q_i-1}} u_i(\lambda_i y)$. Since interior blow ups are ruled out we have that $d(\hat{p}_i, \partial M) \to 0$. Fix some small positive constant $\delta > 0$ independent of i such that $\partial M \cap B_\delta(\hat{p}_i) \neq \emptyset$. We may assume without loss of generality, by taking δ smaller, that $\exp_{\hat{p}_i}^{-1}(\partial M) \cap B_\delta(0)$ has only one connected component, and may arrange to let the closest point on $\exp_{\hat{p}_i}^{-1}(\partial M) \cap B_\delta(0)$ to 0 be at $(0, \ldots, 0, -t_i)$ and

$$\exp_{\hat{p}_i}^{-1}(\partial M) \cap B_\delta(0) = \partial \mathbb{R}^n_+ \cap B_\delta^M(0)$$

is a graph over $(x^1, \ldots, x^{n-1})$ with horizontal tangent plane at $(0, \ldots, -t_i)$ and uniformly bounded second derivatives. In $\exp_{\hat{p}_i}^{-1}(B_\delta(\hat{p}_i))$ we write $g(x) = g_{ab}(x)\, dx^a dx^b$. Define

$$g^{(i)}(y) = g_{ab}(\lambda_i^{-1} y)\, dy^a dy^b.$$

Then $\hat{v}_i$ satisfies

$$\begin{cases} -L_{g^{(i)}} \hat{v}_i = \hat{K}_i \hat{v}_i{}^{q_i}, \quad \hat{v}_i > 0, \\ B_{g^{(i)}} \hat{v}_i = 0. \end{cases}$$

Note that $\lambda_i d(\hat{p}_i, C_i) \to \infty$ and, for $|y| \leq \frac{1}{4}\lambda_i d(\hat{p}_i, C_i)$ with $x = \lambda_i^{-1} y \in \exp_{\hat{p}_i}^{-1}(B_\delta(\hat{p}_i))$, we have

$$d(x, C_i) \geq \frac{1}{2} d(\hat{p}_i, C_i),$$

and therefore

$$\left(\frac{1}{2} d(\hat{p}_i, C_i)\right)^{\frac{1}{q_i-1}} u_i(x) \leq d(x, C_i)^{\frac{1}{q_i-1}} u_i(x) \leq d(\hat{p}_i, C_i)^{\frac{1}{q_i-1}} u_i(\hat{p}_i)$$

which implies that, for all $|y| \leq \frac{1}{4}\lambda_i d(\hat{p}_i, C_i)$ with $\lambda_i^{-1} y \in \exp_{\hat{p}_i}^{-1}(B_\delta(\hat{p}_i))$,

$$\hat{v}_i(y) \leq 2^{\frac{1}{q_i-1}}.$$

Standard elliptic theory, see [40], imply that there exists a subsequence, still denoted by $\hat{v}_i$, such that, for $T = \lim_i \lambda_i d(\hat{p}_i, \partial M) \in [0, +\infty]$, $\hat{v}_i$ converges to a limit $\hat{v}$ in C^2-norm on any compact set of $\{y = (y^1, \ldots, y^n) \in \mathbb{R}^n : \ y^n \geq -T\}$, where $\hat{v} > 0$ satisfies

$$\begin{cases} -\Delta \hat{v} = K(\overline{x}) \hat{v}^{\frac{n+2}{n-2}}, & \text{in } \{y^n > -T\}, \\ -\frac{\partial \hat{v}}{\partial y^n} = 0, \quad \text{on } \{y^n = -T\}, & \text{if } T < +\infty. \end{cases}$$

Now arguing as we did in the proof of Proposition 4.1, we derive that that $T < +\infty$, and, from the Liouville-type theorem of Caffarelli, Gidas and Spruck, [22], we have that

$$\hat{v}(x', x^n) = \left(\frac{1}{(1 + \kappa(T^2 + |x' - x_0'|^2)}\right)^{\frac{1}{2}},$$

where $\kappa := \frac{K(\overline{x})}{24}$. Setting now $\hat{y} = (\hat{y}', -T)$, it follows from the explicit form of $\hat{v}_i$ that there exist $y_i \to \hat{y}$ which are local maximum points of $\hat{v}_i$ such that $\hat{v}_i(y_i) \to \max \hat{v}$.

Define $p_i = \exp_{\hat{p}_i}(\lambda_i^{-1} y_i)$, then $p_i \in M \setminus C_i$ is a local maximum point of u_i, and if we repeat the scaling with p_i replacing $\hat{p}_i$, we still obtain a new limit v. Due to our choice, $v(0) = 1$ is a local maximum. So $T = 0$ and

$$\left\| u_i^{-1}(p_i) u_i(\exp_{p_i}(y u_i^{1-q_i}(p_i)) - \left(\frac{1}{(1+|x'|^2)} \right)^{\frac{1}{2}} \right\|_{C^2(B_{2R}^M(0))} < \varepsilon$$

which leads to a contradiction. □

Proof of Proposition 5.1. First we apply Lemma 5.1, by taking $C = \emptyset$ and $d(p, C) \equiv 1$, to obtain $p_1 \in \partial M$ which is a maximum point of u and (i) of Lemma 5.1 holds. If

$$\max_{p \in M \setminus C_1} d^{\frac{1}{q-1}}(p, C_1) u(p) \leq C_0,$$

where $C_1 = \overline{B_{\bar{r}_1}(p_1)}$, we stop. Otherwise we apply again Lemma 5.1 to obtain $p_2 \in \partial M$. It is clear that we have $\overline{B_{\bar{r}_1}(p_1)} \cap \overline{B_{\bar{r}_2}(p_2)} = \emptyset$ by taking ε small from the beginning. We continue the process. Since there exists $a(n) > 0$ such that $\int_{B_{\bar{r}_i}(p_i)} u_i^{q_i+1} \geq a(n)$, our process will stop after a finite number of steps. Thus we obtain $\mathcal{S} = \{p_1, \ldots, p_N\} \subset \partial M$ as in (ii) and

$$d^{\frac{1}{q-1}}(p, \mathcal{S}) u(p) \leq C_0,$$

for any $p \in M \setminus \mathcal{S}$. Clearly, we have that item (iii) holds. □

Though Proposition 5.1 states that u is very well approximated in strong norms by standard bubbles in disjoint balls $B_{\bar{r}_1}(p_1), \ldots, B_{\bar{r}_N}(p_N)$, it is far from the compactness result we wish to prove. Interactions between all these bubbles have to be analyzed to rule out the possibility of blowing-ups.

The next Proposition rules out possible accumulations of these bubbles, and this implies that only isolated blow-up points may occur to a blowing-up sequence of solutions.

Proposition 5.2 *Let (M, g) be a smooth compact three dimensional Riemannian manifold with umbilic boundary. For suitably large R and small $\varepsilon > 0$, there exist δ_1 and d depending only on M, g, $\|a\|_{C^2(\partial M)}$, $\|K\|_{C^2(\partial M)}$, $\inf_{y \in \partial M} K(y)$, R, and ε, such that for all u in*

$$\bigcup_{5-\delta_1 \leq q \leq 5} \mathcal{M}_{K,q}$$

with $\max_M u \geq c_0$, we have

$$\min\{d(p_i, p_j) : \ i \neq j, \ 1 \leq i, j \leq N\} \geq d$$

where $c_0, p_1, \ldots, p_N$ are given by Proposition 5.1.

Proof. By contradiction, suppose that the conclusion does not hold. Then there exist sequences $5 - \frac{1}{i} \leq q_i \leq 5$, $u_i \in \mathcal{M}_{q_i}$ such that $\min\{d(p_{i,a}, p_{i,b}),\ 1 \leq a, b \leq N\} \to 0$ as $i \to +\infty$, where $p_{i,1}, \ldots, p_{i,N}$ are the points given by Proposition 5.1. Notice that when we apply Proposition 5.1 to determine these points, we fix some large constant R, and then some small constant $\varepsilon > 0$ (which may depend on R), and in all the arguments i will be large (which may depend on R and ε). Let

$$d_i = d(p_{i,1}, p_{i,2}) = \min_{a \neq b} d(p_{i,a}, p_{i,b})$$

and

$$p_0 = \lim_{i \to +\infty} p_{i,1} = \lim_{i \to +\infty} p_{i,2} \in \partial M.$$

Since M has an umbilic boundary, one choose y to be a geodesic normal coodinate sytem in a neighborhood of p_0 given by $exp_{p_0}^{-1}$. We denote $g_i = (g_i)_{\alpha\beta}(x)\, dx^\alpha dx^\beta = g_{\alpha\beta}(u_i^{1-q_i}(y_i)x)\, dx^\alpha dx^\beta$, the scaled metric.

We can assume without loss of generality that $y_{i,a} := exp_{p_0}^{-1}(p_{i,a})$ are local maxima of $v_i(y) := u_i(exp_{p_0}(y))$. So it is easy to see that

$$v_i(y_{i,a}) \longrightarrow +\infty, \tag{5.38}$$

$$d\bigg(x, \bigcup_a \{x_{i,a}\}\bigg)^{\frac{1}{q_i-1}} v_i(x) \le c_1, \quad \forall\, x \in B_1^+, \tag{5.39}$$

$$0 < \sigma_i := |y_{i,1} - y_{i,2}| \longrightarrow 0, \tag{5.40}$$

$$\sigma_i^{\frac{1}{q_i-1}} v_i(y_i, y) \ge \frac{R^{\frac{n-2}{2}}}{c_2} \quad \text{for } a = 1, 2, \tag{5.41}$$

where $c_1, c_2 > 0$ are some constants independent of i, ε, R. Without loss of generality, we assume that $y_{i,1} = (0, \dots, y_{i,1}^n)$. Consider

$$w_i(y) = \sigma_i^{\frac{1}{q_i-1}} v_i(\sigma_i y)$$

and set, for $y_{i,a} \in \overline{B_1^+}$, $z_{i,a} = \frac{y_{i,a} - y_{i,1}}{\sigma_i}$ and $T_i = \frac{1}{\sigma_i} y_{i,a}^n$. Clearly, w_i satisfies

$$\begin{cases} -\Delta_{g_i} w_i(y) \,+\, \frac{1}{8}\, \sigma_i^{2(1-q_i)} R_{g_i}(\sigma_i^{1-q_i} y) \\ = K_i \left(\sigma_i^{\frac{q_i-1}{2}}\, y\right) w_i(x)^{q_i}, \text{ in } \left\{|y| < \frac{1}{\sigma_i},\ y^n > -T_i\right\}, \\ \frac{\partial w_i}{\partial y^n} = 0, \quad \text{on } \left\{|y| < \frac{1}{\sigma_i},\ y^n = -T_i\right\}. \end{cases} \tag{5.42}$$

It follows that

$$|z_{i,a} - z_{i,b}| \ge 1, \quad \forall\, a \neq b, \quad y_{i,1} = 0, \quad |y_{i,2}| = 1. \tag{5.43}$$

After passing to a subsequence, we have

$$\bar{z} = \lim_{i \to +\infty} z_{i,2}, \quad |\bar{z}| = 1.$$

It follows easily from (5.38), (5.39), (5.40), (5.41) that

$$w_i(0) \ge c_0' \quad w_i(z_{i,2}) \ge c_0',$$

$$\text{each } z_{i,a} \text{ is a local maximum point of } w_i, \tag{5.44}$$

$$\min_a |y - z_{i,a}|^{\frac{1}{q_i-1}} w_i(y) \le c_1, \tag{5.45}$$

$$|y| \le \tfrac{1}{2\sigma_i}, \quad y^n \ge -T_i \tag{5.46}$$

where $c_0' > 0$ is independent of i.

At this point we need the following Lemma which is a direct consequence of Lemma 4.1

Lemma 5.2 *If along some subsequence both $\{z_{i,a_i}\}$ and $w_i(z_{i,a_i})$ remain bounded, then along the same subsequence*

$$\limsup_{i\to+\infty} \max_{B_{1/4}^{-T_i}(y_{i,a_i})} w_i < \infty,$$

where $B_{1/4}^{-T_i}(z_{i,a_i}) = \{y : \ |y - z_{i,a_i}| < 1/4, \ y^n > -T_i\}$.

Due to Proposition 4.4 and Lemma 5.2, all the points y_{i,a_i} are either regular points of w_i or isolated simple blow-up points. We deduce, using Proposition 4.2, Lemma 5.2 and (5.42), (5.43) that

$$w_i(0) \longrightarrow +\infty, \quad w_i(y_{i,2}) \longrightarrow +\infty.$$

It follows that $\{0\}$, $\{y_{i,2} \to \bar{y}\}$ are both isolated simple blow-up points. Let $\tilde{w}_i = w_i(0)w_i$. It follows from Proposition 4.2 that there exists $\widetilde{\mathcal{S}}_1$ such that $\{0, \bar{y}\} \subset \widetilde{\mathcal{S}}_1 \subset \mathcal{S}$,

$$\min\{|x - y| : \ x, y \in \widetilde{\mathcal{S}}_1, \ x \neq y\} \geq 1,$$

and

$$w_i(0)w_i \underset{i\to\infty}{\longrightarrow} h \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n_{-T} \setminus \widetilde{\mathcal{S}}_1)$$

where h satisfies

$$\begin{cases} \Delta h = 0, & \text{in } \mathbb{R}^n_{-T} \setminus \widetilde{\mathcal{S}}_1, \\ \frac{\partial h}{\partial y^n} = 0, & \text{on } \partial\mathbb{R}^n_{-T} \setminus \widetilde{\mathcal{S}}_1. \end{cases}$$

Making an even extension of h across the hyperplane $\{y^n = -T\}$, we obtain $\tilde{h}$ satisfying $\Delta\tilde{h} = 0$ on $\mathbb{R}^n \setminus \widetilde{\mathcal{S}}_1$. Using Böcher's Theorem, the fact that $\{0, \bar{y}\} \subset \widetilde{\mathcal{S}}_1$, and the Maximum Principle, we obtain some nonnegative function $b(y)$ and some positive constants $a_1, a_2 > 0$ such that

$$\begin{cases} b(y) \geq 0, & y \in \mathbb{R}^n \setminus \{\widetilde{\mathcal{S}}_1 \setminus \{0, \bar{y}\}\}, \\ \Delta b(y) = 0, & y \in \mathbb{R}^n \setminus \{\widetilde{\mathcal{S}}_1 \setminus \{0, \bar{y}\}\}, \\ \frac{\partial b}{\partial \nu} = 0, & \text{on } \partial\mathbb{R}^n_+ \setminus \{\widetilde{\mathcal{S}}_1 \setminus \{0, \bar{y}\}\}, \end{cases}$$

and $h(y) = a_1|x|^{2-n} + a_2|x - \bar{y}|^{2-n} + b$, $y \in \mathbb{R}^n \setminus \widetilde{\mathcal{S}}_1$. Therefore there exists $A > 0$ such that

$$h(y) = a_1|y|^{2-n} + A + O(|y|)$$

for y close to zero. Using Pohozaev identity, see Lemma 4.3 and Corollary 1.1in the Appendix, we obtain a contradiction as in Proposition 4.4. The proof of our Proposition is thereby complete. □

The previous two propositions imply that any blow-up point is in fact an isolated blow-up point. Thanks to Proposition 4.4, any blow-up point is in fact an isolated simple blow-up point.

Proof of Theorem 2.1. Arguing by contradiction, suppose that there exist some sequences $q_i \to q \in]1,5]$, $u_i \in \mathcal{M}_{K_i,q_i}$ such that $\|u_i\|_{H^1(M)} \to +\infty$ as $i \to \infty$, which, in view of standard elliptic estimates, implies that $\max_M u_i \to +\infty$.

From Gidas-Spruck Liouville type theorem [39] (see also [50]), we know that $q = 5$. By Proposition 5.2, we have that for some small $\varepsilon > 0$, large $R > 0$, and some $N \geq 1$ there exist $y_i^{(1)}, \dots, y_i^{(N)} \in \partial M$ such that (i-iii) of Proposition 5.1 hold. $\{y_i^{(1)}\}_i, \dots, \{y_i^{(N)}\}_i$ are isolated blow-up points and hence, by Proposition 4.4, isolated simple blow-up points. Since the manifold is compact there could be only finitely many such points. We notice first that Proposition 4.2 tells us that outside arbitrarly small neigborhoods of such points the H^1- norm is bounded, now from (4.6) in the statement of Proposition 4.1 and 4.15 in the statement of Corollay 4.1 we deduce that such an H^1- norm in those small neighborhoods is bounded as well, actually it is arbitrarly close to the H^1- norm of one bubble. In the neighborhood of each bubble this computation is already included in Lemma 4.7. Therefore we have that $\{\|u_i\|_{H^1(M)}\}_i$ is bounded, thus finding a contradiction. Theorem 2.1 is thereby established. □

6 Compactness of the solutions

Before proving Theorem 2.2, we state the following result about the compactness of solutions of $(P_{K,q})$ when q stays strictly below the critical exponent. The proof of this theorem is, up to minor modifications, similiar to Theorem 1.1 of [41].

Theorem 6.1 *Let (M, g) be a smooth, compact, three dimensional Riemannian manifold with umbilic boundary. Then for any $\delta_1 > 0$ there exists a constant $C > 0$ depending only on M, g, δ_1, $\|K\|_{C^2(\partial M)}$, and the positive lower bound of K on ∂M such that for all $u \in \bigcup_{1+\delta_1 \leq q \leq 5-\delta_1} \mathcal{M}_{K,q}$ we have*

$$\frac{1}{C} \leq u(x) \leq C, \quad \forall\, x \in M; \quad \|u\|_{C^2(M)} \leq C.$$

Now we prove Theorem 2.2.

Proof of Theorem 2.2. Due to elliptic estimates and Lemma 3.1, we have to prove just the L^∞ bound, i.e. $u \leq C$. Suppose the contrary. Then there exist a sequence $q_i \to q \in]1,5]$ with

$$u_i \in \mathcal{M}_{K,q_i}, \quad \text{and } \max_M u_i \to +\infty,$$

where $\bar{c}$ is some positive constant independent of i. From Theorem 6.1, we have that q must be 5. It follows from Proposition 4.4 and Proposition 5.2 that, after passing to a subsequence, $\{u_i\}_i$ has N $(1 \leq N < \infty)$ isolated simple blow-up points denoted by $y^{(1)}, \dots, y^{(N)}$. Let $y_i^{(\ell)}$ denotes the local maximum points as in Definition 3.1. It follows from Proposition 4.2 that

$$u_i(y_i^{(1)})u_i \underset{i\to+\infty}{\longrightarrow} h(y) = \sum_{j=1}^{N} b_j G(y, y^{(j)}) + E(y) \quad \text{in } C^2_{\text{loc}}\big(M \setminus \{y^{(1)}, \dots, y^{(N)}\}\big),$$

where $b_j > 0$ and $E \in C^2(M)$ satisfies

$$\begin{cases} -L_g E = 0, & \text{in } M, \\ \dfrac{\partial E}{\partial \nu} = 0, & \text{on } \partial M. \end{cases} \tag{6.47}$$

Since the manifold is of positive type, we have that $E \equiv 0$. Therefore,

$$u_i(y_i^{(1)})u_i \underset{i \to +\infty}{\longrightarrow} h(y) = \sum_{j=1}^{N} b_j G_a(y, y^{(j)}) \quad \text{in } C^2_{\text{loc}}\big(M \setminus \{y^{(1)}, \dots, y^{(N)}\}\big).$$

Let $x = (x^1, x^2, x^3)$ be some geodesic normal coordinate system centered at $y_i^{(1)}$. From Lemma 1.2 of the Appendix, the Positive Mass Theorem, and the assumption that the manifold is not conformally equivalent to the standard ball, we derive that there exists a positive constant A such that

$$h(x) = h(\exp_{y_i^{(1)}}(x)) = c|x|^{-1} + A_i + O(|x|^{-\alpha}) \quad \text{for } |x| \text{ close to } 0$$

and $A_i \geq A > 0$. This contradicts the result of Proposition 4.3. The compactness part of Theorem 2.2 is proved. Since we have compactness, we can proceed as in section 6 of [41], see also section 4 of [37] to prove that the total degree of the solutions is -1. Theorem 2.2 is established. □

7 Appendix

In this Appendix, we recall some well known results and provide some description of singular behaviour of positive solutions to some boundary value elliptic equations in punctured half balls. It is extracted from our paper in collaboration with V, Felli, see [38] for the proofs of the main results cited here.

For $n \geq 3$ let B_r^+ denote the set $\{x = (x', x^n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} : |x| < r \text{ and } x^n > 0\}$ and set $\Gamma_1(B_r^+) := \partial B_r^+ \cap \partial \mathbb{R}^n_+$, $\Gamma_2(B_r^+) := \partial B_r^+ \cap \mathbb{R}^n_+$. Throughout this section, let $g = g_{ij}\, dx^i dx^j$ denote some smooth Riemannian metric in B_1^+ and $a \in C^1(\Gamma_1(B_1^+))$.

First of all, we recall the following Maximum Principle; for the proof see [41].

Theorem 1.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\partial\Omega = \Gamma \cup \Sigma$, $V \in L^\infty(\Omega)$, and $h \in L^\infty(\Sigma)$ such that there exists some $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $\psi > 0$ in $\overline{\Omega}$ satisfying*

$$\begin{cases} \Delta_g \psi + V\psi \leq 0, & \text{in } \Omega, \\ \dfrac{\partial \psi}{\partial \nu} \geq h\psi, & \text{on } \Sigma. \end{cases}$$

If $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta_g v + Vv \leq 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} \geq hv, & \text{on } \Sigma, \\ v \geq 0, & \text{on } \Gamma, \end{cases}$$

then $v \geq 0$ *in* $\overline{\Omega}$.

Now we state the following Maximum Principle which holds for the operator T defined by

$$Tu = v \quad \text{if and only if} \quad \begin{cases} L_g u = 0, & \text{in } \overset{\circ}{M}, \\ \dfrac{\partial u}{\partial \nu} = v, & \text{on } \partial M. \end{cases}$$

Proposition 1.1 ([37] [35]) *Let* (M, g) *be a Riemannian manifold with boundary of positive type. Then for any* $u \in C^2(\overset{\circ}{M}) \cap C^1(M)$ *satisfying*

$$\begin{cases} L_g u \geq 0, & \text{in } \overset{\circ}{M}, \\ \dfrac{\partial u}{\partial \nu} \leq 0, & \text{on } \partial M, \end{cases}$$

we have $u \leq 0$ *in* M.

Lemma 1.1 ([37]) *Suppose that* $u \in C^2(B_1^+ \setminus \{0\})$ *is a solution of*

$$\begin{cases} -L_g u = 0, & \text{on } B_1^+, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1(B_1^+ \setminus \{0\}), \end{cases} \tag{1.1}$$

and $u(x) = o\left(|x|^{2-n}\right)$ *as* $|x| \to 0$. *Then* $u \in C^{2,\alpha}(B_{1/2}^+)$ *for any* $0 < \alpha < 1$.

Lemma 1.2 ([37]) *There exists some constant* $\delta_0 > 0$ *depending only on* n, $\|g_{ij}\|_{C^2(B_1^+)}$ *and* $\|K\|_{L^\infty(B_1^+)}$ *such that for all* $0 < \delta < \delta_0$ *there exists some function* G *satisfying*

$$\begin{cases} -L_g G = 0, & \text{in } B_\delta^+, \\ \dfrac{\partial G}{\partial \nu} = 0, & \text{on } \Gamma_1(B_\delta^+) \setminus \{0\}, \\ \lim_{|x| \to 0} |x|^{-1} G(x) = 1 \end{cases} \tag{1.2}$$

such that, for some A *constant and some* $\alpha \in (0,1)$, $\forall\, x \in B_\delta^+$

$$G(x) = |x|^{-1} + A + O(|x|^\alpha).$$

Making again reflection across $\Gamma_1(B_1^+)$, we derive from Lemma 9.3 in [51], the following:

Lemma 1.3 *If* $u \in C^2(B_1^+ \setminus \{0\})$ *satisfies*

$$\begin{cases} -L_g u = 0, & \text{in } B_1^+, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1(B_1^+) \setminus \{0\}, \end{cases}$$

then

$$\alpha = \limsup_{r \to 0^+} \max_{x \in \Gamma_2(B_r^+)} u(x)|x|^{n-2} < +\infty.$$

Proposition 1.2 ([37]) *Suppose that* $u \in C^2(B_1^+ \setminus \{0\})$ *satisfies*

$$\begin{cases} -L_g u = 0, & \text{in } B_1^+, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1(B_1^+) \setminus \{0\}. \end{cases} \tag{1.3}$$

Then there exists some constant $b \geq 0$ *such that*

$$u(x) = b\, G(x) + E(x) \quad \text{in } B_{1/2}^+ \setminus \{0\},$$

where G *is defined in Lemma 1.2, and* $E \in C^2(B_1^+)$ *satisfies*

$$\begin{cases} -L_g E = 0, & \text{in } B_{1/2}^+, \\ \dfrac{\partial E}{\partial \nu} = 0, & \text{on } \Gamma_1(B_{1/2}^+). \end{cases} \tag{1.4}$$

Finally we prove the following corollary.

Corollary 1.1 *Let* u *be a solution of* (1.3) *which is singular at* 0*. Then*

$$\lim_{r \to 0^+} \int_{\Gamma_2(B_r^+)} \frac{\partial u}{\partial \nu}\, d\sigma = b \cdot \lim_{r \to 0^+} \int_{\Gamma_2(B_r^+)} \frac{\partial G}{\partial \nu}\, d\sigma = -\frac{n-2}{2}\, b\, |\mathbb{S}^{n-1}|,$$

where $\mathbb{S}^{n-1}$ *denotes the standard* n*-dimensional sphere and* $b > 0$ *is given by Proposition 1.2.*

Proof. From the previous proposition, we know that

$$u(x) = b\, G(x) + E(x), \quad \text{in } B_{1/2}^+(0) \setminus \{0\}, \quad b \geq 0.$$

Since u is singular at 0, b must be strictly positive. From (1.4), we have

$$0 = -\int_{B_r^+} \Delta_g E\, dV - \int_{\Gamma_2(B_r^+)} \frac{\partial E}{\partial \nu}\, d\sigma + \frac{1}{8} \int_{B_r^+} R_g E.$$

Hence, since E is regular, we obtain

$$\int_{\Gamma_2(B_r^+)} \frac{\partial E}{\partial \nu}\, d\sigma = \frac{1}{8} \int_{B_r^+} R_g(x) E(x)\, d\sigma \underset{r \to 0^+}{\longrightarrow} 0$$

and so

$$\lim_{r \to 0^+} \int_{\Gamma_2(B_r^+)} \frac{\partial u}{\partial \nu}\, d\sigma = \lim_{r \to 0^+} b \int_{\Gamma_2(B_r^+)} \frac{\partial G}{\partial \nu}\, d\sigma.$$

From Lemma 1.2 we know that G is of the form

$$G(x) = |x|^{-1} + \mathcal{R}(x)$$

where $\mathcal{R}$ is regular. Since

$$\int_{\Gamma_2(B_r^+)} \frac{\partial}{\partial \nu} |x|^{-1}\, d\sigma = -\frac{1}{2}\, |\mathbb{S}^{n-1}|$$

and
$$\int_{\Gamma_2(B_r^+)} \frac{\partial \mathcal{R}}{\partial \nu}\, d\sigma \underset{r\to +0^+}{\longrightarrow} 0,$$
we conclude that
$$\lim_{r\to 0^+} \int_{\Gamma_2(B_r^+)} \frac{\partial u}{\partial \nu}\, d\sigma = -\frac{1}{2}\, b\, |\mathbb{S}^{n-1}|$$
thus getting the conclusion. □

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