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# Some Existence Results on Nontrivial Solutions of the Prescribed Mean Curvature Equation

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**Abstract**

The paper is concerned with an eigenvalue problem for the prescribed mean curvature equation. We formulate the problem as a variational inequality and show that under some growth conditions on the lower order term, the relaxed problem has at least two nontrivial solutions in a space of functions of bounded variation when the parameter is small.

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## 1 Introduction

We are concerned here with an eigenvalue problem for the prescribed mean curvature equation, namely, the equation

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(x,u) \ \text{ in } \ \Omega, \tag{1.1}$$

with Dirichlet boundary condition

$$u = 0 \ \text{ on } \ \partial\Omega, \tag{1.2}$$

here, $\Omega$ is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with sufficiently smooth boundary, $\nabla u = (\partial_1 u, \ldots, \partial_N u)$ and $\lambda$ is a positive parameter, and we are interested in the existence of nontrivial solutions (eigenfunctions) of the boundary value problem (1.1)-(1.2), and furthermore in nonnegative solutions, that is solutions $u$ such that

$$u \geq 0 \text{ in } \Omega. \tag{1.3}$$

The prescribed mean curvature problem and, in particular, the minimal surface problem, have been studied extensively with different approaches. Classical existence theorems and gradient estimates are presented in [22] with references to the original works by Finn, Bombieri/De Giorgi/Miranda, Jenkins, Serrin, etc. In this paper, we start with the variational approach for the problem in the space of functions of bounded variation. This approach was developed in e.g. [5, 38, 19, 20, 21]. In the existence theorems established in most of those works, the solutions of the prescribed mean curvature problem are considered as global minimizers of the corresponding energy functionals.

We propose here a new formulation of the relaxed problem for (1.1)-(1.2) as a variational inequality which can be seen as a "hybrid" from equations and minimization problems. This allows us to study other types of solutions for the problem, such as saddle points, as well. This formulation is simple and elementary, yet it gives some new insights to problem (1.1)-(1.2). It also seems suitable for the application of recent results in Nonsmooth Analysis and the theory of variational inequalities (cf. e.g. [43, 15, 12, 26, 16, 33, 30, 34, 31, 35] and the references therein) to the classical problem of prescribed mean curvature. We consider here homogeneous Dirichlet boundary conditions; hence the question of the existence of nontrivial solutions is crucial.

Our main goal here is the inequality formulation and the existence of nontrivial solutions of problem (1.1)-(1.2) as local minimizers and as saddle points of the potential functional. The existence of nonzero local minimizers is proved by using appropriate estimates. On the other hand, the existence of saddle points is based on a version of the Moutain Pass theorem for variational inequalities. Mountain Pass theorems for inequalities seem to have been first developed in [40] and [10] to study unstable minimal surfaces. The inequalities considered in [40] and [10] consist of the Dirichlet integral over appropriate closed and convex sets. We need here a different version of the Mountain Pass theorem for inequalities with convex functionals which are not indicator functions of convex sets. Also, since it seems rather difficult to verify the Palais–Smale (PS) condition for inequalities in our case, we shall prove and use a version of the theorem without the (PS) condition.

Concerning prescribed mean curvature equations with parameters (eigenvalue problems), the existence of radial solutions has been studied extensively in the case that $\Omega$ is radially symmetric (cf. e.g. [39, 4, 27] and the references therein). The general case of nonsymmetric domains and nonradial solutions seems not to have been investigated to such an extent. An existence result for nontrivial solutions of the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - \mu u + \lambda f(u) = 0 \ \text{ in } \ \Omega \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1.4}$$

where $\mu \geq 0$, $f \in C[0,\infty)$ satisfies the Nehari condition, was obtained in [11]. By using an extension of Nehari's method to partial differential equations, Coffman and Ziemer showed that under certain growth conditions, for each $\mu$ fixed, (1.4) has a nontrivial nonnegative $C^1$-solution for $\lambda$ sufficiently large.

In this paper, we concentrate on equation (1.1) with a general lower-order term $f(x,u)$. Using a different approach, we show that under certain growth conditions on $f$, problem (1.1)-(1.2) has at least two nontrivial solutions for $\lambda$ sufficiently small.

The paper is organized as follows. In the next section, we formulate the relaxed problem for (1.1)-(1.2) as a variational inequality in an appropriate space of functions of bounded variation. In section 3, we show the existence of nontrivial solutions of the relaxed problem of (1.1)-(1.2) as local minimizers of certain potential functionals. Section 4 is devoted to the existence of saddle points of Mountain Pass type of the problem.

## 2 Variational inequality formulation

In this section, we shall formulate (1.1)-(1.2) as a variational inequality which is more convenient for using variational arguments. First, note that the operator

$$u \mapsto -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

has $J_0(u) = \int_\Omega (\sqrt{1+|\nabla u|^2} - 1)\, dx$ as a potential functional (with $J_0(0) = 0$), at least for $u$ with $\nabla u$ satisfying certain growth conditions (for example, for $u \in W^{1,1}(\Omega)$). Let $F$ be the anti-derivative of $f$ (with respect to the second variable) with $F(\cdot, 0) = 0$,

$$F(x,u) = \int_0^u f(x,s)\, ds,\ \ x \in \Omega, u \in \mathbb{R}, \tag{2.1}$$

and

$$\mathcal{F}(u) = \int_\Omega F(x, u(x))\, dx.$$

We have

$$\langle \mathcal{F}'(u), v\rangle = \int_\Omega f(x,u) v\, dx$$

for all $u, v$ in some appropriate function space (to be specified later). Therefore, solutions of (1.1) are critical points of the functional

$$I_0 = J_0 - \lambda \mathcal{F},$$

subject to zero boundary conditions. As already well-known, the choice of a suitable function space is very important for the solving of problem (1.1)-(1.2). A choice of such space should be such that the functional $J_0$ is coercive and has certain appropriate continuity properties.

A simple and natural choice of such a function space is $W_0^{1,1}(\Omega)$ with the usual norm

$$\|u\|_{W^{1,1}} = \|u\|_{L^1} + \|\,|\nabla u|\,\|_{L^1}.$$

Then, $J_0$ is coercive in the sense that

$$J_0(u) \to \infty \text{ as } \|u\|_{W^{1,1}} \to \infty.$$

However, $J_0$ is not lower semicontinuous with respect to the weak topology in this space. To overcome this difficulty, a popular way is to consider the relaxed functional of $J_0$ in the space of functions of bounded variation (cf. e.g. [17, 13, 6, 14]). As usual, $BV(\Omega)$ denotes the set of all functions in $L^1(\Omega)$ with bounded variation, that is, $\nabla u$ (in the distributional sense) is a (vector) bounded Radon measure. In other words, $BV(\Omega)$ is the set of all $u \in L^1(\Omega)$ such that

$$\int_\Omega |\nabla u| := \sup\left\{\int_\Omega u \operatorname{div} g\,dx : g = (g_1, \dots, g_N) \in C_0^1(\Omega, \mathbb{R}^N) \text{ and } \max_{x\in\Omega} |g(x)| \le 1\right\} < \infty, \tag{2.2}$$

($\operatorname{div} g = \sum_{i=1}^N \partial_i g_i$). The usual norm on $BV(\Omega)$ is defined as

$$\|u\| = \|u\|_{BV(\Omega)} = \int_\Omega |u|dx + \int_\Omega |\nabla u|, \tag{2.3}$$

where $\int_\Omega |\nabla u|$ is defined in (2.2). Note that $\int_\Omega |\nabla u|$ is the total variation of the measure $\nabla u$. $BV(\Omega)$ with the norm (2.3) is a Banach space. The relaxed functional associated with $J_0(u)$ and the boundary condition $u = 0$ on $\partial\Omega$ is given by (cf. [6, 7, 8, 25, 24, 1, 17]):

$$J_1(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} + \int_{\partial\Omega} |u|_{\partial\Omega}|\, d\mathcal{H}^{N-1}, \tag{2.4}$$

for $u \in BV(\Omega)$, where $d\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$ and $\int_\Omega \sqrt{1 + |\nabla u|^2}$ is given by

$$\int_\Omega \sqrt{1 + |\nabla u|^2} = \sup\left\{\int_\Omega (g_{n+1} + u \operatorname{div} g)dx : g = (g_1, g_2, \dots, g_{n+1}) \in C_0^1(\Omega, \mathbb{R}^{n+1}), \max_{x\in\Omega} |g(x)| \le 1\right\}. \tag{2.5}$$

As shown in [25, 8], $\int_\Omega \sqrt{1 + |\nabla u|^2}$ can be defined equivalently as

$$\int_\Omega \sqrt{1 + |\nabla u|^2} = \int_\Omega \sqrt{1 + |(\nabla u)_a|^2} dx + \int_\Omega |(\nabla u)_s|, \tag{2.6}$$

where $(\nabla u)_a \in [L^1(\Omega)]^N$ and $(\nabla u)_s \in BV(\Omega)$ are, respectively, the absolutely continuous part and the singular part of $\nabla u$ with respect to the $N$-dimensional Lebesgue measure $dx = d\mathcal{L}^N$.

This leads us to the study of critical points of the relaxed functional associated to $I_0$:

$$I_1 = J_1 - \lambda \mathcal{F} \tag{2.7}$$

in $BV(\Omega)$. However, the continuous differentiability of $J_1$ cannot be established in $BV(\Omega)$. Since $J_1$ is convex in $BV(\Omega)$, we can consider critical points $u$ of $I_1$ as points such that

$$0 \in \partial J_1 - \lambda \mathcal{F}', \tag{2.8}$$

where $\partial J_1$ is the subdifferential of $J_1$ in $BV(\Omega)$ in the sense of convex analysis (and $\mathcal{F}'$ is the usual Fréchet derivative of $\mathcal{F}$). The inclusion (2.8) is equivalent to the variational inequality

$$\begin{cases} J_1(v) - J_1(u) - \lambda \displaystyle\int_\Omega f(x,u)(v-u)dx \geq 0, \ \forall v \in BV(\Omega) \\ u \in BV(\Omega). \end{cases} \tag{2.9}$$

Our goal here is to study (2.9) (and its equivalent inclusion) by using variational methods. We note that (2.9) is, in some sense, a "hybrid" from an equation and a minimization problem. In fact, (2.9) can be seen as the Euler–Lagrange equation for the minimization problem of $J_1 - \lambda \mathcal{F}$. On the other hand, if $f$ does not depend on $u$, then (2.9) is equivalent to the minimization of $J_1(u) - \lambda \displaystyle\int_\Omega fu$. Moreover, if $J_1$ is Gâteaux differentiable in a direction $h$, then from (2.9) with $v = u + th$ (and let $t \to 0^+$), we have that

$$\langle J_1'(u), h\rangle - \lambda \int_\Omega f(x,u)h dx = 0.$$

Another step is needed for our final formulation of the problem. Let $\mathcal{B}$ be a ball (or a bounded region with smooth boundary) in $\mathbb{R}^N$ such that $\overline{\Omega} \subset \mathcal{B}$. For $u \in BV(\Omega)$, put

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathcal{B} \setminus \Omega. \end{cases}$$

Then, $\tilde{u} \in BV(\Omega)$. As in Section 14.4 of [24], we have

$$\int_\mathcal{B} \sqrt{1 + |\nabla \tilde{u}|^2} = \int_\Omega \sqrt{1 + |\nabla u|^2} + \int_{\partial\Omega} |u|_{\partial\Omega}|\, d\mathcal{H}^{N-1} = J_1(u). \tag{2.10}$$

Let us put

$$X = \{u \in BV(\mathcal{B}) : u(x) = 0 \text{ for a.e. } x \in \mathcal{B} \setminus \overline{\Omega}\} \tag{2.11}$$

and

$$J(u) = \int_\mathcal{B} \sqrt{1 + |\nabla u|^2}, \ u \in BV(\mathcal{B}). \tag{2.12}$$

Then, in view of (2.10) and (2.12), the inequality (2.9) is equivalent to the following inequality in $X$:

$$\begin{cases} J(v) - J(u) - \lambda \displaystyle\int_\Omega f(x,u)(v-u)dx \geq 0, \ \forall x \in X \\ u \in X. \end{cases} \tag{2.13}$$

By moving from (2.9) to (2.13) and replacing $J_1$ in (2.4) by $J$ in (2.12), we get a simpler representation for the relaxed functional of $J_0$ (without involving the integral on the boundary of $\Omega$). Moreover, $X$ is a closed subspace of $BV(\Omega)$. In fact, assume $\{u_n\} \subset X$ and $u_n \to u$ in $BV(\mathcal{B})$. We have $u_n \to u$ in $L^1(\mathcal{B})$ and by passing to a subsequence, if necessary, we can assume that

$$u_n \to u \text{ a.e. in } \mathcal{B}.$$

It follows that $u = 0$ a.e. in $\mathcal{B} \setminus \overline{\Omega}$, i.e., $u \in X$. Thus, (2.13) is equivalent to the following variational inequality in $BV(\mathcal{B})$:

$$\begin{cases} (J + I_X)(v) - (J + I_X)(u) - \lambda \int_\Omega f(x,u)(v-u)dx \geq 0, \ \forall v \in BV(\mathcal{B}) \\ u \in BV(\mathcal{B}). \end{cases} \tag{2.14}$$

We also use an equivalent form of (2.13) with $J$ given, instead of (2.13), by

$$\begin{aligned} J(u) &= \int_{\mathcal{B}} (\sqrt{1+|\nabla u|^2} - 1) \\ &= \int_{\mathcal{B}} [\sqrt{1+|(\nabla u)_a|^2} - 1]dx + \int_{\mathcal{B}} |(\nabla u)_s|. \end{aligned} \tag{2.15}$$

From this formula, we see that $J(u) \geq 0$ for all $u \in BV(\mathcal{B})$, $J(0) = 0$, and moreover

$$J(u) = 0, u \in X \iff u = 0.$$

In fact, assume $J(u) = 0$ for some $u \in X$. We have $|(\nabla u)_s|(\mathcal{B}) = 0$ and $\sqrt{1+|\nabla u|^2} - 1 = 0$ a.e. in $\mathcal{B}$. The first equation implies that $(\nabla u)_s$ is a zero measure on $\mathcal{B}$. Since $\sqrt{1+|\nabla u|^2} - 1 \geq 0$ on $\mathcal{B}$, it follows from the second equation that $(\nabla u)_a = 0$ a.e. in $\mathcal{B}$. Thus, $\nabla u = 0$ in $\mathcal{B}$ (as a measure). This shows that $u$ is a constant. Since $u \in X$, this constant must be zero.

Assume that

$$f(x,0) = 0 \text{ for a.e. } x \in \mathcal{B}. \tag{2.16}$$

Hence, $u = 0$ is a trivial solution of (2.13) for all $\lambda (> 0)$. We are interested here in the existence of nonzero solutions of (2.13) in $X$.

# 3 Existence of nontrivial solutions as local minimizers

We study the existence of nontrivial critical points of the functional

$$I(u)(= I_\lambda(u)) = J(u) - \lambda \int_{\mathcal{B}} F(x,u)dx, \ u \in X,$$

where $J$ is given by (2.15) and $F$ given by (2.1). We assume the following conditions.

(A1) $f : \mathcal{B} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

(A2) There exists $q \in \left(1, \dfrac{N}{N-1}\right)$ such that

$$|f(x,\xi)| \leq d_1 \xi^{q-1} + d_2, \quad \text{for a.e. } x \in \mathcal{B}, \text{ all } \xi \in \mathbb{R}, \tag{3.1}$$

with $d_1, d_2 > 0$. Assume furthermore that $q < 2$ (this holds if $N \geq 2$) and there exist $r \in (1,2)$, $d_3, \xi_0 > 0$ such that

$$f(x,\xi) \geq d_3 \xi^{r-1} \tag{3.2}$$

for a.e. $x \in \mathcal{B}$, all $\xi \in [0, \xi_0)$.

First, we need an estimate for $\displaystyle\int_{\mathcal{B}} \sqrt{1+|\nabla u|^2}$.

**Lemma 3.1** *We have*

$$\int_{\mathcal{B}} \sqrt{1+|\nabla u|^2} \geq \sqrt{|\mathcal{B}|^2 + \left(\int_{\mathcal{B}} |\nabla u|\right)^2}, \ \forall u \in BV(\mathcal{B}). \tag{3.3}$$

*Proof.* First, note that (3.3) holds for all $u \in W^{1,1}(\mathcal{B})$. In fact, for $u \in W^{1,1}(\mathcal{B})$, we have $|\nabla u| \in L^1(\mathcal{B})$ and Jensen's inequality, applied to the convex function $\Phi(x) = \sqrt{1+x^2}$, gives

$$\Phi\left(\frac{\displaystyle\int_{\mathcal{B}} |\nabla u| dx}{\displaystyle\int_{\mathcal{B}} dx}\right) \leq \frac{\displaystyle\int_{\mathcal{B}} \Phi(|\nabla u|) dx}{\displaystyle\int_{\mathcal{B}} dx},$$

that is,

$$\sqrt{1 + \frac{1}{|\mathcal{B}|^2}\left(\int_{\mathcal{B}} |\nabla u|\right)^2} \leq \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \sqrt{1+|\nabla u|^2},$$

which is the same as (3.3). Now, assume $u \in BV(\mathcal{B})$. From Theorem 3.3, [8], there exists a sequence $\{u_n\}$ in $W^{1,1}(\mathcal{B})$ such that $u_n \to u$ in $L^1(\mathcal{B})$ and

$$\int_{\mathcal{B}} \sqrt{1+|\nabla u_n|^2} \to \int_{\mathcal{B}} \sqrt{1+|\nabla u|^2}. \tag{3.4}$$

It follows from the lower semicontinuity of the total variation $\displaystyle\int_{\mathcal{B}} |\nabla u|$ (cf. e.g. [24], Theorem 1.9) that

$$\int_{\mathcal{B}} |\nabla u| \leq \liminf \int_{\mathcal{B}} |\nabla u_n|.$$

Therefore,

$$\sqrt{|\mathcal{B}|^2 + \left(\int_{\mathcal{B}} |\nabla u|\right)^2} \leq \liminf \sqrt{|\mathcal{B}|^2 + \left(\int_{\mathcal{B}} |\nabla u_n|\right)^2}. \tag{3.5}$$

Applying (3.3) to each $u_n$ and using (3.3), (3.4), we get

$$\begin{aligned}\int_{\mathcal{B}} \sqrt{1+|\nabla u|^2} &= \lim \int_{\mathcal{B}} \sqrt{1+|\nabla u_n|^2} \\ &\geq \liminf \sqrt{|\mathcal{B}|^2 + \left(\int_{\mathcal{B}} |\nabla u_n|\right)^2} \\ &\geq \sqrt{|\mathcal{B}|^2 + \left(\int_{\mathcal{B}} |\nabla u|\right)^2}.\end{aligned}$$

Hence, (3.3) is proved for $u \in BV(\mathcal{B})$. □

We need the following Poincaré inequality for functions in $X$ (based on e.g. Theorem 1.28, [24]), whose proof is included for completeness.

**Theorem 3.1** *For each* $\beta \in \left[1, \dfrac{N}{N-1}\right]$, *there exists* $C_\beta > 0$ *such that*

$$\left(\int_{\mathcal{B}} |u|^\beta dx\right)^{\frac{1}{\beta}} \leq C_\beta \int_{\mathcal{B}} |\nabla u|, \ \forall u \in X. \tag{3.6}$$

*Proof.* For $u \in X$, we extend $u$ to a function in $BV(\mathbb{R}^N)$ as

$$\tilde{u} = \begin{cases} u & \text{in} \quad \mathcal{B} \\ 0 & \text{in} \quad \mathbb{R}^N \setminus \mathcal{B}. \end{cases}$$

Then, $\tilde{u} \in BV(\mathbb{R}^N)$ and the traces of $\tilde{u}$ on $\partial\mathcal{B}$ both inside and outside of $\mathcal{B}$ are 0 (cf. Chapter 2, [24]), i.e.,

$$\tilde{u}^+|_{\partial\mathcal{B}} = \tilde{u}^-|_{\partial\mathcal{B}} = u^-|_{\partial\mathcal{B}} = 0.$$

From Theorem 1.28, [24], it follows that

$$\left(\int_{\mathbb{R}^N} |\tilde{u}|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} \leq C_1 \int_{\mathbb{R}^N} |\nabla \tilde{u}|, \tag{3.7}$$

and from Remark 2.14, [24], we conclude that

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}| = \int_{\mathcal{B}} |\nabla u| + \int_{\partial\mathcal{B}} |u^-|_{\partial\mathcal{B}}| d\mathcal{H}^{N-1} = \int_{\mathcal{B}} |\nabla u|. \tag{3.8}$$

It is clear that

$$\int_{\mathbb{R}^N} |\tilde{u}|^{\frac{N}{N-1}} = \int_{\mathcal{B}} |u|^{\frac{N}{N-1}}. \tag{3.9}$$

Combining (3.7), (3.8), (3.9), we get (3.6) for the case $\beta = \frac{N}{N-1}$. For $\beta \in [1, \frac{N}{N-1})$, using Hölder's inequality, we get

$$\int_{\mathcal{B}} |u|^\beta dx \leq |\Omega|^{1-\beta\frac{N-1}{N}} \left(\int_{\mathcal{B}} |u|^{\frac{N}{N-1}}\right)^{\frac{\beta(N-1)}{N}}.$$

Therefore,

$$\left(\int_{\mathcal{B}} |u|^{\beta}\right)^{1/\beta} \leq C_{\beta}\left(\int_{\mathcal{B}} |u|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}}.$$

This and (3.6) in the case $\beta = \frac{N}{N-1}$ imply (3.6) for $\beta \in [1, \frac{N}{N-1}]$. □

From the Poincaré inequality (3.6) (with $\beta = 1$), we know that, within $X$, the norm $\|\cdot\|_{BV(\mathcal{B})}$ defined above is equivalent to the norm

$$\|u\|_0 = \int_{\mathcal{B}} |\nabla u|.$$

Now, we are ready to prove the existence of a nonzero local minimizer of $I_\lambda$.

**Theorem 3.2** *Under assumptions (A1)-(A2) and (3.2), there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, the functional $I_\lambda$ has a nonzero local minimum point $u$ in the open ball $\{u \in X : \|u\|_0 < \lambda^\alpha\}$ with $0 < \alpha < 1$.*

*Proof.* From (3.1), there exists $C_1 > 0$ such that

$$|F(x,\xi)| \leq C_1(|\xi|^q + |\xi|) \tag{3.10}$$

for a.e. $x \in \mathcal{B}$, all $\xi \in \mathbb{R}$. Using (3.6) with $\beta = q$ and $\beta = 1$, one gets

$$\begin{aligned} \int_{\mathcal{B}} |F(x,u)|dx &\leq C_1\left(\int_{\mathcal{B}} |u|^q + \int_{\mathcal{B}} |u|\right) \\ &\leq \left[\left(\int_{\mathcal{B}} |\nabla u|\right)^q + \int_{\mathcal{B}} |\nabla u|\right], \ \forall u \in X, \end{aligned} \tag{3.11}$$

for some $C_2 > 0$. For $\rho > 0$, let us consider the sphere

$$S_\rho = \{u \in X : \|u\|_0 = \rho\}. \tag{3.12}$$

From (3.3) and (3.11), we have, for $u \in S_\rho$,

$$\begin{aligned} I_\lambda(u) &= J(u) - \lambda \int_{\mathcal{B}} F(x,u)dx \\ &\geq (|\mathcal{B}|^2 + \rho^2)^{1/2} - |\mathcal{B}| - C_2\lambda(\rho^q + \rho). \end{aligned} \tag{3.13}$$

Choosing $\rho = \lambda^\alpha$ ($\alpha > 0$ will be chosen later), we have, for $u \in S_{\lambda^\alpha}$,

$$I_\lambda(u) \geq (B|^2 + \lambda^{2\alpha})^{1/2} - |\mathcal{B}| - C_2(\lambda^{q\alpha+1} + \lambda^{\alpha+1}). \tag{3.14}$$

By taking $\alpha \in (0,1)$, we have

$$2\alpha < \alpha + 1 \leq q\alpha + 1.$$

Since

$$(|\mathcal{B}|^2 + \lambda^{2\alpha})^{1/2} - |\mathcal{B}| = \frac{\lambda^{2\alpha}}{(|\mathcal{B}|^2 + \lambda^{2\alpha})^{1/2} + |\mathcal{B}|} \simeq \frac{\lambda^{2\alpha}}{2|\mathcal{B}|},$$

as $\lambda \to 0$, there exists $\lambda^* > 0$ such that the right hand side of (3.14) is positive for all $\lambda \in (0, \lambda^*]$. Consequently,

$$I_\lambda(u) > 0 \text{ on } S_{\lambda^\alpha} \text{ for all } \lambda \in (0, \lambda^*]. \tag{3.15}$$

Let $B_\rho = \{u \in X : \|u\|_0 \le \rho\}$. From (3.13),

$$\inf\{I_\lambda(u) : u \in B_{\lambda^\alpha}\} > -\infty.$$

Let us choose a function $\phi = \phi_\lambda \in C_0^\infty(\Omega)$ such that

$$\|\phi\|_0 \le \lambda^\alpha,$$

and

$$0 \le \phi(x) < \xi_0, \ \forall x \in \Omega,$$

where $\xi_0$ is given in (3.2). For $\epsilon \in (0, 1)$, we have $\epsilon\phi(x) \in [0, \xi_0)$ and

$$F(x, \epsilon\phi(x)) = \int_0^{\epsilon\phi(x)} f(x, \xi) d\xi \ge C_3 \epsilon^r [\phi(x)]^r \ (C_3 = d_3 r^{-1}). \tag{3.16}$$

Also, by extending $\phi$ to $\mathcal{B}$ (by putting $\phi(x) = 0$ for $x \in \mathcal{B} \setminus \Omega$), one has $\phi \in C_0^\infty(\mathcal{B}) \cap X$ and furthermore $\phi \in B_{\lambda^\alpha}$.

Since $\epsilon\phi \in C_0^\infty(\mathcal{B})$,

$$\begin{aligned} J(\epsilon\phi) &= \int_{\mathcal{B}} \left( \sqrt{1 + |\nabla(\epsilon\phi)(x)|^2} - 1 \right) dx \\ &\le \frac{1}{2}\epsilon^2 \int_{\mathcal{B}} |\nabla\phi(x)|^2 \, dx, \end{aligned}$$

(because $\sqrt{1+\xi^2} - 1 \le \frac{1}{2}\xi^2, \ \forall \xi \in \mathbb{R}$). Thus, from (3.16), we may deduce that

$$\begin{aligned} I_\lambda(\epsilon\phi) &= \int_{\mathcal{B}} (\sqrt{1 + |\nabla(\epsilon\phi)|^2} - 1) - \lambda \int_{\mathcal{B}} F(x, \epsilon\phi) dx \\ &\le \frac{\epsilon^r}{2} \left( \epsilon^{2-r} \int_{\mathcal{B}} |\nabla\phi|^2 dx - \lambda C_3 \int_{\mathcal{B}} \phi^r dx \right). \end{aligned} \tag{3.17}$$

Since $r < 2$, we have $\epsilon^{2-r} \to 0$ as $\epsilon \to 0$. Therefore, the right hand side of (3.17) is negative for $\epsilon > 0$ sufficiently small. This proves that $\inf\{I_\lambda(u) : u \in B_{\lambda^\alpha}\} < 0$. Now, since the embedding

$$BV(\mathcal{B}) \hookrightarrow L^\beta(\mathcal{B})$$

is compact for $\beta \in [1, \frac{N}{N-1})$ (cf. Section 1.19, [24]), (3.10) shows that the mapping

$$u \mapsto \int_{\mathcal{B}} F(x, u) dx$$

is completely continuous from $BV(\mathcal{B})$ to $\mathbb{R}$ in the sense that if $u_n \to u$ in $L^1(\mathcal{B})$ and $\{u_n\}$ is bounded in $BV(\mathcal{B})$ then

$$\int_{\mathcal{B}} F(x, u_n) dx \to \int_{\mathcal{B}} F(x, u) dx. \tag{3.18}$$

Let $\{u_n\}$ be a sequence in $B_{\lambda^\alpha}$ such that

$$I_\lambda(u_n) \to \inf\{I_\lambda(u) : u \in B_{\lambda^\alpha}\}.$$

Since $\{u_n\}$ is bounded in $BV(\mathcal{B})$, by passing to a subsequence, if necessary, we have $u \in BV(\mathcal{B})$ such that

$$\begin{aligned} &u_n \to u \text{ in } L^1(\mathcal{B}), \text{ and} \\ &\nabla u_n \rightharpoonup^* \nabla u \text{ in } M(\mathcal{B}). \end{aligned} \tag{3.19}$$

It is clear from (3.19) that $u \in X$. From the lower semicontinuity of the norm $\|\cdot\|_0$ with respect to the $L^1$-topology (Theorem 1.9, [24]), we have

$$\int_{\mathcal{B}} |\nabla u| \leq \liminf \int_{\mathcal{B}} |\nabla u_n| \leq \lambda^\alpha.$$

Hence, $u \in B_{\lambda^\alpha}$. We also note that $J$ is lower semicontinuous with respect to the $L^1$-topology in $BV(\Omega)$ (Theorem 14.2, [24]). It follows that

$$J(u) \leq \liminf J(u_n).$$

Together with (3.18), this shows that

$$I_\lambda(u) \leq \liminf I_\lambda(u_n).$$

Consequently, $I_\lambda$ has a minimum at $u$ in $B_{\lambda^\alpha}$. We have $I_\lambda(u) < 0$. This implies that $u$ cannot be on the boundary $S_{\lambda^\alpha}$ of $B_{\lambda^\alpha}$ (by (3.15)). Also, $u \neq 0$ (since $I_\lambda(0) = 0$). Thus, $u$ is a nontrivial, local minimum point of $I_\lambda$. □

**Remark 3.1** Since $\|u\|_0 \leq \lambda^\alpha$, we immediately have

$$\int_{\mathcal{B}} |\nabla u| \to 0 \quad \text{as } \lambda \to 0^+.$$

That the local minimum point $u$ of the above theorem (Theorem 3.2) is a nontrivial solution of (2.13) follows from the following lemma.

**Lemma 3.2** *Assume $V$ is a Banach space, $J : V \to \mathbb{R} \cup \{\infty\}$ is convex, and $\mathcal{F} : V \to \mathbb{R}$ is Gâteaux differentiable. If $u \in V$ is a local minimum point of $I = J - \mathcal{F}$, then $u$ is a solution of the variational inequality*

$$J(v) - J(u) - \langle \mathcal{F}'(u), v - u\rangle \geq 0, \ \forall v \in V. \tag{3.20}$$

The proof is straightforward and is omitted. From the growth condition (3.10), it follows that $\mathcal{F}$ is Fréchet differentiable from $L^q(\mathcal{B})$ to $\mathbb{R}$. Since the embedding mapping of $BV(\mathcal{B})$ into $L^q(\mathcal{B})$ is continuous, $\mathcal{F}$ is also Fréchet and thus Gâteaux differentiable from $BV(\mathcal{B})$ to $\mathbb{R}$. As a consequence of Theorem 3.2 and Lemma 3.2, one has the following existence result for nontrivial solutions of (2.13).

**Corollary 3.1** *Under assumptions (A1), (A2), and (3.2), there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, the variational inequality (2.13) has a solution $u_\lambda \neq 0$, which is a local minimizer of the functional $I_\lambda$. Moreover,*

$$\int_{\mathcal{B}} |\nabla u_\lambda| \to 0 \quad \textit{as } \lambda \to 0. \tag{3.21}$$

# 4 Existence of nontrivial solutions as saddle points

Under certain conditions, we can find another nontrivial solution of (2.13) as a saddle point of the energy functional. For this purpose, we shall use a version of the Mountain Pass Theorem for variational inequalities (cf. [2, 43]), without the Palais–Smale (compactness) condition. This modification is crucial for our purpose, since, different from Sobolev spaces which are normally reflexive (and uniformly convex), $BV(\mathcal{B})$ is not. Hence, compactness properties for functions of bounded variation seem less convenient than their counterparts in Sobolev spaces (more comments are given in Remark 4.1).

Our tools here consist of some concepts of slopes of nonsmooth functionals, together with a critical point theory for nonsmooth functionals developed recently by M. Degiovanni and his collaborators (cf. e.g. [15, 12]), also independently by Ioffe and Schwartzman ([26]). First, we recall some definitions (cf. [15, 12, 26]).

**Definition 4.1** Let $(X, d)$ be a metric space, $u \in X$, and $f : X \to \mathbb{R}$ be continuous. One denotes by $|df|(u)$ (in $[0, \infty]$, called the weak slope of $f$ at $u$) the supremum of the $\sigma$'s in $[0, \infty)$ such that there exist $\delta > 0$, an open neighborhood $U$ of $u$ in $X$, and a continuous function $\mathcal{H} : U \times [0, \delta] \to X$ such that

(1) $d(\mathcal{H}(v, t), v) \leq t$, and

(2) $f(\mathcal{H}(v, t)) \leq f(v) - \sigma t,\ \forall (v, t) \in U \times [0, \delta]$.

**Definition 4.2** Let $(X, d)$ be as in Definition 4.1 and $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinous function.

(a) The effective domain of $f$ is $D(f) = \{u \in X : f(u) < \infty\}$.

(b) The epigraph of $f$ is the set $\text{epi}(f) = \{(u, \xi) \in X \times \mathbb{R} : f(u) \leq \xi\}$. $\text{epi}(f)$ is a metric space with the metric $d((u, \xi), (v, \eta)) = [d(u, v)^2 + (\xi - \eta)^2]^{1/2}$.

(c) We denote by $\mathcal{G}_f$ the projection of $\text{epi}(f)$ to $\mathbb{R}$:

$$\mathcal{G}_f : \text{epi}(f) \to \mathbb{R},\ \mathcal{G}_f(u, \xi) = \xi.$$

Note that $\mathcal{G}_f$ is Lipschitz continuous on $\text{epi}(f)$ with the Lipschitz constant 1 and it is easy to check that $|d\mathcal{G}_f|(u, \xi) \leq 1$. Hence, in the case $f$ is merely lower semicontinous, one can define $|df|$ (in terms of $|d\mathcal{G}_f|$) as follows.

**Definition 4.3** Let $(X, d)$ be a metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous. We set

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}} & \text{if} \quad |d\mathcal{G}_f|(u, f(u)) < 1 \\ \infty & \text{if} \quad |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases} \tag{4.1}$$

As shown in [15], this definition extends that in Definition 4.1. For functionals that are perturbations of convex functionals by $C^1$ functionals, we propose some simpler, more direct measurements for their slopes. Assume now that $(X, \|\cdot\|)$ is a Banach space and $\phi = J - F : X \to \mathbb{R} \cup \{\infty\}$, where $J : X \to \mathbb{R} \cup \{\infty\}$ is a convex, lower semicontinuous, proper functional and $F \in C^1(X, \mathbb{R})$. It is clear that $D(\phi) = D(J)$.

**Definition 4.4** For $u \in D(J)$, we define

$$|\nabla\phi|(u) = \max\left\{0, \sup_{v\in X\setminus\{u\}} \frac{J(u) - J(v) - \langle F'(u), u - v\rangle}{\|u - v\|}\right\}, \tag{4.2}$$

and

$$\|\nabla\phi\|(u) = \sup_{v\in X, \|u-v\|\leq 1} [J(u) - J(v) - \langle F'(u), u - v\rangle], \tag{4.3}$$

($\langle\cdot,\cdot\rangle$ denotes the pairing between $X$ and its dual $X^*$). The above definitions of slopes for $\phi$ are closely related to the following variational inequality:

$$\begin{cases} J(v) - J(u) - \langle F'(u), v - u\rangle \geq 0, \ \forall v \in X \\ u \in X. \end{cases} \tag{4.4}$$

Some properties of the slopes defined above are given in the following result.

**Proposition 4.1** *(a) For any $R > 0$, we have*

$$\begin{aligned} |\nabla\phi|(u) &= \max\left\{0, \sup_{\|v-u\|\leq R, v\neq u} \frac{J(u) - J(v) - \langle F'(u), u - v\rangle}{\|u - v\|}\right\} \\ &= \begin{cases} \limsup\limits_{v\to u} \dfrac{\phi(u) - \phi(v)}{\|u - v\|} & \text{if } u \text{ is not a local minimum of } \phi \\ 0 & \text{if } u \text{ is a local minimum of } \phi. \end{cases} \end{aligned} \tag{4.5}$$

*In particular, $|\nabla\phi|(u)$ defined in (4.2) coincides with the concept of strong slope of $\phi$ defined in [23].*

*(b) For every $u \in D(J)$,*

$$\|\nabla\phi\|(u) \leq |\nabla\phi|(u) = |d\phi|(u). \tag{4.6}$$

*(c) For $u \in D(J)$, $\|\nabla\phi\|(u) = 0$ if and only if $|\nabla\phi|(u) = 0$. We call such $u$ a critical point of $\phi$.*

*(d) $u$ is a critical point of $\phi$ if and only if $u$ is a solution of the variational inequality (4.4).*

*(e) For every $u \in D(J)$,*

$$J(v) - J(u) - \langle F'(u), v - u\rangle \geq -|\nabla\phi|(u)\|v - u\|, \ \forall v \in X, \tag{4.7}$$

*and*

$$J(v) - J(u) - \langle F'(u), v - u\rangle \geq -\|\nabla\phi\|(u)\max\{1, \|v - u\|\}, \ \forall v \in X. \tag{4.8}$$

*Proof.* (a) Let $R > 0$. For $v$ such that $\|v - u\| > R$, we put $z = x + \frac{R(v-u)}{\|v-u\|}$. Then, $\|z - u\| = R$ and $J(z) \leq \|v - u\|^{-1}RJ(v) + (1 - \|v - u\|^{-1}R)J(u)$. Thus,

$\|z-u\|^{-1}[J(z)-J(u)] \leq \|v-u\|^{-1}[J(v)-J(u)]$. Because $\|u-v\|^{-1}\langle F'(u), u-v\rangle = \|u-z\|^{-1}\langle F'(u), u-z\rangle$, we have

$$\begin{aligned}\frac{J(u)-J(v)-\langle F'(u), u-v\rangle}{\|u-v\|} &\leq \frac{J(u)-J(z)-\langle F'(u), u-z\rangle}{\|u-z\|} \\ &\leq \sup_{\|w-u\|\leq R, w\neq u} \frac{J(u)-J(w)-\langle F'(u), u-w\rangle}{\|u-w\|}.\end{aligned}$$

Since this holds for every $v \in X$ with $\|v-u\| > R$, we have the first identity in (4.5). As a consequence, one gets

$$\sup_{v\neq u} \frac{J(u)-J(v)-\langle F'(u), u-v\rangle}{\|u-v\|} = \limsup_{v\to u(v\neq u)} \frac{J(u)-J(v)-\langle F'(u), u-v\rangle}{\|u-v\|}.$$

Now, since

$$\lim_{v\to u} \frac{F(u)-F(v)-\langle F'(u), u-v\rangle}{\|u-v\|} = 0,$$

one has

$$\begin{aligned}\limsup_{v\to u} \frac{J(u)-J(v)-\langle F'(u), u-v\rangle}{\|u-v\|} &= \limsup_{v\to u} \frac{J(u)-J(v)-[F(u)-F(v)]}{\|u-v\|} \\ &= \limsup_{v\to u} \frac{\phi(u)-\phi(v)}{\|u-v\|}.\end{aligned}$$

We have shown that

$$\sup_{v\neq u} \frac{J(u)-J(v)-\langle F'(u), u-v\rangle}{\|u-v\|} = \limsup_{v\to u} \frac{\phi(u)-\phi(v)}{\|u-v\|}. \tag{4.9}$$

If $\phi$ has a local minimum at $u$ then

$$\limsup_{v\to u} \frac{\phi(u)-\phi(v)}{\|u-v\|} = \sup_{v\neq u} \frac{J(u)-J(v)-\langle F'(u), u-v\rangle}{\|u-v\|} \leq 0.$$

Hence, $|\nabla\phi|(u) = 0$ and the second equality in (4.5) is proved in this case. Assume now that $\phi$ does not have a local minimum at $u$. Thus, there exists a sequence $\{v_n\}$ converging to $u$ such that $\phi(v_n) < \phi(u), \ \forall n$. We have

$$\sup_{v\neq u} \frac{J(u)-J(v)-\langle F'(u), u-v\rangle}{\|u-v\|} \geq \limsup_{n\to\infty} \frac{\phi(u)-\phi(v_n)}{\|u-v_n\|} \geq 0.$$

Therefore, $|\nabla\phi|(u) = \limsup_{v\to u} \dfrac{\phi(u)-\phi(v)}{\|u-v\|}$. The second equality in (4.5) is proved.

(b) Since $|\nabla\phi|(u)$ is the same as the strong slope of $\phi$ at $u$, Theorem 2.11 in [15] shows that $|\nabla\phi|(u) = |d\phi|(u)$ for all $u \in D(J)$.

If $\|\nabla\phi\|(u) = 0$, then $J(u)-J(v)-\langle F'(u), u-v\rangle \leq 0$ for all $v$ with $\|v-u\| \leq 1$. It follows from the definition of $|\nabla\phi|(u)$ that $|\nabla\phi|(u) = 0 = \|\nabla\phi\|(u)$. Assume now that

$\|\nabla\phi\|(u) > 0$. Consider $v$ such that $v \neq u$ and $\|v-u\| \leq 1$. If $J(u)-J(v)-\langle F'(u), u-v\rangle \geq 0$ then

$$J(u)-J(v)-\langle F'(u),u-v\rangle \leq \frac{J(u)-J(v)-\langle F'(u),u-v\rangle}{\|u-v\|}.$$

Thus,

$$\begin{aligned}
\|\nabla\phi\|(u) &= \sup\{J(u)-J(v)-\langle F'(u),u-v\rangle : \\
&\qquad \|v-u\| \leq 1, J(u)-J(v)-\langle F'(u),u-v\rangle \geq 0\} \\
&\leq \sup\{\frac{J(u)-J(v)-\langle F'(u),u-v\rangle}{\|u-v\|} : \\
&\qquad \|v-u\| \leq 1, v \neq u, J(u)-J(v)-\langle F'(u),u-v\rangle \geq 0\} \\
&= \sup\{\frac{J(u)-J(v)-\langle F'(u),u-v\rangle}{\|u-v\|} : \|v-u\| \leq 1, v \neq u\} \;=\; |\nabla\phi|(u).
\end{aligned}$$

(c) If $\|\nabla\phi\|(u) = 0$ then, as shown in (b), $|\nabla\phi|(u) = 0$. Conversely, assume $|\nabla\phi|(u) = 0$. Since $0 \leq \|\nabla\phi\|(u) \leq |\nabla\phi|(u) = 0$, one must have $\|\nabla\phi\|(u) = 0$.

(d) and (e) follow immediately from the definition of $\|\nabla\phi\|(u)$ and $|\nabla\phi|(u)$. □

We now prove the following version of the Mountain Pass Theorem for the inequality (4.4).

**Theorem 4.1** *Let $X, \phi, J, F$ be as above. Assume $K$ is a compact metric space and $K_0$ is a closed proper subset of $K$. Let $\chi \in C(K,X)$ and define*

$$M = \{g \in C(K,X) : g(s) = \chi(s),\ \forall s \in K_0\}, \tag{4.10}$$

$$c = \inf_{g\in M}\sup_{s\in K}\phi(g(s)), \tag{4.11}$$

*and*

$$c_0 = \sup_{s\in K_0}\phi(\chi(s)).$$

*Assume that there exists a function $h \in C(K,\mathbb{R})$ such that*

$$\phi(\chi(s)) \leq h(s),\ \forall s \in K, \tag{4.12}$$

*and*

$$\sup_{s\in K_0} h(s) = \sup_{s\in K_0}\phi(\chi(s)). \tag{4.13}$$

*If $c_0 < c < \infty$ then there exists a sequence $\{u_n\}$ in $D(J)$ such that*

$$\lim_{n\to\infty}\phi(u_n) = \bar{c} \geq c, \tag{4.14}$$

*and*

$$\lim_{n\to\infty}|\nabla\phi|(u_n) = \lim_{n\to\infty}\|\nabla\phi\|(u_n) = 0. \tag{4.15}$$

*Proof.* First, we need the following version of the Mountain Pass Theorem for continuous functionals.

**Theorem 4.2** *Assume $X_1$ is a complete metric space and $\phi_1 \in C(X_1, \mathbb{R})$. Let $K$ be a compact metric space, $K_0$ be a closed proper subset of $K$ and $\chi_1 \in C(K, \mathbb{R})$. Put*

$$M_1 = \{g \in C(K, X_1) : g(s) = \chi_1(s),\ \forall s \in K_0\},$$

$$c_1 = \inf_{g \in M_1} \sup_{s \in K} \phi_1(g(s)),$$

*and*

$$c_{01} = \sup_{s \in K_0} \phi_1(\chi_1(s)).$$

*If $c_{01} < c_1$, then there exists a sequence $\{u_n\}$ in $X_1$ such that*

$$\lim_{n \to \infty} \phi_1(u_n) = c_1,$$

*and*

$$\lim_{n \to \infty} |d\phi_1|(u_n) = 0.$$

The proof of Theorem 4.2 was basically presented in [28] (Theorem 5.1). It is an adaptation of the proof given in Theorem 4.2, [37], for the $C^1$ version of the Mountain Pass Theorem. In [28], the proof is for the classical case where $K = [0,1]$, $K_0 = \{0,1\}$ and the (PS) condition is assumed. However, it is clear from the arguments there and from the original proof in [37] that we can extend it to the more general case stated in Theorem 4.2 without any substantial modifications.

Now, assume that $X$ and $\phi$ satisfy the conditions in Theorem 4.1. We show that the metric space $X_1 = \text{epi}(\phi)$ and the continuous functional $\phi_1 = \mathcal{G}_\phi$ satisfy the conditions in Theorem 4.2. It is clear that $X_1$ is a complete metric space with the metric defined in Definition 4.2. Let us define

$$\chi_1(s) = (\chi(s), h(s)),\ s \in K. \tag{4.16}$$

From (4.12), $\chi_1(s) \in \text{epi}(\phi)$, for all $s \in K$, and also $\chi_1 \in C(K, X_1)$. Furthermore, from (4.13),

$$\max_{s \in K_0} \phi_1(\chi_1(s)) = \max_{s \in K_0} \mathcal{G}_\phi((\chi(s), h(s)) = \max_{s \in K_0} h(s) = \sup_{s \in K_0} \phi(\chi(s)) = c_0. \tag{4.17}$$

Now, we observe that

$$(c =) \inf_{g \in M} \sup_{s \in K} \phi(g(s)) \le \inf_{g_1 \in M_1} \max_{s \in K} \phi_1(g_1(s)). \tag{4.18}$$

First, note that $M_1 \neq \emptyset$ since $\chi_1 \in M_1$. Assume $g_1$ belongs to $M_1$,

$$g_1(s) = (\alpha(s), \beta(s)),\ s \in K.$$

Then, $\alpha \in C(K, X)$, $\beta \in C(K, \mathbb{R})$, and

$$\beta(s) \geq \phi(\alpha(s)),\ \forall s \in K. \tag{4.19}$$

Because $g_1 = \chi_1$ on $K_0$, we have $\alpha = \chi$ on $K_0$. Hence, $\alpha \in M$. (4.19) implies that

$$\sup_{s\in K} \phi_1(g_1(s)) \geq \sup_{s\in K} \phi(\alpha(s)) \geq \inf_{g\in M} \sup_{s\in K} \phi(g(s)) = c.$$

Since this holds for every $g_1 \in M_1$, we obtain (4.18). Since $c > c_0$, (4.17) and (4.18) show that

$$\inf_{g_1\in M_1} \max_{s\in K} \phi_1(g_1(s)) > \max_{s\in K_0} \phi_1(\chi_1(s)). \tag{4.20}$$

This means that $X_1, \phi_1, M_1$, and $\chi_1$ satisfy the conditions in Theorem 4.2. According to this theorem, there exists a sequence $\{(u_n, \xi_n)\}$ in epi($\phi$) such that

$$\mathcal{G}_\phi(u_n, \xi_n) = \xi_n \to c_1(\geq c), \tag{4.21}$$

and

$$|d\mathcal{G}_\phi|(u_n, \xi_n) \to 0. \tag{4.22}$$

We have $u_n \in D(J)$ and $\xi_n \geq \phi(u_n),\ \forall n$. Moreover, according to Theorem 3.13 in [15], one always has $|d\mathcal{G}_\phi|(u, \xi) = 1$ whenever $u \in D(\phi)$ and $\xi > \phi(u)$. (4.22) thus implies that

$$\xi_n = \phi(u_n), \quad \text{for all } n \text{ sufficiently large.} \tag{4.23}$$

It follows from (4.1), (4.21), (4.22), and (4.23) that

$$\phi(u_n) \to c_1(\geq c), \tag{4.24}$$

and

$$|d\phi|(u_n) = \frac{|d\mathcal{G}_\phi|(u_n, \phi(u_n))}{\sqrt{1 - |d\mathcal{G}_\phi|(u_n, \phi(u_n))^2}} \to 0. \tag{4.25}$$

From Proposition 4.1, it immediately follows that $|\nabla\phi|(u_n), \|\nabla\phi\|(u_n) \to 0$ as $n \to \infty$. □

In the classical case where $K = [0, 1]$ and $K_0 = \{0, 1\}$, we have the following result, which is a direct consequence of Theorem 4.1 and Proposition 4.1(e). We just note that conditions (4.12)–(4.13) are clearly satisfied in this case.

**Corollary 4.1** *Suppose $(X, \|\cdot\|)$ is a Banach space and $F \in C^1(X, \mathbb{R})$, $J : X \to \mathbb{R} \cup \{\infty\}$ is a convex, proper, lower semicontinuous functional. Put $\phi = J - F$. Assume that*

*(i) $\phi(0) = 0$ and there exist $\alpha, \rho > 0$ such that*

$$\phi(x) \geq \alpha, \quad \textit{for all } x \in X \ \textit{ with } \ \|x\| = \rho. \tag{4.26}$$

*(ii) There exists $e \in X$ such that*

$$\|e\| > \rho \ \ \textit{and} \ \ \phi(e) \leq 0. \tag{4.27}$$

*Let*

$$c = \inf_{f\in\Gamma} \sup_{t\in[0,1]} I(f(t))(\geq \alpha), \tag{4.28}$$

*where*

$$\Gamma = \{f \in C([0,1], X) : f(0) = 0, f(1) = e\}. \tag{4.29}$$

*Then, there exist sequences* $\{u_n\} \subset X$ *and* $\{\epsilon_n\} \subset (0,\infty)$ *such that*

$$\epsilon_n \to 0^+,\ \phi(u_n) \to \bar{c}(\geq c \geq \alpha), \tag{4.30}$$

*and*

$$J(v) - J(u_n) - \langle F'(u_n), v - u_n\rangle \geq -\epsilon_n \|v - u_n\|,\ \forall v \in X, \forall n \in \mathbb{N}. \tag{4.31}$$

Some remarks are in order to put the above concepts and results in perspective.

**Remark 4.1** (a) In the case $J = I_K$, the indicator function of a close convex set $K \subset X$, the inequality (4.4) becomes

$$\begin{cases} \langle F'(u), v - u\rangle \geq 0,\ \forall v \in K \\ u \in K, \end{cases} \tag{4.32}$$

and the norm in (4.3) is

$$\|\nabla\phi\|(u) = \sup_{v\in K, \|u-v\|\leq 1} \langle F'(u), v - u\rangle. \tag{4.33}$$

The inequality (4.32), which is a suitable model for the minimal surface problem, was studied in [40] and [9, 10], where the norm (4.33) was introduced (see also [42] and [36]). Versions of the Mountain Pass theorem were derived in these papers for (4.32), which resulted in the existence of unstable solutions of the minimal surface problem. The definition in (4.3) is motivated by that in [40, 10] and extends it to more general convex functionals (not necessarily over a convex set). Although related, it seems that the approach used in the above works for the Plateau problem cannot be applied to our problem here, in which the mean curvature depends on both the position $x$ and the displacement $u$. This extension is necessary for the equation of prescribed mean curvature (1.1) that we consider here since in the case the curvature is not zero, a variational inequality with the convex functional being the area functional seems more appropriate for the formulation.

(b) Using the slopes in (4.2) or (4.3), one can define in a straightforward manner the (PS) conditions for inequalities of type (4.4), by requiring that if a sequence $\{u_n\}$ in $X$ satisfies

$$\phi(u_n) \to c_0,$$

and

$$|\nabla\phi|(u_n) \to 0 \text{ (or } \|\nabla\phi\|(u_n) \to 0),$$

then $\{u_n\}$ has a convergent subsequence (cf. [40, 10], also [36]). In Theorem 4.1 or Corollary 4.1, if a (PS) condition is assumed then one immediately has the existence of a critical point. Corollary 4.1 and Theorem 4.1 are extended versions of the result in [43] and [32], without the (PS) condition. Furthermore, we use here a different approach for the proof which is based on recent developments in nonsmooth analysis and seems to require fewer technical arguments than that in [43]. The norm $|\nabla\phi|$ is defined here directly from the functional $\phi$, which makes the (PS) condition and the condition for (PS) sequences in inequalities more like the classical case of $C^1$ functionals.

(c) In [41], Struwe studied the generic existence of multiple solutions of the prescribed mean curvature problem. In [41], Dirichlet problems with nonhomogeneous boundary conditions were considered and the curvature does not depend on the displacement as we consider here. The problem and approach we consider here are different from those in [41].

(d) The potential (energy) functional here is a $C^1$ perturbation of a convex functional (the area functional). Since the area functional is also continuous (in fact, Lipschitz continuous) on the space of functions of bounded variation, we can prove and use versions of the Mountain Pass Theorem without the (PS) condition for continuous or Lipschitz functionals. As proved above, this will be equivalent to what we consider here. However, we adopt the convex functional version due to the simpler and more direct representations of the norms $\|\nabla\phi\|$ and $|\nabla\phi|$ rather than that of $|d\phi|$ in Definitions 4.1 and 4.3. Furthermore, convexity of the area functional and the formulation of the problem as a variational inequality, rather than as a pointwise inclusion, will be extensively exploited later to obtain critical points from (PS) sequences.

We are now ready for the proof of the existence of saddle points. Assume, in addition to (A1)–(A2), the following growth condition ("super-linear" condition): There exists $\gamma > 1$ and $\xi_1 > 0$ such that

$$f(x,\xi)\xi \geq \gamma \int_0^\xi f(x,\eta)d\eta, \tag{4.34}$$

for a.e. $x \in \mathcal{B}$, all $\xi \in [\xi_1, \infty)$. For $\xi > 0$, let us put

$$g_\xi(t) = \frac{t}{[(1+t^2)^{1/2}+1](t^{q-1}+\xi)} \ (t > 0). \tag{4.35}$$

It is easy to check that $\sup_{0<t<\infty} g_\xi(t) \in (0,\infty)$ for every $\xi \in (0,\infty)$. We define the function $M : (0,\infty) \to (0,\infty)$ by

$$M(\xi) = \sup_{0<t<\infty} g_\xi(t). \tag{4.36}$$

**Theorem 4.3** *Assume (A1), (A2), and the growth condition (4.34). Suppose furthermore that*

$$M\left(\frac{C_1 d_2 q}{d_1 C_q^q |\mathcal{B}|^{q-1}}\right) > \frac{\lambda}{q} d_1 C_q^q |\mathcal{B}|^{q-1}, \tag{4.37}$$

*where $d_1, d_2, q$ are given in (A2) and $C_1, C_q$ are in Theorem 3.1. Then, there exists a nontrivial solution $u$ of the inequality (2.13) which is the weak$^*$-limit of a sequence $\{u_n\}$ satisfying*

$$\lim_{n\to\infty} I_\lambda(u_n) > 0 \quad \textit{and} \quad \lim_{n\to\infty} |\nabla I_\lambda|(u_n) = 0.$$

*Proof.* We show that under the assumptions stated above in Theorem 4.3, all conditions in Corollary 4.1 are satisfied, with $\phi = J - \lambda\mathcal{F}$. First, let us check condition (i) in Corollary4.1. From (3.1), we have

$$|F(x,\xi)| \le \frac{d_1}{q}|\xi|^q + d_2|\xi| \ \ (\text{for a.e. } x \in \mathcal{B}, \text{all } \xi \in \mathbb{R}), \tag{4.38}$$

and thus,

$$\begin{aligned} \int_{\mathcal{B}} |F(x,u)|dx &\le \frac{d_1}{q}\int_{\mathcal{B}} |u|^q + d_2\int_{\mathcal{B}} |u| \\ &\le \frac{d_1}{q}C_q^q\left(\int_{\mathcal{B}} |\nabla u|\right)^q + d_2C_1\int_{\mathcal{B}} |\nabla u|, \end{aligned} \tag{4.39}$$

($C_q$ and $C_1$ are given in Theorem 3.1). Let us consider the function

$$\phi(\rho) = \frac{(|\mathcal{B}|^2+\rho^2)^{1/2} - |\mathcal{B}|}{D_1\rho^q + D_2\rho} \ (\rho > 0),$$

where $D_1 = \dfrac{d_1}{q}C_q^q(>0), D_2 = d_2C_1(>0)$. For $\rho = t|\mathcal{B}|$, one has

$$\phi(\rho) \quad = \quad \frac{1}{D_1|\mathcal{B}|^{q-1}}\frac{t}{[(1+t^2)^{1/2}+1](t^{q-1}+D_3)},$$

with $D_3 = D_2D_1^{-1}|\mathcal{B}|^{1-q}(>0)$. Then,

$$\phi(\rho) = \phi(t|\mathcal{B}|) = D_1^{-1}|\mathcal{B}|^{1-q}g_{D_2D_1^{-1}|\mathcal{B}|^{1-q}}(t). \tag{4.40}$$

From the definitions of $M$ and $g_\xi$, one gets

$$\begin{aligned} \sup_{0<\rho<\infty} \phi(\rho) &= \sup_{0<t<\infty} \phi(t|\mathcal{B}|) \\ &= \frac{1}{D_1|\mathcal{B}|^{q-1}}M\left(\frac{D_2}{D_1|\mathcal{B}|^{q-1}}\right) \\ &= \frac{q}{d_1C_q^q|\mathcal{B}|^{q-1}}M\left(\frac{C_1d_2q}{d_1C_q^q|\mathcal{B}|^{q-1}}\right). \end{aligned}$$

It follows from (4.37) that

$$\sup_{0<\rho<\infty} \phi(\rho) > \lambda. \tag{4.41}$$

Therefore, there exist $\rho_0$ and $\lambda_1 > \lambda$ such that

$$(|\mathcal{B}|^2 + \rho_0^2)^{1/2} - |\mathcal{B}| > \lambda_1(D_1\rho_0^q + D_2\rho_0). \tag{4.42}$$

For $u \in X$ such that $\|u\|_0 = \rho_0$, one has, from (3.3), (4.39) and (4.42),

$$\begin{aligned} I_\lambda(u) &= J(u) - \lambda \int_{\mathcal{B}} F(x,u)dx \\ &\geq (|\mathcal{B}|^2 + \rho_0^2)^{1/2} - |\mathcal{B}| - \lambda(D_1\rho_0^q + D_2\rho_0) \\ &\geq (\lambda_1 - \lambda)(D_1\rho_0^q + D_2\rho_0)(> 0). \end{aligned} \tag{4.43}$$

This shows (4.26) with $\rho = \rho_0$ and $\alpha = (\lambda_1 - \lambda)(D_1\rho_0^q + D_2\rho_0)$.

To check (ii), we fix a function $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$ and $\phi \not\equiv 0$. By assigning $\phi(x) = 0$ for $x \in \mathcal{B} \setminus \Omega$, we can consider $\phi$ as a function on $\mathcal{B}$. It is clear that $r\phi \in X$ for all $r > 0$. On the other hand, it follows from (4.34) that

$$f(x,\xi) \geq d_4\xi^{\gamma-1} - d_5,$$

for a.e. $x \in \mathcal{B}$, all $\xi \geq 0$ (with some $d_4, d_5 > 0$). It follows that

$$F(x,\xi) \geq \frac{d_4}{\gamma}\xi^\gamma - d_5\xi,$$

for a.e. $x \in \mathcal{B}$, all $\xi \geq 0$. Then, for $r > 0$,

$$\begin{aligned} I_\lambda(r\phi) &= \int_{\mathcal{B}} \frac{r^2|\nabla\phi|^2}{\sqrt{1 + r^2|\nabla\phi|^2} + 1}dx - \lambda \int_{\mathcal{B}} F(x,r\phi)dx \\ &\leq r\int_{\mathcal{B}} |\nabla\phi|dx - \frac{d_4 r^\gamma}{\gamma}\int_{\mathcal{B}} |\phi|^\gamma dx + \lambda r d_5 \int_{\mathcal{B}} |\phi|\, dx. \end{aligned}$$

Since $\gamma > 1$, $I(r\phi) \to -\infty$ as $r \to \infty$. Hence, $I_\lambda(r\phi) < 0$ for $r > 0$ sufficiently large. We have checked condition (ii) in Corollary 4.1.

From Corollary 4.1 it follows that there exist sequences $\{u_n\}$ in $BV(\mathcal{B})$ and $\{\epsilon_n\}$ in $(0,\infty)$ such that $\epsilon_n \to 0$ and

$$I_\lambda(u_n) \to c(\geq \alpha > 0), \tag{4.44}$$

and

$$J(v) - J(u_n) - \lambda \int_{\mathcal{B}} f(x,u_n)(v - u_n)dx \geq \epsilon_n\|v - u_n\|_{BV(\mathcal{B})}, \tag{4.45}$$

for all $v \in BV(\mathcal{B})$, $n \in \mathbb{N}$.

Let us prove that the sequence $\{u_n\}$ is bounded in $BV(\mathcal{B})$. In fact, (4.30) implies

$$\int_{\mathcal{B}} \sqrt{1 + |\nabla u_n|^2} - \lambda \int_{\mathcal{B}} F(x,u_n)dx = c + |\mathcal{B}| + s_n, \tag{4.46}$$

with $\{s_n\}$ being a bounded sequence in $\mathbb{R}$. From (4.31), one has

$$\begin{aligned} &\int_{\mathcal{B}} \sqrt{1 + |\nabla v|^2} - \int_{\mathcal{B}} \sqrt{1 + |\nabla u_n|^2} - \lambda \int_{\mathcal{B}} f(x,u_n)(v - u_n)dx \\ \geq\ & -\epsilon_n \int_{\mathcal{B}} |\nabla(v - u_n)|,\ \forall v \in X. \end{aligned} \tag{4.47}$$

Note that for $v = 2u_n$, we have the estimate

$$\begin{aligned}\int_{\mathcal{B}} \sqrt{1+|\nabla v|^2} &= \int_{\mathcal{B}} \sqrt{1+4|(\nabla u_n)_a|^2}dx + 2\int_{\mathcal{B}} |(\nabla u_n)_s| \\ &\leq 2\int_{\mathcal{B}} \sqrt{1+|(\nabla u_n)_a|^2}dx + 2\int_{\mathcal{B}} |(\nabla u_n)_s| \\ &= 2\int_{\mathcal{B}} \sqrt{1+|\nabla u_n|^2}.\end{aligned}$$

Letting $v = 2u_n$ in (4.47) and using this estimate, one gets

$$\begin{aligned}\int_{\mathcal{B}} \sqrt{1+|\nabla u_n|^2} &\geq \int_{\mathcal{B}} \sqrt{1+|\nabla (2u_n)|^2} - \int_{\mathcal{B}} \sqrt{1+|\nabla u_n|^2} \\ &\geq \lambda \int_{\mathcal{B}} f(x,u_n)u_n dx - \epsilon_n \int_{\mathcal{B}} |\nabla u_n|.\end{aligned}$$

Using (4.46), we have

$$\lambda \int_{\mathcal{B}} F(x,u_n)dx + c + |\mathcal{B}| + s_n \geq \lambda \int_{\mathcal{B}} f(x,u_n)u_n dx - \epsilon_n \int_{\mathcal{B}} |\nabla u_n|,$$

that is,

$$\lambda \int_{\mathcal{B}} [f(x,u_n)u_n - F(x,u_n)]dx \leq C + |\mathcal{B}| + s_n + \epsilon_n \int_{\mathcal{B}} |\nabla u_n|. \tag{4.48}$$

(4.34) implies that there exists $d_6 > 0$ such that

$$\int_{\mathcal{B}} f(x,u_n)u_n dx \geq \gamma \int_{\mathcal{B}} F(x,u_n) - d_6, \ \forall n.$$

Thus,

$$\int_{\mathcal{B}} [f(x,u_n)u_n - F(x,u_n)]dx \geq (\gamma - 1) \int_{\mathcal{B}} F(x,u_n)dx - d_6. \tag{4.49}$$

It follows from (4.46), (4.48), and (4.49) that

$$\begin{aligned}\int_{\mathcal{B}} |\nabla u_n| &\leq \int_{\mathcal{B}} \sqrt{1+|\nabla u_n|^2} \\ &= \lambda \int_{\mathcal{B}} F(x,u_n)dx + c + |\mathcal{B}| + s_n \\ &\leq \frac{\lambda}{\gamma-1} \int_{\mathcal{B}} [f(x,u_n)u_n - F(x,u_n)]dx + c + |\mathcal{B}| + s_n + \frac{d_6}{\gamma-1} \\ &\leq \left(1 + \frac{\lambda}{\gamma-1}\right)(c + |\mathcal{B}| + s_n) + \frac{\epsilon_n}{\gamma-1} \int_{\mathcal{B}} |\nabla u_n| + \frac{d_6}{\gamma-1}.\end{aligned}$$

Therefore,

$$\left(1 - \frac{\epsilon_n}{\gamma-1}\right) \int_{\mathcal{B}} |\nabla u_n| \leq \frac{d_6}{\gamma-1} + \left(\frac{\lambda}{\gamma-1}\right)(c + |\mathcal{B}| + \max_{n\in N} |s_n|).$$

Since $\epsilon_n \to 0$, this estimate shows that the sequence $\left\{ \int_{\mathcal{B}} |\nabla u_n| \right\}$ is bounded. Thus, $\{u_n\}$ is bounded in $BV(\mathcal{B})$. By passing to a subsequence, if necessary, we can assume that

$$u_n \to u \ \text{ in } L^\beta(\Omega), \ \beta \in \left[1, \frac{N}{N-1}\right), \tag{4.50}$$

$$u_n \to u \ \text{ a.e. on } \Omega, \tag{4.51}$$

and

$$\nabla u_n \rightharpoonup \nabla u \ \text{ in } \ [M(\Omega)]^N(-\text{weak}^*). \tag{4.52}$$

Let us show that $u$ is a solution of (2.13). From (4.50)–(4.52) and the lower semi-continuity of $J$ (cf. e.g. [24]), one has

$$J(u) \leq \liminf J(u_n). \tag{4.53}$$

From the compactness of the embedding $BV(\mathcal{B}) \hookrightarrow L^\beta(\mathcal{B})$, for $1 \leq \beta < \frac{N}{N-1}$, and the growth condition (3.1), we have $u_n \to u$ in $L^q(\mathcal{B})$ and $f(\cdot, u_n) \to f(\cdot, u)$ in $L^{q'}(\mathcal{B})$. Therefore,

$$\int_{\mathcal{B}} f(\cdot, u_n)(v - u_n)dx \to \int_{\mathcal{B}} f(\cdot, u)(v - u)dx, \ \forall v \in BV(\mathcal{B}). \tag{4.54}$$

It is clear that

$$\epsilon_n \|v - u_n\|_{BV(\mathcal{B})} \to 0. \tag{4.55}$$

Letting $n \to \infty$ in (4.45) and using (4.53)-(4.55), we see that $u$ satisfies (2.13) for all $v \in BV(\mathcal{B})$.

Now, let us show that $u \neq 0$. Assume otherwise that $u = 0$ in (4.50)-(4.52). Letting $v = 0$ in (2.13), we get

$$\int_{\mathcal{B}} f(\cdot, u_n)u_n dx \geq J(u_n)(\geq 0), \ \forall n.$$

Letting $n \to \infty$ in this inequality, we have $\lim J(u_n) = 0$. On the other hand, it follows from (3.18) that

$$\int_{\mathcal{B}} F(x, u_n)dx \to \int_{\mathcal{B}} F(x, 0)dx = 0.$$

Thus, $I_\lambda(u_n) = J(u_n) - \lambda \int_{\mathcal{B}} F(\cdot, u_n)dx \to 0$, contradicting (4.44). Hence, (2.13) has a solution $u \neq 0$. □

**Remark 4.2** Note that for a given function $F$, inequality (4.37) always holds when $\lambda$ is sufficiently small. That is, there exists $\lambda_* > 0$ such that (4.26) is true for all $\lambda \in (0, \lambda_*)$. $\lambda_*$ can be chosen as

$$\lambda_* = \frac{q}{d_1 C_q^q |\mathcal{B}|^{q-1}} M\left(\frac{C_1 d_2 q}{d_1 C_q^q |\mathcal{B}|^{q-1}}\right).$$

On the other hand, if $\lambda$ is fixed, for example, $\lambda = 1$ (the equation does not depend on a parameter), then (4.37) holds if

$$g_{\frac{C_1 d_2 q}{d_1 C_q^q |\mathcal{B}|^{q-1}}}(t_0) > \frac{1}{q} d_1 C_q^q |\mathcal{B}|^{q-1},$$

for some $t_0 > 0$ (with $g_\xi$ defined in (4.35)). In the case $t_0 = 1$, this inequality becomes

$$\frac{d_1 C_q^q |\mathcal{B}|^{q-1}}{(1+\sqrt{2})(C_1 d_2 q + d_1 C_q^q |\mathcal{B}|^{q-1})} > \frac{1}{q} d_1 C_q^q |\mathcal{B}|^{q-1},$$

that is,

$$C_1 d_2 q + d_1 C_q^q |\mathcal{B}|^{q-1} < \frac{q}{1+\sqrt{2}}. \tag{4.56}$$

This sufficient condition is satisfied if the coefficients $d_1$ and $d_2$ in (3.1) are small, i.e., $f$ is sufficiently small.

As a consequence of Theorem 4.3 and the above remark, one has the following existence t of eigenfunctions of (2.13) for small values of $\lambda$:

**Corollary 4.2** *Under assumptions (A1), (A2), and the growth condition (4.34), there exists $\lambda_* > 0$ such that for all $\lambda \in (0, \lambda_*)$, (2.13) has at least one nontrivial solution $u_\lambda \in X$. Moreover, if $f(x,u)$ satisfies:*

*(A3) For a.e. $x \in \Omega$,*

$$f(x,u) \begin{cases} \geq 0 & \text{if } u \in [0,\infty) \\ = 0 & \text{if } u \in (-\infty, 0], \end{cases}$$

*then*

$$\int_{\mathcal{B}} |\nabla u_\lambda| \to \infty \quad \text{as } \lambda \to 0^+. \tag{4.57}$$

*Proof.* The existence of $u_\lambda$ follows directly from Theorem 4.3 and Remark 4.2. Let us check the limit (4.57). It follows from (A3) that, for a.e. $x \in \Omega$,

$$F(x,\xi) \geq 0 \ \text{ for } \ \xi \geq 0, \ F(x,\xi) = 0 \ \text{ for } \ \xi \leq 0. \tag{4.58}$$

For $\lambda \in (0, \lambda_*)$, we denote by $u_\lambda$ the nontrivial solution of (2.13) and by $\{u_{\lambda n}\}$ the sequence that converges weakly* to $u_\lambda$ as mentioned in Theorem 4.3. We have, from (4.58) and (4.43),

$$\begin{aligned} (\lambda_* - \lambda)(D_1 \rho_0^q + D_2 \rho_0) &\leq \lim_{n\to\infty} I_\lambda(u_{\lambda n}) \\ &\leq \limsup_{n\to\infty} J(u_{\lambda n}). \end{aligned} \tag{4.59}$$

For simplicity of notation, we put $u_n = u_{\lambda n}$. Since $u_n$ satisfies (4.45), one can let $v = 0$ in (4.45) to get

$$-J(u_n) + \lambda \int_{\mathcal{B}} f(x,u_n) u_n \geq -\epsilon_n \|u_n\|_{BV(\mathcal{B})}.$$

Since $\{u_n\}$ is bounded, $\epsilon_n \|u_n\|_{BV(\mathcal{B})} \to 0$. (4.54) implies that

$$\begin{aligned} \lambda \int_{\mathcal{B}} f(x, u_\lambda) u_\lambda dx &= \lim_{n\to\infty} \int_{\mathcal{B}} \lambda f(x, u_n) u_n dx \\ &\geq \limsup J(u_n). \end{aligned} \tag{4.60}$$

Hence, from (3.1), (4.59), (4.60), it follows that

$$\begin{aligned} (\lambda_* - \lambda)(D_1 \rho_0^q + D_2 \rho_0) &\leq \lambda \int_{\mathcal{B}} f(x, u_\lambda) u_\lambda dx \\ &\leq \lambda \int_{\mathcal{B}} (d_1 |u_\lambda|^q + d_2 |u_\lambda|)\, dx \\ &\leq \lambda \left[ (d_1 + d_2) \int_{\mathcal{B}} |u_\lambda|^q + d_2 |\mathcal{B}| \right]. \end{aligned}$$

Consequently,

$$\int_{\mathcal{B}} |u_\lambda|^q \quad \geq \frac{(\lambda_* - \lambda)(D_1 \rho_0^q + D_2 \rho_0)}{\lambda (d_1 + d_2)} - \frac{d_2 |\mathcal{B}|}{d_1 + d_2}.$$

This holds for every $\lambda \in (0, \lambda_*)$, implying that

$$\lim_{\lambda \to 0^+} \int_{\mathcal{B}} |u_\lambda|^q = \infty.$$

Using Poincaré's inequality, we obtain

$$\|u_\lambda\|_{BV(\mathcal{B})} = \int_{\mathcal{B}} |\nabla u_\lambda| \to \infty \ \text{ as } \ \lambda \to 0^+. \tag{4.61}$$

□

Note that this last estimate on $\|u_\lambda\|_{BV(\mathcal{B})}$ (cf. (4.61) and (3.21)) shows that for all $\lambda > 0$ sufficiently small, the solution $u_\lambda$ obtained by the min-max arguments in Theorem 4.3 is different from the local minimizer solution in Corollary 3.1. We thus have the following result.

**Corollary 4.3** *Assume conditions (A1), (A2), (A3), (3.2), and (4.34) are satisfied. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, the inequality (2.13) has at least two nontrivial solutions.*

### Notes on nontrivial nonnegative solutions

In this last part, we study the existence of nonnegative solutions of the inequality (2.13). Here, we also assume that $f$ satisfies (A3). Suppose that $u$ is a nontrivial solution of (2.13) whose existence is shown in Theorem 4.3 or Corollary 3.1. We show that $u^+ = \max\{u, 0\}$ is also a nontrivial solution of (2.13). First, let us check that $u^+ \in BV(\mathcal{B})$ and

$$J(u^+) \leq J(u). \tag{4.62}$$

In fact, from Theorem 3.3, [8], there exists a sequence $\{u_n\}$ in $W^{1,1}(\Omega)$ such that

$$u_n \to u \ \text{ in } \ L^1(\mathcal{B}) \ \text{ and } \ J(u_n) \to J(u). \tag{4.63}$$

Moreover, as noted in [3], by using an approximation result by Anzellotti and Giaquinta, one can show that $u^+ \in BV(\mathcal{B})$ and

$$|\nabla(u^+)|(E) \leq |\nabla u|(E),$$

for all Borel subsets $E$ of $\mathcal{B}$. From (4.63) and Stampachia's theorem (cf. e.g. [29]), one has $u_n^+ \in W^{1,1}(\mathcal{B})$ and

$$u_n^+ \to u^+ \ \text{ in } \ L^1(\mathcal{B}).$$

We have

$$J(u^+) \leq \liminf J(u_n^+). \tag{4.64}$$

Since $u_n^+ \in W^{1,1}(\mathcal{B})$, $\nabla u_n^+ = (\nabla u_n)\chi_{\{x:u_n(x)>0\}}$, and thus

$$\begin{aligned} J(u_n^+) &= \int_{\{x:u_n(x)>0\}} \left(\sqrt{1+|\nabla u_n|^2}-1\right) dx \\ &\leq \int_{\mathcal{B}} \left(\sqrt{1+|\nabla u_n|^2}-1\right) dx \ = \ J(u_n), \ \forall n \in \mathbb{N}. \end{aligned}$$

Therefore, by (4.63),

$$\liminf J(u_n^+) \leq \liminf J(u_n) = J(u).$$

Combining this inequality with (4.64), one obtains (4.62).

Since $u^+ = 0$ a.e. in $\mathcal{B} \setminus \overline{\Omega}$, we have $u^+ \in X$. Letting $v = u^+$ in (2.13), one gets

$$J(u^+) - J(u) \geq \lambda \int_{\mathcal{B}} f(x,u)(u^+ - u)dx \geq 0.$$

Together with (4.62), this implies that

$$J(u^+) = J(u). \tag{4.65}$$

Let $v \in X$. Because $f(x,0) = 0$,

$$\begin{aligned} \int_{\mathcal{B}} f(x,u^+)(v-u^+)dx &= \int_{\{x\in\mathcal{B}:u(x)>0\}} f(x,u^+)(v-u^+)dx \\ &= \int_{\{x\in\mathcal{B}:u(x)>0\}} f(x,u)(v-u)dx \\ &= \int_{\mathcal{B}} f(x,u)(v-u)dx. \end{aligned} \tag{4.66}$$

It follows from (4.65) and (4.66) that $u^+$ also satisfies (2.13), whenever $u$ does. Moreover, $u^+ \not\equiv 0$. In fact, if $u^+ \equiv 0$ then $J(u) = J(u^+) = 0$. Therefore,

$$\int_{\mathcal{B}} (\sqrt{1+|(\nabla u)_a|^2}-1)dx = \int_{\mathcal{B}} |(\nabla u)_s| = 0.$$

We thus have $(\nabla u)_s = 0$, i.e. $\nabla u = (\nabla u)_a \in [L^1(\mathcal{B})]^N$ and

$$\int_{\mathcal{B}} (\sqrt{1+|\nabla u|^2}-1)dx = 0.$$

This implies that $\nabla u = 0$ a.e. on $\mathcal{B}$ and therefore $u = 0$ (since $u = 0$ on $\mathcal{B} \setminus \overline{\Omega}$). Using the arguments in the proof of Theorem 4.3, we obtain from this a contradiction. Hence, $u^+$ is not identically zero. We have thus shown that if $u$ is a nontrivial solution of (2.13) then $u^+$ is also a nonnegative, nontrivial solution of (2.13).

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