

Research Article

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General solutions of weakly delayed discrete systems in 3D

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Abstract: Discrete systems $x(k+1) = Ax(k) + Bx(k-m)$, $k = 0, 1, \dots$ are analyzed, where m is a fixed positive integer, A, B are constant 3 by 3 matrices and $x : \{-m, -m+1, \dots\} \rightarrow \mathbb{R}^3$. Assuming that the system is weakly delayed, its general solution is constructed for every case of the Jordan form of the matrix A . It is shown that, for $k \geq 3m$ or for $k \geq 2m$, these formulas reduce to simple forms depending on only the three independent parameters generated by the initial values. Formulas connecting these parameters with the initial ones are found. The results are illustrated by examples. Open problems for future research are discussed, and comparisons are given with the previous results.

Keywords: general solution, discrete system, delay, weakly delayed system, independent initial values

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1 Introduction

The article deals with three-dimensional discrete systems of weakly delayed equations. Weakly delayed systems form a class of systems such that their solutions have a special “merging” property, meaning that some solutions, being defined by different initial data, continue, after an interval, as a single solution. In the case considered, the set of all initial data splits into three mutually different subsets with the initial data of each subset generating solutions that merge into a single one after passing an interval. Then, it is possible to find simple formulas for general solutions, which is the aim of the present article.

Weakly delayed discrete planar systems are investigated in [15], where, using $\mathcal{Z}$ -transform, all possible cases of such systems are considered. This approach was used in [16] as well to study planar weakly delayed systems with multiple delays. Unfortunately, this method is not so well applicable if the number of equations is greater than two because the computations of exact inverses of functional matrices with dimensions greater than two, being cumbersome, complicate further analysis. For a three-dimensional discrete system, the present article develops a method that transforms discrete systems into high-dimensional nondelayed ones, overcoming the aforementioned obstacle. Weakly delayed planar differential systems are considered in a recent article [12]. Some phenomena are mathematically modeled by a class of systems called finite-dimension reducible systems (as a subset of weakly delayed systems). We refer to [7–10], where a mathematical model

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of endocrine regulation is considered, or to [6,22,26,27], where a model of the Mach number dynamics concerning the time-optimal control of a high-speed closed-circuit wind tunnel is discussed. The aforementioned applications serve as an additional motivation to study further properties of solutions of WD-systems.

1.1 Weakly delayed systems

Throughout the article, the following notation is used: Θ is the zero 3 by 3 matrix, E is the unit 3 by 3 matrix, I is a $3(m+1)$ by $3(m+1)$ unit matrix, θ is the zero 3 by 1 vector, m is a fixed positive integer, and $\mathbb{Z}_{q_1}^{q_2} = \{q_1, q_1 + 1, \dots, q_2\}$, where q_1 and q_2 are integers, $q_1 \leq q_2$. Similarly, a set $\mathbb{Z}_{q_1}^\infty$ is defined. We also denote by Θ_s an s by s zero matrix if $s \neq 3$, by θ^* a zero $(3m+3)$ by 1 vector, and by I_ℓ an ℓ by ℓ unit matrix if $\ell \neq 3$, $\ell \neq 3(m+1)$. As customary, by $\text{rank } \mathcal{A}^*$, we denote the rank of a matrix $\mathcal{A}^*$ of arbitrary size. If $\mathcal{A}^*$ is a square matrix and has an eigenvalue λ , then, whenever it is reasonable, we will denote by $m_a(\mathcal{A}^*, \lambda)$ and by $m_g(\mathcal{A}^*, \lambda)$ its algebraic multiplicity and geometric multiplicity, respectively.

In this article, we investigate discrete systems with constant coefficients and with a single delay

$$x(k+1) = Ax(k) + Bx(k-m), \quad k \in \mathbb{Z}_0^\infty, \quad (1)$$

where $x: \mathbb{Z}_{-m}^\infty \rightarrow \mathbb{R}^3$ is a vector of unknown variables. The positive fixed integer m in (1) is a delay, $A = \{a_{ij}\}_{i,j=1}^3$ and $B = \{b_{ij}\}_{i,j=1}^3$ are constant 3 by 3 matrices such that $\det A \neq 0$, $B \neq \Theta$, and $x: \mathbb{Z}_{-m}^\infty \rightarrow \mathbb{R}^3$. For a fixed system of $(m+1)$ vectors $x_l = (x_{l1}, x_{l2}, x_{l3})^T \in \mathbb{R}^3$, where $l \in \mathbb{Z}_{-m}^0$ and the symbol T stands for the transpose, an initial problem to (1) is given by the relation

$$x(k) = x_k, \quad k \in \mathbb{Z}_{-m}^0. \quad (2)$$

The initial problem (1), (2) defines a unique solution of (1) on $\mathbb{Z}_{-m}^\infty$, i.e., a unique function $x: \mathbb{Z}_{-m}^\infty \rightarrow \mathbb{R}^3$ satisfying (1) and (2).

The characteristic equation to (1),

$$D(\lambda) = \det(A + \lambda^{-m}B - \lambda E) = 0 \quad (3)$$

is derived in the usual way if we look for a solution to (1) in the form $x(k) = \xi \lambda^k$, where ξ is a 3 by 1 nonzero vector and a constant $\lambda \in \mathbb{C} \setminus \{0\}$. Definition 1 and Lemma 1 are much the same as in [15].

Definition 1. System (1) is called weakly delayed if, for every $\lambda \in \mathbb{C} \setminus \{0\}$,

$$D(\lambda) = \det(A - \lambda E). \quad (4)$$

In the sequel, rather than repeating the term weakly delayed system, we will use the abbreviation WD-system, etc. An important property formulated by the below lemma enables regular transformations of WD-systems.

Lemma 1. If the system (1) is a WD-system, then its arbitrary transformation $x(k) = Sy(k)$ with a 3 by 3 nonsingular matrix S leads again to a WD-system with respect to a 3 by 1 new unknown vector $y(k)$.

1.2 Problem under consideration

The article treats WD-systems (1) solving the problem of finding sharp formulas for the general solutions that solely depend on three parameters when a transient interval has been passed. The problem is solved completely, for every possible Jordan form of the matrix A . Such problem is new and, for its solution, an original approach is suggested consisting of several steps described in the following part.

1.3 Structure of the article

After the general solutions are constructed for all possible WD-systems (1), Section 1.4 introduces some preliminaries. Here, the general coefficient condition for a system (1) to be a WD-system is given with some properties of the matrix B listed. Specific coefficient criteria are deduced in Section 2 characterizing WD-systems, if the system (1) is transformed into a new WD-system with Jordan form of the matrix of nondelayed terms. Also, from this section, without loss of generality, we assume that A in (1) is given in its Jordan form. In Section 3, the initial system with delay is transformed into a high-dimensional nondelayed associated one and some necessary connections between matrices A , B and the matrix of the aforementioned high-dimensional system are proved. The Jordan forms of this matrix, along with their powers, are found in Section 4 by Weyr's algorithm. These are useful to express explicitly the general solutions of the associated nondelayed system. The results of this part are used in Section 5 where explicit formulas solving problem (1), (2) are derived. Some of the solutions of WD-systems, although defined by different initial values, continue (if $k \geq 3m$ or $k \geq 2m$) as a single solution. This "merging" phenomenon, being a characteristic property of WD-systems, is discussed in Section 6 where new independent initial values are defined as suitable linear combinations of old initial values. By formulas connecting old and new initial values, it is easy to solve the problem when different sets of old initial values define solutions which will be merged into a single one. The results are illustrated by examples in Section 7. Relevant mathematical models are, among others, mentioned in Section 8, with a relation being discussed between the so-called finite-dimension reducible differential delayed systems and WD-systems. Some comments and suggestions are added in this section as well for further research.

1.4 Coefficient criterion for WD-systems and properties of matrix B

Here, we formulate a general coefficient criterion indicating when system (1) is a WD-system. It directly follows from a detailed analysis of equation (4) (we refer to [34]).

Theorem 1. *System (1) is a WD-system if and only if*

$$b_{11} + b_{22} + b_{33} = 0, \quad (5)$$

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = 0, \quad (6)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0, \quad (7)$$

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} = 0, \quad (8)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0, \quad (9)$$

$$\begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ a_{32} & a_{33} \end{vmatrix} = 0. \quad (10)$$

The following properties are obvious. If (5), (6), and (8) hold, then all eigenvalues of matrix B equal zero. If (1) is a WD-system, then B is a nilpotent matrix. System (1) is a WD-system if and only if B is a nilpotent matrix and (7), (9), and (10) hold.

2 Specific criteria for WD-systems

In this section, we formulate specific criteria for WD-systems derived from Theorem 1. These will depend on the Jordan forms of the matrix A . Denote by S the nonsingular matrix that transforms A into its Jordan form $A_J = S^{-1}AS$, where A_J has the seven possible forms (denoted below by Λ_i , $i = 1, \dots, 7$), depending on the roots of the characteristic equation (3), that is, on the roots of the equation:

$$\det(A - \lambda E) = 0. \quad (11)$$

Recall that all eigenvalues of A are (due to $\det A \neq 0$) nonzero. The transformed systems

$$x^*(k) = A_J x^*(k) + S^{-1} B S x^*(k - m), \quad (12)$$

that can be obtained from (1) by transformation

$$x(k) = S x^*(k), \quad (13)$$

are, by Lemma 1, again WD-systems. The initial data for (12) derived from (2) are

$$x^*(k) = S^{-1} x(k) S, \quad k \in \mathbb{Z}_{-m}^0. \quad (14)$$

Remark 1. Now, in the sequel, without loss of generality, we can always assume that the matrix A in (1) is written in one of the possible Jordan forms, i.e.,

$$A \equiv A_J = \Lambda_i, \quad i = 1, \dots, 7,$$

where Λ_i are specified in Section 2.1.

2.1 Jordan canonical forms of A and their powers

In this section, we recall all possible Jordan forms of matrix A and give formulas for their arbitrary powers as they will be needed later on.

If (11) has real distinct roots $\lambda_1, \lambda_2, \lambda_3$, then

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{and} \quad \Lambda_1^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}, \quad n \geq 1. \quad (15)$$

If (11) has two distinct roots – a single root λ_1 and a two-fold real root λ_2 , then

$$\Lambda_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad \Lambda_2^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_2^n \end{pmatrix}, \quad n \geq 1 \quad (16)$$

if $m_g(A, \lambda_2) = 2$, and

$$\Lambda_3 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad \Lambda_3^n = \begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & n\lambda_2^{n-1} \\ 0 & 0 & \lambda_2^n \end{pmatrix}, \quad n \geq 1 \quad (17)$$

if $m_g(A, \lambda_2) = 1$. In the case of one triple real root λ , the following forms are possible:

$$\Lambda_4 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \Lambda_4^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & \lambda^n \end{pmatrix}, \quad n \geq 1 \quad (18)$$

if $m_g(A, \lambda) = 3$,

$$\Lambda_5 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \Lambda_5^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}, \quad n \geq 1 \quad (19)$$

if $m_g(A, \lambda) = 2$,

$$\Lambda_6 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \Lambda_6^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}, \quad n \geq 1 \quad (20)$$

if $m_g(A, \lambda) = 1$. Finally, if one root is real and two roots are complex conjugate having the form $p \pm iq$, $q \neq 0$, then

$$\Lambda_7 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & p & q \\ 0 & -q & p \end{pmatrix} \quad \text{and} \quad \Lambda_7^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \operatorname{Re}(p + iq)^n & \operatorname{Im}(p + iq)^n \\ 0 & -\operatorname{Im}(p + iq)^n & \operatorname{Re}(p + iq)^n \end{pmatrix}, \quad n \geq 1. \quad (21)$$

It is easy to see that

$$\operatorname{Re}(p + iq)^n = \sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s \binom{n}{2s} p^{n-2s} q^{2s}, \quad \operatorname{Im}(p + iq)^n = \sum_{s=0}^{\lfloor n/2 \rfloor} (-1)^s \binom{n}{2s+1} p^{n-2s-1} q^{2s+1},$$

where $\lfloor \cdot \rfloor$ is the floor function and, for the whole numbers n, ℓ , we have

$$\binom{n}{\ell} = \begin{cases} \frac{n!}{\ell!(n-\ell)!} & \text{if } n \geq \ell \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If the complex conjugate roots are given in the exponential form $re^{i\varphi}$ and $re^{-i\varphi}$, where $r > 0$ and $\varphi \in (0, \pi)$, then

$$\begin{aligned} \operatorname{Re}\lambda_2^n &= \operatorname{Re}(re^{i\varphi})^n = \operatorname{Re} r^n e^{ni\varphi} = r^n \cos n\varphi, \\ \operatorname{Im}\lambda_2^n &= \operatorname{Im}(re^{i\varphi})^n = \operatorname{Im} r^n e^{ni\varphi} = r^n \sin n\varphi, \\ \operatorname{Re}\lambda_3^n &= \operatorname{Re}\lambda_2^n, \\ \operatorname{Im}\lambda_3^n &= -\operatorname{Im}\lambda_2^n \end{aligned}$$

and

$$\Lambda_7^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & r^n \cos n\varphi & r^n \sin n\varphi \\ 0 & -r^n \sin n\varphi & r^n \cos n\varphi \end{pmatrix}, \quad n \geq 1. \quad (22)$$

2.2 Specific criteria

The following formulated criteria are consequences of conditions (5)–(10) in Theorem 1. Their proofs can be done by computing the relevant determinants and analyzing the arising expressions.

2.2.1 Criterion for WD-systems in case (15)

Consider system (1) with the matrix $A = \Lambda_1$, i.e.,

$$x(k+1) = \Lambda_1 x(k) + Bx(k-m). \quad (23)$$

Theorem 2. *System (23) is a WD-system if and only if*

$$b_{11} = b_{22} = b_{33} = 0, \quad (24)$$

$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} = 0, \quad (25)$$

$$b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32} = 0, \quad (26)$$

$$\lambda_3 b_{12}b_{21} + \lambda_2 b_{13}b_{31} + \lambda_1 b_{23}b_{32} = 0. \quad (27)$$

2.2.2 Criterion for WD-systems in case (16)

Consider system (1) with the matrix $A = \Lambda_2$, i.e.,

$$x(k+1) = \Lambda_2 x(k) + Bx(k-m). \quad (28)$$

Theorem 3. *System (28) is a WD-system if and only if*

$$b_{11} = 0, \quad (29)$$

$$b_{22} + b_{33} = 0, \quad (30)$$

$$b_{12}b_{21} + b_{13}b_{31} = 0, \quad (31)$$

$$b_{22}b_{33} - b_{23}b_{32} = 0, \quad (32)$$

$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \quad (33)$$

2.2.3 Criterion for WD-systems in case (17)

Consider system (1) with the matrix $A = \Lambda_3$, i.e.,

$$x(k+1) = \Lambda_3 x(k) + Bx(k-m). \quad (34)$$

Theorem 4. *System (34) is a WD-system if and only if*

$$b_{11} = 0, \quad (35)$$

$$b_{22} + b_{33} = 0, \quad (36)$$

$$b_{32} = 0, \quad (37)$$

$$b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} = 0, \quad (38)$$

$$(\lambda_1 - \lambda_2)b_{22}b_{33} + b_{12}b_{31} = 0, \quad (39)$$

$$b_{12}b_{23}b_{31} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \quad (40)$$

2.2.4 Criterion for WD-systems in case (18)

Consider system (1) with the matrix $A = \Lambda_4$, i.e.,

$$x(k+1) = \Lambda_4 x(k) + Bx(k-m). \quad (41)$$

Theorem 5. *System (41) is a WD-system if and only if*

$$b_{11} + b_{22} + b_{33} = 0, \quad (42)$$

$$b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} - b_{23}b_{32} = 0, \quad (43)$$

$$b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32} = 0. \quad (44)$$

2.2.5 Criterion for WD-systems in case (19)

Consider system (1) with the matrix $A = \Lambda_5$, i.e.,

$$x(k+1) = \Lambda_5 x(k) + Bx(k-m). \quad (45)$$

Theorem 6. *System (45) is a WD-system if and only if*

$$b_{11} + b_{22} + b_{33} = 0, \quad (46)$$

$$b_{32} = 0, \quad (47)$$

$$b_{12}b_{31} = 0, \quad (48)$$

$$b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} = 0, \quad (49)$$

$$b_{11}b_{22}b_{33} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \quad (50)$$

2.3 Criterion for WD-systems in case (20)

Consider system (1) with the matrix $A = \Lambda_6$, i.e.,

$$x(k+1) = \Lambda_6 x(k) + Bx(k-m). \quad (51)$$

Theorem 7. *System (51) is a WD-system if and only if*

$$b_{11} + b_{22} + b_{33} = 0, \quad (52)$$

$$b_{21} + b_{32} = 0, \quad (53)$$

$$b_{31} = 0, \quad (54)$$

$$b_{21}b_{33} + b_{11}b_{32} = 0, \quad (55)$$

$$b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21} - b_{23}b_{32} = 0, \quad (56)$$

$$b_{11}b_{22}b_{33} + b_{13}b_{21}b_{32} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32} = 0. \quad (57)$$

2.3.1 Criterion for WD-systems in case (21)

Consider system (1) with the matrix $A = \Lambda_7$, i.e.,

$$x(k+1) = \Lambda_7 x(k) + Bx(k-m). \quad (58)$$

Theorem 8. *System (58) is a WD-system if and only if*

$$b_{11} = 0, \quad (59)$$

$$b_{22} + b_{33} = 0, \quad (60)$$

$$b_{23} - b_{32} = 0, \quad (61)$$

$$b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} - b_{23}b_{32} = 0, \quad (62)$$

$$(\lambda - p)(b_{12}b_{21} + b_{13}b_{31}) + q(b_{12}b_{31} - b_{13}b_{21}) = 0, \quad (63)$$

$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} = 0. \quad (64)$$

3 Associated nondelayed system

In this section, we will transform system (1) into a higher dimensional nondelayed system to find its general solution. Define new dependent functions $y_i(k)$, $i = 1, \dots, 3(m+1)$ by the formulas

$$y_s(k) = x_s(k), \quad (65)$$

$$y_{s+3}(k) = x_s(k-1),$$

$$y_{s+6}(k) = x_s(k-2),$$

$$\dots$$

$$y_{s+3m}(k) = x_s(k-m),$$

where $s = 1, 2, 3$. It is easy to see that system (1) is transformed into a higher dimensional system

$$y(k+1) = \mathcal{A}y(k), \quad k \in \mathbb{Z}_0^\infty, \quad (66)$$

where $y = (y_1, \dots, y_{3(m+1)})^T$ and

$$\mathcal{A} = \begin{pmatrix} A & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} \quad (67)$$

is a $3(m+1)$ by $3(m+1)$ matrix. The initial values for the system (66), in terms of initial values (2), are

$$y(0) = y_0 = (x_0^T, \dots, x_{-m}^T)^T. \quad (68)$$

Let us transform system (66) using a suitable transformation

$$y(k) = \mathcal{T}w(k), \quad (69)$$

where $\mathcal{T}$ is a regular transition $3(m+1)$ by $3(m+1)$ matrix and $w(k)$ is a new dependent $3(m+1)$ -dimensional vector, into a system with a matrix of the Jordan form (possible Jordan forms will be derived in Section 4). We obtain $\mathcal{T}w(k+1) = \mathcal{A}\mathcal{T}w(k)$ or

$$w(k+1) = \mathcal{G}w(k), \quad k \in \mathbb{Z}_0^\infty, \quad (70)$$

where

$$\mathcal{G} = \mathcal{T}^{-1}\mathcal{A}\mathcal{T}. \quad (71)$$

The initial data for system (70) are, as it follows from (68), (69),

$$w(0) = \mathcal{T}^{-1}y(0) = \mathcal{T}^{-1}(x_0^T, \dots, x_{-m}^T)^T. \quad (72)$$

The solution of initial problems (70) and (72) is given by formula

$$w(k) = \mathcal{G}^k w(0), \quad k \in \mathbb{Z}_1^\infty \quad (73)$$

and the solution of initial problems (66) and (68) is

$$y(k) = \mathcal{T}\mathcal{G}^k w(0), \quad k \in \mathbb{Z}_1^\infty. \quad (74)$$

If the matrix A in (67) is in its Jordan form, that is, $A = \Lambda_i$, $i \in \{1, \dots, 7\}$, we will denote the matrix $\mathcal{A}$ as $\mathcal{A}_i$, $i \in \{1, \dots, 7\}$, that is,

$$\mathcal{A}_i = \begin{pmatrix} \Lambda_i & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix}. \quad (75)$$

3.1 Properties of matrices used

Before studying the possible Jordan forms of the matrix $\mathcal{G}$ in (69), we will clarify some properties of the matrices used.

3.1.1 Geometric multiplicities of identical eigenvalues of $\mathcal{A}$ and $\mathcal{G}$

Recall some obvious facts. Eigenvalues of matrices $\mathcal{A}$ and $\mathcal{G}$ are identical. Considering system (66), suppose that an eigenvalue λ of the matrix $\mathcal{A}$ has a geometric multiplicity of $m_g(\mathcal{A}, \lambda)$. Then, also the eigenvalue λ of the matrix $\mathcal{G}$ in (70) has the same geometric multiplicity, i.e., $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}, \lambda)$.

3.1.2 Relationship between the eigenvalues of matrices A , B , and $\mathcal{A}$

Theorem 9. *Let (1) be a WD system. Then, the set of all the eigenvalues of the matrix $\mathcal{A}$ equals the union of all the eigenvalues of the matrix A with the remaining $3m$ eigenvalues of $\mathcal{A}$ being zero.*

Proof. Let

$$\Delta(\mu) := \det(\mathcal{A} - \mu I) = \begin{vmatrix} A - \mu E & \Theta & \Theta & \dots & \Theta & \Theta & B \\ E & -\mu E & \Theta & \dots & \Theta & \Theta & \Theta \\ \Theta & E & -\mu E & \dots & \Theta & \Theta & \Theta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Theta & \Theta & \Theta & \dots & E & -\mu E & \Theta \\ \Theta & \Theta & \Theta & \dots & \Theta & E & -\mu E \end{vmatrix}, \quad (76)$$

where $\mu \in \mathbb{R}$. Computing this determinant, we obtain

$$\Delta(\mu) = (-1)^m \det(\mu^m(A - \mu E) + B). \quad (77)$$

Consider equation

$$\Delta(\mu) = 0. \quad (78)$$

As it follows from (76), equation (78) has exactly $3(m+1)$ roots (every s -multiple root being counted s -times). If there exists a nonzero root $\mu = \mu^*$ of (78), then, from (77),

$$\begin{aligned} \Delta(\mu^*) &= (-1)^m \det((\mu^*)^m(A - \mu^*E) + B) \\ &= (-1)^m (\mu^*)^m \det(A - \mu^*E + (\mu^*)^{-m}B) = 0 \end{aligned} \quad (79)$$

and nonzero roots satisfy $\det(A - \mu^*E + (\mu^*)^{-m}B) = 0$. By (3) and (4), the latter equation is equivalent with $\det(A - \mu^*E) = 0$. Then, the nonzero roots of equation (79) and, consequently, of the equation (77), coincide with the eigenvalues of the matrix A . Obviously, the remaining roots of equation (77) are zeros. $\square$

As a consequence of the conclusion of Theorem 9, one can expect that the respective geometric multiplicities of eigenvalues of the matrices Λ_i , $i = 2, 4, 5$ will equal those of the same eigenvalues of the matrix $\mathcal{A}_i$. The following example shows that, in the general case, this is not true.

Example 1. Consider weakly delayed system (41) with $m = 1$ and with matrices

$$A = \Lambda_4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & -1 \\ -3 & -2 & 2 \\ -3 & -2 & -1 \end{pmatrix}.$$

Then, $m_g(\Lambda_4, 2) = 3$, the corresponding matrix $\mathcal{A}_4$ is

$$\mathcal{A}_4 = \begin{pmatrix} 2 & 0 & 0 & 3 & 2 & -1 \\ 0 & 2 & 0 & -3 & -2 & 2 \\ 0 & 0 & 2 & -3 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and $m_g(\mathcal{A}_4, 2) = 1 \neq m_g(\Lambda_4, 2)$.

Nevertheless, contrary to the property described in Example 1, the following theorem can be proved.

Theorem 10. *The geometric multiplicity of the zero eigenvalue of the matrix $\mathcal{A}$ equals the geometric multiplicity of the zero eigenvalue of the matrix B .*

Proof. Let us find the geometric multiplicity of the zero eigenvalue of the matrix $\mathcal{A}$. That is, we look for the maximal number of linearly independent eigenvectors $\xi = (\xi_1, \dots, \xi_{3(m+1)})^T$ of the matrix $\mathcal{A}$. Consider the system

$$\mathcal{A}\xi = \theta^*, \quad (80)$$

where $\mathcal{A}$ is given by (67). Immediately from (80), we obtain $\xi_1 = \dots = \xi_{3m} = 0$ and, therefore,

$$\begin{aligned} b_{11}\xi_{3m+1} + b_{12}\xi_{3m+2} + b_{13}\xi_{3m+3} &= 0, \\ b_{21}\xi_{3m+1} + b_{22}\xi_{3m+2} + b_{23}\xi_{3m+3} &= 0, \\ b_{31}\xi_{3m+1} + b_{32}\xi_{3m+2} + b_{33}\xi_{3m+3} &= 0. \end{aligned} \quad (81)$$

Now it is clear that the number of linearly independent eigenvectors of matrix $\mathcal{A}$ equals the number of linearly independent eigenvectors of matrix B since system (81) is the same as a system used to determine the coordinates of the eigenvectors of matrix B . $\square$

4 Jordan forms of the associated-system matrix

When using formula (73), it is necessary to compute the powers of the matrix $\mathcal{G}$. Below we assume that the matrix $\mathcal{T}$ in (71) is such that $\mathcal{G} = \mathcal{T}^{-1}\mathcal{A}\mathcal{T}$ takes its Jordan form. Note that such the transient matrix $\mathcal{T}$ can be found by a standard procedure as $\mathcal{T} = (v_1, v_2, \dots, v_{3m+3})$, where v_i , $i = 1, 2, \dots, 3m + 3$ are eigenvectors and generalized eigenvectors associated to all eigenvalues of the matrix $\mathcal{A}$. Due to Theorems 9 and 10 and Weyr's algorithm described, it is possible to exactly find the Jordan forms of $\mathcal{A}$. When Weyr's algorithm is applied, the coefficient criteria characterizing WD-systems formulated in Theorems 2–8 are crucial for determining the ranks of auxiliary matrices (in Sections 4.3–4.9).

The results of the computations depend on the geometric multiplicity of the zero eigenvalue of matrix B and, in some cases, on some auxiliary matrices B_i^* , where $i = 2, 3, 5, 6$, which will be introduced if considering cases $\mathcal{A} = \mathcal{A}_i$. The solution of problems (66) and (68), as can be seen from (69), (70), (72), and (73), is given by (74), that is, $y(k) = \mathcal{T}w(k) = \mathcal{T}\mathcal{G}^k w(0)$, $k \in \mathbb{Z}_1^\infty$.

Next we give formulas for powers of the relevant matrix $\mathcal{G}$ for all previously considered Jordan forms of the matrix A in $\mathcal{A}$. The forms of the transition matrices depend on matrices $\mathcal{A} = \mathcal{A}_i$, $i = 1, \dots, 7$, on matrices B and, if $i = 2, 3, 5, 6$, on the aforementioned matrices B_i^* . Therefore, to differentiate all possible cases, we will use the notation

$$\mathcal{T}_i(m_g(B, 0)) \quad \text{if } i = 1, 4, 7$$

and

$$\mathcal{T}_i(m_g(B, 0), m_g(B^*, 0)) \quad \text{if } i = 2, 3, 5, 6.$$

Similarly, instead of $\mathcal{G}$, the relevant Jordan forms of $\mathcal{A}$ will be denoted by $\mathcal{G}_i(m_g(B, 0))$ if $i = 1, 4, 7$ and $\mathcal{G}_i(m_g(B, 0), m_g(B^*, 0))$ if $i = 2, 3, 4, 5$. The following statements use the powers of matrices Λ_i , $i = 1, \dots, 7$, computed in Section 2.1, formulas (15)–(22).

4.1 Jordan forms of matrices by Weyr's algorithm

For each high-dimensional matrix in Sections 4.3–4.9, its Jordan form will be established using Weyr's algorithm [4,11,23,35]. We will use only the part of this algorithm described below (following the explanation given in [23]). Let D be an ℓ by ℓ real matrix. Denote $h = \text{rank } D$. Let v be the matrix nullity, computed as $v = \ell - h$. Assuming that λ is an eigenvalue of an ℓ by ℓ real matrix C with its algebraic multiplicity $m_a(C, \lambda) = r$, consider the sequence of matrices

$$(C - \lambda I_\ell)^0 = I_\ell, \quad (C - \lambda I_\ell)^1 = C - \lambda I_\ell, \quad (C - \lambda I_\ell)^2, \quad \dots, \quad (C - \lambda I_\ell)^p, \quad (82)$$

their ranks

$$h_0 > h_1 > h_2 > \dots > h_p,$$

and nullities

$$0 = v_0 < v_1 < v_2 < \dots < v_p = r, \quad (83)$$

where the number p in (82) and (83) is determined from the equation $h_p = \ell - r$. For the characteristic numbers $\sigma_1, \sigma_2, \dots, \sigma_p$ of the matrix C , given as

$$\sigma_1 = v_1 - v_0, \quad \sigma_2 = v_2 - v_1, \dots, \sigma_p = v_p - v_{p-1},$$

we have $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p$. There exists a system of r , $r = \sigma_1 + \sigma_2 + \dots + \sigma_p$, linearly independent eigenvectors and generalized eigenvectors of the matrix C ,

$$\begin{aligned} v_{s1}, \dots, v_{s\sigma_p}, & \quad s = 1, 2, \dots, p-2, p-1, p, \\ v_{s\sigma_p+1}, \dots, v_{s\sigma_{p-1}}, & \quad s = 1, 2, \dots, p-2, p-1, \\ v_{s\sigma_{p-1}+1}, \dots, v_{s\sigma_{p-2}}, & \quad s = 1, 2, \dots, p-2, \\ \dots & \quad \dots \\ v_{s\sigma_3+1}, \dots, v_{s\sigma_2}, & \quad s = 1, 2, \\ v_{s\sigma_2+1}, \dots, v_{s\sigma_1}, & \quad s = 1, \end{aligned}$$

associated with the eigenvalue λ . A table with p rows and σ_1 columns of the above vectors can be constructed, see Table 1. The first row is formed by all linearly independent eigenvectors of C and every eigenvector in a

Table 1: The system of eigenvectors and generalized eigenvectors to an eigenvalue of λ

v_{11}	v_{12}	$\dots$	$v_{1\sigma_p}$	v_{1,σ_p+1}	$\dots$	$v_{1,\sigma_{p-1}}$	$\dots$	$v_{1\sigma_2}$	$\dots$	$v_{1\sigma_1}$
v_{21}	v_{22}	$\dots$	$v_{2\sigma_p}$	v_{2,σ_p+1}	$\dots$	$v_{2,\sigma_{p-1}}$	$\dots$	$v_{2\sigma_2}$		
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
$v_{p-1,1}$	$v_{p-1,2}$	$\dots$	v_{p-1,σ_p}	v_{p-1,σ_p+1}	$\dots$	$v_{p-1,\sigma_{p-1}}$				
v_{p1}	v_{p2}	$\dots$	$v_{p\sigma_p}$							

given column is followed by the related generalized eigenvectors. Note that the number σ_1 is the geometrical multiplicity of the root λ (i.e., the number of linearly independent eigenvectors related to λ). All vectors in Table 1 are assumed to be nonzero and linearly independent.

4.2 Auxiliary matrices and their powers

Below we use some auxiliary matrices. Define an $s \times s$ matrix

$$G_s = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

and a frequently used in the article matrix $G = \text{diag}(G_{2m}, G_m)$. Obviously, for powers of G_s , we have

$$G_s^k = \begin{pmatrix} \overbrace{0 \dots 0}^k & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{if } 1 \leq k \leq s-1 \text{ and } G_s^k = \Theta_s \text{ if } k \geq s.$$

In particular, $G_{2m}^k = \Theta_{2m}$ if $k \geq 2m$ and $G_m^k = \Theta_m$ if $k \geq m$. For the powers of the matrix G , we have $G^k = \text{diag}(G_{2m}^k, G_m^k)$ and $G^k = \Theta_{3m}$ if $k \geq 2m$.

4.3 The form and powers of $\mathcal{G}$ if $\mathcal{A} = \mathcal{A}_1$

4.3.1 The case of $m_g(B, 0) = 1$

In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_1(1)$ in (69) transforming $\mathcal{A}_1$ into its Jordan form $\mathcal{G} = \mathcal{G}_1(1) = \mathcal{T}^{-1}\mathcal{A}_1\mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_1$ for eigenvalue $\lambda = 0$, compute $h_0 = \text{rank } \mathcal{A}_1^0 = \text{rank } I = 3(m+1)$ and

$$h_1 = \text{rank } \mathcal{A}_1^1 = \text{rank} \begin{pmatrix} \Lambda_1 & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = 3m + 2.$$

Then, $v_0 = \ell - h_0 = 3(m+1) - 3(m+1) = 0$ and $v_1 = \ell - h_1 = 3(m+1) - 3m - 2 = 1$. Since $\sigma_1 = v_1 - v_0 = 1$, Table 1 will have only one column taking the same form as in Table 2. Because the eigenvalues λ_i , $i = 1, 2, 3$ of Λ_1 are simple, the associated Jordan form of $\mathcal{A}_1$ is

$$\mathcal{G} = \mathcal{G}_1(1) = \left(\begin{array}{c|ccc} \Lambda_1 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_1^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m}^k & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \quad \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_1^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 3m.$$

4.3.2 The case of $m_g(B, 0) = 2$

In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_1(2)$ in (69) transforming $\mathcal{A}_1$ into its Jordan form $\mathcal{G} = \mathcal{G}_1(2) = \mathcal{T}^{-1}\mathcal{A}_1\mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_1$ for the eigenvalue $\lambda = 0$, we need ranks $h_i = \text{rank } \mathcal{A}_1^i$, $i = 0, \dots, m+1$. We have $h_0 = \text{rank } \mathcal{A}_1^0 = \text{rank } I = 3(m+1)$, and, by elementary line operations,

$$\begin{aligned} h_1 = \text{rank } \mathcal{A}_1 &= \text{rank} \begin{pmatrix} \Lambda_1 & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = \text{rank} \begin{pmatrix} \Theta & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = 3m+1, \\ h_2 = \text{rank } \mathcal{A}_1^2 &= \text{rank} \begin{pmatrix} \Lambda_1^2 & \Theta & \dots & \Theta & B & \Lambda_1 B \\ \Lambda_1 & \Theta & \dots & \Theta & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta & \Theta \end{pmatrix} = \text{rank} \begin{pmatrix} \Theta & \Theta & \dots & \Theta & B & \Theta \\ \Theta & \Theta & \dots & \Theta & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta & \Theta \end{pmatrix} = 3m-1, \\ h_3 = \text{rank } \mathcal{A}_1^3 &= \text{rank} \begin{pmatrix} \Lambda_1^3 & \Theta & \dots & \Theta & \Theta & B & \Lambda_1 B & \Lambda_1^2 B \\ \Lambda_1^2 & \Theta & \dots & \Theta & \Theta & \Theta & B & \Lambda_1 B \\ \Lambda_1 & \Theta & \dots & \Theta & \Theta & \Theta & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta & \Theta & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta & \Theta & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta & \Theta & \Theta & \Theta \end{pmatrix} = \text{rank} \begin{pmatrix} \Theta & \Theta & \dots & \Theta & \Theta & B & \Theta & \Theta \\ \Theta & \Theta & \dots & \Theta & \Theta & \Theta & B & \Theta \\ \Theta & \Theta & \dots & \Theta & \Theta & \Theta & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta & \Theta & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta & \Theta & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta & \Theta & \Theta & \Theta \end{pmatrix} = 3m-3, \end{aligned}$$

and, as the formula for determining the rank h_s , $s = 0, 1, \dots, m$, where $0 \leq s \leq m$ is

$$h_s = \text{rank } \mathcal{A}_1^s = 3(m+1) - 2s,$$

we obtain

Table 2: Reduction of Table 1 if $\mathcal{A} = \mathcal{A}_1$, $m_g(B, 0) = 1$, $\lambda = 0$

v_{11}
 $\dots$
 $v_{3m,1}$

$$h_m = \text{rank } \mathcal{A}_1^m = \text{rank} \begin{pmatrix} \Lambda_1^m & B & \Lambda_1 B & \dots & \Lambda_1^{m-1} B \\ \Lambda_1^{m-1} & \Theta & B & \dots & \Lambda_1^{m-2} B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_1 & \Theta & \Theta & \dots & B \\ E & \Theta & \Theta & \dots & \Theta \end{pmatrix} = \text{rank} \begin{pmatrix} \Theta & B & \Theta & \dots & \Theta \\ \Theta & \Theta & B & \dots & \Theta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \Theta & \dots & B \\ E & \Theta & \Theta & \dots & \Theta \end{pmatrix} = m + 3.$$

To compute the next rank h_{m+1} , it is necessary to determine

$$h_{m+1} = \text{rank } \mathcal{A}_1^{m+1} = \text{rank} \begin{pmatrix} \Lambda_1^{m+1} + B & \Lambda_1 B & \Lambda_1^2 B & \dots & \Lambda_1^m B \\ \Lambda_1^m & B & \Lambda_1 B & \dots & \Lambda_1^{m-1} B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_1^2 & \Theta & \Theta & \dots & \Lambda_1 B \\ \Lambda_1 & \Theta & \Theta & \dots & B \end{pmatrix} = \text{rank} \begin{pmatrix} B & \Theta & \dots & \Theta & \Theta \\ \Theta & B & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & B & \Theta \\ \Lambda_1 & \Theta & \dots & \Theta & B \end{pmatrix}.$$

Let us begin with the determination of an auxiliary rank

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_1 & B \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & \lambda_2 & 0 & b_{21} & 0 & b_{23} \\ 0 & 0 & \lambda_3 & b_{31} & b_{32} & 0 \end{pmatrix}.$$

We perform the following operations on rows, where each number in round brackets refers to the number of a row in the last matrix:

$$(1) - \frac{b_{12}}{\lambda_2}(5) - \frac{b_{13}}{\lambda_3}(6), \quad (2) - \frac{b_{21}}{\lambda_1}(4) - \frac{b_{23}}{\lambda_3}(6), \quad (3) - \frac{b_{31}}{\lambda_1}(4) - \frac{b_{32}}{\lambda_2}(5).$$

Since $\text{rank } B = 1$, all determinants of the second order of B equal zero, and we obtain

$$h_{m+1}^* = \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & \lambda_2 & 0 & b_{21} & 0 & b_{23} \\ 0 & 0 & \lambda_3 & b_{31} & b_{32} & 0 \end{pmatrix} = 3.$$

Therefore, $h_{m+1} = h_{m+1}^* + m - 1 = m + 2$. Then, for the nullities, we derive

$$\begin{aligned} v_0 &= \ell - h_0 = 3(m+1) - 3(m+1) = 0, \\ v_1 &= \ell - h_1 = 3(m+1) - (3m+1) = 2, \\ v_2 &= \ell - h_2 = 3(m+1) - (3m-1) = 4, \\ v_3 &= \ell - h_3 = 3(m+1) - (3m-3) = 6, \\ &\dots \\ v_m &= \ell - h_m = 3(m+1) - (m+3) = 2m, \\ v_{m+1} &= \ell - h_{m+1} = 3(m+1) - (m+2) = 2m+1. \end{aligned}$$

The characteristic numbers are

$$\begin{aligned} \sigma_1 &= v_1 - v_0 = 2, \\ \sigma_2 &= v_2 - v_1 = 2, \\ \sigma_3 &= v_3 - v_2 = 2 \\ &\dots \\ \sigma_m &= v_m - v_{m-1} = 2m - 2(m-1) = 2, \\ \sigma_{m+1} &= v_{m+1} - v_m = 2m+1 - 2m = 1, \end{aligned}$$

and, therefore, Table 1 will have two columns (with the first one consisting of $2m$ vectors and with the second one consisting of m vectors) taking the same form as in Table 3.

Because the roots λ_i , $i = 1, 2, 3$ are simple, the associated Jordan form of $\mathcal{A}_1$ is

$$\mathcal{G} = \mathcal{G}_1(2) = \left(\begin{array}{c|ccc} \Lambda_1 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_1^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_1^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 2m.$$

4.4 The form and powers of $\mathcal{G}$ if $\mathcal{A} = \mathcal{A}_2$

4.4.1 Preliminaries

Consider an auxiliary matrix

$$B_2^* = \begin{pmatrix} (\lambda_1 - \lambda_2)\lambda_2^m & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

By (30)–(33), $\det B_2^* = 0$ and $0 < \text{rank} B_2^* \leq 2$. Below we construct the Jordan form of the matrix $\mathcal{G}$ where, in addition to the submatrix Λ_2 (as one can expect), the submatrix Λ_3 is used as well. The correctness of the forms of $\mathcal{G}$ is proved as a consequence of combinations of ranks of matrices B and B_2^* .

Theorem 11. *If $\text{rank} B_2^* = 2$, then $m_g(\mathcal{A}_2, \lambda_2) = 1$, if $\text{rank} B_2^* = 1$, then $m_g(\mathcal{A}_2, \lambda_2) = 2$.*

Proof. Let us find $m_g(\mathcal{A}_2, \lambda_2)$. That is, we look for the maximal number of linearly independent eigenvectors $\xi = (\xi_1, \xi_2, \dots, \xi_{3(m+1)})^T$ of the matrix $\mathcal{A}_2 - \lambda_2 I$. Consider the system

$$(\mathcal{A}_2 - \lambda_2 I)\xi = \theta^*, \quad (84)$$

where, see (75),

$$\mathcal{A}_2 - \lambda_2 I = \left(\begin{array}{cccc|c} \Lambda_2 - \lambda_2 E & \Theta & \dots & \Theta & B \\ E & -\lambda_2 E & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & -\lambda_2 E \end{array} \right), \quad \Lambda_2 - \lambda_2 E = \begin{pmatrix} \lambda_1 - \lambda_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Table 3: Reduction of Table 1 if $\mathcal{A} = \mathcal{A}_1$, $m_g(B, 0) = 2$, $\lambda = 0$

v_{11}	v_{12}
...	...
v_{m1}	v_{m2}
$v_{m+1,1}$	
...	
$v_{2m,1}$	

We obtain $\xi_i = \lambda_2 \xi_{i+3}$, $i = 1, \dots, 3m$, and

$$\xi_1 = \lambda_2 \xi_4 = \lambda_2^2 \xi_7 = \dots = \lambda_2^m \xi_{3m+1}, \quad (85)$$

$$\xi_2 = \lambda_2 \xi_5 = \lambda_2^2 \xi_8 = \dots = \lambda_2^m \xi_{3m+2}, \quad (86)$$

$$\xi_3 = \lambda_2 \xi_6 = \lambda_2^2 \xi_9 = \dots = \lambda_2^m \xi_{3m+3}. \quad (87)$$

Therefore, taking into account that, by (29), $b_{11} = 0$, from (84) to (87), it follows

$$(\lambda_1 - \lambda_2) \lambda_2^m \xi_{3m+1} + b_{12} \xi_{3m+2} + b_{13} \xi_{3m+3} = 0, \quad (88)$$

$$b_{21} \xi_{3m+1} + b_{22} \xi_{3m+2} + b_{23} \xi_{3m+3} = 0, \quad (89)$$

$$b_{31} \xi_{3m+1} + b_{32} \xi_{3m+2} + b_{33} \xi_{3m+3} = 0, \quad (90)$$

that is, $B_2^*(\xi_{3m+1}, \xi_{3m+2}, \xi_{3m+3})^T = \theta$. Now it is clear that if $\text{rank} B_2^* = 2$, the general solution of system (88)–(90) depends on one arbitrary parameter only, relations (85)–(87) imply that, among the coordinates of the vector ξ , there is only one arbitrary parameter and $m_g(\mathcal{A}_2, \lambda_2) = 1$. If $\text{rank} B_2^* = 1$, the general solution of system (88)–(90) depends on two arbitrary parameters, and relations (85)–(87) imply that, among the coordinates of the vector ξ , there are exactly two arbitrary parameters. Then, $m_g(\mathcal{A}_2, \lambda_2) = 2$. $\square$

4.4.2 The case of $m_g(B, 0) = 1$

• **Assume** $\text{rank} B_2^* = 2$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_2(1, 1)$ in (69) transforming $\mathcal{A}_2$ into its Jordan form $\mathcal{G} = \mathcal{G}_2(1, 1) = \mathcal{T}^{-1} \mathcal{A}_2 \mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_2$ for the eigenvalue $\lambda = 0$, we compute $h_0 = \text{rank } \mathcal{A}_2^0 = \text{rank } I = 3(m+1)$ and, by elementary line operations,

$$h_1 = \text{rank } \mathcal{A}_2 = \text{rank} \begin{pmatrix} \Lambda_2 & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = 3m + 2.$$

Then, $v_0 = \ell - h_0 = 3(m+1) - 3(m+1) = 0$ and $v_1 = \ell - h_1 = 3(m+1) - 3m - 2 = 1$. Since $\sigma_1 = v_1 - v_0 = 1$, Table 1 will have only one column and takes the same form as in Table 2 (in this case for $\mathcal{A} = \mathcal{A}_2$, $m_g(B, 0) = 1$, $\lambda = 0$). By Theorem 11, $m_g(\mathcal{A}_2, \lambda_2) = 1$ and, by a comment in Section 3.1.1, $m_g(\mathcal{A}_2, \lambda_2) = m_g(\mathcal{G}, \lambda_2) = 1$. The aforementioned information is sufficient for the construction of the Jordan form of $\mathcal{G}$, and it is not necessary to apply Weyr's algorithm with $C = \mathcal{A}_2$ for the eigenvalue $\lambda = \lambda_2$. The associated Jordan form of $\mathcal{A}_2$ is given as follows:

$$\mathcal{G} = \mathcal{G}_2(1, 1) = \left(\begin{array}{c|ccc} \Lambda_3 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m} & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m}^k & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \quad \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & \Theta_{3m} & \end{array} \right) \quad \text{if } k \geq 3m.$$

• **Let** $\text{rank } B_2^* = 1$. By Theorem 11, $m_g(\mathcal{A}_2, \lambda_2) = 2$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda_2) = m_g(\mathcal{A}_2, \lambda_2) = 2$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_2(1, 2)$ in (69) transforming $\mathcal{A}_2$ into its Jordan form $\mathcal{G} = \mathcal{G}_2(1, 2) = \mathcal{T}^{-1} \mathcal{A}_2 \mathcal{T}$ by formula (71). The application of Weyr's algorithm with $C = \mathcal{A}_2$ for the eigenvalue $\lambda = 0$ is identical

as in the previous section, where $\text{rank } B_2^* = 2$. This information is sufficient for the construction of the Jordan form $\mathcal{G}$ which is the following:

$$\mathcal{G} = \mathcal{G}_2(1, 2) = \left(\begin{array}{c|ccc} \Lambda_2 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 3m.$$

4.4.3 The case of $m_g(B, 0) = 2$

• **Assume** $\text{rank } B_2^* = 2$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_2(2, 1)$ in (69) transforming $\mathcal{A}_2$ into its Jordan form $\mathcal{G} = \mathcal{G}_2(2, 1) = \mathcal{T}^{-1} \mathcal{A}_2 \mathcal{T}$ by formula (71). By Theorem 11, $m_g(\mathcal{A}_2, \lambda_2) = 1$ and, by Section 3.1.1, $m_g(\mathcal{A}_2, \lambda_2) = m_g(\mathcal{G}, \lambda_2) = 1$. To apply Weyr's algorithm with $C = \mathcal{A}_2$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as in Section 4.3.2. Then, we derive

$$\begin{aligned} h_0 &= \text{rank } \mathcal{A}_2^0 = 3m + 3, \\ &\dots \\ h_m &= \text{rank } \mathcal{A}_2^m = m + 3. \end{aligned}$$

To compute $h_{m+1} = \text{rank } \mathcal{A}_2^{m+1}$, consider an auxiliary rank

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_2 & B \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & \lambda_2 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda_2 & b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

We perform the following operations on rows, where each number in round brackets refers to the number of a row in the last matrix:

$$(1) - \frac{b_{12}}{\lambda_2}(5) - \frac{b_{13}}{\lambda_2}(6), \quad (2) - \frac{b_{21}}{\lambda_1}(4) - \frac{b_{22}}{\lambda_2}(5) - \frac{b_{23}}{\lambda_2}(6), \quad (3) - \frac{b_{31}}{\lambda_1}(4) - \frac{b_{32}}{\lambda_2}(5) - \frac{b_{33}}{\lambda_2}(6).$$

By using formula (30) and the fact that all determinants of the second order of B are zeros, implied by the property $\text{rank } B = 1$, we obtain

$$h_{m+1}^* = \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & \lambda_2 & 0 & b_{21} & 0 & b_{23} \\ 0 & 0 & \lambda_2 & b_{31} & b_{32} & 0 \end{pmatrix} = 3.$$

Therefore, $h_{m+1} = h_{m+1}^* + m - 1 = m + 2$. Then

$$\begin{aligned}
v_0 &= \ell - h_0 = 3(m+1) - 3(m+1) = 0, \\
v_1 &= \ell - h_1 = 3(m+1) - (3m+1) = 2, \\
&\dots \\
v_m &= \ell - h_m = 3(m+1) - (m+3) = 2m, \\
v_{m+1} &= \ell - h_{m+1} = 3(m+1) - (m+2) = 2m+1.
\end{aligned}$$

Since

$$\begin{aligned}
\sigma_1 &= v_1 - v_0 = 2, \\
&\dots \\
\sigma_m &= v_m - v_{m-1} = 2m - 2(m-1) = 2, \\
\sigma_{m+1} &= v_{m+1} - v_m = 2m+1 - 2m = 1,
\end{aligned}$$

Table 1 will have two columns taking the same form as in Table 3 if $\mathcal{A} = \mathcal{A}_2$, $m_g(B, 0) = 2$, $m_g(B_2^*, 0) = 1$ and $\lambda = 0$. This suffices for the construction of the Jordan form $\mathcal{G}$ and the associated Jordan form of $\mathcal{G}$ is

$$\mathcal{G} = \mathcal{G}_2(2, 1) = \left(\begin{array}{c|ccc} \Lambda_3 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 2m.$$

• **Assume** $\text{rank} B_2^* = 1$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_2(2, 2)$ in (69) transforming $\mathcal{A}_2$ into its Jordan form $\mathcal{G} = \mathcal{G}_2(2, 2) = \mathcal{T}^{-1} \mathcal{A}_2 \mathcal{T}$ by formula (71). By Theorem 11, $m_g(\mathcal{A}_2, \lambda_2) = 2$ and, by Section 3.1.1, $m_g(\mathcal{A}_2, \lambda_2) = m_g(\mathcal{G}, \lambda_2) = 2$. To apply Weyr's algorithm with $C = \mathcal{A}_2$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as earlier with the same results. Reduction of Table 1 if $\mathcal{A} = \mathcal{A}_2$, $m_g(B, 0) = 2$, $m_g(B^*, 0) = 2$, $\lambda = 0$ leads to Table 3 and

$$\mathcal{G} = \mathcal{G}_2(2, 2) = \left(\begin{array}{c|ccc} \Lambda_2 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 2m.$$

4.5 The form and powers of $\mathcal{G}$ if $\mathcal{A} = \mathcal{A}_3$

4.5.1 Preliminaries

Consider an auxiliary matrix

$$B_3^* = \begin{pmatrix} (\lambda_1 - \lambda_2)\lambda_2^m & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} + \lambda_2^m \\ b_{31} & 0 & b_{33} \end{pmatrix}.$$

By (39) and (40), we have $\det B_3^* = 0$ and $0 < \text{rank} B_3^* \leq 2$. If $\text{rank} B_3^* = 1$, then $m_g(B_3^*, 0) = 2$ and, if $\text{rank} B_3^* = 2$, then $m_g(B_3^*, 0) = 1$. We will construct the Jordan form of the matrix $\mathcal{G}$ using, except for the submatrix Λ_3 (as one can anticipate), the submatrix Λ_2 as well.

Theorem 12. *If $\text{rank} B_3^* = 2$, then $m_g(\mathcal{A}_3, \lambda_2) = 1$, if $\text{rank} B_3^* = 1$, then $m_g(\mathcal{A}_3, \lambda_2) = 2$.*

Proof. Let us find $m_g(\mathcal{A}_3, \lambda_2)$. That is, we look for the maximal number of linearly independent eigenvectors $\xi = (\xi_1, \xi_2, \dots, \xi_{3(m+1)})^T$ of the matrix $\mathcal{A}_3 - \lambda_2 I$. Consider the system

$$(\mathcal{A}_3 - \lambda_2 I)\xi = \theta^*, \quad (91)$$

where, see (75),

$$\mathcal{A}_3 - \lambda_2 I = \begin{pmatrix} \Lambda_3 - \lambda_2 E & \Theta & \dots & \Theta & B \\ E & -\lambda_2 E & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & -\lambda_2 E \end{pmatrix}, \quad \Lambda_3 - \lambda_2 E = \begin{pmatrix} \lambda_1 - \lambda_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (92)$$

We obtain $\xi_i = \lambda_2 \xi_{i+3}$, $i = 1, \dots, 3m$, and

$$\xi_1 = \lambda_2 \xi_4 = \lambda_2^2 \xi_7 = \dots = \lambda_2^m \xi_{3m+1}, \quad (93)$$

$$\xi_2 = \lambda_2 \xi_5 = \lambda_2^2 \xi_8 = \dots = \lambda_2^m \xi_{3m+2}, \quad (94)$$

$$\xi_3 = \lambda_2 \xi_6 = \lambda_2^2 \xi_9 = \dots = \lambda_2^m \xi_{3m+3}. \quad (95)$$

Therefore, taking into account that, by (35), $b_{11} = 0$ and, by (37), $b_{32} = 0$, from (91) and (92), it follows:

$$(\lambda_1 - \lambda_2)\lambda_2^m \xi_{3m+1} + b_{12}\xi_{3m+2} + b_{13}\xi_{3m+3} = 0, \quad (96)$$

$$b_{21}\xi_{3m+1} + b_{22}\xi_{3m+2} + (b_{23} + \lambda_2^m)\xi_{3m+3} = 0, \quad (97)$$

$$b_{31}\xi_{3m+1} + b_{32}\xi_{3m+2} + b_{33}\xi_{3m+3} = 0, \quad (98)$$

that is, $B_3^*(\xi_{3m+1}, \xi_{3m+2}, \xi_{3m+3})^T = \theta$. Now it is clear that, if $\text{rank} B_3^* = 2$, system (96)–(98) depends on one arbitrary parameter only, relations (93)–(95) imply that among the coordinates of the vector ξ , there is only one arbitrary parameter and $m_g(\mathcal{A}_3, \lambda_2) = 1$. If $\text{rank} B_3^* = 1$, the general solution of system (96)–(98) depends on two arbitrary parameters, and relations (93)–(95) imply that, among the coordinates of the vector ξ , there are only two arbitrary parameters. Then, $m_g(\mathcal{A}_3, \lambda_2) = 2$. $\square$

4.5.2 The case of $m_g(B, 0) = 1$

• **Assume** $\text{rank} B_3^* = 2$. By Theorem 12, we have $m_g(\mathcal{A}_3, \lambda_2) = 1$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda_2) = m_g(\mathcal{A}_3, \lambda_2) = 1$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_3(1, 1)$ in (69) transforming $\mathcal{A}_3$ into its Jordan form $\mathcal{G} = \mathcal{G}_3(1, 1) = \mathcal{T}^{-1}\mathcal{A}_3\mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_3$ for the eigenvalue $\lambda = 0$, we compute $h_0 = \text{rank } \mathcal{A}_3^0 = 3(m+1)$ and

$$h_1 = \text{rank } \mathcal{A}_3 = \text{rank} \begin{pmatrix} \Lambda_3 & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = 3m + 2.$$

Then, $v_0 = \ell - h_0 = 3(m+1) - 3(m+1) = 0$ and $v_1 = \ell - h_1 = 3(m+1) - 3m - 2 = 1$. The aforementioned information is sufficient for the construction of the Jordan form $\mathcal{G}$, and it is not necessary to apply Weyr's algorithm with $C = \mathcal{A}_3$ for the eigenvalue $\lambda = \lambda_2$. The associated Jordan form of $\mathcal{A}_3$ is

$$\mathcal{G} = \mathcal{G}_3(1, 1) = \left(\begin{array}{c|ccc} \Lambda_3 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 3m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 3m.$$

• **Assume** $\text{rank} B_3^* = 1$. By Theorem 12, we have $m_g(\mathcal{A}_3, \lambda_2) = 2$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda_2) = m_g(\mathcal{A}_3, \lambda_2) = 2$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_3(1, 2)$ in (69) transforming $\mathcal{A}_3$ into its Jordan form $\mathcal{G} := \mathcal{G}_3(1, 2) = \mathcal{T}^{-1} \mathcal{A}_3 \mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C := \mathcal{A}_3$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as in Section 4.4.2 with the same results.

Then, the associated Jordan form of $\mathcal{G}$ is

$$\mathcal{G} = \mathcal{G}_3(1, 2) = \left(\begin{array}{c|ccc} \Lambda_2 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 3m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 3m.$$

4.5.3 The case of $m_g(B, 0) = 2$

• **Assume** $\text{rank} B_3^* = 2$. By Theorem 12, we have $m_g(\mathcal{A}_3, \lambda_2) = 1$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda_2) = m_g(\mathcal{A}_3, \lambda_2) = 1$. To apply Weyr's algorithm with $C := \mathcal{A}_3$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as in Section 4.4.3 with much the same results. We will only comment on the crucial detail when it is necessary to compute

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_3 & B \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & 0 & b_{33} & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & \lambda_2 & 1 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda_2 & b_{31} & 0 & b_{33} \end{pmatrix}.$$

By using formulas (36) and the property that all determinants of the second order of B are zero, we conclude that $b_{22} = b_{33} = 0$ and

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_3 & B \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & 0 & b_{23} & 0 & 0 & 0 \\ b_{31} & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & \lambda_2 & 1 & b_{21} & 0 & b_{23} \\ 0 & 0 & \lambda_2 & b_{31} & 0 & 0 \end{pmatrix}.$$

We perform the following operations on the rows, where each number in round brackets refers to the number of a row in the last matrix:

$$(1) - \frac{b_{12}}{\lambda_2}(5) - \left(b_{13} - \frac{b_{12}}{\lambda_2}\right)\frac{1}{\lambda_2}(6), \quad (2) - \frac{b_{21}}{\lambda_1}(4) - \frac{b_{23}}{\lambda_2}(6), \quad (3) - \frac{b_{31}}{\lambda_1}(4),$$

and then

$$h_{m+1}^* = \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & \lambda_2 & 0 & b_{21} & 0 & b_{23} \\ 0 & 0 & \lambda_2 & b_{31} & 0 & 0 \end{pmatrix} = 3.$$

Now we continue as in Section 4.4.3. Reduction of Table 1 if $\mathcal{A} = \mathcal{A}_3$, $m_g(B, 0) = 2$, $m_g(B_3^*, 0) = 1$, $\lambda = 0$ leads to Table 3. Then, there exists a transition matrix $\mathcal{T} = \mathcal{T}_3(2, 1)$ in (69) such that the associated Jordan form of $\mathcal{G} = \mathcal{G}_3(2, 1)$ is

$$\mathcal{G} = \mathcal{G}_3(2, 1) = \left(\begin{array}{c|ccc} \Lambda_3 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_3^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

• **Assume** $\text{rank} B_3^* = 1$. By Theorem 12, we have $m_g(\mathcal{A}_3, \lambda_2) = 2$ and, by Section 3.1.1, $m_g(\mathcal{A}_3, \lambda_2) = m_g(\mathcal{G}, \lambda_2) = 2$. To apply Weyr's algorithm with $C = \mathcal{A}_3$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as mentioned earlier with the same results. Reduction of Table 1 if $\mathcal{A} = \mathcal{A}_3$, $m_g(B, 0) = 2$, $m_g(B^*, 0) = 2$, $\lambda = 0$ leads to Table 3. Then, there exists a transition matrix $\mathcal{T} = \mathcal{T}_3(2, 2)$ in (69) such that the associated Jordan form $\mathcal{G} = \mathcal{G}_3(2, 2)$ of $\mathcal{A}_3$ is

$$\mathcal{G} = \mathcal{G}_3(2, 2) = \left(\begin{array}{c|ccc} \Lambda_2 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_2^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

4.6 The form and powers of $\mathcal{G}$ if $\mathcal{A} = \mathcal{A}_4$

4.6.1 Preliminaries

Here, we explain why, in this case, the matrix of type B_4^* (constructed in much the same way as matrices B_2^* or B_3^* above) coincides with the matrix B . We will construct the Jordan forms of the matrix $\mathcal{G}$ using, except for the submatrix Λ_4 (as one can anticipate), the submatrices Λ_5 and Λ_6 as well.

Theorem 13. If $\text{rank} B = 2$, then $m_g(\mathcal{A}_4, \lambda) = 1$, if $\text{rank} B = 1$, then $m_g(\mathcal{A}_4, \lambda) = 2$

Proof. Let us find $m_g(\mathcal{A}_4, \lambda)$. That is, we will look for the maximal number of linearly independent eigenvectors $\xi = (\xi_1, \xi_2, \dots, \xi_{3(m+1)})^T$ of the matrix $\mathcal{A}_4 - \lambda I$. Consider the system

$$(\mathcal{A}_4 - \lambda I)\xi = \theta^*, \quad (99)$$

where, see (75),

$$\mathcal{A}_4 - \lambda I = \begin{pmatrix} \Lambda_4 - \lambda E & \Theta & \dots & \Theta & B \\ E & -\lambda E & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & -\lambda E \end{pmatrix}, \quad \Lambda_4 - \lambda E = \Theta. \quad (100)$$

Immediately from (99) and (100), we obtain $\xi_i = \lambda \xi_{i+3}$, $i = 1, \dots, 3m$,

$$\xi_1 = \lambda \xi_4 = \lambda^2 \xi_7 = \dots = \lambda^m \xi_{3m+1}, \quad (101)$$

$$\xi_2 = \lambda \xi_5 = \lambda^2 \xi_8 = \dots = \lambda^m \xi_{3m+2}, \quad (102)$$

$$\xi_3 = \lambda \xi_6 = \lambda^2 \xi_9 = \dots = \lambda^m \xi_{3m+3}. \quad (103)$$

and

$$b_{11}\xi_{3m+1} + b_{12}\xi_{3m+2} + b_{13}\xi_{3m+3} = 0, \quad (104)$$

$$b_{21}\xi_{3m+1} + b_{22}\xi_{3m+2} + b_{23}\xi_{3m+3} = 0, \quad (105)$$

$$b_{31}\xi_{3m+1} + b_{32}\xi_{3m+2} + b_{33}\xi_{3m+3} = 0. \quad (106)$$

In the previous sections, the matrix of linear terms of the system similar to (104)–(106) was denoted by B^* . Here, we have $B^* = B_4^* = B$, and $B(\xi_{3m+1}, \xi_{3m+2}, \xi_{3m+3})^T = \theta$.

Now it is clear that, if $\text{rank} B = 2$, the general solution of system (104)–(106) only depends on a single parameter with relations (101)–(103) implying that, among the coordinates of the vector ξ , there is only one arbitrary parameter. Then, $m_g(\mathcal{A}_4, \lambda) = 1$. If $\text{rank} B = 1$, the general solution of system (104)–(106) depends on two arbitrary parameters, and relations (101)–(103) imply that, among the coordinates of the vector ξ , there are just two arbitrary parameters. Then, $m_g(\mathcal{A}_4, \lambda) = 2$. $\square$

4.6.2 The case of $m_g(B, 0) = 1$

In this case, by Theorem 13, we have $m_g(\mathcal{A}_4, \lambda) = 1$. By Section 3.1.1, $m_g(\mathcal{A}_4, \lambda) = m_g(\mathcal{G}, \lambda) = 1$. To apply Weyr's algorithm with $C = \mathcal{A}_4$ for the eigenvalue $\lambda = 0$, we compute $h_0 = \text{rank } \mathcal{A}_4^0 = 3(m+1)$ and

$$h_1 = \text{rank } \mathcal{A}_4 = \text{rank} \begin{pmatrix} \Lambda_4 & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = 3m + 2.$$

Then, $v_0 = \ell - h_0 = 3(m+1) - 3(m+1) = 0$ and $v_1 = \ell - h_1 = 3(m+1) - 3m - 2 = 1$. Since $\sigma_1 = v_1 - v_0 = 1$, Table 1 will have only one column and, for the case considered, takes the same form as in Table 2. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_4(1)$ in (69) such that the associated Jordan form of $\mathcal{G} = \mathcal{G}_4(1)$ satisfying all aforementioned restrictions is

$$\mathcal{G} = \mathcal{G}_4(1) = \left(\begin{array}{c|ccc} \Lambda_6 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \hline \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \hline \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \quad \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \hline \Theta & & & \end{array} \right) \quad \text{if } k \geq 3m.$$

4.6.3 The case of $m_g(B, 0) = 2$

In the case considered, by Theorem 13, we have $m_g(\mathcal{A}_4, \lambda) = 2$. By Section 3.1.1, we have $m_g(\mathcal{A}_4, \lambda) = m_g(\mathcal{G}, \lambda) = 2$. To apply Weyr's algorithm with $C = \mathcal{A}_4$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as in Section 4.3.2. Then,

$$\begin{aligned} h_0 &= \text{rank } \mathcal{A}_4^0 = 3m + 3, \\ &\dots \\ h_m &= \text{rank } \mathcal{A}_4^m = m + 3. \end{aligned}$$

To compute $h_{m+1} = \text{rank } \mathcal{A}_4^{m+1}$, consider an auxiliary rank

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_4 & B \end{pmatrix} = \text{rank} \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\ \lambda & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & \lambda & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda & b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

We perform the following operations on rows, where each number in round brackets refers to the number of a row in the last matrix:

$$\begin{aligned} (1) &- \frac{b_{11}}{\lambda}(4) - \frac{b_{12}}{\lambda}(5) - \frac{b_{13}}{\lambda}(6), \\ (2) &- \frac{b_{21}}{\lambda}(4) - \frac{b_{22}}{\lambda}(5) - \frac{b_{23}}{\lambda}(6), \\ (3) &- \frac{b_{31}}{\lambda}(4) - \frac{b_{32}}{\lambda}(5) - \frac{b_{33}}{\lambda}(6). \end{aligned}$$

By using formula (42) and the property $\text{rank } B = 1$, we derive

$$h_{m+1}^* = \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & \lambda & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda & b_{31} & b_{32} & b_{33} \end{pmatrix} = 3.$$

Therefore, $h_{m+1} = h_{m+1}^* + m - 1 = m + 2$. Then

$$\begin{aligned} v_0 &= \ell - h_0 = 3(m+1) - 3(m+1) = 0, \\ v_1 &= \ell - h_1 = 3(m+1) - (3m+1) = 2, \\ &\dots \\ v_m &= \ell - h_m = 3(m+1) - (m+3) = 2m, \\ v_{m+1} &= \ell - h_{m+1} = 3(m+1) - (m+2) = 2m+1. \end{aligned}$$

Since

$$\begin{aligned} \sigma_1 &= v_1 - v_0 = 2, \\ &\dots \\ \sigma_m &= v_m - v_{m-1} = 2m - 2(m-1) = 2, \\ \sigma_{m+1} &= v_{m+1} - v_m = 2m+1 - 2m = 1, \end{aligned}$$

Table 1 will have only two columns and takes the same form as in Table 3 if $\mathcal{A} = \mathcal{A}_4$, $m_g(B, 0) = 2$, and $\lambda = 0$. This property is sufficient for the construction of the Jordan form $\mathcal{G} = \mathcal{G}_4(2)$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_4(2)$ in (69) such that

$$\mathcal{G} = \mathcal{G}_4(2) = \left(\begin{array}{c|ccc} \Lambda_5 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

4.7 The form and powers of $\mathcal{G}$ if $\mathcal{A} = \mathcal{A}_5$

4.7.1 Preliminaries

Define an auxiliary matrix

$$B_5^* = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} + \lambda^m \\ b_{31} & 0 & b_{33} \end{pmatrix}.$$

Obviously, $\det B_5^* = 0$ and, therefore, $0 \leq \text{rank} B_5^* \leq 2$. In the following, in the Jordan forms of the matrix $\mathcal{G}$ are used except for the submatrix Λ_5 , the submatrices Λ_4 and Λ_6 as well.

Theorem 14. *If $\text{rank} B_5^* = 2$, then $m_g(\mathcal{A}_5, \lambda) = 1$, if $\text{rank} B_5^* = 1$, then $m_g(\mathcal{A}_5, \lambda) = 2$ and, if $\text{rank} B_5^* = 0$, then $m_g(\mathcal{A}_5, \lambda) = 3$.*

Proof. Let us find $m_g(\mathcal{A}_5, \lambda)$. That is, we look for the maximal number of linearly independent eigenvectors $\xi = (\xi_1, \xi_2, \dots, \xi_{3(m+1)})^T$ of the matrix $\mathcal{A}_5 - \lambda I$. Consider the system

$$(\mathcal{A}_5 - \lambda I)\xi = \theta^*, \quad (107)$$

where, see (75),

$$\mathcal{A}_5 - \lambda I = \begin{pmatrix} \Lambda_5 - \lambda E & \Theta & \dots & \Theta & B \\ E & -\lambda E & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & -\lambda E \end{pmatrix}, \quad \Lambda_5 - \lambda E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (108)$$

Immediately from (107), we obtain $\xi_i = \lambda \xi_{i+3}$, $i = 1, \dots, 3m$ and

$$\xi_1 = \lambda \xi_4 = \lambda^2 \xi_7 = \dots = \lambda^m \xi_{3m+1}, \quad (109)$$

$$\xi_2 = \lambda \xi_5 = \lambda^2 \xi_8 = \dots = \lambda^m \xi_{3m+2}, \quad (110)$$

$$\xi_3 = \lambda \xi_6 = \lambda^2 \xi_9 = \dots = \lambda^m \xi_{3m+3}. \quad (111)$$

Therefore, taking into account that, by (47), $b_{32} = 0$, from (107) and (108), it follows

$$b_{11}\xi_{3m+1} + b_{12}\xi_{3m+2} + b_{13}\xi_{3m+3} = 0, \quad (112)$$

$$b_{21}\xi_{3m+1} + b_{22}\xi_{3m+2} + (b_{23} + \lambda^m)\xi_{3m+3} = 0, \quad (113)$$

$$b_{31}\xi_{3m+1} + b_{33}\xi_{3m+3} = 0, \quad (114)$$

that is, $B_5^*(\xi_{3m+1}, \xi_{3m+2}, \xi_{3m+3})^T = \theta$. Now it is clear that if $\text{rank} B_5^* = 2$, the general solution of system (112)–(114) depends on one arbitrary parameter and relations (109)–(111) imply that, among the coordinates of the vector ξ , there is only one arbitrary parameter. Then, $m_g(\mathcal{A}_5, \lambda) = 1$. If $\text{rank} B_5^* = 1$, the general solution of system (112)–(114) depends on two arbitrary parameters, and relations (109)–(111) imply that, among the coordinates of the vector ξ , there are two arbitrary parameters and $m_g(\mathcal{A}_5, \lambda) = 2$ and, if $\text{rank} B_5^* = 0$, the general solution of system (112)–(114) depends on three arbitrary parameters, and relations (109)–(111) imply that, among the coordinates of the vector ξ , there are three arbitrary parameters and $m_g(\mathcal{A}_5, \lambda) = 3$. $\square$

4.7.2 The case of $m_g(B, 0) = 1$

• **Assume** $\text{rank} B_5^* = 2$. Then, by Theorem 14, we have $m_g(\mathcal{A}_5, \lambda) = 1$. By Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_5, \lambda) = 1$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_5(1, 1)$ in (69) transforming $\mathcal{A}_5$ into its Jordan form $\mathcal{G} = \mathcal{G}_5(1, 1) = \mathcal{T}^{-1}\mathcal{A}_5\mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_5$ for the eigenvalue $\lambda = 0$, we compute $h_0 = \text{rank } \mathcal{A}_5^0 = 3(m+1)$ and

$$h_1 = \text{rank } \mathcal{A}_5 = \text{rank} \begin{pmatrix} \Lambda_5 & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = 3m + 2.$$

Then, $v_0 = \ell - h_0 = 3(m+1) - 3(m+1) = 0$ and $v_1 = \ell - h_1 = 3(m+1) - 3m - 2 = 1$. Since $\sigma_1 = v_1 - v_0 = 1$, Table 1 will only have a single column taking the same form as in Table 2 (in this case, for $\mathcal{A} = \mathcal{A}_5$, $m_g(B, 0) = 1$, $\lambda = 0$). The aforementioned information is sufficient for the construction of the Jordan form $\mathcal{G}$, and it is not necessary to apply Weyr's algorithm with $C = \mathcal{A}_5$ for the eigenvalue λ . The associated Jordan form of $\mathcal{A}_5$ is

$$\mathcal{G} = \mathcal{G}_5(1, 1) = \left(\begin{array}{c|ccc} \Lambda_6 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \hline \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m}^k & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & \Theta_{3m} & \end{array} \right) \quad \text{if } k \geq 3m.$$

• **Assume** $\text{rank} B_5^* = 1$. By Theorem 14, we have $m_g(\mathcal{A}_5, \lambda) = 2$. By Section 3.1.1, then $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_5, \lambda) = 2$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_5(1, 2)$ in (69) transforming $\mathcal{A}_5$ into its Jordan form $\mathcal{G} = \mathcal{G}_5(1, 2) = \mathcal{T}^{-1} \mathcal{A}_5 \mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_5$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as in the aforementioned section with much the same results. We get

$$\mathcal{G} = \mathcal{G}_5(1, 2) = \left(\begin{array}{c|ccc} \Lambda_5 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m} & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m}^k & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & \Theta_{3m} & \end{array} \right) \quad \text{if } k \geq 3m.$$

• **Assume** $\text{rank} B_5^* = 0$. By Theorem 14, we have $m_g(\mathcal{A}_5, \lambda) = 3$, and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_5, \lambda) = 3$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_5(1, 3)$ in (69) transforming $\mathcal{A}_5$ into its Jordan form $\mathcal{G} = \mathcal{G}_5(1, 3) = \mathcal{T}^{-1} \mathcal{A}_5 \mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_5$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as earlier with much the same results and

$$\mathcal{G} = \mathcal{G}_5(1, 3) = \left(\begin{array}{c|ccc} \Lambda_4 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m} & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & G_{3m}^k & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & & \\ \Theta & & \Theta_{3m} & \end{array} \right) \quad \text{if } k \geq 3m.$$

4.7.3 The case of $m_g(B, 0) = 2$

• **Assume** $\text{rank} B_5^* = 2$. Then, by Theorem 14, $m_g(\mathcal{A}_5, \lambda) = 1$ and by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_5, \lambda) = 1$. To apply Weyr's algorithm with $C = \mathcal{A}_5$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as in Section 4.3.2. Then

$$\begin{aligned} h_0 &= \text{rank } \mathcal{A}_5^0 = 3m + 3, \\ &\dots \\ h_m &= \text{rank } \mathcal{A}_5^m = m + 3. \end{aligned}$$

To compute $h_{m+1} = \text{rank } \mathcal{A}_5^{m+1}$, consider an auxiliary rank

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_5 & B \end{pmatrix} = \text{rank} \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & 0 & b_{33} & 0 & 0 & 0 \\ \lambda & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & \lambda & 1 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda & b_{31} & 0 & b_{33} \end{pmatrix}.$$

We perform the following operations on rows, where each number in round brackets refers to the number of a row in the last matrix:

$$\begin{aligned} (1) & - \frac{b_{11}}{\lambda}(4) - \frac{b_{12}}{\lambda}(5) - \frac{b_{13} - b_{12}/\lambda}{\lambda}(6), \\ (2) & - \frac{b_{21}}{\lambda}(4) - \frac{b_{22}}{\lambda}(5) - \frac{b_{23} - b_{22}/\lambda}{\lambda}(6), \\ (3) & - \frac{b_{31}}{\lambda}(4) - \frac{b_{33}}{\lambda}(6). \end{aligned}$$

By using the formula (46) and the property $\text{rank } B = 1$, we derive

$$h_{m+1}^* = \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & \lambda & 1 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda & b_{31} & 0 & b_{33} \end{pmatrix} = 3.$$

Therefore, $h_{m+1} = h_{m+1}^* + m - 1 = m + 2$. Then

$$\begin{aligned} v_0 &= \ell - h_0 = 3(m+1) - 3(m+1) = 0, \\ v_1 &= \ell - h_1 = 3(m+1) - (3m+1) = 2, \\ &\dots \\ v_m &= \ell - h_m = 3(m+1) - (m+3) = 2m, \\ v_{m+1} &= \ell - h_{m+1} = 3(m+1) - (m+2) = 2m+1. \end{aligned}$$

Since

$$\begin{aligned} \sigma_1 &= v_1 - v_0 = 2, \\ &\dots \\ \sigma_m &= v_m - v_{m-1} = 2m - 2(m-1) = 2, \\ \sigma_{m+1} &= v_{m+1} - v_m = 2m+1 - 2m = 1, \end{aligned}$$

Table 1 (where $\mathcal{A} = \mathcal{A}_5$, $m_g(B, 0) = 2$, and $\lambda = 0$ is applied) will have two columns taking the same form as in Table 3. This is sufficient for the construction of the Jordan form $\mathcal{G} = \mathcal{G}_5(2, 1)$. There exists a transition matrix $\mathcal{T} = \mathcal{T}_5(2, 1)$ in (69) such that $\mathcal{G} = \mathcal{G}_5(2, 1) = \mathcal{T}^{-1}\mathcal{A}_5\mathcal{T}$ and

$$\mathcal{G} = \mathcal{G}_5(2, 1) = \left(\begin{array}{c|ccc} \Lambda_6 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

• **Assume** $\text{rank} B_5^* = 1$. Then, by Theorem 14, we have $m_g(\mathcal{A}_5, \lambda) = 2$. By Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_5, \lambda) = 2$. To apply Weyr's algorithm with $C = \mathcal{A}_5$ for the zero eigenvalue, we proceed in much the same way as in the previous section to obtain the same conclusion. This information suffices for the construction of the Jordan form $\mathcal{G} = \mathcal{G}_5(2, 2)$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_5(2, 2)$ in (69) such that $\mathcal{G} = \mathcal{G}_5(2, 2) = \mathcal{T}^{-1} \mathcal{A}_5 \mathcal{T}$ and we obtain

$$\mathcal{G} = \mathcal{G}_5(2, 2) = \left(\begin{array}{c|ccc} \Lambda_5 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

• **Assume** $\text{rank} B_5^* = 0$. Then, by Theorem 14, we have $m_g(\mathcal{A}_5, \lambda) = 3$. By Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_5, \lambda) = 3$. To apply Weyr's algorithm with $C = \mathcal{A}_5$ for the zero eigenvalue, we proceed in much the same way as earlier considering the case $\text{rank} B_5^* = 2$ to obtain the same conclusion. This information suffices for the construction of the Jordan form $\mathcal{G} = \mathcal{G}_5(2, 3)$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_5(2, 3)$ in (69) such that $\mathcal{G} = \mathcal{G}_5(2, 3) = \mathcal{T}^{-1} \mathcal{A}_5 \mathcal{T}$ and we obtain

$$\mathcal{G} = \mathcal{G}_5(2, 3) = \left(\begin{array}{c|ccc} \Lambda_4 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

4.8 The form and powers of $\mathcal{G}$ if $\mathcal{A} = \mathcal{A}_6$

4.8.1 Preliminaries

Define an auxiliary matrix

$$B_6^* = \begin{pmatrix} b_{11} & b_{12} + \lambda^m & b_{13} \\ b_{21} & b_{22} & b_{23} + \lambda^m \\ 0 & b_{32} & b_{33} \end{pmatrix}.$$

Obviously, $\det B_6^* = 0$ and, therefore, $0 \leq \text{rank} B_6^* \leq 2$. The Jordan form of the matrix $\mathcal{G}$ involves, except for the submatrix Λ_5 , the submatrices Λ_4 and Λ_6 as well.

Theorem 15. *If $\text{rank} B_6^* = 2$, then $m_g(\mathcal{A}_6, \lambda) = 1$, if $\text{rank} B_6^* = 1$, then $m_g(\mathcal{A}_6, \lambda) = 2$ and, if $\text{rank} B_6^* = 0$, then $m_g(\mathcal{A}_6, \lambda) = 3$.*

Proof. Let us find $m_g(\mathcal{A}_6, \lambda)$. That is, we look for the maximal number of linearly independent eigenvectors $\xi = (\xi_1, \xi_2, \dots, \xi_{3(m+1)})^T$ of the matrix $\mathcal{A}_6 - \lambda I$. Consider the system

$$(\mathcal{A}_6 - \lambda I)\xi = \theta, \quad (115)$$

where, see (75),

$$\mathcal{A}_6 - \lambda I = \begin{pmatrix} \Lambda_6 - \lambda E & \Theta & \dots & \Theta & B \\ E & -\lambda E & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & -\lambda E \end{pmatrix}, \quad \Lambda_6 - \lambda E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (116)$$

Immediately from (115) and (116), we obtain $\xi_i = \lambda \xi_{i+3}$, $i = 1, \dots, 3m$ and

$$\xi_1 = \lambda \xi_4 = \lambda^2 \xi_7 = \dots = \lambda^m \xi_{3m+1}, \quad (117)$$

$$\xi_2 = \lambda \xi_5 = \lambda^2 \xi_8 = \dots = \lambda^m \xi_{3m+2}, \quad (118)$$

$$\xi_3 = \lambda \xi_6 = \lambda^2 \xi_9 = \dots = \lambda^m \xi_{3m+3}. \quad (119)$$

Therefore, taking into account that, by (54), $b_{31} = 0$, from (115) and (116), it follows:

$$b_{11}\xi_{3m+1} + (b_{12} + \lambda^m)\xi_{3m+2} + b_{13}\xi_{3m+3} = 0, \quad (120)$$

$$b_{21}\xi_{3m+1} + b_{22}\xi_{3m+2} + (b_{23} + \lambda^m)\xi_{3m+3} = 0, \quad (121)$$

$$b_{32}\xi_{3m+2} + b_{33}\xi_{3m+3} = 0, \quad (122)$$

that is, $B_6^*(\xi_{3m+1}, \xi_{3m+2}, \xi_{3m+3})^T = \theta$. Now it is clear that if $\text{rank} B_6^* = 2$, the general solution of system (120)–(122) only depends on a single parameter and relations (117)–(119) imply that, among the coordinates of the vector ξ , there is only one arbitrary parameter. Then, $m_g(\mathcal{A}_6, \lambda) = 1$. If $\text{rank} B_6^* = 1$, the general solution of system (120)–(122) depends on two arbitrary parameters, and relations (117)–(119) imply, that, among the coordinates of the vector ξ , there are two arbitrary parameters. Then, $m_g(\mathcal{A}_6, \lambda) = 2$. If $\text{rank} B_6^* = 0$, the general solution of system (120)–(122) depends on three arbitrary parameters with relations (117)–(119) implying that, among the coordinates of the vector ξ , there are three arbitrary parameters and $m_g(\mathcal{A}_6, \lambda) = 3$. $\square$

4.8.2 The case of $m_g(B, 0) = 1$

• **Assume** $\text{rank} B_6^* = 2$. Then, by Theorem 15, we have $m_g(\mathcal{A}_6, \lambda) = 1$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_6, \lambda) = 1$. To apply Weyr's algorithm with $C = \mathcal{A}_6$ for the eigenvalue $\lambda = 0$, we compute $h_0 = \text{rank } \mathcal{A}_6^0 = \text{rank } I = 3(m+1)$ and

$$h_1 = \text{rank } \mathcal{A}_6 = \text{rank} \begin{pmatrix} \Lambda_6 & \Theta & \dots & \Theta & B \\ E & \Theta & \dots & \Theta & \Theta \\ \Theta & E & \dots & \Theta & \Theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta & \Theta & \dots & E & \Theta \end{pmatrix} = 3m + 2.$$

Then, $v_0 = \ell - h_0 = 3(m+1) - 3(m+1) = 0$ and $v_1 = \ell - h_1 = 3(m+1) - 3m - 2 = 1$. Since $\sigma_1 = v_1 - v_0 = 0$, Table 1 will only have a single column taking the same form as given in Table 2 (in this case for $\mathcal{A} = \mathcal{A}_6$, $m_g(B, 0) = 1$, $\lambda = 0$). The aforementioned information is sufficient for the construction of the Jordan form and it is not necessary to apply Weyr's algorithm with $C = \mathcal{A}_6$ and with the eigenvalue λ . In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_6(1, 1)$ in (69) transforming $\mathcal{A}_6$ into its Jordan form $\mathcal{G} = \mathcal{G}_6(1, 1) = \mathcal{T}^{-1}\mathcal{A}_6\mathcal{T}$ by formula (71) and

$$\mathcal{G} = \mathcal{G}_6(1, 1) = \left(\begin{array}{c|ccc} \Lambda_6 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \quad \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 3m.$$

• **Assume** $\text{rank} B_6^* = 1$. Then, by Theorem 15, we have $m_g(\mathcal{A}_6, \lambda) = 2$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_6, \lambda) = 2$. Proceeding in much the same way as earlier, we conclude that the information is sufficient for the construction of the Jordan form and it is not necessary to apply Weyr's algorithm with $C = \mathcal{A}_6$ and with the eigenvalue λ . In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_6(1, 2)$ in (69) transforming $\mathcal{A}_6$ into its Jordan form $\mathcal{G} = \mathcal{G}_6(1, 2) = \mathcal{T}^{-1} \mathcal{A}_6 \mathcal{T}$ by formula (71) and

$$\mathcal{G} = \mathcal{G}_6(1, 2) = \left(\begin{array}{c|ccc} \Lambda_5 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \quad \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 3m.$$

• **Assume** $\text{rank} B_6^* = 0$. By Theorem 15, we have $m_g(\mathcal{A}_6, \lambda) = 3$, and by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_6, \lambda) = 3$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_6(1, 3)$ in (69) transforming $\mathcal{A}_6$ into its Jordan form $\mathcal{G} = \mathcal{G}_6(1, 3) = \mathcal{T}^{-1} \mathcal{A}_6 \mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_6$ for the zero eigenvalue, we proceed in much the same way as above with much the same results and

$$\mathcal{G} = \mathcal{G}_5(1, 3) = \left(\begin{array}{c|ccc} \Lambda_4 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \Theta & & & \end{array} \right) \quad \text{if } 1 \leq k < 3m, \text{ and } \quad \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \quad \text{if } k \geq 3m.$$

4.8.3 The case of $m_g(B, 0) = 2$

• **Assume** $\text{rank} B_6^* = 2$. By Theorem 15, we have $m_g(\mathcal{A}_6, \lambda) = 1$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_6, \lambda) = 1$. To apply Weyr's algorithm with $C = \mathcal{A}_6$ for the eigenvalue $\lambda = 0$, we proceed in much the same way as in Section 4.3.2. Then

$$\begin{aligned} h_0 &= \text{rank } \mathcal{A}_6^0 = 3m + 3, \\ &\dots \\ h_m &= \text{rank } \mathcal{A}_6^m = m + 3. \end{aligned}$$

To compute $h_{m+1} = \text{rank } \mathcal{A}_6^{m+1}$, consider an auxiliary rank

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_6 & B \end{pmatrix} = \text{rank} \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ 0 & b_{32} & b_{33} & 0 & 0 & 0 \\ \lambda & 1 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & \lambda & 1 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda & 0 & b_{32} & b_{33} \end{pmatrix}.$$

We perform the following operations on rows, where each number in round brackets refers to the number of a row in the last matrix:

$$\begin{aligned} (1) &- \frac{b_{11}}{\lambda}(4) - \frac{b_{12} - b_{11}/\lambda}{\lambda}(5) - \frac{b_{13} - b_{12}/\lambda + b_{11}/\lambda^2}{\lambda}(6), \\ (2) &- \frac{b_{21}}{\lambda}(4) - \frac{b_{22} - b_{21}/\lambda}{\lambda}(5) - \frac{b_{23} - b_{22}/\lambda + b_{21}/\lambda^2}{\lambda}(6), \\ (3) &- \frac{b_{32}}{\lambda}(5) - \frac{b_{33} - b_{32}/\lambda}{\lambda}(6). \end{aligned}$$

Using formulas (52) and (53) and the property $\text{rank } B = 1$, we derive

$$h_{m+1}^* = \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & \lambda & 1 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & \lambda & 0 & b_{32} & b_{33} \end{pmatrix} = 3.$$

Therefore, $h_{m+1} = h_{m+1}^* + m - 1 = m + 2$. Then

$$\begin{aligned} v_0 &= \ell - h_0 = 3(m+1) - 3(m+1) = 0, \\ v_1 &= \ell - h_1 = 3(m+1) - (3m+1) = 2, \\ &\dots \\ v_m &= \ell - h_m = 3(m+1) - (m+3) = 2m, \\ v_{m+1} &= \ell - h_{m+1} = 3(m+1) - (m+2) = 2m+1. \end{aligned}$$

Since

$$\begin{aligned} \sigma_1 &= v_1 - v_0 = 2, \\ &\dots \\ \sigma_m &= v_m - v_{m-1} = 2m - 2(m-1) = 2, \\ \sigma_{m+1} &= v_{m+1} - v_m = 2m+1 - 2m = 1. \end{aligned}$$

Table 1 will only have two columns taking the same form as in Table 3 (where $\mathcal{A} = \mathcal{A}_6$, $m_g(B, 0) = 2$, $\lambda = 0$ is applied). There exists a transition matrix $\mathcal{T} = \mathcal{T}_6(2, 1)$ in (69) transforming $\mathcal{A}_6$ into its Jordan form $\mathcal{G} = \mathcal{G}_6(2, 1) = \mathcal{T}^{-1}\mathcal{A}_6\mathcal{T}$ by formula (71) and

$$\mathcal{G} = \mathcal{G}_6(2, 1) = \left(\begin{array}{c|ccc} \Lambda_6 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_6^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

• **Assume** $\text{rank} B_6^* = 1$. Then, by Theorem 15, we have $m_g(\mathcal{A}_6, \lambda) = 2$ and, by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_6, \lambda) = 2$. To apply Weyr's algorithm with $C = \mathcal{A}_6$ for the zero eigenvalue, we proceed in much the same way as in the previous section, where the variant $\text{rank} B_6^* = 2$ was considered, to obtain the same conclusion. There exists a transition matrix $\mathcal{T} = \mathcal{T}_6(2, 2)$ in (69) transforming $\mathcal{A}_6$ into its Jordan form $\mathcal{G} = \mathcal{G}_6(2, 2) = \mathcal{T}^{-1} \mathcal{A}_6 \mathcal{T}$ by formula (71) and

$$\mathcal{G} = \mathcal{G}_6(2, 2) = \left(\begin{array}{c|ccc} \Lambda_5 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_5^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

• **Assume** $\text{rank} B_6^* = 0$. By Theorem 15, we have $m_g(\mathcal{A}_6, \lambda) = 3$, and by Section 3.1.1, $m_g(\mathcal{G}, \lambda) = m_g(\mathcal{A}_6, \lambda) = 3$. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_6(2, 3)$ in (69) transforming $\mathcal{A}_6$ into its Jordan form $\mathcal{G} = \mathcal{G}_6(2, 3) = \mathcal{T}^{-1} \mathcal{A}_6 \mathcal{T}$ by formula (71). To apply Weyr's algorithm with $C = \mathcal{A}_6$ for the zero eigenvalue, we proceed in much the same way as earlier with much the same results and

$$\mathcal{G} = \mathcal{G}_5(2, 3) = \left(\begin{array}{c|ccc} \Lambda_4 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_4^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

4.9 The form and powers of $\mathcal{G}$ if $\mathcal{A} = \mathcal{A}_7$

4.9.1 The case of $m_g(B, 0) = 1$

In this case, we can proceed in much the same way as in Section 4.3. We conclude that there exists a transition matrix $\mathcal{T} = \mathcal{T}_7(1)$ in (69) such that the associated Jordan form of $\mathcal{G} = \mathcal{G}_7(1) = \mathcal{T}^{-1} \mathcal{A}_7 \mathcal{T}$ is

$$\mathcal{G} = \mathcal{G}_7(1) = \left(\begin{array}{c|ccc} \Lambda_7 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m} & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_7^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G_{3m}^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 3m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_7^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 3m.$$

4.9.2 The case of $m_g(B, 0) = 2$

In this case, we can proceed in much the same way as in Section 4.3.2. By Section 3.1.1 and Theorem 10, we have $m_g(\mathcal{G}, 0) = m_g(\mathcal{A}_2, 0) = m_g(B, 0) = 2$. To apply Weyr's algorithm for $C := \mathcal{A}_7$ and $\lambda = 0$, we proceed in much the same way as in Section 4.3.2. Then

$$\begin{aligned} h_0 &= \text{rank } \mathcal{A}_2^0 = 3m + 3, \\ &\dots \\ h_m &= \text{rank } \mathcal{A}_2^m = m + 3. \end{aligned}$$

To compute $h_{m+1} = \text{rank } \mathcal{A}_7^{m+1}$, consider an auxiliary rank

$$h_{m+1}^* = \text{rank} \begin{pmatrix} B & \Theta \\ \Lambda_7 & B \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & b_{12} & b_{13} & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & p & q & b_{21} & b_{22} & b_{23} \\ 0 & -q & p & b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

We perform the following operations among rows, where each number in round brackets refers to the number of a row in the last matrix:

$$\begin{aligned} (1) &- \frac{b_{12}p + b_{13}q}{\Delta}(5) - \frac{b_{13}p - b_{12}q}{\Delta}(6), \\ (2) &- \frac{b_{21}}{\lambda_2}(4) - \frac{b_{22}p + b_{23}q}{\Delta}(5) - \frac{b_{23}p - b_{22}q}{\Delta}(6), \\ (3) &- \frac{b_{31}}{\lambda_2}(4) - \frac{b_{32}p + b_{33}q}{\Delta}(5) - \frac{b_{33}p - b_{32}q}{\Delta}(6), \end{aligned}$$

where $\Delta = p^2 + q^2 > 0$. Using formulas (60) and (61), implied by the property $\text{rank } B = 1$ relations, we have

$$h_{m+1}^* = \text{rank} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & b_{12} & b_{13} \\ 0 & p & q & b_{21} & b_{22} & b_{23} \\ 0 & -q & p & b_{31} & b_{32} & b_{33} \end{pmatrix} = 3.$$

Therefore, $h_{m+1} = h_{m+1}^* + m - 1 = m + 2$. Then

$$\begin{aligned} v_0 &= \ell - h_0 = 3(m+1) - 3(m+1) = 0, \\ v_1 &= \ell - h_1 = 3(m+1) - (3m+1) = 2, \\ &\dots \\ v_m &= \ell - h_m = 3(m+1) - (m+3) = 2m, \\ v_{m+1} &= \ell - h_{m+1} = 3(m+1) - (m+2) = 2m+1. \end{aligned}$$

Since

$$\begin{aligned}\sigma_1 &= v_1 - v_0 = 2, \\ &\dots \\ \sigma_m &= v_m - v_{m-1} = 2m - 2(m-1) = 2, \\ \sigma_{m+1} &= v_{m+1} - v_m = 2m + 1 - 2m = 1,\end{aligned}$$

Table 1 takes the form of Table 3 (constructed for $\mathcal{A} = \mathcal{A}_7$, $m_g(B, 0) = 2$ and $\lambda = 0$) with two columns. In this case, there exists a transition matrix $\mathcal{T} = \mathcal{T}_7(2)$ in (69) such that the associated Jordan form of $\mathcal{G} = \mathcal{G}_7(2) = \mathcal{T}^{-1}\mathcal{A}_7\mathcal{T}$ is

$$\mathcal{G} = \mathcal{G}_7(2) = \left(\begin{array}{c|ccc} \Lambda_7 & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G & \\ \Theta & & & \end{array} \right).$$

Then

$$\mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_7^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & G^k & \\ \Theta & & & \end{array} \right) \text{ if } 1 \leq k < 2m, \text{ and } \mathcal{G}^k = \left(\begin{array}{c|ccc} \Lambda_7^k & \Theta & \dots & \Theta \\ \hline \Theta & & & \\ \vdots & & \Theta_{3m} & \\ \Theta & & & \end{array} \right) \text{ if } k \geq 2m.$$

5 Formulas for the general solution of problem (1), (2)

The initial data for system (1) are defined by (2). The solution of the initial problem (66), (68), that is, the problem

$$y(k+1) = \mathcal{A}y(k), \quad k \in \mathbb{Z}_0^\infty, \quad y(0) = y_0 = (x_0^T, \dots, x_m^T)^T$$

is (we refer to formulas (69)–(74))

$$y(k) = \mathcal{A}^k y(0) = (\mathcal{T}\mathcal{G}\mathcal{T}^{-1})^k y(0) = \mathcal{T}\mathcal{G}^k \mathcal{T}^{-1} y(0) = \mathcal{T}\mathcal{G}^k w(0), \quad k \in \mathbb{Z}_1^\infty, \quad (123)$$

where the relation between $\mathcal{A}$ and $\mathcal{G}$ is given by (71) and $w(0)$ is given by (72). In (66), the matrix $\mathcal{A}$ is defined by (67) and takes seven different forms (75) depending on the Jordan forms Λ_i , $i = 1, \dots, 7$ of A . The matrix $\mathcal{G}$ defined by (71) also takes different forms depending on the Jordan form of $\mathcal{A}$, on the geometrical multiplicity of B and, in some cases, on the geometrical multiplicity of B_l^* , $v = 2, 3, 5, 6$. Recall that the results derived in Section 4 imply the following.

- Let $i = 1, 4, 7$. Then, the transition matrix transforming $\mathcal{A} = \mathcal{A}_i$ into its Jordan form $\mathcal{G} = \mathcal{G}_i(j)$ is $\mathcal{T} = \mathcal{T}_i(j)$, where $j = 1$ if $m_g(B, 0) = 1$ and $j = 2$ if $m_g(B, 0) = 2$.
- Let $i = 2, 3$. Then, the transition matrix transforming $\mathcal{A} = \mathcal{A}_i$ into its Jordan form $\mathcal{G} = \mathcal{G}_i(j, l)$ is $\mathcal{T} = \mathcal{T}_i(j, l)$, where $j = 1$ if $m_g(B, 0) = 1$, $j = 2$ if $m_g(B, 0) = 2$, and $l = 1$ if $m_g(B_l^*, 0) = 1$, $l = 2$ if $m_g(B_l^*, 0) = 2$.
- Let $i = 5, 6$. Then, the transition matrix transforming $\mathcal{A} = \mathcal{A}_i$ into its Jordan form $\mathcal{G} = \mathcal{G}_i(j, l)$ is $\mathcal{T} = \mathcal{T}_i(j, l)$, where $j = 1$ if $m_g(B, 0) = 1$, $j = 2$ if $m_g(B, 0) = 2$, and $l = 1$ if $m_g(B_l^*, 0) = 1$, $l = 2$ if $m_g(B_l^*, 0) = 2$, $l = 3$ if $m_g(B_l^*, 0) = 2$.

Using an auxiliary matrix

$$Q = (E, \underbrace{\Theta, \dots, \Theta}_m),$$

formula (65) (i.e., $x_s(k) = y_s(k)$, $s = 1, 2, 3$) and (123), we can write the solution of initial problem (1), (2) as follows:

$$x(k) = Q\mathcal{T}\mathcal{G}^k w(0), \quad k \in \mathbb{Z}_1^\infty, \quad (124)$$

where the initial data $w(0)$ are defined by (72). Therefore, the following theorems can be proved. The forms and powers of matrices $\mathcal{G}_i(j)$, $i = 1, 4, 7$, $j = 1, 2$, $\mathcal{G}_i(j, l)$, $i = 2, 3$, $j = 1, 2$, and $\mathcal{G}_i(j, l)$, $i = 5, 6$, $j = 1, 2$, $l = 1, 2, 3$ are listed in Section 4.

Theorem 16. Let the Jordan form of A be Λ_1 and let the entries of the matrix B satisfy (24)–(27). Then, the solution of the initial problem (1), (2) is given by the formula

$$x(k) = Q\mathcal{T}_1(j)\mathcal{G}_1^k(j)w(0), \quad k \in \mathbb{Z}_1^\infty, \quad w(0) = \mathcal{T}_1^{-1}(j)(x_0^T, \dots, x_{-m}^T)^T, \quad (125)$$

where $j = 1, 2$. Formula (125) can be simplified as follows. If $j = 1$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_1(1)\mathcal{G}_1^k(1)w(0) = Q\mathcal{T}_1(1) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_3^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (126)$$

If $j = 2$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_1(2)\mathcal{G}_1^k(2)w(0) = Q\mathcal{T}_1(2) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_3^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (127)$$

Theorem 17. Let the Jordan form of A be Λ_2 and let the entries of the matrix B satisfy (29)–(33). Then, the solution of the initial problem (1), (2) is given by the formula:

$$x(k) = Q\mathcal{T}_2(j, l)\mathcal{G}_2^k(j, l)w(0), \quad k \in \mathbb{Z}_1^\infty, \quad w(0) = \mathcal{T}_2^{-1}(j, l)(x_0^T, \dots, x_{-m}^T)^T, \quad (128)$$

where $j, l = 1, 2$. Formula (128) can be simplified as follows. If $(j, l) = (1, 1)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_2(1, 1)\mathcal{G}_2^k(1, 1)w(0) = Q\mathcal{T}_2(1, 1) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1}w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (129)$$

if $(j, l) = (1, 2)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_2(1, 2)\mathcal{G}_2^k(1, 2)w(0) = Q\mathcal{T}_2(1, 2) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (130)$$

if $(j, l) = (2, 1)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_2(2, 1)\mathcal{G}_2^k(2, 1)w(0) = Q\mathcal{T}_2(2, 1) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1}w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (131)$$

if $(j, l) = (2, 2)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_2(2, 2)\mathcal{G}_2^k(2, 2)w(0) = Q\mathcal{T}_2(2, 2) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (132)$$

Theorem 18. Let the Jordan form of A be Λ_3 and let the entries of the matrix B satisfy (35)–(40). Then, the solution of the initial problem (1), (2) is given by the formula

$$x(k) = Q\mathcal{T}_3(j, l)\mathcal{G}_3^k(j, l)w(0), \quad k \in \mathbb{Z}_1^\infty, \quad w(0) = \mathcal{T}_3^{-1}(j, l)(x_0^T, \dots, x_{-m}^T)^T, \quad (133)$$

where $j, l = 1, 2$. Formula (133) can be simplified as follows. If $(j, l) = (1, 1)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_3(1, 1)\mathcal{G}_3^k(1, 1)w(0) = Q\mathcal{T}_3(1, 1) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1} w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (134)$$

if $(j, l) = (1, 2)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_3(1, 2)\mathcal{G}_3^k(1, 2)w(0) = Q\mathcal{T}_3(1, 2) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (135)$$

if $(j, l) = (2, 1)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_3(2, 1)\mathcal{G}_3^k(2, 1)w(0) = Q\mathcal{T}_3(2, 1) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) + k\lambda_2^{k-1} w_3(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (136)$$

if $(j, l) = (2, 2)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_3(2, 2)\mathcal{G}_3^k(2, 2)w(0) = Q\mathcal{T}_3(2, 2) \begin{pmatrix} \lambda_1^k w_1(0) \\ \lambda_2^k w_2(0) \\ \lambda_2^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (137)$$

Theorem 19. Let the Jordan form of A be Λ_4 and let the entries of the matrix B satisfy (42)–(44). Then, the solution of the initial problem (1), (2) is given by the formula

$$x(k) = Q\mathcal{T}_4(j)\mathcal{G}_4^k(j)w(0), \quad k \in \mathbb{Z}_1^\infty, \quad w(0) = \mathcal{T}_4^{-1}(j)(x_0^T, \dots, x_{-m}^T)^T \quad (138)$$

where $j = 1, 2$. Formula (138) can be simplified as follows. If $j = 1$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_4(1)\mathcal{G}_4^k(1)w(0) = Q\mathcal{T}_4(1) \begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{1}{2}k(k-1)\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (139)$$

If $j = 2$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_4(2)\mathcal{G}_4^k(2)w(0) = Q\mathcal{T}_4(2) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (140)$$

Theorem 20. Let the Jordan form of A be Λ_5 and let the entries of the matrix B satisfy (46)–(50). Then, the solution of the initial problem (1), (2) is given by the formula

$$x(k) = Q\mathcal{T}_5(j, l)\mathcal{G}_5^k(j, l)w(0), \quad k \in \mathbb{Z}_1^\infty, \quad w(0) = \mathcal{T}_5^{-1}(j, l)(x_0^T, \dots, x_{-m}^T)^T, \quad (141)$$

where $j = 1, 2$ and $l = 1, 2, 3$. Formula (141) can be simplified as follows. If $(j, l) = (1, 1)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_5(1, 1)\mathcal{G}_5^k(1, 1)w(0) = Q\mathcal{T}_5(1, 1) \begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{1}{2}k(k-1)\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (142)$$

if $(j, l) = (1, 2)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_5(1, 2)\mathcal{G}_5^k(1, 2)w(0) = Q\mathcal{T}_5(1, 2) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (143)$$

if $(j, l) = (1, 3)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_5(1, 3)\mathcal{G}_5^k(1, 3)w(0) = Q\mathcal{T}_5(1, 3) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (144)$$

if $(j, l) = (2, 1)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_5(2, 1)\mathcal{G}_5^k(2, 1)w(0) = Q\mathcal{T}_5(2, 1) \begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{1}{2}k(k-1)\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (145)$$

if $(j, l) = (2, 2)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_5(2, 2)\mathcal{G}_5^k(2, 2)w(0) = Q\mathcal{T}_5(2, 2) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (146)$$

if $(j, l) = (2, 3)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_5(2, 3)\mathcal{G}_5^k(2, 3)w(0) = Q\mathcal{T}_5(2, 3) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (147)$$

Theorem 21. Let the Jordan form of A be Λ_6 and let the entries of the matrix B satisfy (52)–(57). Then, the solution of the initial problem (1), (2) is given by the formula

$$x(k) = Q\mathcal{T}_6(j, l)\mathcal{G}_6^k(j, l)w(0), \quad k \in \mathbb{Z}_1^\infty, \quad w(0) = \mathcal{T}_6^{-1}(j, l)(x_0^T, \dots, x_m^T)^T, \quad (148)$$

where $j = 1, 2$ and $l = 1, 2, 3$. Formula (148) can be simplified as follows. If $(j, l) = (1, 1)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_6(1, 1)\mathcal{G}_6^k(1, 1)w(0) = Q\mathcal{T}_6(1, 1) \begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{1}{2}k(k-1)\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (149)$$

if $(j, l) = (1, 2)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_6(1, 2)\mathcal{G}_6^k(1, 2)w(0) = Q\mathcal{T}_6(1, 2) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (150)$$

if $(j, l) = (1, 3)$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_6(1, 3)\mathcal{G}_6^k(1, 3)w(0) = Q\mathcal{T}_6(1, 3) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (151)$$

if $(j, l) = (2, 1)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_6(2, 1)\mathcal{G}_6^k(2, 1)w(0) = Q\mathcal{T}_6(2, 1) \begin{pmatrix} \lambda^k w_1(0) + k\lambda^{k-1}w_2(0) + \frac{1}{2}k(k-1)\lambda^{k-2}w_3(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (152)$$

if $(j, l) = (2, 2)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_6(2, 2)\mathcal{G}_6^k(2, 2)w(0) = Q\mathcal{T}_6(2, 2) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) + k\lambda^{k-1}w_3(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (153)$$

if $(j, l) = (2, 3)$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_6(2, 3)\mathcal{G}_6^k(2, 3)w(0) = Q\mathcal{T}_6(2, 3) \begin{pmatrix} \lambda^k w_1(0) \\ \lambda^k w_2(0) \\ \lambda^k w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (154)$$

Theorem 22. Let the Jordan form of A be Λ_7 and let the entries of the matrix B satisfy (59)–(64). Then, the solution of the initial problem (1), (2) is given by the formula

$$x(k) = Q\mathcal{T}_7(j)\mathcal{G}_7^k(j)w(0), \quad k \in \mathbb{Z}_1^\infty, \quad w(0) = \mathcal{T}_7^{-1}(j)(x_0^T, \dots, x_{-m}^T)^T \quad (155)$$

where $j = 1, 2$. Formula (155) can be simplified as follows. If $j = 1$ and $k \in \mathbb{Z}_{3m}^\infty$, then

$$x(k) = Q\mathcal{T}_7(1)\mathcal{G}_7^k(1)w(0) = Q\mathcal{T}_7(1) \begin{pmatrix} \lambda^k w_1(0) \\ (r^k \cos k\varphi)w_2(0) + (r^k \sin k\varphi)w_3(0) \\ (-r^k \sin k\varphi)w_2(0) + (r^k \cos k\varphi)w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (156)$$

If $j = 2$ and $k \in \mathbb{Z}_{2m}^\infty$, then

$$x(k) = Q\mathcal{T}_7(2)\mathcal{G}_7^k(2)w(0) = Q\mathcal{T}_7(2) \begin{pmatrix} \lambda^k w_1(0) \\ (r^k \cos k\varphi)w_2(0) + (r^k \sin k\varphi)w_3(0) \\ (-r^k \sin k\varphi)w_2(0) + (r^k \cos k\varphi)w_3(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (157)$$

6 Initial values defining the merging of solutions

If $k \geq 3m$ (or $k \geq 2m$), then the formulas describing the solutions of the initial problem, as it is visible from Theorems 16–22, are reduced and depend only on three initial values

$$w_i(0), \quad i = 1, 2, 3. \quad (158)$$

More exactly, if $k \geq 3m$ and the geometric multiplicity of the zero root of the matrix B is 1, it follows from formulas (126), (129), (130), (134), (135), (139), (142)–(144), (149)–(151), (156) that the solution $x(k)$ depends only on three independent initial values (158), that is, $3(m+1)$ initial values (2) generate only 3 independent initial values (158). If different initial values (2) define the same numbers $w_i(0)$, $i = 1, 2, 3$, the solutions generated by these values are the same (are merged into a single one) for $k \geq 3m$.

Similarly, if $k \geq 2m$ and the geometric multiplicity of the zero root of the matrix B is 2, the same conclusion follows from formulas (127), (131) (132), (136), (137), (140), (145)–(147), (152)–(154), (157). If different initial values (2) define identical numbers $w_i(0)$, $i = 1, 2, 3$, the solutions generated by such initial values merge into a single one for $k \geq 2m$.

In the following theorem, formulas for computing the above-mentioned three initial values (158) through initial values $x_0, \dots, x_{-m}$ in (2), are given.

Define vectors

$$Q_1 := (1 \ 0 \ 0), \quad Q_2 := (0 \ 1 \ 0), \quad Q_3 := (0 \ 0 \ 1).$$

Theorem 23. *Initial data (158) are defined by formulas*

$$w_i(0) = Q_i Q \mathcal{T}^{-1} (x_0^T, \dots, x_{-m}^T)^T, \quad i = 1, 2, 3, \quad (159)$$

where $x_0, \dots, x_{-m}$ are given by (2) and $\mathcal{T}$ is a relevant transposition matrix, that is

$$\mathcal{T} = \mathcal{T}_v(p) \quad \text{if } v = 1, 4, 7, \quad p = 1, 2,$$

$$\mathcal{T} = \mathcal{T}_v(p, q) \quad \text{if } v = 2, 3 \quad p, q = 1, 2,$$

and

$$\mathcal{T} = \mathcal{T}_v(p, q) \quad \text{if } v = 5, 6, \quad p = 1, 2, \quad q = 1, 2, 3.$$

Proof. The initial data (158) are defined as suitable linear combinations of $3(m+1)$ initial values $x(k)$, $k = 0, -1, \dots, -m$ in (2). By (72),

$$\begin{pmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \end{pmatrix} = Q w(0) = Q \mathcal{T}^{-1} y(0) = Q \mathcal{T}^{-1} (x_0^T, \dots, x_{-m}^T)^T. \quad (160)$$

Multiplying (160) successively by vectors Q_i , $i = 1, 2, 3$ from the left, we derive formulas (159). The transposition matrices $\mathcal{T}$ are different, matrices $\mathcal{T} = \mathcal{T}_v(p)$ if $v = 1, 4, 7$ and $p = 1, 2$ are mentioned in Theorems 16 and 19 and 22, matrices $\mathcal{T} = \mathcal{T}_v(p, q)$, if $v = 2, 3$ and $p, q = 1, 2$ in Theorems 17 and 18 and matrices $\mathcal{T} = \mathcal{T}_v(p, q)$, if $v = 5, 6$, $p = 1, 2$ and $q = 1, 2, 3$ in Theorems 20 and 21. $\square$

From Theorem 23, formula (159), we conclude that there exist linear combinations involving initial values $x_0, \dots, x_{-m}$ with the property that each fixed set of initial values $x_0, \dots, x_{-m}$ satisfying the same linear combination defines the same solution after the interval $\mathbb{Z}_0^{3m-1}$ (or $\mathbb{Z}_0^{2m-1}$) has been passed, i.e., such solutions are merged on $\mathbb{Z}_{3m}^\infty$ (or on $\mathbb{Z}_{2m}^\infty$) into a single one. This phenomenon is explained in the following theorem, which reformulates the statement of Theorem 23.

Let real numbers w_i^* , $i = 1, 2, 3$ be given. Consider three linear systems of equations:

$$Q_i Q \mathcal{T}_v^{-1}(p) (x_0^T, \dots, x_{-m}^T)^T = w_i^*, \quad i = 1, 2, 3, \quad (161)$$

where v, p are fixed, $v \in \{1, 4, 7\}$, $p \in \{1, 2\}$,

$$Q_i Q \mathcal{T}_v^{-1}(p, q)(x_0^T, \dots, x_{-m}^T)^T = w_i^*, \quad i = 1, 2, 3, \quad (162)$$

where v, p, q are fixed, $v \in \{2, 3\}$, $p, q \in \{1, 2\}$, and

$$Q_i Q \mathcal{T}_v^{-1}(p, q)(x_0^T, \dots, x_{-m}^T)^T = w_i^*, \quad i = 1, 2, 3, \quad (163)$$

where v, p, q are fixed, $v \in \{5, 6\}$, $p \in \{1, 2\}$, $q \in \{1, 2, 3\}$.

Theorem 24. (i) Every solution $x_s = x_s^*$, $s = 0, -1, \dots, -m$ of system (161) defines,

(i₁) for $k \geq 3m$, the same solution of the initial problem (1), (2)

- by formula (126) if $(v, p) = (1, 1)$,
- by formula (139) if $(v, p) = (4, 1)$,
- by formula (156) if $(v, p) = (7, 1)$.

(i₂) for $k \geq 2m$, the same solution of the initial problem (1), (2)

- by formula (127) if $(v, p) = (1, 2)$,
- by formula (140) if $(v, p) = (4, 2)$,
- by formula (157) if $(v, p) = (7, 2)$.

(ii) Every solution $x_s = x_s^*$, $s = 0, -1, \dots, -m$ of the system (162) defines,

(ii₁) for $k \geq 3m$, the same solution of the initial problem (1), (2)

- by formula (129) if $(v, p, q) = (2, 1, 1)$,
- by formula (130) if $(v, p, q) = (2, 1, 2)$,
- by formula (134) if $(v, p, q) = (3, 1, 1)$,
- by formula (135) if $(v, p, q) = (3, 1, 2)$.

(ii₂) for $k \geq 2m$, the same solution of the initial problem (1), (2)

- by formula (131) if $(v, p, q) = (2, 2, 1)$,
- by formula (132) if $(v, p, q) = (2, 2, 2)$,
- by formula (136) if $(v, p, q) = (3, 2, 1)$,
- by formula (137) if $(v, p, q) = (3, 2, 2)$.

(iii) Every solution $x_s = x_s^*$, $s = 0, -1, \dots, -m$ of system (163) defines,

(iii₁) for $k \geq 3m$, the same solution of the initial problem (1), (2)

- by formula (142) if $(v, p, q) = (5, 1, 1)$,
- by formula (143) if $(v, p, q) = (5, 1, 2)$,
- by formula (144) if $(v, p, q) = (5, 1, 3)$,
- by formula (149) if $(v, p, q) = (6, 1, 1)$,
- by formula (150) if $(v, p, q) = (6, 1, 2)$,
- by formula (151) if $(v, p, q) = (6, 1, 3)$.

(iii₂) for $k \geq 2m$, the same solution of the initial problem (1), (2)

- by formula (145) if $(v, p, q) = (5, 2, 1)$,
- by formula (146) if $(v, p, q) = (5, 2, 2)$,
- by formula (147) if $(v, p, q) = (5, 2, 3)$,
- by formula (152) if $(v, p, q) = (6, 2, 1)$,
- by formula (153) if $(v, p, q) = (6, 2, 2)$,
- by formula (154) if $(v, p, q) = (6, 2, 3)$.

7 Examples

The results of the article will be now illustrated by three examples. In the first one, computations are performed in detail while in the remaining ones, parts of computations are omitted.

Example 2. Let system (1) be reduced to system (23), where $k \in \mathbb{Z}_0^\infty$, $m = 1$,

$$A = \Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & -2 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix},$$

that is, to the system

$$x_1(k+1) = x_1(k) + x_2(k-1) - 2x_3(k-1), \quad (164)$$

$$x_2(k+1) = 2x_2(k) + 2x_1(k-1) + 2x_3(k-1), \quad (165)$$

$$x_3(k+1) = 3x_3(k) + 2x_1(k-1) + x_2(k-1). \quad (166)$$

Conditions (24)–(27) are fulfilled, system (164)–(166) is a WD-system and $m_g(B, 0) = 1$. By formula (126) in Theorem 16, where $j = m_g(B, 0) = 1$, the general solution of the system (164)–(166) is $x(k) = Q\mathcal{T}_1(1)\mathcal{G}_1^k(1)w(0)$, $k \in \mathbb{Z}_1^\infty$, where

$$\mathcal{T}_1(1) = \begin{pmatrix} -2 & -2 & -3 & 0 & -1 & -2 \\ 2 & 0 & 6 & 0 & 2 & 1 \\ 1 & 2 & 12 & 0 & 1 & 0 \\ -2 & -1 & -1 & -1 & -2 & 0 \\ 2 & 0 & 2 & 2 & 1 & 1 \\ 1 & 1 & 4 & 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{G}_1(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{T}_1^{-1}(1) = \frac{1}{108} \begin{pmatrix} 0 & 108 & -108 & 0 & -108 & 216 \\ -27 & -54 & 27 & -27 & 0 & -27 \\ 4 & 4 & 8 & 8 & 4 & 0 \\ 11 & -70 & 49 & -5 & 92 & -81 \\ 6 & -48 & 66 & -42 & 60 & -162 \\ -36 & -36 & 36 & 36 & 72 & -108 \end{pmatrix}.$$

The general solution (164)–(166) is expressed by the formula:

$$x(k) = Q\mathcal{T}_1(1) \begin{pmatrix} 1^k w_1(0) \\ 2^k w_2(0) \\ 3^k w_3(0) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2w_1(0) - 2 \cdot 2^k w_2(0) - 3 \cdot 3^k w_3(0) \\ 2w_1(0) + 6 \cdot 3^k w_3(0) \\ w_1(0) + 2 \cdot 2^k w_2(0) + 12 \cdot 3^k w_3(0) \end{pmatrix}, \quad k \in \mathbb{Z}_3^\infty, \quad (167)$$

where, by (68) and (72),

$$\begin{aligned} w(0) &= \mathcal{T}_1^{-1}(1)y(0) = \mathcal{T}_1^{-1}(1)(x_0^T, x_{-1}^T)^T = \mathcal{T}_1^{-1}(1)(x_{0,1}, x_{0,2}, x_{0,3}, x_{-1,1}, x_{-1,2}, x_{-1,3})^T \\ &= \frac{1}{108} \begin{pmatrix} 108x_{0,2} - 108x_{0,3} - 108x_{-1,2} + 216x_{-1,3} \\ -27x_{0,1} - 54x_{0,2} + 27x_{0,3} - 27x_{-1,1} - 27x_{-1,3} \\ 4x_{0,1} + 4x_{0,2} + 8x_{0,3} + 8x_{-1,1} + 4x_{-1,2} \\ 11x_{0,1} - 70x_{0,2} + 49x_{0,3} - 5x_{-1,1} + 92x_{-1,2} - 81x_{-1,3} \\ 6x_{0,1} - 48x_{0,2} + 66x_{0,3} - 42x_{-1,1} + 60x_{-1,2} - 162x_{-1,3} \\ -36x_{0,1} - 36x_{0,2} + 36x_{0,3} + 36x_{-1,1} + 72x_{-1,2} - 108x_{-1,3} \end{pmatrix}. \end{aligned}$$

For $w_i(0)$, $i = 1, 2, 3$, we derive

$$w_1(0) = (108x_{0,2} - 108x_{0,3} - 108x_{-1,2} + 216x_{-1,3})/108, \quad (168)$$

$$w_2(0) = (-27x_{0,1} - 54x_{0,2} + 27x_{0,3} - 27x_{-1,1} - 27x_{-1,3})/108, \quad (169)$$

$$w_3(0) = (4x_{0,1} + 4x_{0,2} + 8x_{0,3} + 8x_{-1,1} + 4x_{-1,2})/108. \quad (170)$$

The general solution (167) depends only on parameters $w_1(0), w_2(0), w_3(0)$ whose form is defined by the six initial values $x_{0,1}, x_{0,2}, x_{0,3}, x_{-1,1}, x_{-1,2}, x_{-1,3}$ in (168)–(170) and it is only a 3-parametric rather than the expected 6-parameter dependence. Therefore, in accordance with Theorem 24, the case $(l_1), (v, p) = (1, 1)$, solutions of the system (164)–(166), defined by different sets of the initial data

$$(x_{0,1}^*, x_{0,2}^*, x_{0,3}^*, x_{-1,1}^*, x_{-1,2}^*, x_{-1,3}^*) \quad \text{and} \quad (x_{0,1}^{**}, x_{0,2}^{**}, x_{0,3}^{**}, x_{-1,1}^{**}, x_{-1,2}^{**}, x_{-1,3}^{**}),$$

must be identical for $k \geq 3$ if such initial data satisfy formula (161), which in our case is true for formulas (168)–(170). For example, if

$$(x_{0,1}^*, x_{0,2}^*, x_{0,3}^*, x_{-1,1}^*, x_{-1,2}^*, x_{-1,3}^*) = (1, 0, 1, 0, 1, 0) \quad (171)$$

or if

$$(x_{0,1}^{**}, x_{0,2}^{**}, x_{0,3}^{**}, x_{-1,1}^{**}, x_{-1,2}^{**}, x_{-1,3}^{**}) = (1, 6, 5, 1, -15, -9), \quad (172)$$

we obtain, by (168)–(170), the same values

$$\begin{aligned} w_1(0) &= w_1^*(0) = w_1^{**}(0) = -2, \\ w_2(0) &= w_2^*(0) = w_2^{**}(0) = 0, \\ w_3(0) &= w_3^*(0) = w_3^{**}(0) = 4/27. \end{aligned}$$

We conclude that the solutions of system (164)–(166) defined by different initial data (171) and (172) are identical for $k \in \mathbb{Z}_3^\infty$.

Example 3. Let system (1) be reduced to system (41), where $k \in \mathbb{Z}_0^\infty, m = 1$,

$$A = \Lambda_4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

that is, to the system

$$x_1(k+1) = 2x_1(k), \quad (173)$$

$$x_2(k+1) = 2x_2(k) - x_1(k-1) + 3x_3(k-1), \quad (174)$$

$$x_3(k+1) = 2x_3(k). \quad (175)$$

Conditions (42)–(44) are fulfilled, system (173)–(175) is a WD-system and $m_g(B, 0) = 2$. By formula (140) in Theorem 19, the general solution of the system (173)–(175) is $x(k) = Q\mathcal{T}_4(2)\mathcal{G}_4^k(2)w(0)$, $k \in \mathbb{Z}_1^\infty$, where

$$\mathcal{T}_4(2) = \begin{pmatrix} -6 & 0 & 4 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 2 & 0 & 2 & 3 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{G}_4(2) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{T}_4^{-1}(2) = \frac{1}{4} \begin{pmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ -1 & -2 & 3 & 1 & 0 & -3 \\ 1 & 0 & -3 & 0 & 0 & 0 \\ -1 & -2 & 3 & 1 & 4 & -3 \\ -1 & 0 & 3 & 2 & 0 & -6 \\ 0 & 0 & -2 & 0 & 0 & 4 \end{pmatrix}.$$

Then, the general solution of the system (173)–(175) is expressed by the formula

$$x(k) = Q\mathcal{T}_4(2) \begin{pmatrix} 2^k w_1(0) \\ 2^k w_2(0) + k 2^{k-1} w_3(0) \\ 2^k w_3(0) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \cdot 2^k w_1(0) + 4 \cdot 2^k w_3(0) \\ -2 \cdot 2^k w_2(0) - 2 \cdot k \cdot 2^{k-1} w_3(0) - 2^k w_3(0) \\ -2 \cdot 2^k w_1(0) \end{pmatrix},$$

where $k \in \mathbb{Z}_2^\infty$. We use formulas (68) and (72) to obtain

$$\begin{aligned} w(0) &= \mathcal{T}_4^{-1}(2)y(0) = \mathcal{T}_4^{-1}(2)(x_0^T, x_{-1}^T)^T = \mathcal{T}_4^{-1}(2)(x_{0,1}, x_{0,2}, x_{0,3}, x_{-1,1}, x_{-1,2}, x_{-1,3})^T \\ &= \frac{1}{4} \begin{pmatrix} -2x_{0,3} \\ -x_{0,1} - 2x_{0,2} + 3x_{0,3} + x_{-1,1} - 3x_{-1,3} \\ x_{0,1} - 3x_{0,3} \\ -x_{0,1} - 2x_{0,2} + 3x_{0,3} + x_{-1,1} + 4x_{-1,2} - 3x_{-1,3} \\ -x_{0,1} + 3x_{0,3} + 2x_{-1,1} - 6x_{-1,3} \\ -2x_{0,3} + 4x_{-1,3} \end{pmatrix}. \end{aligned}$$

The aforementioned general solution of system (173)–(175) depends on only three initial data $w_i(0)$, $i = 1, 2, 3$, where

$$\begin{aligned} w_1(0) &= (-2x_{0,3})/4, \\ w_2(0) &= (-x_{0,1} - 2x_{0,2} + 3x_{0,3} + x_{-1,1} - 3x_{-1,3})/4, \\ w_3(0) &= (x_{0,1} - 3x_{0,3})/4. \end{aligned}$$

Now, we will demonstrate (in accordance with Theorem 24, case (i_2) , $(v, p) = (4, 2)$) that a different choice of the six initial values $x_{0,1}, x_{0,2}, x_{0,3}, x_{-1,1}, x_{-1,2}, x_{-1,3}$, for $k \geq 2$, defines identical solutions if such initial data satisfy formula (161). For instance, initial data

$$(x_{0,1}^*, x_{0,2}^*, x_{0,3}^*, x_{-1,1}^*, x_{-1,2}^*, x_{-1,3}^*) = (2, 0, 2, 0, 2, 0) \quad (176)$$

or

$$(x_{0,1}^{**}, x_{0,2}^{**}, x_{0,3}^{**}, x_{-1,1}^{**}, x_{-1,2}^{**}, x_{-1,3}^{**}) = (2, 2, 2, 10, 2, 2) \quad (177)$$

give the same values

$$\begin{aligned} w_1(0) &= w_1^*(0) = w_1^{**}(0) = -1, \\ w_2(0) &= w_2^*(0) = w_2^{**}(0) = 1, \\ w_3(0) &= w_3^*(0) = w_3^{**}(0) = -1 \end{aligned}$$

and the solutions of system (173)–(175) defined by (176) and (177) are identical for $k \in \mathbb{Z}_2^\infty$.

Example 4. Let system (1) be reduced to system (51), where $k \in \mathbb{Z}_0^\infty$, $m = 1$,

$$A = \Lambda_6 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that is, to the system

$$x_1(k+1) = 2x_1(k) + x_2(k) - 2x_2(k-1) - 2x_3(k-1), \quad (178)$$

$$x_2(k+1) = 2x_2(k) + x_3(k), \quad (179)$$

$$x_3(k+1) = 2x_3(k). \quad (180)$$

Conditions (52)–(57) are fulfilled, system (178)–(180) is a WD-system and $m_g(B, 0) = 2$. In this case, we have (by Section 4.8)

$$B_6^* = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and $m_g(B_6^*, 0) = 2$. By formula (153) in Theorem 21, the general solution of (178)–(180) is $x(k) = Q\mathcal{T}_6(2, 2)\mathcal{G}_6^k(2, 2)w(0)$, $k \in \mathbb{Z}_1^\infty$, with

$$\mathcal{T}_6(2, 2) = \begin{pmatrix} 0 & -2 & -1 & 0 & 1 & 0 \\ 2 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{G}_6(2, 2) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{T}_6^{-1}(2, 2) = \frac{1}{4} \begin{pmatrix} 4 & 4 & 1 & 0 & -4 & -4 \\ -2 & -1 & -1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & 4 & 2 & 2 \\ 0 & -2 & -1 & 0 & 4 & 4 \\ 0 & 0 & -2 & 0 & 0 & 4 \end{pmatrix}.$$

Then, the general solution of the system (178)–(180) is expressed by the formula

$$x(k) = Q\mathcal{T}_6(2, 2) \begin{pmatrix} 2^k w_1(0) \\ 2^k w_2(0) + k 2^{k-1} w_3(0) \\ 2^k w_3(0) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2^{k+1} w_2(0) - (k+1) 2^k w_3(0) \\ 2^{k+1} w_1(0) + 2^{k+2} w_2(0) + (k+1) 2^{k+1} w_3(0) \\ 2^{k+2} w_3(0) \end{pmatrix},$$

where $k \in \mathbb{Z}_2^\infty$. We use formulas (68) and (72) to obtain

$$\begin{aligned} w(0) &= \mathcal{T}_6^{-1}(2, 2)y(0) \\ &= \mathcal{T}_6^{-1}(2, 2)(x_0^T, x_{-1}^T)^T = \mathcal{T}_6^{-1}(2, 2)(x_{0,1}, x_{0,2}, x_{0,3}, x_{-1,1}, x_{-1,2}, x_{-1,3})^T \\ &= \frac{1}{4} \begin{pmatrix} 4x_{0,1} + 4x_{0,2} + x_{0,3} - 4x_{-1,2} - 4x_{-1,3} \\ -2x_{0,1} - x_{0,2} - x_{0,3} + 2x_{-1,2} + 2x_{-1,3} \\ x_{0,3} \\ -2x_{0,1} - x_{0,2} - x_{0,3} + 4x_{-1,1} + 2x_{-1,2} + 2x_{-1,3} \\ -2x_{0,2} - x_{0,3} + 4x_{-1,2} + 4x_{-1,3} \\ -2x_{0,3} + 4x_{-1,3} \end{pmatrix}. \end{aligned}$$

The aforementioned general solution of system (178)–(180) depends on only three initial data $w_i(0)$, $i = 1, 2, 3$, where

$$\begin{aligned} w_1(0) &= \frac{1}{4}(4x_{0,1} + 4x_{0,2} + x_{0,3} - 4x_{-1,2} - 4x_{-1,3}), \\ w_2(0) &= \frac{1}{4}(-2x_{0,1} - x_{0,2} - x_{0,3} + 2x_{-1,2} + 2x_{-1,3}), \\ w_3(0) &= \frac{1}{4}x_{0,3}. \end{aligned}$$

Now, we will demonstrate (in accordance with Theorem 24, case (iii₂), $(v, p, q) = (6, 2, 2)$) that different choices of the six initial values $x_{0,1}, x_{0,2}, x_{0,3}, x_{-1,1}, x_{-1,2}, x_{-1,3}$ can define solutions that are identical for $k \geq 2$. For example, initial data

$$(x_{0,1}^*, x_{0,2}^*, x_{0,3}^*, x_{-1,1}^*, x_{-1,2}^*, x_{-1,3}^*) = (1, 0, 4, 0, 1, 0) \quad (181)$$

and

$$(x_{0,1}^{**}, x_{0,2}^{**}, x_{0,3}^{**}, x_{-1,1}^{**}, x_{-1,2}^{**}, x_{-1,3}^{**}) = (-3, 0, 4, 6, -3, 0) \quad (182)$$

give the same new initial values

$$\begin{aligned} w_1(0) &= w_1^*(0) = w_1^{**}(0) = 1, \\ w_2(0) &= w_2^*(0) = w_2^{**}(0) = -1, \\ w_3(0) &= w_3^*(0) = w_3^{**}(0) = 1. \end{aligned}$$

Initial data (181) and (182) define two solutions of the system (178)–(180) that are identical for $k \in \mathbb{Z}_2^\infty$.

8 Remarks, conclusions, and open problems

Since Section 2, Remark 1, we have assumed, without loss of generality, that the matrix A in (1) has its Jordan form (we refer to Remark 1). In the general case, when A is an arbitrary matrix, transformation (13) leads to system (12) with the matrix of linear nondelayed terms having a Jordan form. Solutions of systems (12), i.e., of systems

$$x^*(k) = A_J x^*(k) + S^{-1} B S x^*(k - m)$$

defined by initial data (14), that is,

$$x^*(k) = S^{-1} x(k) S, \quad k \in \mathbb{Z}_{-m}^0$$

then can be transformed into solutions of system (1) by the substitution (13), that is, by the formula

$$x(k) = S x^*(k),$$

where (see (124))

$$x^*(k) = Q \mathcal{T} \mathcal{G}^k w(0), \quad k \in \mathbb{Z}_1^\infty,$$

and the initial data $w(0)$ are defined by the formula (72):

$$w(0) = \mathcal{T}^{-1} y(0) = \mathcal{T}^{-1} (x_0^{*T}, \dots, x_{-m}^{*T})^T.$$

The article considers WD-systems (1) and their explicit general solutions are found by transforming them into nondelayed discrete higher dimensional systems. Simple formulas are found that describe general solutions depending on three arbitrary parameters only (rather than on the given $3m + 3$ “old” initial values) and are valid if $k \geq 3m$ or $k \geq 2m$. This means that there exist solutions defined by different initial values such that these solutions are merged into a single one after $3m$ steps if $m_g(B, 0) = 1$, or after $2m$ steps if $m_g(B, 0) = 2$. The results are demonstrated by examples.

The property of the geometric multiplicities described in Section 3.1, the properties of auxiliary matrices B_i^* , $i = 2, 3, 5, 6$, and Weyr’s algorithm, are basic theoretical findings for the results derived. With them, the proper Jordans forms of higher dimensional matrices $\mathcal{A}$ are derived.

Discrete planar WD-systems are investigated in [15,16,20]. The criteria formulated in Theorems 2–7 (i.e., the cases of $A = \Lambda_i$, $i = 1, \dots, 6$) are reduced, formally substituting zero for all entries of matrices with at least one index equaling 3, in the well known and in the case of Theorem 8 (the case of $A = \Lambda_7$), formally zeroing all entries of matrices with at least one index equaling 1, then we arrive at a situation that contradicts the existence of a WD-system. Problems similar to those in the article are treated in [31].

Papers [7–10] consider mathematical models of endocrine regulation applying the so-called finite-dimension (FD) reducible differential delayed systems

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau).$$

Let us compare FD-reducible systems and WD-systems. FD-reducible systems are characterized by there existing a constant matrix D such that any solution $x(t)$ defined for $t > 0$ satisfies the ordinary differential system $\dot{x}(t) = Dx(t)$ for all $t > \tau$, so that its solutions are indistinguishable from those of a finite-dimensional system. For an analysis of WD and FD-reducible systems, we use the properties of FD-reducible systems listed in the above articles. If a system is FD-reducible, then, e.g., the matrix product $A_1 A_0 A_1$ is a zero matrix (the property (i) in [9] for $k = 1$). But, for WD-systems, this cannot be true. As an example, consider $A_0 = A$ and $A_1 = B$ used in Example 2 with $A_1 A_0 A_1 = BAB$ being a nonzero matrix. Moreover, each FD-reducible system is a WD-system because the same “characteristic” equations are satisfied in both cases (we refer to equation (15) in [9] and to equation (6) in [12]). For completeness, we point out that, for differential and difference systems, the “characteristic” equations require the same coefficient conditions. Papers [26,27] as well as books [6,22], study the delayed three-dimensional differential system

$$\dot{x}_1(t) = -ax_1(t) + asx_2(t - h), \quad (183)$$

$$\dot{x}_2(t) = x_3(t), \quad (184)$$

$$\dot{x}_3(t) = -w^2 x_2(t) - 2\xi w x_3(t) + w^2 u(t). \quad (185)$$

It describes a linearized model of the Mach number dynamics concerning the time-optimal control of a high-speed closed-circuit wind tunnel. In system (183)–(185), x_1 is the Mach number, x_2 gives the actuator position, x_3 is the actuator rate, $a = 1/1.964$, $s = -0.117$, $w = 6$, $\xi = 1.6$, $h = 0.33$, and $u(t)$ is a control function. In [14], this system is discretized showing that the homogeneous part of the discrete system is a WD-system. In the present article, all possible cases are considered of three dimensional WD-systems, and the results can be directly applied not only to the discrete systems derived in the aforementioned articles, but also to arbitrary three-dimensional discrete WD-systems derived by discretizing differential ones. Since all general solutions of the homogeneous system are derived, it is easy to find a solution of the nonhomogeneous discrete system related to (183)–(185). This is presented in [14].

In [15,16], a concept of conditional stability for planar WD-systems is formulated with relevant results proved. Generalizing this concept to three-dimensional discrete WD-systems may be a challenge for further research. Another problem is to construct nondelayed discrete systems having, on an interval, general solutions coinciding with general solutions of WD-systems. For a planar system, such results can be found in [15]. In addition, as our approach gives exact results, it might be useful to write a computer program detecting WD-systems and producing explicit forms of general solutions. Another important challenge is to consider WD-systems of arbitrary dimension, developing the coefficient criteria and giving exact formulas for their solutions. Also, in the future, it might be useful to apply the known concrete formulas to representing the solutions of delayed systems of discrete equations, such as those derived and used, for instance, in papers [21,25,28,29,32], to obtain concrete formulas representing general solutions for higher dimensional discrete delayed systems. Attention deserves a generalization of the results for the case of the independent variable changing over a time-scale (for time-scales theory we refer to [3]) and as well as an analysis of connections between the solutions of WD-systems and the discrete variants of systems obtained by modifying the Euler-Poincaré equations [2]. Let us briefly comment on the recent papers [13,17,18]. In [13], the asymptotic behavior of solutions of fractional discrete equations is studied by the retract apparatus while, in [17], this technique is applied to studying the asymptotic behavior of solutions of Emden-Fowler second-order discrete equations. An original geometrical method is developed in [18] to investigate the behavior of solutions of complex differential equations in a neighborhood of a singular point. All the results derived in the aforementioned three articles have qualitative character and do not give exact formulas describing solutions as the present paper does. Nevertheless, it is an open question whether, for some particular cases of fractional discrete equations considered in [13], it will be possible to develop a WD-system classification and derive exact formulas for their solutions. Finally, for rudiments of theory of discrete equations, we refer to [1,5,19,24,30,33].

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