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Some Elliptic Problems With Degenerate Coercivity

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Abstract

In this paper we are interested in existence of solutions for some nonlinear elliptic equations with principal part having degenerate coercivity. The model case is

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{1+a(x)|u|}\right) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

with Ω bounded open subset of $I\!R^N$, $N \geq 2$ and f a datum in some Lebesgue space.

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1 Introduction

In this paper we study the existence of solutions of the following elliptic problem (with degenerate coercivity):

$$\begin{cases} \mathrm{div}(a(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

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where Ω is a bounded, open subset of $I\!R^N$, with $N > 2$, and $a(x,s) : \Omega \times I\!R \to I\!R$ is a Caratheodory function (that is, measurable with respect to x for every $s \in I\!R$, and continuous with respect to s for almost every $x \in \Omega$) satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)} \le a(x,s) \le \beta\,, \tag{1.3}$$

for almost every $x \in \Omega$, for every $s \in I\!R$, where α and β are positive constants.

The main difficulty in dealing with problem (1.2) is the fact that, because of assumption (1.3), the differential operator $A(v) = -\mathrm{div}(a(x,v)\nabla v)$ is not coercive on $W_0^{1,2}(\Omega)$. This implies that the classical methods used in order to prove the existence of a solution for problem (1.2) cannot be applied in general, even if the datum f is regular.

In [4], it is proved that, if the datum f belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$, then there exists a weak solution of (1.2) u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Here, we study the existence of solutions under the assumption $f \in L^m(\Omega)$, $1 \le m \le \frac{N}{2}$, with particular care to the case $f(x) = \frac{A}{|x|^2}$, which is an important borderline case, as Theorems 3.1, 4.1 and following Remark show.

Remark 1.1 Consider the model problem (0.1) in $\Omega = B(0,1)$, with $a(x) = 1$, $f(x) = \frac{A}{|x|^2}$:

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{1+|u|}\right) = \dfrac{A}{|x|^2} & \text{in } B(0,1), \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Then, if we look for radial solutions of the type $u(x) = |x|^{-\rho} - 1$, our differential problem has a solution if $\rho(N-2) = A$. Thus $u(x) = |x|^{-\frac{A}{N-2}} - 1$, so that the regularity of u and ∇u depends on A. Moreover, this example shows that the assumptions of Theorems 3.1 and 4.1 are optimal.

Furthermore, if we consider the model problem (0.1) in $\Omega = B(0,1)$, with $a(x) = 1$, $f(x) = \frac{A}{|x|^{\frac{N}{m}}}$, $m < \frac{N}{2}$:

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{1+|u|}\right) = \dfrac{A}{|x|^{\frac{N}{m}}} & \text{in } B(0,1), \\ u = 0 & \text{on } \partial B(0,1), \end{cases}$$

and we look for radial solutions of the type $u(x) = e^{\mu|x|^\rho} - 1$, our differential problem has a solution if

$$\begin{cases} \rho - 2 = -\frac{N}{m} \\ -\mu\rho(\rho-1) = A. \end{cases}$$

Thus we have $\log(1+|u|) = \mu|x|^{2-\frac{N}{m}}$ (see Theorem 4.1).

If we replace the assumption (1.3) with

$$\frac{\alpha}{(1+|s|)^\theta} \le a(x,s) \le \beta\,, \quad 0 < \theta < 1$$

existence results can be found in [5], [4], [1] (in this paper is also studied a problem with $\theta > 1$).

Our results can be generalized, in the direction of a nonlinear operator with respect to the gradient, thanks to a combined use of our a priori estimate and of some techniques of [1]. Uniqueness results are studied in [8].

The proofs of the existence results will be obtained by approximation. Let f be a function in $L^m(\Omega)$, with $1 \le m \le \frac{N}{2}$ and $\{f_n\}$ be a sequence of smooth functions converging to f in $L^m(\Omega)$, such that

$$\|f_n\|_{L^m(\Omega)} \le \|f\|_{L^m(\Omega)}, \qquad \forall n \in I\!N . \tag{1.4}$$

Recall the following definition of truncation:

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k \\ k\dfrac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

Let us define the following sequence of problems:

$$\begin{cases} -\text{div}(a(x, T_n(u_n))\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Take, for instance, $f_n = T_n(f)$.

The existence of weak solutions u_n in $W_0^{1,2}(\Omega)$ of the Dirichlet problem (1.5) is classical, since the differential operator in (1.5) is uniformly elliptic. Moreover, for any $n \in I\!N$, u_n is a bounded function.

The use of $T_k(u_n)$ as test function in (1.5) yields the following main estimate.

Lemma 1.1

$$\frac{\alpha}{1+k} \int\limits_{\{x\in\Omega : |u_n(x)| \le k\}} |\nabla u_n|^2 \le \int\limits_{\Omega} T_k(u) f_n \le k\|f\|_1, \quad \forall k > 0. \tag{1.6}$$

2 Weak solutions

Theorem 2.1 *Assume (1.3). If f belongs to $L^{\frac{N}{2}}(\Omega)$, there exists a weak solution of the boundary value problem (1.2); that is*

$$u \in W_0^{1,2}(\Omega) : \int\limits_{\Omega} a(x, u)\, \nabla u \cdot \nabla v = \int\limits_{\Omega} f\, v\,, \qquad \forall v \in W_0^{1,2}(\Omega)\,. \tag{2.7}$$

Moreover $u \in L^p(\Omega)$, for any $1 \le p < \infty$.

Proof. First of all, use $[\log(1+|u_n|)]\text{sgn}(u_n)$ as test function in (1.5). Then, if S is Sobolev constant,

$$S^2\alpha\Big[\int\limits_{\Omega} \log(1+|u_n|)^{2^*}\Big]^{\frac{2}{2^*}} \le \alpha \int\limits_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^2} \le \|f\|_{\frac{2N}{N+2}} \Big[\int\limits_{\Omega} \log(1+|u_n|)^{2^*}\Big]^{\frac{1}{2^*}}$$

and

$$S^2\alpha\Big[\int_\Omega \log(1+|u_n|)^{2^*}\Big]^{\frac{1}{2^*}} \le \|f\|_{\frac{2N}{N+2}}$$

which implies

$$\text{meas}\{x\in\Omega : |u_n|\ge k\} \le \left(\frac{\|f\|_{\frac{2N}{N+2}}}{S^2\alpha}\cdot\frac{1}{\log(1+k)}\right)^{2^*}. \tag{2.8}$$

Thus, so far, we cannot say that the sequence $\{u_n\}$ is bounded in $L^1(\Omega)$. However, we have $\lim_{k\to\infty}\text{meas}\{x\in\Omega : |u_n|\ge k\} = 0$.

Let $M > 1$. We shall use the following inequality.

$$t^2 - 1 \le M(t-1)^2 + \frac{1}{M-1}, \quad t\in I\!R \tag{2.9}$$

Use

$$[(1+|u_n|)^{2(\gamma+1)} - (1+k)^{2(\gamma+1)}]^+ sgn(u_n)$$

as test function. We have

$$2(\gamma+1)\alpha \int_{\{x\in\Omega:k\le|u_n(x)|\}} (1+|u_n|)^{2\gamma}|\nabla u_n|^2$$

$$\le \int_{\{x\in\Omega:k\le|u_n(x)|\}} |f||(1+|u_n|)^{2(\gamma+1)} - (1+k)^{2(\gamma+1)}|$$

$$\le M \int_{\{x\in\Omega:k\le|u_n(x)|\}} |f||(1+|u_n|)^{\gamma+1} - (1+k)^{\gamma+1}|^2 + \frac{1}{(M-1)}\int_\Omega |f|$$

that is

$$\left[2(\gamma+1)\alpha - \frac{M}{S^2}\Bigg(\int_{\{x\in\Omega:k\le|u_n(x)|\}} |f|^{\frac{N}{2}}\Bigg)^{\frac{N}{2}}\right] \int_{\{x\in\Omega:k\le|u_n(x)|\}} |\nabla u_n|^2(1+|u_n|)^{2(\gamma+1)}$$

$$\le \frac{1}{(M-1)}\int_\Omega |f|.$$

Thanks to the absolute continuity of the integral, we can take k_0 such that

$$\frac{M}{S^2}\Bigg(\int_{\{x\in\Omega:k_0\le|u_n(x)|\}} |f|^{\frac{N}{2}}\Bigg)^{\frac{N}{2}} \le \alpha(\gamma+1).$$

Then

$$\int_{\{x\in\Omega:k_0\le|u_n(x)\}} |\nabla u_n|^2(1+|u_n|)^{2\gamma} \le \frac{1}{\alpha(\gamma+1)(M-1)}\int_\Omega |f|, \quad k_0 = k_0(\|f\|_{\frac{N}{2}},\gamma) \tag{2.10}$$

Now we use the previous estimate for $\gamma = 0$. On one hand (1.6) gives, for any $k > 0$,

$$\int\limits_{\{x\in\Omega:|u_n(x)|\le k\}} |\nabla u_n|^2 \le \frac{2k^2}{\alpha}\|f\|_1.$$

Then the sequence $\{u_n\}$ is bounded in the Sobolev spaces $W_0^{1,2}(\Omega)$. Thus, up to a subsequence, it converges weakly to some function $u \in W_0^{1,2}(\Omega)$. Moreover, u_n converges to u almost everywhere in Ω as a consequence of the Rellich theorem. Let v be a function in $W_0^{1,2}(\Omega)$, and take v as test function in (1.5). We obtain

$$\int\limits_\Omega a(x, T_k(u_n))\, \nabla u_n \cdot \nabla v = \int\limits_\Omega f_n\, v\,.$$

The left hand side side passes to the limit as n tends to infinity since $a(x, T_k(u_n)) \rightharpoonup a(x, u)$ in $L^\infty(\Omega)$ (and almost everywhere in Ω) and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$, so that we have $a(x, T_k(u_n))\,\nabla u_n \rightharpoonup a(x, u)\,\nabla u$ in $L^2(\Omega)$. Hence, since $\int\limits_\Omega f_n v \to \int\limits_\Omega f v$, u is a weak solution of (1.2).

Now use (2.10) for $\gamma > 0$. Sobolev inequality implies

$$\Big[\int\limits_\Omega |G_{k_0}(u_n)|^{\gamma 2^*}\Big]^{\frac{2}{2^*}} \le \frac{1}{\alpha(\gamma+1)(M-1)} \int\limits_\Omega |f|, \quad k_0 = k_0(\|f\|_{\frac{N}{2}}, \gamma).$$

Use the previous estimate together with (1.6) in order to prove that u belongs to $L^p(\Omega)$, for any $p < \infty$.

Now, if 0 belongs to Ω, on the datum f, we put a slightly weaker assumption:

$$|f| \le \frac{A}{|x|^2}.$$

Recall that $\frac{A}{|x|^2}$ does not belong to the Lebesgue space $L^{\frac{N}{2}}(\Omega)$, but, for any $v \in W_0^{1,2}(\Omega)$

$$\int\limits_\Omega \frac{|v|^2}{|x|^2} < \infty,$$

thanks to the following proposition (see [6], [7]).

Proposition 2.1 [Hardy inequality] *Assume that* 0 *belongs to* Ω. *If* $v \in W_0^{1,2}(\Omega)$, *then*

$$\int\limits_\Omega \frac{|v|^2}{|x|^2} \le \Big(\frac{N-2}{2}\Big)^2 \int\limits_\Omega |\nabla v|^2. \tag{2.11}$$

Moreover, $H^2 = \big(\frac{N-2}{2}\big)^2$ *is optimal.*

Lemma 2.1 *Assume that* 0 *belongs to* Ω,

$$|f| \le \frac{A}{|x|^2}, \tag{2.12}$$

and $1 \le \gamma < \frac{2\alpha H^2}{A}$. *Then there exixts* $L > 0$ *such that*

$$\|u_n\|_{\gamma 2^*} \le L, \;\; \|u_n\|_{W_0^{1,2}(\Omega)} \le L.$$

Proof. Use $[(1+|u_n|)^{2\gamma} - 1]\mathrm{sgn}(u_n)$, $\gamma > 1$, as test function. Let $M > 1$ be such that $2\alpha > \gamma \frac{AM}{H^2}$. We have

$$2\alpha\gamma \int\limits_\Omega |\nabla u_n|^2 (1+|u_n|)^{2\gamma-2} \le A \int\limits_\Omega \frac{[(1+|u_n|)^{2\gamma} - 1]}{|x|^2}$$

$$\le AM \int\limits_\Omega \frac{((1+|u_n|)^{\gamma} - 1)^2}{|x|^2} + \frac{A}{(M-1)} \int\limits_\Omega \frac{1}{|x|^2}$$

$$\le \gamma^2 \frac{AM}{H^2} \int\limits_\Omega |\nabla u_n|^2 (1+|u_n|)^{2\gamma-2} + \frac{A}{(M-1)} \int\limits_\Omega \frac{1}{|x|^2}.$$

Then

$$(2\alpha - \gamma\frac{AM}{H^2}) \int\limits_\Omega |\nabla u_n|^2 (1+|u_n|)^{2\gamma-2} \le \frac{A}{(M-1)} \int\limits_\Omega \frac{1}{|x|^2}$$

$$S(2\alpha - \gamma\frac{AM}{H^2}) \Big[\int\limits_\Omega ((1+|u_n|)^{\gamma} - 1)^{2^*}\Big]^{\frac{2}{2^*}} \le \frac{A}{(M-1)} \int\limits_\Omega \frac{1}{|x|^2}.$$

So

$$S(2\alpha - \gamma\frac{AM}{H^2}) \Big[\int\limits_\Omega |u_n|^{2^*\gamma}\Big]^{\frac{2}{2^*}} \le C_0.$$

Now it is possible to repeat the proof of previous theorem in order to obtain a second existence results:

Theorem 2.2 *Assume that* 0 *belongs to* Ω, *(2.12) and* $1 \le \gamma < \frac{2\alpha H^2}{A}$. *Then there exists a weak solution* u *of the boundary value problem (1.2). Moreover,* u *belongs to* $L^{\gamma 2^*}(\Omega)$.

3 Distributional solutions

Lemma 3.1 *Assume that* 0 *belongs to* Ω, *(2.12) and*

$$\begin{cases} \dfrac{N-2}{2(N-1)} < \theta \le 1 \\ \theta < \dfrac{2\alpha H^2}{A}. \end{cases}$$

Then there exixts $L > 0$ such that

$$\|u_n\|_{W_0^{1,q}(\Omega)} \leq L, \quad q = \frac{2N\theta}{N-2(1-\theta)}.$$

Proof. Use $[(1+|u_n|)^{2\theta} - 1]\text{sgn}(u_n)$, $0 < \theta < 1$, as test function. Let $M > 1$ be such that $2\alpha > \theta \frac{AM}{H^2}$. Then

$$2\alpha\theta \int_\Omega |\nabla u_n|^2 (1+|u_n|)^{2\theta-2} \leq A \int_\Omega \frac{[(1+|u_n|)^{2\theta} - 1]}{|x|^2}$$

$$\leq AM \int_\Omega \frac{((1+|u_n|)^\theta - 1)^2}{|x|^2} + \frac{A}{(M-1)} \int_\Omega \frac{1}{|x|^2}$$

$$\leq \theta^2 \frac{AM}{H^2} \int_\Omega |\nabla u_n|^2 (1+|u_n|)^{2\theta-2} + \frac{A}{(M-1)} \int_\Omega \frac{1}{|x|^2}.$$

Hence

$$(2\alpha - \theta\frac{AM}{H^2}) \int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{2(1-\theta)}} \leq \frac{A}{(M-1)} \int_\Omega \frac{1}{|x|^2}. \tag{3.13}$$

So, if $q < 2$,

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{q(1-\theta)}} . (1+|u_n|)^{q(1-\theta)}$$

$$\leq C_1 \Big[\int_\Omega (1+|u_n|)^{\frac{2q(1-\theta)}{2-q}} \Big]^{\frac{2-q}{2}}.$$

Definition of q gets

$$q^* = \frac{2q(1-\theta)}{2-q}$$

so that

$$\int_\Omega |\nabla u_n|^q \leq C_1 \Big[\int_\Omega (1+|u_n|)^{\frac{2q(1-\theta)}{2-q}} \Big]^{\frac{2-q}{2q(1-\theta)}[q(1-\theta)]} \leq C_2 + C_3 \Big[\int_\Omega |\nabla u_n|^q \Big]^{\frac{1}{q}[q(1-\theta)]}$$

that is

$$\int_\Omega |\nabla u_n|^q \leq L^q,$$

which implies the following existence theorem.

Theorem 3.1 *Assume that 0 belongs to Ω, (2.12) and*

$$\begin{cases} \dfrac{N-2}{2(N-1)} < \theta \leq 1 \\ \theta < \dfrac{2\alpha H^2}{A}. \end{cases}$$

Then there exists a distributional solution $u \in W_0^{1,q}(\Omega)$, $q = \frac{2N\theta}{N-2(1-\theta)}$, *of the boundary value problem (1.2), that is*

$$u \in W_0^{1,q}(\Omega): \int_\Omega a(x,u)\,\nabla u \cdot \nabla v = \int_\Omega f\,v\,, \qquad \forall v \in C_0^\infty(\Omega)\,. \tag{3.14}$$

4 Entropy solutions

Lemma 4.1 *Assume that* 0 *belongs to* Ω, *(2.12) and*

$$\begin{cases} 0 \le \theta \le \dfrac{N-2}{2(N-1)} \\ \theta < \dfrac{2\alpha H^2}{A}. \end{cases}$$

Then there exixts $L > 0$ *such that*

$$\int_\Omega |u_n|^{\theta 2^*} \le L.$$

Proof. Estimate (3.13) yields (recall that any u_n is a bounded function)

$$\Big(2\alpha - \theta\frac{AM}{H^2}\Big)\frac{1}{\theta^2}\int_\Omega \Big|\nabla[(1+|u_n|)^\theta - 1]\Big|^2 = \int_\Omega \Big|\frac{|\nabla u_n|}{(1+|u_n|)^{(1-\theta)}}\Big|^2 \le C_1,$$

so that

$$\Big(2\alpha - \theta\frac{AM}{H^2}\Big)\frac{1}{\theta^2}\Big[\int_\Omega [(1+|u_n|)^\theta - 1]^{2^*}\Big]^{\frac{2}{2^*}} \le C_1.$$

Lemma 4.2 *Assume* $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m < \frac{N}{2}$. *Then*

$$\|\log(1+|u_n|)\|_{m^{**}} \le C_{\alpha,m}\|f\|_m.$$

Proof. Use $[\log(1+|u_n|)]^{2\gamma-1} sgn(u_n)$ as test function in (1.5), with $\gamma = \frac{m^{**}}{2^*}$. Then

$$S^2\alpha(2\gamma-1)\Big[\int_\Omega \log(1+|u_n|)^{2^*\gamma}\Big]^{\frac{2}{2^*}} \le \alpha(2\gamma-1)\int_\Omega \frac{|\nabla u_n|^2\log(1+|u_n|)^{2\gamma-2}}{(1+|u_n|)^2}$$

$$\le \|f\|_m\Big[\int_\Omega \log(1+|u_n|)^{(2\gamma-1)m'}\Big]^{\frac{1}{m'}}.$$

Lemma 4.3 *Assume* $f \in L^m(\Omega)$, $1 < m < \frac{2N}{N+2}$. *Then*

$$\|\log(1+|u_n|)\|_{m^{**}} \le C_{\alpha,m}\|f\|_m.$$

Proof. Use $\Big[\log\dfrac{1+|u_n|}{1+k}\Big]_+^{2\theta-1} sgn(u_n)$ as test function in (1.5), with $\theta = \frac{m^{**}}{2^*}$.

$$S^2\alpha(2\theta-1)\Big[\int_\Omega\Big[\log\frac{1+|u_n|}{1+k}\Big]_+^{\theta 2^*}\Big]^{\frac{2}{2^*}}$$

$$\le \alpha(2\theta-1)\int_{\{x\in\Omega:k\le|u_n(x)|\}}\frac{|\nabla u_n|^2\log(1+|u_n|)^{2\theta-2}}{(1+|u_n|)^2}$$

$$\int_\Omega|f|\Big[\log\frac{1+|u_n|}{1+k}\Big]_+^{2\theta-1}\le\|f\|_m\Big(\int_\Omega|f|\Big[\log\frac{1+|u_n|}{1+k}\Big]_+^{(2\theta-1)m'}\Big)^{\frac{1}{m'}}.$$

Lemma 4.4 *Assume* $f\in L^1(\Omega)$. *Then*

$$|\{x:|u_n(x)|>k\}^{\frac{2}{2^*}}|\le\frac{1}{\log(1+k)}\int_\Omega|f|. \tag{4.15}$$

Proof. Use $\log(1+|T_k(u_n)|)sgn(u_n)$ as test function in (1.5).

$$S^2\Big[\int_{\{x\in\Omega:k\le|u_n(x)|\}}\log(1+k)^{2^*}\Big]^{\frac{2}{2^*}}\le S^2\Big[\int_\Omega\log(1+|T_k(u_n)|)^{2^*}\Big]^{\frac{2}{2^*}}$$

$$\le\int_\Omega\frac{|\nabla T_k(u_n)|^2}{\log(1+|T_k(u_n)|)^2}\le\int_\Omega\log(1+|T_k(u_n)||f|\le\log(1+k)\int_\Omega|f|.$$

Recall (see (1.6)) that the truncation $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$, and the following result (see [2], Lemma 2.1).

Proposition 4.1 *Let u be a measurable function such that $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$ for every $k>0$. Then there exists a unique measurable function $v:\Omega\to I\!R^N$ such that*

$$v\chi_{\{|u|<k\}}=\nabla T_k(u),\qquad \textit{for almost every } x\in\Omega,\ \forall k>0, \tag{4.16}$$

where χ_E is the characteristic function of a measurable set E. If, moreover, u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the standard distributional gradient of u.

In view of the above result, for every measurable function u such that $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$ for every $k>0$, we define ∇u, the weak gradient of u, as the unique function v which satisfies (4.16). The definition of weak gradient allows us to give the following definition of entropy solution for problem (1.2) (see [2]).

Definition 4.1 (see [2]) Let f be in $L^m(\Omega)$, $m\ge 1$. A measurable function u is an entropy solution of (1.2) if $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$ for every $k>0$, and if

$$\int_\Omega a(x,u)\nabla u\cdot\nabla T_k(u-\varphi)\le\int_\Omega f\,T_k(u-\varphi)\,, \tag{4.17}$$

for every $k>0$ and for every $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Proposition 4.2 *Let $\{v_n\}$ be a sequence of measurable functions such that $T_k(v_n)$ is bounded in $W_0^{1,2}(\Omega)$ for every $k > 0$. Then there exists a measurable function v, with $T_k(v)$ belonging to $W_0^{1,2}(\Omega)$, for every $k > 0$, and a subsequence, still denoted by $\{v_n\}$, such that $v_n(x) \to v(x)$ a. e. in Ω and $\nabla T_k(v_n) \rightharpoonup \nabla T_k(v)$, in $W_0^{1,2}(\Omega)$.*

Thanks to Main Estimate (1.6) and the previous Proposition, we can state the following

Corollary 4.1 *There exists a measurable function u, with $T_k(u)$ belonging to $W_0^{1,2}(\Omega)$, for every $k > 0$, and a subsequence, still denoted by $\{u_n\}$, such that $u_n(x) \to u(x)$ a. e. in Ω, and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $W_0^{1,2}(\Omega)$.*

Note that the left hand side is well defined, since the integral is only on the set $\{x \in \Omega : |u(x) - \varphi(x)| \leq k\}$ and on this set $|u| \leq k + \|\varphi\|_{L^\infty(\Omega)}$.

Theorem 4.1 *Under the assumptions either of Lemma 4.1, or of Lemma 4.2, or of Lemma 4.3, or of Lemma 4.4, there exists an entropy solution of the boundary value problem (1.2).*

Proof. Let $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Choosing $T_k[u_n - \varphi]$ as test function in (1.5), we have

$$\int_\Omega a(x, T_k(u_n)) \nabla u_n \cdot \nabla T_k[u_n - \varphi] = \int_\Omega f_n \, T_k[u_n - \varphi] \, .$$

The right hand side easily passes to the limit as n tends to infinity. As for the left hand side, we can write it as

$$\int_\Omega a(x, T_k(u_n)) \, |\nabla(u_n - \varphi)|^2 + \int_\Omega a(x, T_k(u_n)) \, \nabla\varphi \cdot \nabla T_k[u_n - \varphi] \, .$$

In the first term we use the weak lower semicontinuty of the quadratic forms; while the second converges to

$$\int_\Omega a(x, u) \, \nabla\varphi \cdot \nabla T_k[u_n - \varphi] \, ,$$

as n tends to infinity. Putting together the terms, we thus have

$$\int_\Omega a(x, u) \, \nabla u \cdot \nabla T_k[u - \varphi] \leq \int_\Omega f \, T_k[u - \varphi] \, ,$$

for every $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and so u is an entropy solution of (1.2).

Corollary 4.2 *Under the assumptions of Lemma 4.1 entropy solution u is such that*

$$\int_\Omega |u|^{\theta 2^*} \leq L$$

which implies that

$$\mathrm{meas}\{x \in \Omega : |u(x)| > k\} \leq \frac{L}{k^{\frac{1}{\theta 2^*}}}.$$

Moreover, we can repeat the proof of Lemma 3.2 in [5] (see also [2]) to prove that there exists $\Lambda > 0$ such that, for any $h > 0$ we have

$$\mathrm{meas}\{x \in \Omega : |\nabla u(x)| > h\} \leq \frac{\Lambda}{h^{\frac{2N\theta}{2N\theta+N-2}}}.$$

Corollary 4.3 *Under the assumptions of Lemmas 4.2, 4.3, 4.4, the decay with respect to k of the measure of the sets $\{x \in \Omega : |u(x)| > k\}$, $\{x \in \Omega : |\nabla u(x)| > h\}$ of entropy solution u is of logarithmic type.*

Proof. We follow the lines of the proof of [2], Lemma 4.2 (see also [5]). Lemmas 4.2, 4.3, 4.4 imply that

$$\mathrm{meas}\{x \in \Omega : |u(x)| > k\} \leq \frac{C_1}{[\log k]^{m^{**}}},$$

Let h be a fixed positive real number. We have, for every $k > 0$,

$$|\{x \in \Omega : |\nabla u(x)| > h\}| \leq |\{|\nabla u| > h\,, |u| \leq k\}| + |\{|u| > k\}| \leq C_2\Big(\frac{k^2}{h^2} + \frac{1}{[\log k]^{m^{**}}}\Big).$$

Now, if we take $k = h^\lambda$, $\lambda < 1$, we have

$$|\{x \in \Omega : |\nabla u(x)| > h\}| \leq C_3 \frac{1}{[\log k]^{m^{**}}}.$$

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