Advanced Nonlinear Studies **3** (2003), 1–23

Multiple Solutions for Perturbed Elliptic Equations in Unbounded Domains

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Received 18 September 2002
Communicated by Donato Fortunato

Abstract

We look for solutions of a nonlinear perturbed Schrödinger equation with non-homogeneous Dirichlet boundary conditions. By using a perturbation method introduced by Bolle, we prove the existence of multiple solutions in spite of the lack of the symmetry of the problem.

2000 Mathematics Subject Classification. 35Q55, 35J20, 35J25.
Key words. Nonlinear Schrödinger equation, critical point theory, perturbation arguments, multiplicity results.

1 Introduction

The aim of this paper is to look for solutions of the elliptic problem

$$\begin{cases} -\Delta u + \sigma(x)u = \lambda u + f(x,u) + g(x) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \\ |u| \to 0 & \text{if } |x| \to \infty \end{cases} \tag{1.1}$$

where $\lambda \in \mathbf{R}$, Ω is an open unbounded subset of $\mathbf{R}^N$, $N \geq 2$, with a C^2 bounded boundary $\partial\Omega$, and $f(x,u)$ is superlinear in u.

*Supported by M.I.U.R. (research funds ex 40 % and 60%)

Assume for a moment that Ω=$\mathbf{R}^N$ and $g = 0$. In this case, the main difficulty is caused by the lack of compactness of the problem; indeed, if $f(x, u)$ grows subcritically, setting $F(x,u) = \int_0^u f(x,s)ds$, the action functional

$$\frac{1}{2}\int_{\mathbf{R}^N} \left(|\nabla u|^2 + \sigma(x)u^2 - \lambda u^2\right) dx - \int_{\mathbf{R}^N} F(x,u)dx$$

defined on the set $\left\{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} \sigma(x)u^2dx < +\infty\right\}$ does not satisfy in general the classical Palais-Smale condition or one of its variants. In order to overcome this problem, it is possible to assume, besides suitable assumptions on $F(x, u)$, that the function $\sigma(x)$ has a "good" behaviour at infinity in such a way that the Schrödinger operator $-\Delta + \sigma$ on $L^2(\mathbf{R}^N)$ has a discrete spectrum and then the Palais-Smale condition holds. Hence, a direct application of the mountain pass theorem or one of its generalizations (see [2] and [18]) allows us to find a solution of the problem (see [20], [4] if $\lambda = 0$ and [16] for a class of elliptic systems) while, if $f(x, u)$ is odd in u, the symmetric mountain pass theorem proved in [2] implies the existence of infinitely many solutions (see [4] and [21]).Those results still hold, with small modifications in the proof, even if Ω is an open unbounded subset of $\mathbf{R}^N$ having a smooth boundary $\partial\Omega$ and $g = \varphi = 0$. On the contrary, if g or φ are different from 0, the problem loses its symmetry and existence and multiplicity results in general do not hold.

However, if Ω is bounded and $\varphi = 0$, perturbation methods introduced by Bahri and Berestycki, Rabinowitz, and Struwe give the existence of infinitely many solutions if the growth of the nonlinearity $f(x, u)$ is not too large (see [3], [19], [22] and [24]). Moreover, a multiplicity result holds for any subcritical growth of $f(x, u)$ if the forcing term $g(x)$ is "small enough" (see [1] and [3]). Later, in [10] and [11] those results have been extended to the case of elliptic equations with non-homogeneous boundary conditions, i. e. to the case $\varphi \neq 0$. Analogous results for elliptic systems are contained in [12].

More recently, a different perturbation method introduced by Bolle in [7] allows us to improve, in the case $\varphi \neq 0$, the previous results about the multiplicity of solutions for non-homogeneous elliptic equations in bounded domains (see [8], [14] and, for the systems, [13] and [15]).

Now, our aim is to use the Bolle's perturbation method in order to state the existence of multiple solutions for the problem (1.1) when Ω is unbounded.

Troughout the paper, we suppose that f satisfies the following hypotheses:

(f_1) $f \in C(\Omega \times \mathbf{R})$ and $f(x, \cdot)$ is odd;

(f_2) there exists $\mu > 2$ such that for every $x \in \Omega$, $u \in \mathbf{R}\setminus\{0\}$

$$0 < \mu F(x,u) \leq uf(x,u);$$

(f_3) there exists $\bar{u} \neq 0$ such that $\inf_{\Omega} F(x, \bar{u}) > 0$;

(f_4) there are constants $\alpha_1 > 0$ and $2 < p < \mu + 1$, $p < 2^* = \frac{2N}{N-2}$ if $N \geq 3$, such that for every $(x, u) \in \Omega \times \mathbf{R}$

$$|f(x,u)| \leq \alpha_1 |u|^{p-1};$$

(f_5) $F(\cdot, u)$ has the partial derivatives in Ω and, setting $\frac{\partial F}{\partial x} = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_N}\right)$, there exist $\alpha_2, \alpha_3, q > 0$ such that for every $(x, u) \in \Omega \times \mathbf{R}$ it is

$$\left|\frac{\partial F}{\partial x}(x,u)\right| \leq \alpha_2 |u|^q + \alpha_3,$$

where $q < 2^*$ if $N \geq 4$ and $q < 2\mu + N - \frac{N}{2}\mu$ if $N = 2, 3$.

Moreover, for overcoming the lack of compactness we assume

(σ_1) $\sigma \in C(\bar{\Omega})$ is such that $\inf_\Omega \sigma(x) > -\infty$, and for every $b > 0$,

$$m(\{x \in \Omega : \sigma(x) \leq b\}) < +\infty,$$

where m denotes the Lebesgue measure in $\mathbf{R}^N$.

As it will be proved in the proposition 3.3, this assumption implies that the spectrum of the self-adjoint realization of $-\Delta + \sigma(x)$ with Dirichlet boundary conditions in $L^2(\Omega)$ is discrete; then, from now on, we will denote by $(\lambda_n)_n$ the divergent sequence of its eigenvalues (counting multiplicity). Let $\mu' = \frac{\mu}{\mu-1}$ the coniugate exponent of μ and, for $t \geq 1$, let $|\cdot|_{L^t(A)}$ be the norm in $L^t(A)$. Denoted by $H^1(\Omega, \sigma)$ a suitable weigthed Sobolev space (see Section 3 for its definition), the following multiplicity theorem holds:

Theorem 1.1 *Let us assume that f and σ satisfy $(f_1) - (f_5)$ and (σ_1). Moreover, let $\varphi \in H^{\frac{1}{2}}(\partial\Omega) \cap C^1(\partial\Omega)$, $\sigma \in C^1(N_\delta)$ and $g \in L^{\mu'}(\Omega) \cap C^1(N_\delta)$, where N_δ denotes a suitable small neighbourhoord of $\partial\Omega$. Then, for fixed $\lambda \in \mathbf{R}$, the following results hold:*
i) for any integer k there exists $\theta_k > 0$ such that problem (1.1) has at least k solutions provided $|g|_{L^{\mu'}(\Omega)} \leq \theta_k$ and $|\varphi|_{L^p(\partial\Omega)} \leq \theta_k$.
ii) Taking $\bar{\alpha} = \frac{1}{2}$ if $q \leq \mu$ and $\bar{\alpha} = \frac{2N - N\mu + 2q}{2(2N - N\mu + 2\mu)}$ otherwise, if for n large

$$\lambda_n > n^{\frac{2(p-2)}{2p-(p-2)N}\frac{1}{1-\bar{\alpha}}}, \tag{1.2}$$

then the problem (1.1) has an unbounded sequence of solutions $\{u_n\} \subset H^1(\Omega, \sigma)$ with higher and higher energy, i.e. such that as $n \to \infty$

$$\frac{1}{2}\int_\Omega (|\nabla u_n|^2 + \sigma(x)u_n^2 - \lambda u_n^2)\,dx - \int_\Omega F(x, u_n)dx - \int_\Omega g\,u_n dx \to +\infty.$$

Remark 1.2 The condition (1.2) is satisfied if $\sigma(x)$ is so large that $\{\lambda_n\}$ rapidly grows or, equivalently, if the exponent p is greater than 2 but not too large.

Remark 1.3 Let us point out that, by integration, it is easy to prove that assumptions $(f_2) - (f_3)$ imply the existence of a constant $\alpha_4 > 0$ such that for any $u \in \mathbf{R}$

$$\alpha_4 |u|_\mu^\mu \le \int_\Omega F(x,u)dx \le \frac{1}{\mu} \int_\Omega uf(x,u)dx.$$

Clearly, by the previous inequalities and (f_4), $\mu \le p$.

In some particular cases, the hypothesis (σ_1) can be weakened as follows (see Proposition 3.1):

(σ_2) $\sigma \in L^2_{loc}(\Omega)$ is such that $\inf_\Omega \operatorname{ess} \sigma(x) > -\infty$ and

$$\int_{S(x)} \frac{1}{\sigma(y)} dy \to 0 \qquad \text{if } |x| \to +\infty,$$

where $S(x)$ is the unit sphere of $\mathbf{R}^N$ centered at x. Indeed, the following results will be proved:

Theorem 1.4 *Let us assume that all the hypotheses of Theorem 1.1 are satisfied excepting (σ_1) replaced by (σ_2). Then, the statements i) and ii) of Theorem 1.1 still hold for any $\lambda < \lambda_1$.*

Theorem 1.5 *Let us assume that all the hypotheses of Theorem 1.4 are satisfied and, moreover, let $q \le \mu = p$. Then, for any $\lambda \in \mathbf{R}$ the statements i) and ii) still hold with $\bar{\alpha} = \frac{\mu-1}{\mu}$.*

If we consider problem (1.1) with $\varphi = 0$, in particular if $\Omega = \mathbf{R}^N$, the regularity assumptions in the previous theorems can be improved as follows.

Corollary 1.6 *Let us assume that F and σ satisfy $(f_1)-(f_4)$ and (σ_1) (respectively (σ_2)). Let $g \in L^{\mu'}(\Omega)$ and $\varphi = 0$. Then, for any $\lambda \in \mathbf{R}$ (respectively for any $\lambda < \lambda_1$ if $\mu < p$ or for any $\lambda \in \mathbf{R}$ if $\mu = p$) the statements i) and ii) of Theorem 1.1 still hold with $\bar{\alpha} = \frac{1}{\mu}$.*

Remark 1.7 Since in the equation (1.1) λ is any real number, without loss of generality we can assume that in the assumption (σ_1) (respectively (σ_2)), $\inf_{x\in\Omega} \sigma > 0$ (respectively $\inf_{x\in\Omega} \operatorname{ess} \sigma > 0$). So, from now on we replace (σ_1), respectively (σ_2), by

$(\sigma_1)'$ $\sigma \in C(\bar{\Omega})$ is such that $\inf_\Omega \sigma(x) > 0$ and for every $b > 0$,

$$m(\{x \in \Omega : \sigma(x) \le b\}) < +\infty,$$

respectively

$(\sigma_2)'$ $\sigma \in L^2_{loc}(\Omega)$ is such that $\inf_{\Omega} ess\, \sigma(x) > 0$ and

$$\int_{S(x)} \frac{1}{\sigma(y)} dy \to 0 \qquad \textit{if } |x| \to +\infty.$$

Remark 1.8 Assumption $(\sigma_1)'$ has been introduced by Bartsch and Wang in [4] in order to study a superlinear Scrödinger equation in $\mathbf{R}^N$ like (1.1) with $\lambda = 0$, while assumption $(\sigma_2)'$ has been introduced by Benci and Fortunato in [5] for proving some compact imbedding theorems for weigthed Sobolev spaces. It is easy to see that $(\sigma_1)'$, and therefore $(\sigma_2)'$, hold in particular if $\sigma(x)$ is a continuous positive function on $\mathbf{R}^N$ which diverges at infinity. This condition on $\sigma(x)$ has been assumed by Rabinowitz in [20] in order to prove the existence of two solutions, the first one positive, the second one negative, for a superlinear Schrödinger equation like (1.1) with $\lambda = 0$. Later, always if $\sigma(x) \to \infty$ at infinity, in [16] Costa has stated the existence of solutions for a class of elliptic systems in $\mathbf{R}^N$ (see also [4] and [21] for related results under the weaker assumptions $(\sigma_1)'$ or $(\sigma_2)'$).

Lastly, we give a simple application of the previous results by taking into account a particular function $\sigma(x)$ such that the distribution of the eigenvalues of $-\Delta + \sigma(x)$ is known, and so condition (1.2) can be explicited.

Example Let us consider the elliptic equation

$$\begin{cases} -\Delta u + |x|^2 u = \lambda u + f(x,u) + g(x) & \text{in } \mathbf{R}^3 \\ |u| \to 0 & \text{if } |x| \to \infty \end{cases} \tag{1.3}$$

where f satisfies $(f_1)-(f_4)$. As the operator $-\Delta + |x|^2$ in $L^2(\mathbf{R}^3)$ has the eigenvalues $\lambda_j = 2j+3$ with multiplicities $\frac{1}{2}(j+1)(j+2)$, $j \in \mathbf{N}$ (see [17], pp. 514), Corollary 1.6 applies. Then, given $g \in L^{\mu'}(\Omega)$ and $\lambda \in \mathbf{R}$, it follows that:
i) for any integer k there exists $\theta_k > 0$ such that problem (1.3) has at least k solutions provided $|g|_{L^{\mu'}(\Omega)} \leq \theta_k$.
ii) if

$$\frac{6-p}{6(p-2)} > \frac{\mu}{\mu-1},$$

then the problem (1.3) has un unbounded sequence of solutions $\{u_n\}$ with higher and higher energy. In particular, if $\mu = p$, (1.3) has infinitely many solutions for any $p \in]2, \bar{p}[$, $\bar{p}$ being the largest rooth of the equation $7p^2 - 19p + 6 = 0$ ($\bar{p} \simeq 2,34$).

2 Stability of critical points under deformation of an even functional

In order to apply the method introduced by Bolle for dealing with problems with broken symmetry, let us recall the main theorem as stated in [14].

The idea is to consider a continuous path of functionals starting from a symmetric functional J_0 and to prove a preservation result for min-max critical levels in order to get critical points also for the end-point functional J_1 which is the real functional associated to the non-symmetric problem.

Consider two continuous functions $\rho_1, \rho_2 : [0,1] \times \mathbf{R} \to \mathbf{R}$ which are Lipschitz continuous with respect to the second variable. Assume $\rho_1 \leq \rho_2$ and denote by $\psi_1, \psi_2 : [0,1] \times \mathbf{R} \to \mathbf{R}$ the scalar fields solutions of the Cauchy problems

$$\begin{cases} \frac{\partial}{\partial\theta}\psi_i(\theta,s) = \rho_i(\theta, \psi_i(\theta,s)) \\ \psi_i(0,s) = s. \end{cases}$$

Note that ψ_1 and ψ_2 are continuous, increasing in s and $\psi_1 \leq \psi_2$.

Definition 2.1 Let J_0 be a C^1-functional on a Hilbert space E with norm $\|\cdot\|$. We say that a C^1-functional $J : [0,1] \times E \to \mathbf{R}$ is a good path of funtionals starting from J_0 and controlled by ρ_1, ρ_2 if $J(0,\cdot) = J_0$ and if it satisfies $(H_1) - (H_4)$ below, where $J_\theta = J(\theta, \cdot)$.

(H_1) J satisfies the following Palais-Smale condition: any sequence $\{(\theta_n, u_n)\}_n$ in $[0,1] \times E$ such that

$$\{J(\theta_n, u_n)\}_n \text{ is bounded} \quad \text{and } \lim_n J'_{\theta_n}(u_n) = 0$$

has a convergent subsequence;

(H_2) for any $b > 0$ there exists $C_b > 0$ such that if $(\theta, u) \in [0,1] \times E$ then

$$|J_\theta(u)| \leq b \Rightarrow \left|\frac{\partial J}{\partial\theta}(\theta, u)\right| \leq C_b(\|J'_\theta(u)\| + 1)(\|u\| + 1);$$

(H_3) for any critical point u of J_θ we have

$$\rho_1(\theta, J_\theta(u)) \leq \frac{\partial}{\partial\theta}J(\theta, u) \leq \rho_2(\theta, J_\theta(u));$$

(H_4) for any finite dimensional subspace W of E and any $\theta \in [0,1]$, we have

$$\lim_{\substack{u \in W \\ \|u\| \to +\infty}} \sup_{\beta \in [0,\theta]} J(\beta, u) = -\infty.$$

Setting

$$\bar{\rho}_i(\theta, s) = \sup_{\beta \in [0,\theta]} |\rho_i(\beta, s)|,$$

the following abstract result can be proved (see Theorem 2.1 in [14]).

Theorem 2.2 *Let $\rho_1 \leq \rho_2$ be two velocity fields and let ψ_1, ψ_2 be the corresponding scalar flows. Assume that the Hilbert space E is decomposed as $E = \bigcup_{n=0}^{\infty} E_n$ where*

$E_0 = E_-$ is a finite dimensional subspace and $(E_n)_n$ is an increasing sequence of subspaces of E such that $E_n = E_{n-1} \bigoplus \mathbf{R}e_n$. Let J_0 be an even C^1−functional on E and consider the levels $c_n = \inf_{h\in\mathcal{H}} \sup_{h(E_n)} J_0$, where

$$\mathcal{H} = \{h \in C(E,E) : h \text{ is odd and } h(u) = u \text{ for } \|u\| > R \text{ for some } R > 0\}.$$

(i) If $\psi_1(\theta, c_n) \uparrow +\infty$ as $n + \infty$, then for every integer k there exists $\theta_k \in (0,1]$, depending only on J_0 and ρ_1, ρ_2, such that for any good path of functionals $J : [0,1] \times \mathbf{E} \to \mathbf{R}$ starting from J_0 and controlled by ρ_1, ρ_2, the functional J_θ has, for any $\theta \in [0, \theta_k]$, at least k distinct critical levels.

(ii) If $c_n \geq B_1 + (B_2(n))^{\bar{\beta}}$ where $\bar{\beta} > 0$, $B_1 \in \mathbf{R}$, $B_2(n) > 0$ and if $\bar{\rho}_i(\theta, s) \leq A_1 + A_2 |s|^{\bar{\alpha}}$ with $0 \leq \bar{\alpha} < 1$ and $A_1, A_2 \geq 0$, then J_1 has an unbounded sequence of critical levels provided $(B_2(n))^{\bar{\beta}} > n^{\frac{1}{1-\bar{\alpha}}}$.

Remark 2.3 Let us point out that in [14] (see also [7]) the functional J is C^2, but this assumption can be weaken by taking J only of class C^1 if in the deformation Lemma we replace the gradient vector field by a pseudo-gradient vector field (see [23]).

3 The variational approach

In the sequel, for any x,$y \in \mathbf{R}^N$ we will denote by $x \cdot y$ the inner product in $\mathbf{R}^N$. Moreover, if A is an open subset of $\mathbf{R}^N$, we shall consider the Banach spaces $L^t\,(A)$ and $H^1(A)$ with the usual norms

$$|u|_{L^t(A)} = \left(\int_A |u(x)|^t dx\right)^{\frac{1}{t}} \qquad \text{and} \qquad \|u\|_{H^1(A)} = \left(\int_A \left(|\nabla u|^2 + u^2\right) dx\right)^{\frac{1}{2}}.$$

For simplicity, if $A = \Omega$ we will write $|u|_t$ and $\|u\|_{H^1}$ instead of $|u|_{L^t(\Omega)}$ and $\|u\|_{H^1(\Omega)}$.

From now on, let us assume that the coefficient $\sigma(x)$ satisfies $(\sigma_1)'$ or $(\sigma_2)'$. Then, we can consider the Hilbert spaces

$$L^2(\Omega, \sigma) = \left\{u \in L^2(\Omega) : \int_\Omega \sigma(x) u^2(x) dx < +\infty\right\}$$

equipped with the inner product $\int_\Omega \sigma(x) u\, v dx$ and

$$H^1(\Omega, \sigma) = H^1(\Omega) \cap L^2(\Omega, \sigma)$$

endowed with the inner product

$$\int_\Omega \left(\nabla u \cdot \nabla v + \sigma(x) u\, v\right) dx \tag{3.1}$$

and the corresponding norm

$$\|u\| = \left(\int\limits_{\Omega} \left(|\nabla u|^2 + \sigma(x)u^2 \right) dx \right)^{\frac{1}{2}} . \tag{3.2}$$

Let us point out that the space $H^1(\Omega, \sigma)$ is a particular case of a class of weighted Sobolev spaces introduced in [5].

Before we state some important properties of this space, we want to analyse the different assumptions made on the function $\sigma(x)$.
Set $\Omega_b = \{x \in \Omega : \sigma(x) \leq b\}$. The following proposition holds:

Proposition 3.1 *i) $\sigma(x)$ satisfies $(\sigma_2)' \iff \sigma(x) \in L^2_{loc}(\Omega)$, $\inf_{\Omega} ess\, \sigma(x) > 0$ and for every $b > 0$:*

$$m\,(S(x) \cap \Omega_b) \to 0 \qquad \textit{if } |x| \to +\infty.$$

ii) $\sigma(x)$ satisfies $(\sigma_1)' \Rightarrow \sigma(x)$ satisfies $(\sigma_2)'$.

Proof i) For the implication from the left to the right, it is enough to observe that, for any $b > 0$,

$$\int_{S(x)} \frac{1}{\sigma(y)} dy \geq \int_{S(x)\cap\Omega_b} \frac{1}{\sigma(y)} dy \geq \frac{1}{b} m\,(S(x) \cap \Omega_b).$$

On the other hand, since

$$\begin{aligned} \int_{S(x)} \frac{1}{\sigma(y)} dy &= \int_{S(x)\cap\Omega_b} \frac{1}{\sigma(y)} dy + \int_{S(x)\setminus\Omega_b} \frac{1}{\sigma(y)} dy \\ &\leq \frac{1}{\inf_{\Omega} ess\ \sigma(x)} m\,(S(x) \cap \Omega_b) + \frac{m(S(x))}{b}, \end{aligned}$$

it follows that

$$\limsup_{|x|\to+\infty} \int_{S(x)} \frac{1}{\sigma(y)} dy \leq \frac{m(S(x))}{b},$$

hence, the other implication follows by the arbitrariness of b.

ii) Thanks to i), this implication is obvious.■

Remark 3.2 Let us point out that conditions $(\sigma_1)'$ and $(\sigma_2)'$ (and therefore (σ_1) and (σ_2)) are not equivalent because it is possible that, for some $b > 0$, $m(\Omega_b) = \infty$ but $m\,(S(x) \cap \Omega_b) \to 0$ if $|x| \to +\infty$.

In order to overcome the lack of compactness of the problem, the following proposition is crucial.

Proposition 3.3 *Let $\sigma(x)$ satisfy assumption $(\sigma_2)'$. Then, the space $H^1(\Omega,\sigma)$ is embedded in L^t for any $t \in [2,2^*]$, and the embedding is compact for any $t \in [2,2^*[$. Moreover, the spectrum of the self-adjoint realization of $-\Delta+\sigma(x)$ with Dirichlet boundary conditions on $\partial\Omega$ in $L^2(\Omega)$ is discrete, i.e. it consists of a denumerable set of eigenvalues of finite multiplicity whose corresponding eigenmanifolds span all the space $L^2(\Omega)\cap\{u=0 \text{ on } \partial\Omega\}$.*

Proof If $\Omega=\mathbf{R}^N$, the result has been already proved in [6] (see also [21]). Now, let Ω be an open smooth subset with bounded boundary. By Theorem IX.7 in [9] there exists a linear extension operator $P: H^1(\boldsymbol{\Omega}) \to H^1(\mathbf{R}^N)$ such that for any $u \in H^1(\boldsymbol{\Omega})$
(*i*) $(Pu)_{|\Omega}=u$
(*ii*) $|Pu|_{L^2(\mathbf{R}^N)} \le C\,|u|_2$
(*iii*) $\|Pu\|_{H^1(\mathbf{R}^N)} \le C\,\|u\|_{H^1}$
where the constant C depends only on Ω. Then, taken a sequence $\{u_n\}$ such that $u_n \rightharpoonup u$ in $H^1(\Omega,\sigma)$, it is $Pu_n \rightharpoonup Pu$ in $H^1(\mathbf{R}^N,\sigma)$ and $Pu_n \to Pu$ in $L^2(\mathbf{R}^N)$ since $H^1(\mathbf{R}^N,\sigma)$ is compactly embedded in $L^2(\mathbf{R}^N)$ (see Theorem 3.1 in [6] and also Remark 3.5 of [4]). Hence, $H^1(\Omega,\sigma)$ is compactly embedded in $L^2(\Omega)$. Moreover, by the Sobolev embeddings it follows that

$$H^1(\Omega) \hookrightarrow L^t(\Omega) \qquad \text{for any } t \in [2,2^*].$$

Now, by the Gagliardo-Nirenberg interpolation inequality (see, e.g., [9]) for any $u \in L^2(\Omega)\cap L^{2^*}(\Omega)$ it is $u \in L^t(\Omega)$ and

$$|u|_t \le |u|_2^{1-a}\,|u|_{2^*}^a \qquad \text{with } \frac{1}{t}=\frac{1-a}{2}+\frac{a}{2^*}, \qquad 0\le a\le 1. \tag{3.3}$$

Hence, the embedding of $H^1(\Omega,\sigma)$ in $L^t(\Omega)$ is compact for any $t\in[2,2^*[$.
Finally, since $-\Delta+\sigma(x)$ with Dirichlet boundary conditions on $\partial\Omega$ is essentially self-adjoint on $C_0^\infty(\Omega)$, the second part of the proposition follows as in Theorem 4.1 of [6].■

Now, let us present the variational formulation of the problem. It is easy to see that the weak solutions of problem (1.1) are the critical points of the functional

$$\frac{1}{2}\int_\Omega \left(|\nabla u|^2+\sigma(x)u^2-\lambda u^2\right)dx - \int_\Omega F(x,u)dx - \int_\Omega gu dx$$

on the linear manifold

$$\mathcal{M}_\varphi=\left\{u\in H^1(\Omega,\sigma): u=\varphi \text{ a.e. on } \partial\Omega\right\}.$$

If $g=\varphi=0$, the previous proposition allows to prove that the action functional verifies the classical Palais-Smale condition, then a direct application of the symmetric mountain pass theorem gives the existence of infinitely many solutions of the problem (see [21] for a detailed proof in the case $F(x,u) = \frac{1}{\mu}|u|^{\mu}$). Now, we want to state a multiplicity result even if g or φ are non trivial.

First of all, if $\varphi \neq 0$, we reduce the original problem to an elliptic problem with homogeneous boundary conditions. Indeed, denoted by Φ the solution of the linear problem

$$\begin{cases} -\Delta u + \sigma(x)u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \\ |u| \to 0 & \text{if } |x| \to \infty, \end{cases}$$

the following result can be easily proved:

Proposition 3.4 *A function $u \in \mathcal{M}_{\varphi}$ is a solution of problem (1.1) if and only if $v \in H^1(\Omega,\sigma)$ is a solution of*

$$\begin{cases} -\Delta v + \sigma(x)v = \lambda(v+\Phi) + f(x, v+\Phi) + g(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \\ |v| \to 0 & \text{if } |x| \to \infty \end{cases}$$

where $u(x) = v(x) + \Phi(x)$ for a.e. $x \in \Omega$.

Hence, our aim is to state the existence of multiple critical points of the functional

$$I_1(u) = \frac{1}{2}\int_{\Omega} \left(|\nabla u|^2 + \sigma(x)u^2\right) dx - \int_{\Omega}\left(\frac{\lambda}{2}(u+\Phi)^2 + F(x, u+\Phi) + gu\right) dx$$

defined on the Hilbert space

$$X = H_0^1(\Omega) \cap L^2(\Omega,\sigma) = \left\{u \in H^1(\Omega,\sigma) : u = 0 \text{ on } \partial\Omega\right\}$$

endowed with the inner product (3.1) and the corresponding norm (3.2). Following the Bolle's perturbation method, let us consider the path of functionals

$$\begin{aligned} I(\theta, u) = I_\theta(u) = \frac{1}{2}\int_{\Omega} \left(|\nabla u|^2 + \sigma(x)u^2\right) dx + \\ - \int_{\Omega}\left(\frac{\lambda}{2}(u+\theta\,\Phi)^2 + F(x, u+\theta\,\Phi) + \theta\, gu\right) dx. \end{aligned}$$

Standard arguments show that I is a C^1 functional and for any $\theta \in [0,1]$ and $u, v \in X$ it is

$$\frac{\partial I}{\partial \theta}(\theta, u) = -\int_{\Omega} \left(\lambda\,(u+\theta\,\Phi)\,\Phi + f(x, u+\theta\,\Phi)\Phi + gu\right) dx \tag{3.4}$$

and

$$\begin{aligned} I'_\theta(u)\,[v] &= \frac{\partial I}{\partial u}(\theta,u)[v] \qquad (3.5) \\ &= \int_{\Omega} \left(\nabla u \cdot \nabla v + \sigma(x)u\,v - \lambda\,(u+\theta\,\Phi)\,v - f(x, u+\theta\,\Phi)v - \theta g v\right) dx. \end{aligned}$$

Remark 3.5 Clearly, if $\varphi = 0$, in particular if $\Omega = \mathbf{R}^N$, the perturbed functional I_θ becomes

$$I_\theta(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \sigma(x)u^2 - \lambda u^2\right) dx - \int_\Omega F(x,u)dx - \theta \int_\Omega gu\, dx.$$

Remark 3.6 Let us point out that the regularity of Φ in particular implies that $\Phi \in L^t(\Omega)$ for all $t \geq 2$.

4 Some preliminary lemmas

In order to apply the abstract Theorem 2.2 to the functional $I(\theta, u)$, the following lemmas will be need. In the sequel, we will denote by a_i some suitable positive constants.

Lemma 4.1 *Assume that* $(f_1) - (f_4)$ *and* $(\sigma_1)'$ *hold. Taken any* $\rho \in]\frac{1}{\mu}, \frac{1}{2}[$ *there exists a positive constant* $\alpha(\rho)$ *such that*

$$\|u\|^2 + |u + \theta\Phi|_\mu^\mu \leq \alpha(\rho)\left(I_\theta(u) - \rho I_\theta'(u)[u] + 1\right).$$

Proof Let $(\theta, u) \in [0,1] \times X$ and $\rho \in]\frac{1}{\mu}, \frac{1}{2}[$. By the definition of I_θ, (3.5), Remark 1.3 and (f_4) it follows that

$$\begin{aligned}
&I_\theta(u) - \rho I_\theta'(u)[u] \\
&= \left(\frac{1}{2} - \rho\right) \int_\Omega \left(|\nabla u|^2 + \sigma(x)u^2 - \lambda u^2\right) dx - \frac{\lambda}{2}\theta^2 \int_\Omega \Phi^2 dx \\
&- \int_\Omega \left(F(x, u + \theta\,\Phi) - \rho f(x, u + \theta\,\Phi)\,(u + \theta\,\Phi)\right) dx - \rho\theta \int_\Omega f(x, u + \theta\,\Phi)\,\Phi\, dx \\
&- \lambda(1-\rho)\theta \int_\Omega \Phi u\, dx - (1-\rho)\,\theta \int_\Omega gu\, dx \\
&\geq \left(\frac{1}{2} - \rho\right)\left(\|u\|^2 - \lambda\, |u|_2^2\right) - |\lambda| \int_\Omega |u + \theta\,\Phi|\,|\Phi|\, dx + (\rho\mu - 1)\,\alpha_4\, |u + \theta\Phi|_\mu^\mu \\
&- \alpha_1 \int_\Omega |u + \theta\,\Phi|^{p-1}\,|\Phi|\, dx - \int_\Omega |u + \theta\,\Phi|\,|g|\, dx - a_1.
\end{aligned}$$

Now, by the Young inequality and Remark 3.6 for any $\varepsilon > 0$, we have

$$\int_\Omega |u + \theta\,\Phi|\,|\Phi|\, dx \leq \varepsilon \int_\Omega |u + \theta\,\Phi|^\mu\, dx + \beta_\mu(\varepsilon) \int_\Omega |\,\Phi|^{\mu'}\, dx, \tag{4.1}$$

$$\int_\Omega |u + \theta\,\Phi|^{p-1}\,|\Phi|\, dx \leq \varepsilon \int_\Omega |u + \theta\Phi|^\mu\, dx + \beta_{\mu,p}(\varepsilon) \int_\Omega |\Phi|^s\, dx, \tag{4.2}$$

$$\int_\Omega |u+\theta\,\Phi|\,|g|\,dx \le \varepsilon \int_\Omega |u+\theta\,\Phi|^\mu\,dx + \beta_\mu(\varepsilon)\int_\Omega |g|^{\mu'}\,dx, \tag{4.3}$$

where μ' is the exponent coniugate to μ and

$$\beta_\mu(\varepsilon) = \frac{\mu-1}{\mu}\left(\frac{1}{\varepsilon\mu}\right)^{\frac{1}{\mu-1}}, \quad \beta_{\mu,p}(\varepsilon) = \frac{\mu-p+1}{\mu}\left(\frac{p-1}{\varepsilon\mu}\right)^{\frac{p-1}{\mu-p+1}}, \quad s = \frac{\mu}{\mu-p+1}.$$

The previous inequalities imply

$$\begin{aligned} I_\theta(u) - \rho I'_\theta(u)\,[u] \ge \left(\frac{1}{2}-\rho\right)\left(\|u\|^2 - \lambda\,|u|_2^2\right) \\ + \left((\rho\mu-1)\,\alpha_4 - (\alpha_1+|\lambda|+1)\,\varepsilon\right)|u+\theta\Phi|_\mu^\mu - a_2. \end{aligned} \tag{4.4}$$

Denoting by λ_1 the first eigenvalue of $-\Delta+\sigma(x)$ in $L^2(\Omega)$, if $\lambda<\lambda_1$ the proof follows, choosing ε small enough. On the other hand, if $\lambda\ge\lambda_1$, by $(\sigma_1)'$ it is easy to deduce that, for fixed $\varepsilon>0$, there exists $a_\varepsilon>0$ such that

$$|u|_2^2 \le a_\varepsilon\,|u|_\mu^2 + \varepsilon\,\|u\|^2. \tag{4.5}$$

Indeed, if we set

$$\Omega_\varepsilon = \left\{x\in\Omega : \sigma(x) < \frac{1}{\varepsilon}\right\},$$

by the assumption $(\sigma_1)'$ the measure of Ω_ε is finite, then, denoted by a_ε the imbedding constant of $L^\mu(\Omega_\varepsilon)$ in $L^2(\Omega_\varepsilon)$, we have

$$\begin{aligned} \int_\Omega u^2dx &= \int_{\Omega_\varepsilon} u^2dx + \int_{\Omega\setminus\Omega_\varepsilon} u^2dx \le \int_{\Omega_\varepsilon} u^2dx + \varepsilon\int_{\Omega\setminus\Omega_\varepsilon}\sigma(x)u^2dx \\ &\le a_\varepsilon\left(\int_{\Omega_\varepsilon}|u|^\mu\,dx\right)^{\frac{2}{\mu}} + \varepsilon\int_\Omega \sigma(x)u^2dx \le a_\varepsilon\,|u|_\mu^2 + \varepsilon\,\|u\|^2. \end{aligned}$$

Finally, since $\mu>2$, by (4.4) and (4.5) we have the conclusion.■

Remark 4.2 If we assume the weaker condition $(\sigma_2)'$ instead of $(\sigma_1)'$, Lemma 4.1 still holds if $\lambda<\lambda_1$ while, if λ is any, by (4.4) it becomes

$$\left(\frac{1}{2}-\rho\right)\left(\|u\|^2 - \lambda\;|u|_2^2\right) + a_3\,|u+\theta\Phi|_\mu^\mu \le I_\theta(u) - \rho I'_\theta(u)\,[u] + a_2.$$

In particular, chosen $\rho=\frac{1}{2}$ we obtain

$$|u+\theta\Phi|_\mu^\mu \le a_4\left(I_\theta(u) - \rho I'_\theta(u)\,[u] + 1\right). \tag{4.6}$$

Lemma 4.3 *Assume that* $(f_1)-(f_4)$ *and* $(\sigma_1)'$ *hold. Then,* I_θ *satisfies the Palais-Smale condition, i.e. any sequence* $\{(\theta_n,u_n)\}_n \subset [0,1]\times \mathbf{X}$ *such that*

$$\{I_{\theta_n}(u_n)\}_n \text{ is bounded} \quad \text{and} \quad \lim_n I'_{\theta_n}(u_n)=0, \tag{4.7}$$

converges up to subsequence.

Proof If $\{(\theta_n,u_n)\}_n$ satisfies (4.7), the previous lemma implies that $\|u_n\|$ is bounded, then, passing to subsequence,

$$u_n \rightharpoonup u \quad \text{in } X.$$

By Proposition 3.3 it follows that

$$u_n \to u \quad \text{in } L^t(\Omega) \quad \text{for any } t\in[2,2^*[.$$

Hence, standard arguments allow us to prove that $u_n\to u$ in X (see, e.g., [4]).■

Lemma 4.4 *Assume that* $(f_1)-(f_4)$ *and* $(\sigma_2)'$ *hold. Then,* I_θ *still satisfies the Palais-Smale condition for any* $\lambda\in\mathbf{R}$ *if* $\mu=p$ *or for* $\lambda<\lambda_1$ *if* $\mu<p$.

Proof The proof follows as above if $\lambda<\lambda_1$ (see Remark 4.2). Now, assume $\lambda\in\mathbf{R}$ and $\mu=p$. Setting $Lu=-\Delta u+\sigma(x)u-\lambda u$, by Proposition 3.3 it follows that $X=X^+\oplus X^-\oplus X^0$, $X^+\oplus X^0\neq\emptyset$, where for any $u^0\in X^0$ it is $Lu^0=0$ and for any $u^\pm\in X^\pm$ it is

$$\langle Lu^+,u^+\rangle \ge \lambda_+\|u^+\|^2, \quad \langle Lu^-,u^-\rangle\le\lambda_-\|u^-\|^2 \quad \text{with } \lambda_+>0>\lambda_-.$$

Let $\{(\theta_n,u_n)\}_n\subset[0,1]\times X$ verifying (4.7) such that $u_n=u_n^++u_n^-+u_n^0$, $n\in\mathbf{N}$. Clearly, it is $I'_{\theta_n}(u_n)[u_n^+]\le\|u_n^+\|$, hence, (f_4), Holder inequality and (4.6) imply

$$\begin{aligned}\lambda_+\|u_n^+\|^2 &\le \theta\int_\Omega(\lambda\Phi+g)\,u_n^+\,dx+\alpha_1\int_\Omega|u_n+\theta\,\Phi|^{\mu-1}|u_n^+|\,dx+\|u_n^+\|\\ &\le \left(|\lambda|\,|\Phi|_{\mu'}+|g|_{\mu'}\right)|u_n^+|_\mu+\alpha_1|u_n+\theta\,\Phi|_\mu^{\mu-1}|u_n^+|_\mu+\|u_n^+\|\\ &\le a_5\left(1+\|u_n\|^{\frac{\mu-1}{\mu}}\right)\|u_n^+\|,\end{aligned}$$

and therefore

$$\|u_n^+\|\le a_6\left(\|u_n\|^{\frac{\mu-1}{\mu}}+1\right). \tag{4.8}$$

Moreover, as $X^-\oplus X^0$ has finite dimension, by (4.6), (4.7) and (4.8) we deduce

$$\|u_n^-+u_n^0\|\le a_7\,|u_n^-+u_n^0|_\mu\le a_8\left(\|u_n\|^{\frac{1}{\mu}}+\|u_n\|^{\frac{\mu-1}{\mu}}+1\right). \tag{4.9}$$

Adding (4.8) and (4.9) we obtain

$$\|u_n\|\le a_9\left(\|u_n\|^{\frac{1}{\mu}}+\|u_n\|^{\frac{\mu-1}{\mu}}+1\right)$$

which imply the boundness of $\|u_n\|$. The conclusion follows as in the proof of Lemma 4.3.■

Lemma 4.5 *Assume that* $(f_1)-(f_4)$ *and* $(\sigma_2)'$ *hold. For any* $b>0$ *there exists* $C_b>0$ *such that if* $(\theta,u)\in[0,1]\times X$ *then*

$$|I_\theta(u)|\le b\Rightarrow\left|\frac{\partial I}{\partial\theta}(\theta,u)\right|\le C_b(\|I'_\theta(u)\|+1)(\|u\|+1).$$

Proof By (3.4), (f_4) and (4.1)-(4.3) with $\varepsilon=1$, it follows that

$$\left|\frac{\partial I}{\partial\theta}(\theta,u)\right|\le(|\lambda|+\alpha_1+1)\int_\Omega|u+\theta\,\Phi|^\mu\,dx+\int_\Omega|\Phi|\,|g|\,dx$$
$$+\beta_\mu(1)\left(\int_\Omega|\Phi|^{\mu'}\,dx+\int_\Omega|g|^{\mu'}\,dx\right)+\alpha_1\beta_{\mu,p}(1)\int_\Omega|\Phi|^s\,dx.$$

Hence, (4.6) gives the conclusion.■

Lemma 4.6 *Assume that* $(f_1)-(f_5)$ *hold and* g, φ *and* σ *are functions regular as in Theorem 1.1. Then, a constant* $L>0$ *exists such that for all critical point* u *of* I_θ,

$$\int_{\partial\Omega}\left|\frac{\partial v}{\partial n}\right|^2d\sigma\le L\left(\|v\|^2+|v|_m^m+\int_\Omega F(x,v)dx+1\right)$$

where $v=u+\theta\Phi$ *and* $m=\max\{\mu,q\}$.

Proof Let u be a critical point of I_θ, i.e. a solution of

$$\begin{cases}-\Delta u+\sigma(x)u=\lambda(u+\theta\Phi)+f(x,u+\theta\Phi)+\theta g(x) & \text{in }\Omega\\ u=0 & \text{on }\partial\Omega\\ |u|\to0 & \text{if }|x|\to\infty.\end{cases}\tag{4.10}$$

Then, $v=u+\theta\Phi$ satisfies

$$\begin{cases}-\Delta v+\sigma(x)v=\lambda v+f(x,v)+\theta g(x) & \text{in }\Omega\\ v=\theta\Phi & \text{on }\partial\Omega\\ |v|\to0 & \text{if }|x|\to\infty.\end{cases}\tag{4.11}$$

Following [8], define $n:\bar\Omega\to\mathbf{R}$ such that n coincides with the outward normal on $\partial\Omega$ as follows: for $x\in\bar\Omega$, let $l(x)=dist(x,\partial\Omega)$ the distance from the boundary of Ω. Since Ω is of class C^2, without loss of generality we can assume that l is C^2 on the set $N_\delta=\{x\in\bar\Omega:l(x)<\delta\}$ where g and σ are C^1. Then, let $n(x)=\nabla l(x)$. Moreover, let $\psi:\mathbf{R}\to[0,1]$ be a smooth function such that $\psi=1$ on $(-\infty,0]$ and $\psi=0$ on $[\delta,+\infty)$. Let $\omega(x)=\psi(l(x))$. Clearly, n and ω are C^1.

Multiply the equation in (4.11) by $\omega\nabla v\cdot n$ and integrate over Ω (or equivalently on N_δ). On the left, we get

$$\begin{aligned}\int_\Omega(-\Delta v+\sigma v)\omega\nabla v\cdot n\,dx &= -\int_{\partial\Omega}\nabla v\cdot(\omega\nabla v\cdot n)n\,d\sigma+\int_\Omega\nabla v\cdot\nabla(\omega\nabla v\cdot n)dx\\ &\quad+\int_\Omega\sigma v\,\omega\nabla v\cdot n\,dx\\ &= -\int_{\partial\Omega}\left|\frac{\partial v}{\partial n}\right|^2 d\sigma+\int_\Omega\nabla v\cdot\nabla(\omega\nabla v\cdot n)\,dx\\ &\quad+\int_\Omega\sigma v\,\omega\nabla v\cdot n\,dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega\nabla v\cdot\nabla(\omega\nabla v\cdot n)dx &= \sum_{i,j=1}^N\int_\Omega v_{x_i}\left(\omega v_{x_j}n_j\right)_{x_i}dx\\ &= \sum_{i,j=1}^N\int_\Omega\Big(v_{x_i}v_{x_j}\left(\omega n_j\right)_{x_i}+v_{x_i}v_{x_jx_i}\left(\omega\, n_j\right)\Big)\,dx\\ &= \sum_{i,j=1}^N\int_\Omega\left(v_{x_i}v_{x_j}\left(\omega n_j\right)_{x_i}+\frac12\left(v_{x_i}\right)^2_{x_j}\left(\omega\, n_j\right)\right)dx\\ &= \sum_{i,j=1}^N\int_\Omega\left(v_{x_i}v_{x_j}\left(\omega n_j\right)_{x_i}-\frac12\left(v_{x_i}\right)^2\left(\omega\, n_j\right)_{x_j}\right)dx\\ &\quad+\sum_{i,j=1}^N\int_{\partial\Omega}\frac12\left(v_{x_i}\right)^2\omega\, n_jn_j\,d\sigma\\ &= \frac12\int_{\partial\Omega}|\nabla v|^2d\sigma+O\left(\int_\Omega|\nabla v|^2dx\right),\end{aligned}$$

where

$$F_1(v)=O(F_2(v))\Leftrightarrow\exists\; c>0 \text{ s.t. } |F_1(v)|\le c\,|F_2(v)| \text{ for any } v\in X.$$

Moreover,

$$\begin{aligned}\int_\Omega \sigma v\,\omega\nabla v\cdot n\,dx &= \sum_{i=1}^N \int_\Omega \sigma\omega\left(\frac{1}{2}v^2\right)_{x_i} n_i\,dx\\ &= \frac{1}{2}\sum_{i=1}^N \int_{\partial\Omega} \sigma v^2\omega n_i\,n_i\,d\sigma - \frac{1}{2}\sum_{i=1}^N \int_\Omega v^2\left(\sigma\omega n_i\right)_{x_i}\,dx\\ &= \frac{1}{2}\int_{\partial\Omega}\sigma v^2 d\sigma + O\left(\int_\Omega |v|^2\,dx\right).\end{aligned}$$

By the previous equalities we obtain

$$\begin{aligned}\int_\Omega(-\Delta v+\sigma v)\omega\nabla v\cdot n\,dx &= \int_{\partial\Omega}\left(\frac{1}{2}|\nabla v|^2+\frac{1}{2}\sigma v^2-\left|\frac{\partial v}{\partial n}\right|^2\right)d\sigma\\ &\quad + O\left(\int_\Omega|\nabla v|^2 dx\right)+O\left(\int_\Omega |v|^2\,dx\right).\end{aligned}\tag{4.12}$$

On the rigth we have

$$\begin{aligned}\int_\Omega \lambda v\,\omega\nabla v\cdot n\,dx &= \lambda\sum_{i=1}^N\int_\Omega\left(\frac{1}{2}v^2\right)_{x_i}\omega\,n_i\,dx\\ &= \frac{\lambda}{2}\int_{\partial\Omega} v^2\,d\sigma + O\left(\int_\Omega |v|^2\,dx\right).\end{aligned}\tag{4.13}$$

Now, let us point out that all the integrals on Ω are really calculated on N_δ, which has a compact closure since $\partial\Omega$ is bounded. Therefore, Holder inequality implies

$$\begin{aligned}\int_\Omega \theta g\,\omega\nabla v\cdot n\,dx &= \theta\int_{\partial\Omega} g\,v\,d\sigma - \theta\sum_{i=1}^N\int_\Omega v\,(g\,\omega\,n_i)_{x_i}\,dx\\ &= \theta\int_{\partial\Omega} gv\,d\sigma + O\left(\left(\int_\Omega |v|^2\,dx\right)^{\frac{1}{2}}\right),\end{aligned}\tag{4.14}$$

while by (f_5) and the imbedding of $L^m(N_\delta)$ in $L^q(N_\delta)$ it follows that

$$\begin{aligned}\int_\Omega f(x,v)\,\omega\nabla v\cdot n\,dx &= \sum_{i=1}^N \int_\Omega \left[((F(x,v(x)))_{x_i} - \frac{\partial F}{\partial x_i}(x,v)\right]\omega\, n_i dx \\ &= \int_{\partial\Omega} F(x,v)\,d\sigma - \sum_{i=1}^N \int_\Omega F(x,v)\,(\omega n_i)_{x_i}\,dx - \int_\Omega \omega\frac{\partial F}{\partial x}(x,v)\cdot n\,dx \qquad (4.15)\\ &\le a_{10} + O\left(\int_\Omega F(x,v)\,dx\right) + O\left(\int_\Omega |v|^m\,dx\right).\end{aligned}$$

Now,

$$\int_{\partial\Omega}\left(\frac{1}{2}|\nabla v|^2 - \left|\frac{\partial v}{\partial n}\right|^2\right)d\sigma = \int_{\partial\Omega}\left(\frac{1}{2}|D_{\partial\Omega}v|^2 - \frac{1}{2}\left|\frac{\partial v}{\partial n}\right|^2\right)d\sigma$$

where $D_{\partial\Omega}v = D_{\partial\Omega}\theta\Phi$ is the component of ∇v along $\partial\Omega$. Thus, putting together relations (4.12)-(4.15), we reach the conclusion.■

Lemma 4.7 *Assume that all the assumptions of Theorem 1.1 are satisfied. Then, a constant $\tilde{L}_1 > 0$ exists such that for all u critical point of I_θ it is*

$$\left|\frac{\partial I_\theta}{\partial\theta}(u)\right| \le \tilde{L}_1\left(1 + I_\theta^2(u)\right)^r$$

where $r = \frac{1}{4}$ if $q \le \mu$ while $r = \frac{2N-\mu N+2q}{4(2N-\mu N+2\mu)}$ if $q > \mu$.

Proof Letting u be a critical point of I_θ, then by (4.10) and (3.4) it follows

$$\begin{aligned}\frac{\partial I_\theta}{\partial\theta}(u) &= \int_\Omega ((\Delta u - \sigma(x)u + \theta g)\,\Phi - gu)\,dx \\ &= \int_{\partial\Omega}\nabla u\cdot\Phi n\,d\sigma - \int_\Omega(\nabla u\cdot\nabla\Phi + \sigma(x)u\Phi - \theta g\Phi + gu)\,dx \\ &= \int_{\partial\Omega}(\Phi\nabla u\cdot n - u\nabla\Phi\cdot n)\,d\sigma + \int_\Omega((\Delta\Phi - \sigma(x)\Phi)\,u)\,dx \\ &- \int_\Omega(u - \theta\Phi)\,g dx.\end{aligned}$$

Recalling that $u = 0$ and $\nabla u = \frac{\partial u}{\partial n}n$ on $\partial\Omega$ while $\Delta\Phi - \sigma(x)\Phi = 0$ on Ω, we obtain

$$\frac{\partial I_\theta}{\partial\theta}(u) = \int_{\partial\Omega}\frac{\partial u}{\partial n}\Phi\,d\sigma - \int_\Omega(u - \theta\Phi)\,g\,dx. \qquad (4.16)$$

Now, Hölder inequality and (4.6) imply

$$\left|\int_\Omega gu\,dx\right| \le |g|_{\mu'}\,|u|_\mu \le a_{11}\left(|I_\theta(u)|+1\right)^{\frac{1}{\mu}} \tag{4.17}$$

while, since $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} - \theta\frac{\partial \Phi}{\partial n}$ on $\partial\Omega$, by Hölder inequality and Lemma 4.6 it is

$$\begin{aligned}\left|\int_{\partial\Omega} \frac{\partial u}{\partial n}\Phi\,d\sigma\right| &\le \left(\int_{\partial\Omega}\left|\frac{\partial v}{\partial n}\right|^2 d\sigma\right)^{\frac{1}{2}} |\Phi|_{L^2(\partial\Omega)} + a_{12} \\ &\le a_{13}\left(\left(\|v\|^2 + |v|_m^m + \int_\Omega F(x,v)dx\right)^{\frac{1}{2}} + 1\right).\end{aligned} \tag{4.18}$$

Let us point out that by the expression of $I_\theta(u)$, Remark 4.2 and (4.17) it is easy to deduce that

$$\begin{aligned}\int_\Omega F(x,v)dx &= \frac{1}{2}\left(\|u\|^2 - \lambda\ |u|_2^2\right) - \theta\int_\Omega\left((\lambda\Phi+g)\,u + \frac{\lambda}{2}\theta\,\Phi^2\right)dx - I_\theta(u) \\ &\le a_{14}\left(|I_\theta(u)|+1\right).\end{aligned} \tag{4.19}$$

If $q \le \mu$, i.e. $m=\mu$, by (4.16)-(4.19) and Lemma 4.1 we have the conclusion. Now, assume $q > \mu$, i.e. $m = q$, and $N \ge 3$ (the proof is similar if $N=2$). Taken $a \in]0,1[$ s.t. $\frac{1-a}{\mu} + \frac{a}{2^*} = \frac{1}{q}$, the inequalities interpolation (3.3) and Lemma 4.1 imply

$$|v|_q^q \le \left(|v|_\mu^{1-a}\,|u|_{2^*}^a\right)^q \le a_{15}\left(|I_\theta(u)|+1\right)^{\left(\frac{1-a}{\mu}+\frac{a}{2}\right)q}. \tag{4.20}$$

Hence, since $\left(\frac{1-a}{\mu}+\frac{a}{2}\right)q \ge 1$, the conclusion follows again by (4.16)-(4.20) and simple calculations.■

Remark 4.8 Thanks to the Remark 4.2, the previous lemma still holds for $\lambda < \lambda_1$ if we assume $(\sigma_2)'$ instead of $(\sigma_1)'$.

Lemma 4.9 *If we assume $(\sigma_2)'$ instead of $(\sigma_1)'$ and $q \le \mu = p$, then, a constant $\tilde{L}_2 > 0$ exists such that for all u, u critical point of I_θ,*

$$\left|\frac{\partial I_\theta}{\partial\theta}(u)\right| \le \tilde{L}_2\left(1+I_\theta^2(u)\right)^{\frac{\mu-1}{2\mu}}.$$

Proof Arguing as above, it is easy to prove that (4.16)-(4.19) hold also in this case. Thus, we need an estimate for $\|v\|^2$. As

$$0 = I_\theta'(u)\left[u^+\right] = \left\langle Lu, u^+\right\rangle - \int_\Omega f(x,v)u^+dx - \theta\int_\Omega(\lambda\Phi+g)\,u^+\,dx,$$

by (f_4) and Holder inequality it follows that

$$\lambda_+ \left\|u^+\right\|^2 \le \alpha_1 |v|_\mu^{\mu-1} \left|u^+\right|_\mu + \left(|\lambda|\, |\Phi|_{\mu'} + |g|_{\mu'}\right) \left|u^+\right|_\mu$$

and therefore

$$\left\|u^+\right\| \le a_{16} \left(|v|_\mu^{\mu-1} + 1\right). \tag{4.21}$$

Since $X^- \oplus X^0$ has finite dimension, by (4.21) we deduce

$$\left\|u^- + u^0\right\| \le a_{17} \left(|u|_\mu + \left\|u^+\right\|\right) \le a_{18} \left(|v|_\mu^{\mu-1} + 1\right), \tag{4.22}$$

hence, by (4.6), (4.21) and (4.22) we have

$$\|v\| \le a_{19} \left(1 + I_\theta^2(u)\right)^{\frac{\mu-1}{2\mu}}.$$

So, the last inequality and (4.16)-(4.19) give the conclusion.■

Remark 4.10 Let us point out that (f_5) and the regularity assumptions on g, φ and σ are used only in the proof of Lemma 4.6, and consequently, Lemmas 4.7 and 4.9. However, in order to get an estimate for $\frac{\partial I_\theta}{\partial \theta}$, those assumptions can be avoided, and $(\sigma_1)'$ can be replaced by $(\sigma_2)'$, if we assume $\varphi = 0$. Indeed, in this case the simpler expression of I_θ (see Remark 3.5) and (4.6) directly imply

$$\left|\frac{\partial I_\theta}{\partial \theta}(u)\right| = \left|\int_\Omega g u dx\right| \le |g|_{\mu'} |u|_\mu \le \tilde{L} \left(I_\theta^2(u) + 1\right)^{\frac{1}{2\mu}}.$$

Lemma 4.11 *If (f_2) and (f_3) hold, then, taking W to be any finite dimensional subspace of X and $\theta \in [0,1]$, we have*

$$\lim_{\substack{u \in W \\ \|u\| \to +\infty}} \sup_{\beta \in [0,\theta]} I(\beta, u) = -\infty.$$

Proof It is enough pointing out that Remark 1.3 implies

$$I(\theta, u) \le \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} |u + \theta\,\Phi|_2^2 - \alpha_4 |u + \theta\,\Phi|_\mu^\mu + |g|_{\mu'} |u|_\mu$$

for any $(\theta, u) \in [0,1] \times X$.■

5 Statement of the results

The aim of this section is to apply Theorem 2.2 to the functional $I_\theta(u)$. Assume that $(f_1) - (f_4)$ and $(\sigma_2)'$ hold. As in section 2, let us introduce a suitable class of minimax values for the even functional I_0. Denoting by X_n the subspace of X

spanned by the first n eigenfunctions of $-\Delta + \sigma(x)$ (see proposition 3.3), let us consider

$$c_n = \inf_{h\in\mathcal{H}} \sup_{h(X_n)} I_0,$$

where

$$\mathcal{H} = \{h \in C(X,X) : h \text{ is odd and } h(u) = u \text{ for } \|u\| > R \text{ for some } R > 0\}.$$

Clearly, for all integer n, c_n is a critical value of I_0 and $c_n \le c_{n+1}$. Now, we need a suitable estimate on the $c_n's$.

First, let us point out that by Lemma 4.11 for all n there exist $R_n > 0$ such that if $\|u\| > R_n$ then $I_0(u) \le I_0(0) = 0$.
Setting

$$D_n = \{u \in X_n : \|u\| \le R_n\}$$

and

$$\mathcal{H}_n = \{h \in C(D_n, X) : h \text{ is odd and } h(u) = u \text{ for } \|u\| = R_n\},$$

we deduce that

$$c_n \ge \inf_{h\in\mathcal{H}_n} \sup_{h(D_n)} I_0.$$

Moreover, by (f_2) and (f_4) it is

$$I_0(u) \ge \frac{1}{2}\,\|u\|^2 - \frac{\lambda}{2}\,|u|_2^2 - \frac{\alpha_1}{\mu}\,|u|_p^p, \tag{5.1}$$

hence, setting $K(u) = \frac{1}{2}\,\|u\|^2 - \frac{\lambda}{2}\,|u|_2^2 - \frac{\alpha_1}{\mu}\,|u|_p^p$, in order to provide a lower bound for c_n it is enough to estimate the $b_n's$, where

$$b_n = \inf_{h\in\mathcal{H}_n} \sup_{h(D_n)} K(u).$$

Arguing as in [19], we shall prove the following result:

Lemma 5.1 *Assume that $(f_1)-(f_4)$ and $(\sigma_2)'$ hold. Then, a constant $\gamma > 0$ exists such that for n large*

$$c_n \ge \gamma\,\lambda_n^{\frac{2p-(p-2)N}{2(p-2)}}.$$

Proof Let $n \in \mathbf{N}$, $h \in \mathcal{H}_n$ and $\rho < R_n$. By Lemma 1.44 in [19], $h(D_n) \cap \partial B_\rho \cap V_{n-1}^\perp \ne \emptyset$. Thus there exists $w \in h(D_n) \cap \partial B_\rho \cap V_{n-1}^\perp$ s.t.

$$\max_{h(D_n)} K(u) \ge K(w) \ge \inf_{u\in\partial B_\rho\cap V_{n-1}^\perp} K(u).$$

Let $u \in \partial B_\rho \cap V_{n-1}^\perp$. Then $|u|_2 \le \lambda_n^{-\frac{1}{2}}\rho$, so the Gagliardo-Nirenberg inequality implies

$$|u|_p \le \rho\lambda_n^{-\frac{(1-a)}{2}} \qquad \text{where } a = \frac{p-2}{p}\cdot\frac{N}{2}.$$

Whence, for n so large that $\lambda_n > 2\lambda$,

$$K(u) \geq \frac{1}{2}\rho^2 - \frac{\lambda}{2}\lambda_n^{-1}\rho^2 - \frac{\alpha_1}{\mu}\lambda_n^{-\frac{(1-a)p}{2}}\rho^p \geq \frac{1}{4}\rho^2 - \frac{\alpha_1}{\mu}\lambda_n^{-\frac{(1-a)p}{2}}\rho^p.$$

Taking $\rho = \rho_n = \left(\frac{\mu}{2p\alpha_1}\right)^{\frac{1}{p-2}} \lambda_n^{\frac{(1-a)}{2}\frac{p}{p-2}}$, we can assume $\rho_n < R_n$ and therefore for n large,

$$K(u) \geq \gamma\lambda_n^{\frac{(1-a)p}{p-2}},$$

which gives the conclusion.■

Remark 5.2 If we assume that $(\sigma_1)'$ holds, arguing as in the proof of (4.5) it follows that $I_0(u) \geq \frac{1}{4}\|u\|^2 - a_{20}|u|_p^p$. So, we suspect that, as in [24] in the case Ω bounded, it is possible to state the following estimate (independent on λ_n):

$$c_n \geq \delta\, n^{\frac{p}{p-2}\frac{2}{N}}.$$

Indeed, denoted by v_n a critical point of $K(u) = \frac{1}{4}\ \|u\|^2 - a_{20}|u|_p^p$ at the level b_n, arguing as in [24] we can prove that

$$b_n \geq a_{21}|v_n|_p^p \qquad \text{and} \qquad |v_n|_{(p-2)\frac{N}{2}} \geq a_{22}\, n^{\frac{1}{p-2}\frac{2}{N}},$$

but, Ω being unbounded, we are not able to directly compare $|v_n|_p$ and $|v_n|_{(p-2)\frac{N}{2}}$.

Finally, we can prove the results of Section 1.

Proof of Theorems 1.1 and 1.4. Assume that all the hypotheses of Theorem 1.1 (respectively Theorem 1.4) hold. Thanks to the Lemmas 4.3, (respectively 4.4) 4.5, 4.7 (respectively Remark 4.8) and 4.11, the functional $I(\theta, u)$ is a good path of functionals starting from I_0 and controlled by ρ_1 and ρ_2 with

$$-\rho_1(s) = \rho_2(s) = \tilde{L}_1\left(1+s^2\right)^r \tag{5.2}$$

where $r = \frac{1}{4}$ if $q \leq \mu$ while $r = \frac{2N-N\mu+2q}{4(2N-N\mu+2\mu)}$ if $q > \mu$.

Straighforward calculations show that

$$\bar{\alpha} = 2r < 1 \tag{5.3}$$

for any $q < 2^*$ if $N \geq 4$ and for $q < 2\mu + N - \frac{N}{2}\mu$ if $N = 2, 3$.
In order to prove i), it is enough to remark that, c_n being unbounded, so is $\psi_1(1, c_n)$. In fact, by the definition of ψ_1, we have that $\psi_1(1,\cdot)$ is increasing and $|\psi_1(1,s) - s| \leq \theta\,\bar{\rho}_1(1,s)$, hence, by (5.2) and (5.3) it follows that

$$\psi_1(1, c_n) \geq c_n - \tilde{L}_1\left(1+c_n^2\right)^r \to +\infty.$$

Then, i) of Theorem 1.1 follows directly by i) of Theorem 2.2.Moreover, by ii) of Theorem 2.2 with $\bar{\alpha} = 2r < 1$ and $(B_2(n))^{\bar{\beta}} = \lambda_n^{\frac{2p-(p-2)N}{2(p-2)}}$, we obtain that I_1 has an infinite number of solutions when

$$\lambda_n^{\frac{2p-(p-2)N}{2(p-2)}} > n^{\frac{1}{1-2r}}. \tag{5.4}$$

In particular, if $q \leq \mu$, (5.4) becomes

$$\lambda_n^{\frac{2p-(p-2)N}{2(p-2)}} > n^2. \blacksquare$$

Proof of Theorem 1.5. The proof follows arguing as above by Lemmas 4.4, 4.5, 4.9 and 4.11.■

Proof of Corollary 1.6. The proof follows by Lemmas 4.3 (respectively 4.4), 4.5, 4.11 and Remark 4.10. ■

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