Research Article

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Qualitative properties of solutions to the viscoelastic beam equation with damping and logarithmic nonlinear source terms

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Abstract: In this article, we consider the initial-boundary value problem for a class of viscoelastic extensible beam equations with logarithmic source term, strong damping term, and weak damping term. We establish the local existence and uniqueness of weak solutions by applying the Faedo-Galerkin method and the contraction mapping principle. Under the framework of potential wells, for subcritical initial energy, we derive the sharp condition for initial data classification, i.e., the global existence and decay estimates of weak solutions in the stable set and the finite-time blow-up of weak solutions in the unstable set. Moreover, we derive sufficient conditions for the finite-time blow-up of weak solutions with negative initial energy and null initial energy. And we obtain an upper bound estimate for the blow-up time.

Keywords: viscoelastic term, damping term, logarithmic source term, decay estimation, blow-up

MSC 2020: 35A01, 35A02, 35D30, 35G31, 46E35

1 Introduction

In this article, we investigate the following initial-boundary value problems governing the viscoelastic beam equation, which incorporates both strong and weak damping mechanisms alongside logarithmic nonlinear source terms

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u - \omega\Delta u_t + \mu u_t - \int_0^t g(t-s)\Delta^2 u(s)\mathrm{d}s = |u|^{p-2}u\ln|u|, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \Delta u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\, u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u_0 \in H = H_0^1(\Omega) \cap H^2(\Omega), u_1 \in L^2(\Omega)$. $\Omega \subset \mathbb{R}^n (n \geq 1)$ denotes a smoothly bounded domain. The function $g(t)$ is a kernel function.

In the framework of continuous dynamics, the evolution of transverse deflections in extendable beam structures can be characterized by their governing equations. The core characteristics of this equation are reflected in the dynamic buckling behavior of articulated beams under axial forces, where the vibration modes are significantly regulated by the beam length and boundary constraint conditions. Woinowsky-

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Krieger [27] first formulated the governing equations for extensible beams in their seminal work, presenting the following mathematical model:

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^4 u}{\partial x^4} - \left(\beta + \int_0^L u_x^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where β denotes the initial axial displacement relative to the unstressed configuration. The function $u(x, t)$ describes the transverse displacement dynamics of an elastic beam with natural length L, where both end-points remain rigidly constrained. For higher high-dimensional cases, Berger [2] derived the following version of the beam model:

$$u_{tt} + \Delta^2 u - \left(Q + \int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = p(u, u_t, x), \tag{1.3}$$

where Q represents the contribution of in-plane forces to the system, and $p(u, u_t, x)$ denotes the transverse load dependent on displacement, velocity, and spatial variables. The extensible beam model has been widely applied in related fields such as micro-machined beams. Fang and Wicket [10] considered the following governing equation:

$$EI\omega_{xxxx} + EA\left(\varepsilon - \frac{1}{2L}\int_0^L \omega_x^2 \mathrm{d}x\right)\omega_{xx} = 0, \tag{1.4}$$

where EI and EA are the bending stiffness and axial stiffness, respectively. The system parameters include L, denoting the beam's characteristic length, and ε, representing the imposed axial strain. In addition, Nayfeh and Younis [23] also established the fundamental dynamical equation characterizing the out-of-plane displacement mechanics in slender microbeam systems

$$\frac{EI}{1-\nu^2}\frac{\partial^4\omega}{\partial x^4} + \rho A\frac{\partial^2\omega}{\partial t^2} + c\frac{\partial\omega}{\partial t} = \left(\frac{EA}{2l(1-\nu^2)}\int_0^l \left(\frac{\partial\omega}{\partial x}\right)^2 \mathrm{d}x + N\right)\frac{\partial^2\omega}{\partial x^2} + \frac{1}{2}\varepsilon_0\varepsilon_r b\frac{(V_p + v(t))^2}{(d-\omega)^2}. \tag{1.5}$$

The system parameters are defined as follows. $\omega(x, t)$ denotes the transverse displacement field, ρ the material density, d the interelectrode gap spacing, N the applied tensile axial force, $v(t)$ the AC component, V_p as the driving voltage comprising a DC component, ε_0 the vacuum permittivity, and ε_r the relative permittivity of the dielectric medium normalized to air.

With a growing understanding of the physical features of extensible beam models, such as stress distributions and boundary constraint conditions, and the engineering application scenarios of these models, these beam equations rooted in practical physics attract many mathematicians due to their rich mathematical connotations and potential application values. Dickey [7] obtained the existence of solutions for standard beam model (1.2) by Fourier sine series. Ma [21] further extended the study to the global existence of weak solutions and decay estimates. Besides, Ma and Narciso [22] derived results on the global attractors and the existence of bounded absorbing sets for standard beam model in high dimensions (1.3) when $p(u, u_t, x) = h - f(u) - g(u_t)$.

Patcheu [24] considered the Cauchy problem for beam equation

$$u'' + A^2 u + M(\|A^{1/2}u\|_H^2)Au + g(u') = 0. \tag{1.6}$$

The equation addresses the physical origin of the problem by studying the dynamic buckling of a hinged, beam under axial tensile or compressive forces. Using the Faedo-Galerkin method, the authors established the local existence and uniqueness of weak solutions. Furthermore, they obtained exponential decay of energy through a special integral inequality. For linear damping, where $g(x) = \delta x$, Henriques de Brito [14] and Biler [3] acquired exponential decay estimates of weak solutions for equation (1.6). Then, Yang et al. [25] researched the initial-boundary value problem (IBVP) for a nonlinear extensible beam model derived from connection mechanics. The equation includes strong and weak damping terms and a power source term

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \Delta u_t + |u_t|^{r-1}u_t = |u|^{p-1}u. \tag{1.7}$$

The authors considered the issue under three distinct initial energy scenarios, including subcritical critical and arbitrarily high levels. Using the contraction mapping principle, they first proved the local existence of solutions. Within the potential well framework, the global existence, non-existence, and decay estimate were deduced for subcritical and critical initial energy levels. Additionally, the global non-existence of solutions was analyzed in both weak and strong damping cases under arbitrarily high initial energy levels.

While researching the equations for extensible beams with damping terms, viscoelastic terms are also taken into consideration. Viscoelastic materials exhibit characteristics that lie between those of elastic solids and those of viscous fluids with differential properties, and their behaviors can be modeled through partial differential equation models. Liu et al. [20] considered the IBVP for viscoelastic Kirchhoff-like beam equations with weak and strong dampings, and power source terms

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\mathrm{d}\tau - \Delta_p u + u_t - \Delta u_t = |u|^{q-2}u. \tag{1.8}$$

Through the utilization of linearization techniques and contraction mapping theorems, they first demonstrated the local existence and uniqueness of solutions to the problem. Subsequently, by implementing the potential well theory, the investigation of global solution existence was conducted under subcritical and critical initial energy conditions. Additionally, the decay estimates of globally existing solutions were analyzed for cases where positive initial energies lie strictly beneath the potential well depth. Finally, a thorough examination of finite-time blow-up scenarios was carried out. This analysis systematically addressed solutions with negative initial energy, zero initial energy, and positive initial energy values strictly below the potential wells depth, alongside solutions characterized by arbitrary positive initial energies. Each case was treated separately to ensure a comprehensive understanding of the dynamical behaviors under varying energy configurations.

When we discuss beam equations with damping terms, power nonlinearity and logarithmic nonlinearity are two important forms of nonlinear source terms. Fang and Li [8] considered the following plate equation:

$$u_{tt} + \Delta^2 u + \alpha\Delta u - \omega\Delta u_t + \beta u_t = |u|^{p-2}u\ln|u|, \tag{1.9}$$

where $\alpha < \lambda_1$, $\omega \geq 0$, $\beta > -\omega\lambda_1$. The authors derived finite-time blow-up results for solutions exhibiting both low and high initial energy states by using the potential well method and enhanced differential inequality techniques. In particular, for high-energy solutions, they considered cases where the initial displacement and initial velocity either have the same or opposite signs in the sense of the L^2 inner product. Additionally, they obtained lower bounds for the blow-up time in four initial energy levels, subcritical, critical, supercritical, and ultra-supercritical.

By analyzing different beam models, readers can clearly identify the unsolved problems in each beam equation. Nonetheless, we believe it remains essential to describe these issues in a clearer manner. To address this, we provide Table 1. In this table, each black dot denotes a specific term within the model, and the description on the far right indicates the conclusions already derived. From this table, we can identify an overarching framework that aims to derive finite-time blow-up of weak solutions for each beam model under different energy levels and asymptotic behavior for solutions. For the beam model considered in [1], which includes viscoelastic, logarithmic source, and strong damping terms, Pereira et al. proved that the model admits a unique weak solution. They also obtained decay estimates and finite-time blow-up results for solutions under the energy condition $0 < E(0) < d$. The cases of negative energy and critical energy were left open. For the beam model considered in [11], which incorporates viscoelastic terms and nonlinear weak damping, the authors proved that weak solutions to this model blow up in finite time for $E(0) < E_1$. The problem of decay estimates for global solutions to this beam equation was left open. Similar issues also arise in [6,9] and others. Motivated by these authors, we should address these unsolved problems. If we tackle each individually, we

Table 1: Mathematical physics models and the constituent elements of the results

Terms included in the function							Results
u_{tt}	$\Delta^2 u$	Δu	Δu_t	u_t	$\int_0^t g(t-s)\Delta^2 u(s)\mathrm{d}s$	$\lvert u\rvert^{p-2}u\ ln\lvert u\rvert$	
•	•	•	•	•			Gloabal existence and exponential decay and finite time blow-up for $E(0) < d$, global existence and exponential decay and finite time blow-up for $E(0) = d$, finite time blow-up for $E(0) > 0$ [17].
•	•	•	•				Global existence and infinite time blow-up for $E(0) < d$, global existence and infinite time blow-up for $E(0) = d$, infinite time blow-up for $E(0) > 0$, blow-up for $E_*(0) > 0$ [6].
•	•		•		•		Gloabal existence and exponential decay [9].
•	•		•		•	•	Gloabal existence and exponential decay and finite time blow-up for $E(0) < d$ [1].
•	•			•			Gloabal existence and asymptotic behavior and finite time blow-up for $E(0) < d$, global existence and asymptotic behavior and finite time blow-up for $E(0) = d$, finite time blow-up for $E(0) > d$ [5].
•		•	•	•		•	Gloabal existence and exponential decay and infinite time blow-up for $E(0) < d$, gloabal existence and exponential decay and infinite time blow-up for $E(0) = d$, infinite time blow-up for $E(0) > 0(\omega = 0)$ [18].
•		•		•		•	Gloabal existence and exponential decay and finite time blow-up for $E(0) < d$, gloabal existence and exponential decay and finite time blow-up for $E(0) = d$, finite time blow-up for $E(0) > 0$ [4].

believe there will be a great deal of repetitive work. To avoid this repetitive work, we combine the structural terms in these equations and study the resulting equation, i.e., model (1.1). The simultaneous occurrence of these terms in this model significantly increases the difficulty of solving the problem of the blow-up of weak solutions. As a nonlinear term, the logarithmic nonlinear term is distinct from the power-type nonlinear term. It has a much weaker effect on the blow-up of weak solutions. This characteristic poses one of the primary challenges in analyzing the blow-up of solutions for this model. Moreover, the energy dissipation induced by damping and viscoelastic terms further exacerbates the difficulties in proving finite-time blow-up of weak solutions. Currently, research on the comprehensive effects of these terms is relatively limited, and classical methods and inequalities cannot be directly applied; we must improve the previous techniques and establish new prior estimations. This is the motivation and contribution of the article.

The following is the organizational framework of this article. In Section 2, we introduce fundamental assumptions, lemmas, and key definitions necessary for the subsequent analysis. Section 3 demonstrates the local existence of weak solutions through a combined application of the Galerkin method and the contraction mapping principle. In Section 4, we established the global existence of solution and decay estimates with subcritical energy condition. In Section 5, we proved the blow-up of solutions in finite time by using the convex method and obtained an upper bound estimate for the blow-up time.

2 Preliminaries

In this article, we assume that $\omega > 0, \mu > -\omega\lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on Ω under homogeneous Dirichlet boundary conditions. We denote

$$\|u\|_p := \|u\|_{L^p(\Omega)}, (u, v) := \int_\Omega uv\mathrm{d}x,$$

and

$$(u, v)_* := \mu(u, v) + \omega(\nabla u, \nabla v), \|u\|_*^2 := \mu \|u\|_2^2 + \omega \|\nabla u\|_2^2.$$

Through simple calculation, we can prove that there exist two positive numbers $C_* < C^*$ such that

$$C_* \|u\|_*^2 < \|u\|_2^2 + \|\nabla u\|_2^2 < C^* \|u\|_*^2, \forall u \in H_0^1(\Omega).$$

We define $\|u\|_H^2 = \|\Delta u\|_2^2 + \|\nabla u\|_2^2$. Besides, let the notation $\langle \cdot, \cdot \rangle$ represent the duality pairing between the space H and its dual space H^{-1}. The spaces involved in this article are all standard Sobolev Spaces.

Regarding Problem (1.1), the following assumptions are put forward for analysis:

(H_1) The exponent p satisfies

$$\begin{cases} 2 < p < \infty, & n \leq 4, \\ 2 < p < \dfrac{2n-6}{n-4}, & n \geq 5. \end{cases}$$

(H_2) The function $g(s) : \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing differentiable and satisfies

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\mathrm{d}s := l > 0.$$

(H_3) The function $\xi : \mathbb{R}^+ \to \mathbb{R}$ is non-increasing differentiable, which satisfies the following formula true:

$$g'(t) \leq -\xi(t)g(t), \quad \forall t > 0,$$

and

$$\int_0^{+\infty} \xi(t)\mathrm{d}t = +\infty.$$

To facilitate our discussion, we define the energy functional

$$\begin{aligned} E(t) &:= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\ &\quad + \frac{1}{2}(g \circ \Delta u)(t) - \frac{1}{p}\int_\Omega |u|^p \ln|u|\mathrm{d}x + \frac{1}{p^2}\|u\|_p^p, \end{aligned} \tag{2.1}$$

where

$$(g \circ \Delta u)(t) := \int_0^t g(t-s)\|\Delta u(x,s) - \Delta u(x,t)\|_2^2 \mathrm{d}s.$$

And the potential energy functional is defined as

$$\begin{aligned} J(u) &:= \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\ &\quad + \frac{1}{2}(g \circ \Delta u)(t) - \frac{1}{p}\int_\Omega |u|^p \ln|u|\mathrm{d}x + \frac{1}{p^2}\|u\|_p^p. \end{aligned} \tag{2.2}$$

Besides, the Nehari functional is defined as

$$I(u) := \left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u(t)\|_2^2 + \|\nabla u\|_2^2 + (g \circ \Delta u)(t) - \int_\Omega u^p \ln|u|\mathrm{d}x. \tag{2.3}$$

According to the definitions of $J(u)$ and $I(u)$, we acquire

$$J(u) := \frac{1}{p}I(u) + \frac{p-2}{2p}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{p-2}{2p}(g \circ \Delta u)(t) + \frac{1}{p^2}\|u\|_p^p. \tag{2.4}$$

We also define the Nehari manifold $\mathcal{N}$, the unstable set V, and the stable set W as follows:

$$\mathcal{N} = \{u \in H\backslash\{0\} | I(u) = 0\},$$

$$V = \{u \in H\backslash\{0\} | J(u) < d, I(u) < 0\},$$

and

$$W = \{u \in H | J(u) < d, I(u) > 0\} \cup \{0\}.$$

Additionally, we define the well depth d as follows:

$$0 < d = \inf_{u \in H\backslash\{0\}}\left\{\sup_{\lambda \geq 0} J(\lambda u) : u \in H\backslash\{0\}\right\},$$

which is alternatively written as

$$0 < d = \inf_{u \in \mathcal{N}} J(u).$$

The proof of $d > 0$ is analogous to that in [13].

Subsequently, we present the definition of the weak solution of Problem (1.1).

Definition 2.1. For a given positive $T > 0$, a function $u(x,t)$ is referred to as the weak solution of Problem (1.1), if

$$u \in C(0,T;H) \cap C^1(0,T;H_0^1(\Omega)) \cap C^2(0,T;H^{-1}),$$

and for every test function $\varphi \in H_0^2(\Omega)$ satisfying

$$\begin{cases} \langle u_{tt}, \varphi\rangle + (\Delta u, \Delta\varphi) + (\nabla u, \nabla\varphi) + \omega(\nabla u_t, \nabla\varphi) \\ + \mu(u_t, \varphi) - \int\limits_0^t g(t-s)(\Delta u(s), \Delta\varphi)\mathrm{d}s \\ = \int\limits_\Omega |u|^{p-2}u\varphi\ln|u|\mathrm{d}x, \ 0 < t < T, \\ u(x,0) = u_0(x) \ \text{ in } H, \\ u_t(x,0) = u_1(x) \ \text{ in } L^2(\Omega). \end{cases}$$

Definition 2.2. For a weak solution $u(x,t)$ of Problem (1.1), we introduce the maximal existence time $T_{\max}$ via the following definition:

$$T_{\max} = \sup\{T > 0;\ u(x,t) \ \text{exists on the interval} \ [0, T]\}.$$

(i) If $T_{\max} = +\infty$, the solution $u(x,t)$ is referred to as a global solution.
(ii) If $T_{\max} < +\infty$, the solution $u(x,t)$ is said to undergo finite-time blow-up, with $T_{\max}$ being designated as the blow-up time.

Next, we will introduce several important lemmas used in subsequent proofs.

Lemma 2.1. *Suppose that* $u \in H\backslash\{0\}$. *Then*
(1) $\lim_{\lambda\to 0^+} J(\lambda u) = 0$ *and* $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$,
(2) *there exist a unique positive real number* $\lambda_* > 0$ *such that* $\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)|_{\lambda=\lambda_*} = 0$,
(3) *the function* $J(\lambda u)$ *is a strictly increasing on* $(0, \lambda_*)$. *It is a strictly decreasing function on* $(\lambda_*, +\infty)$. *Moreover, it reaches its maximum value when* $\lambda = \lambda_*$,
(4) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda_*$, $I(\lambda u) < 0$ for $\lambda > \lambda_*$, *and* $I(\lambda_* u) = 0$.

Proof. From (2.2), we have

$$\begin{aligned} J(\lambda u) = &\frac{\lambda^2}{2}\left(1 - \int\limits_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 - \frac{\lambda^p}{p}\|u\|_p^p\ln\lambda \\ &+ \frac{\lambda^p}{p^2}\int\limits_\Omega u^p\ln|u|\mathrm{d}x + \frac{\lambda^p}{p^2}\|u\|_p^p + \frac{\lambda^2}{2}(g\circ\Delta u)(t) \\ &- \frac{\lambda^p}{p}\int\limits_\Omega u^p\ln|u|\mathrm{d}x + \frac{\lambda^2}{2}\|\nabla u\|_2^2, \end{aligned} \tag{2.5}$$

then it is evident that the conclusion of (1) is valid. By differentiating $J(\lambda u)$, we derive

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) &= \lambda\left(1 - \int\limits_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 - \lambda^{p-1}\|u\|_p^p\ln\lambda - \lambda^{p-1}\int\limits_\Omega u^p\ln|u|\mathrm{d}x + \lambda(g\circ\Delta u)(t) + \lambda\|\nabla u\|_2^2 \\ &= \lambda\left(\left(1 - \int\limits_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 - \lambda^{p-2}\|u\|_p^p\ln\lambda - \lambda^{p-2}\int\limits_\Omega u^p\ln|u|\mathrm{d}x + (g\circ\Delta u)(t) + \|\nabla u\|_2^2\right). \end{aligned} \tag{2.6}$$

Let $K(\lambda u) = \lambda^{-1}\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}K(\lambda u) &= -\lambda^{p-3}\,\|u\|_p^p - (p-2)\lambda^{p-3}\,\|u\|_p^p\ln\lambda - (p-2)\lambda^{p-3}\int_\Omega u^p\ln|u|\mathrm{d}x \\ &= -\lambda^{p-3}\left((p-2)\|u\|_p^p\ln\lambda + \|u\|_p^p + (p-2)\int_\Omega u^p\ln|u|\mathrm{d}x\right).\end{aligned} \tag{2.7}$$

As a result, through taking

$$\lambda_1 = \exp\left(-\frac{(p-2)\int_\Omega |u|^p\ln|u|dx + \|u\|_p^p}{(p-2)\|u\|_p^p}\right)$$

such that $\frac{\mathrm{d}}{\mathrm{d}\lambda}K(\lambda u) > 0$ on $\lambda \in (0,\lambda_1)$, $\frac{\mathrm{d}}{\mathrm{d}\lambda}K(\lambda u) < 0$ on $\lambda \in (\lambda_1,+\infty)$, and $\frac{\mathrm{d}}{\mathrm{d}\lambda}K(\lambda_1 u) = 0$. From $K(\lambda u)|_{\lambda=0} \geq 0$, and $\lim_{\lambda\to+\infty}K(\lambda u) = -\infty$, we obtain that there exists a unique positive real number $\lambda_* > 0$ such that $K(\lambda_* u) = 0$, and $K(\lambda u) > 0$ on $\lambda \in (0,\lambda_*)$, $K(\lambda u) < 0$ on $\lambda \in (\lambda_*,+\infty)$. Thus, $\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) > 0$ on $(0,\lambda_*)$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) < 0$ on $(\lambda_*,+\infty)$, $\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)|_{\lambda=\lambda_*} = 0$. Therefore, the conclusions drawn from (2) and (3) are valid. Moreover, we obtain from (2.3)

$$\begin{aligned}I(\lambda u) &= \lambda^2\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 - \lambda^p\,\|u\|_p^p\ln\lambda - \lambda^p\int_\Omega u^p\ln|u|\mathrm{d}x + \lambda^2(g\circ\Delta u)(t) + \lambda^2\,\|\nabla u\|_2^2 \\ &= \left(\lambda^2\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 - \lambda^{p-2}\,\|u\|_p^p\ln\lambda - \lambda^{p-2}\int_\Omega u^p\ln|u|\mathrm{d}x + (g\circ\Delta u)(t) + \|\nabla u\|_2^2\right) \\ &= \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u),\end{aligned} \tag{2.8}$$

then clearly, the conclusion of (4) holds. Lemma 2.1 demonstrates that the set $\mathcal{N}$ is non-empty. □

Lemma 2.2. [16] *Let $G(t)$ be a positive C^2 function, which satisfies for $t > 0$, inequality*

$$G(t)G''(t) - (1+\alpha)(G'(t))^2 \geq 0,$$

with $\alpha > 0$. If $G(0) > 0$ and $G'(0) > 0$, then there exists a time $T^* \leq \frac{G(0)}{\alpha G'(0)}$ such that

$$\lim_{t\to T^{*-}} G(t) = \infty.$$

Lemma 2.3. [12] *If $g(t)$ satisfies (H_1), the following equation holds:*

$$\int_\Omega\left(\int_0^t g(t-s)(\Delta u(t) - \Delta u(s))\mathrm{d}s\right)^2\mathrm{d}x \leq c(g\circ\Delta u)(t),$$

where $c = 1 - l > 0$.

Lemma 2.4. [19] *Let X be a Banach space, if $f \in L^p(0,T;X)$, $\frac{\partial f}{\partial t} \in L^p(0,T;X)$, then f is a continuous injection from $[0,T]$ on to X when the value is transformed in the set of measure zero in $[0,T]$.*

Lemma 2.5. *From the definition of $E(t)$, we find that $E'(t) \leq 0$.*

Proof. Combining (1.1) and (2.1), we arrive at

$$\begin{aligned}E'(t) &= \int_\Omega u_t u_{tt}\mathrm{d}x - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 + \int_\Omega \Delta u_t \Delta u(t)\mathrm{d}x + \int_\Omega \nabla u_t \nabla u(t)\mathrm{d}x \\ &\quad - \int_0^t g(t-s)\int_\Omega \Delta u_t \Delta u(s)\mathrm{d}x\mathrm{d}s + \frac{1}{2}(g'\circ \Delta u)(t) - \int_\Omega |u|^{p-2}u_t u(t)\ln|u|\mathrm{d}x \\ &= \frac{1}{2}(g'\circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \|u_t\|_*^2 \\ &\le 0.\end{aligned} \tag{2.9}$$

From (2.9), we receive

$$E(t) + \int_0^t \|u_t\|_*^2 \mathrm{d}t \le E(0). \tag{2.10}$$

□

Lemma 2.6. [15] *Suppose that* $q < \frac{np}{n-2p}$, *i.e.,* $q < \infty$ *for* $n \le 2p$ *and* $r \le q < \frac{np}{n-2p}$ *for* $n > 2p$ *and* $r \ge 1$. *Then for any* $u \in W_0^{2,p}(\Omega)$, *it holds that*

$$\|u\|_q \le C\,\|\Delta u\|_p^\theta \|u\|_r^{1-\theta},$$

where $\theta \in (0,1)$ *is determined by*

$$\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{2}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1}$$

and the constant $C > 0$ *depends on* $n.\ p,\ q,$ *and* r.

3 Local existence of weak solutions

Within this section, the local existence of weak solutions is established via the Faedo-Galerkin technique and the contraction mapping principle. For a given positive number T, we examine the function space $\mathcal{H} = C(0,T;\,H) \cap C^1(0,T;\,H_0^1(\Omega))$, which is endowed with a norm defined by $\|v\|_{\mathcal{H}}^2 = \max_{0\le t\le T}(\|\nabla v_t\|_2^2 + l\,\|\Delta v\|_2^2)$.

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold, let* $u_0 \in H$, $u_1 \in L^2(\Omega)$, $v \in \mathcal{H}$, *then there exists a function*

$$u(x,t) \in C(0,T;\,H) \cap C^1(0,T;\,H_0^1(\Omega)) \cap C^2(0,T;\,H^{-1}),$$

satisfying

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u - \omega\Delta u_t + \mu u_t - \int_0^t g(t-s)\Delta^2 u(s)\mathrm{d}s = |v|^{p-2}v\ln|v|, & (x,t)\in\Omega\times(0,T), \\ u(x,t) = 0,\ \ \Delta u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x\in\Omega. \end{cases} \tag{3.1}$$

Proof. Let $\{\varphi_j\}_{j=1}^{+\infty} \subset H$ be the set of all standard orthogonal eigenvectors of $-\Delta$, namely,

$$-\Delta\varphi_j = \lambda_j\varphi_j$$

and

$$(\varphi_k, \varphi_l) = \vartheta_{kl} = \begin{cases} 0, & k \ne l, \\ 1, & k = l, \end{cases}$$

for any $k, l \in N$, where $\lambda_j \in R$. Then, $\{\varphi_j\}_{j=1}^{+\infty}$ constitutes a set of standard orthogonal basis for H and

$$\Delta^2\varphi_j = \lambda_j{}^2\varphi_j.$$

Next, we define the finite dimensional subspace $V_m = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$, taking $u_{0m} \in V_m, u_{1m} \in V_m$. An approximate solution to Problem (3.1) will be sought

$$u_m(x,t) = \sum_{j=1}^{m} h_j^m(t)\varphi_j(x), \; m = 1, 2, \ldots, \tag{3.2}$$

satisfying

$$\begin{cases} (u_{mtt}, \varphi_j) + (\Delta u_m, \Delta\varphi_j) + (\nabla u_m, \nabla\varphi_j) + \omega(\nabla u_{mt}, \nabla\varphi_j) \\ + \mu(u_{mt}, \varphi_j) - \int\limits_0^t g(t-s)(\Delta u_m(s), \Delta\varphi_j)\mathrm{d}s \\ = \int\limits_\Omega |v|^{p-2} v\varphi_j \ln|v|\mathrm{d}x, \\ u_m(0) = u_{0m}(x) = \sum\limits_{j=1}^{m}(u_0, \varphi_j)\varphi_j, \\ u_{mt}(0) = u_{1m}(x) = \sum\limits_{j=1}^{m}(u_1, \varphi_j)\varphi_j, \end{cases} \tag{3.3}$$

and when $m \to \infty$, we have

$$u_{0m}(x) = \sum_{j=1}^{m} h_j^m(0)\varphi_j(x) \to u_0(x) \quad \text{strongly in } H,$$
$$u_{1m}(x) = \sum_{j=1}^{m} h_{jt}^m(0)\varphi_j(x) \to u_1(x) \text{ strongly in } L^2(\Omega).$$

Thus, we have

$$\|u_{0m}\|_H \le C_0 \tag{3.4}$$

and

$$\|u_{1m}\|_2 \le C_0, \tag{3.5}$$

where C_0 represents a positive number whose value does not rely on m. In fact, let $P_j^m(t) = h_{jt}^m(t)$, further simplification of (3.3) allows us to obtain

$$\begin{cases} \dfrac{\mathrm{d}P_j^m(t)}{\mathrm{d}t} = \phi(t, h_j^m(t), P_j^m(t)), \\ P_j^m(0) = h_{jt}^m(0) = \int\limits_\Omega u_1\varphi_j\mathrm{d}x, \end{cases}$$

where

$$\phi(t, h_j^m(t), P_j^m(t)) = \int\limits_\Omega |v|^{p-2}v\ln|v|\varphi_j\mathrm{d}x - (\lambda_j^2 + \lambda_j)h_j^m(t) - (\omega\lambda_j + \mu)P_j^m(t) + \lambda_j^2\int\limits_0^t g(t-s)h_j^m(s)\mathrm{d}s.$$

By Peano's existence theorem, for any $j \ge 1$ and some $T > 0$, we obtain a solution $h_j^m(t) \in C^2[0, T]$ to (3.3). Consequently, we derive an approximate solution $u_m(x,t)$ to Problem (1.1) on the interval $[0, T]$.

Taking the first equation (3.3), we multiply both sides by $h_{jt}^{m}(t)$ and sum the resulting terms over all indices $j = 1, 2, \dots, m$, leading to the following:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_{mt}\|_2^2 + \left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u_m\|_2^2 + \|\nabla u_m\|_2^2 + (g \circ \Delta u_m)(t)\right) + 2\|u_{mt}\|_*^2 - (g' \circ \Delta u_m)(t) + g(t)\|\Delta u_m\|_2^2 \\ &= 2\int_\Omega |v|^{p-2}v\ln|v|u_{mt}\mathrm{d}x.\end{aligned} \tag{3.6}$$

By integrating (3.6) over the interval $(0, t)$ and combining it with (H_2), we obtain

$$\begin{aligned}&\|u_{mt}\|_2^2 + l\|\Delta u_m\|_2^2 + \|\nabla u_m\|_2^2 + (g \circ \Delta u_m)(t) + 2\int_0^t \|u_{mt}\|_*^2\mathrm{d}t \\ &\le \|u_{1m}\|_2^2 + \|\Delta u_{0m}\|_2^2 + \|\nabla u_{0m}\|_2^2 + 2\int_0^t\int_\Omega |v|^{p-2}v\ln|v|u_{mt}\mathrm{d}x\mathrm{d}t \\ &\le 3C_0 + 2\int_0^t\int_\Omega |v|^{p-2}v\ln|v|u_{mt}\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.7}$$

Our assessment focuses on the final term situated on the right-hand side of equation (3.7). By Hölder's inequality, we obtain

$$\int_\Omega |v|^{p-2}v\ln|v|u_{mt}\mathrm{d}x \le \||v|^{p-2}v\ln|v|\|_2\|u_{mt}\|_2. \tag{3.8}$$

Let

$$0 < \mu_1 \le \frac{2}{n-4} - p + 2,$$

we have

$$\begin{aligned}\||v|^{p-2}v\ln|v|\|_2^2 &= \int_{\{x\in\Omega:|v|<1\}} \||v|^{p-2}v\ln|v|\|^2\,\mathrm{d}x + \int_{\{x\in\Omega:|v|\ge1\}} \||v|^{p-2}v\ln|v|\|^2\,\mathrm{d}x \\ &\le (e(p-1))^{-2}|\Omega| + (e\mu_1)^{-2}\|v\|_{2(p-1+\mu_1)}^{2(p-1+\mu_1)} \\ &\le (e(p-1))^{-2}|\Omega| + (e\mu_1)^{-2}B_{2(p-1+\mu_1)}^{2(p-1+\mu_1)}\|\Delta v\|_2^{2(p-1+\mu_1)} \\ &\le C_1,\end{aligned} \tag{3.9}$$

where $B_{2(p-1+\mu_1)}$ denotes the optimal embedding constant for the embedding $H \hookrightarrow L^{2(p-1+\mu_1)}(\Omega)$ and C_1 is a fixed constant. And we have used $|\chi^\sigma\ln\chi| \le (e\sigma)^{-1}$, for $0 < \chi < 1$, while $0 \le \chi^{-\sigma}\ln\chi \le (e\sigma)^{-1}$, for $\chi \ge 1$. Substituting (3.9) into (3.8), we can obtain

$$\begin{aligned}2\int_0^t\int_\Omega |v|^{p-2}v\ln|v|u_{mt}\mathrm{d}x\mathrm{d}t &\le 2\int_0^t \||v|^{p-2}v\ln|v|\|_2\|u_{mt}\|_2\mathrm{d}t \\ &\le 2C_1\int_0^t\|u_{mt}\|_2\mathrm{d}t \\ &\le C_2T + \int_0^t\|u_{mt}\|_*^2\mathrm{d}t,\end{aligned} \tag{3.10}$$

where $C_2 = C_1{}^2C^*$ denotes a positive value that does not depend on m. We combine (3.7) and (3.10) to obtain

$$\|u_{mt}\|_2^2 + l\|\Delta u_m\|_2^2 + \|\nabla u_m\|_2^2 + (g \circ \Delta u_m)(t) + \int_0^t\|u_{mt}\|_*^2\mathrm{d}t \le 3C_0 + C_2T \le C_3, \tag{3.11}$$

where C_3 represents a positive number whose value does not rely on m. Hence, within the sequence $\{u_m\}_{m=1}^{\infty}$, a subsequence can be identified (denoted by the same notation $\{u_m\}_{m=1}^{\infty}$) such that

$$u_m \overset{*}{\rightharpoonup} u \text{ weakly star in } L^{\infty}(0, T; H_0^2(\Omega)), \tag{3.12}$$

$$u_{mt} \overset{*}{\rightharpoonup} u_t \text{ weakly star in } L^{\infty}(0, T; L^2(\Omega)), \tag{3.13}$$

$$u_m \rightharpoonup u \text{ weakly in } L^2(0, T; H_0^2(\Omega)), \tag{3.14}$$

$$u_{mt} \rightharpoonup u_t \text{ weakly in } L^2(0, T; H_0^1(\Omega)). \tag{3.15}$$

From the definition of dual norm, we acquire

$$\begin{aligned}
\|u_{mtt}\|_{H^{-1}} &= \sup_{\varphi\in H} \frac{|\langle u_{mtt}, \varphi\rangle|}{\|\varphi\|_H} \\
&\le \sup_{\varphi\in H} \frac{\int_\Omega |\varphi \cdot |v|^{p-2} v \ln|v|| \mathrm{d}x}{\|\varphi\|_H} + \sup_{\varphi\in H} \frac{\int_\Omega |\Delta\varphi \cdot \Delta u_m| \mathrm{d}x}{\|\varphi\|_H} \\
&\quad + \sup_{\varphi\in H} \frac{\int_\Omega |u_m \cdot \Delta\varphi| \mathrm{d}x}{\|\varphi\|_H} + \sup_{\varphi\in H} \frac{\omega\int_\Omega |u_{mt} \cdot \Delta\varphi| \mathrm{d}x}{\|\varphi\|_H} \\
&\quad + \sup_{\varphi\in H} \frac{|\mu|\int_\Omega |u_{mt} \cdot \varphi| \mathrm{d}x}{\|\varphi\|_H} + \sup_{\varphi\in H} \frac{\int_0^t g(t-s)\int_\Omega |\Delta u_m(s) \cdot \Delta\varphi| \mathrm{d}x\mathrm{d}s}{\|\varphi\|_H}.
\end{aligned} \tag{3.16}$$

In view of Hölder's inequality, hypothesis (H_1), (3.9), and (3.11), we obtain

$$\begin{aligned}
\|u_{mtt}\|_{H^{-1}} &\le \||v|^{p-2} v \ln|v|\|_2 + \|\Delta u_m\|_2 + \|u_m\|_2 \\
&\quad + (\omega + |\mu|)\|u_{mt}\|_2 + \|\Delta u_m\|_2 \int_0^t g(t-s)\mathrm{d}s \\
&\le C,
\end{aligned} \tag{3.17}$$

where C represents a positive number whose value does not rely on m. So we arrive at

$$u_{mtt} \overset{*}{\rightharpoonup} u_{tt}, \text{ weakly star in } L^{\infty}(0, T; H^{-1}). \tag{3.18}$$

Taking $m \to \infty$ in (3.3), and combining with (3.12), (3.13), and (3.18), we derive

$$(u_{tt}, \varphi) + (\Delta u, \Delta\varphi) + (\nabla u, \nabla\varphi) + \omega(\nabla u_t, \nabla\varphi) + \mu(u_t, \varphi) - \int_0^t g(t-s)(\Delta u(s), \Delta\varphi)\mathrm{d}s = \int_\Omega |v|^{p-2} v\varphi \ln|v| \mathrm{d}x.$$

Next, we prove the initial value condition. According to Aubin Lion's lemma (see [26]) and (3.12), (3.15), we have $u_m \to u \text{in} C(0, T; L^2(\Omega))$, then $u_m(x, 0) \to u(x, 0) \text{in} L^2(\Omega)$. In addition, $u_m(0) = u_{0m} \to u_0$ in H; thus, $u(x, 0) = u_0(x)$ in H. From (3.18), we can obtain $u_{tt} \in L^2(0, T; H^{-1})$, combining with $u_t \in L^2(0, T; L^2(\Omega))$ and Lemma 2.4, we can know $u_t \in C(0, T; H^{-1})$, so $u_{mt}(x, 0) \to u_t(x, 0) \text{in} H^{-1}$. Besides, $u_{mt}(x, 0) = u_{1m} \to u_1$; thus, $u_t(x, 0) = u_1(x)$ in $L^2(\Omega)$. The proof is complete. □

The uniqueness of solutions is established through a proof by contradiction. Assume that u and w represent two distinct solutions to Problem (3.1) that possess identical initial data. By subtracting the equations satisfied by u and w and defining the difference function as $\varphi = u - w$, we derive that

$$\varphi_{tt} + \Delta^2\varphi - \Delta\varphi - \omega\Delta\varphi_t + \mu\varphi_t - \int_0^t g(t-s)\Delta^2\varphi(s)\mathrm{d}s = 0.$$

By multiplying the derived equation by φ_t and then integrating the product over the domain $\Omega \times (0, T)$, we arrive at

$$\frac{1}{2}\|\varphi_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\Delta\varphi\|_2^2+\omega\int_0^t\|\nabla\varphi_t\|_2^2\mathrm{d}t+\mu\int_0^t\|\varphi_t\|_2^2\mathrm{d}t+\frac{1}{2}(g\circ\Delta\varphi)(t)$$
$$-\frac{1}{2}\int_0^t(g'\circ\Delta\varphi)(t)\mathrm{d}t+\frac{1}{2}\int_0^t\|\Delta\varphi\|_2^2\mathrm{d}t+\frac{1}{2}\|\nabla\varphi\|_2^2=0. \tag{3.19}$$

Combining this equation with the boundary conditions of (3.1), we can obtain $u=w$.

Theorem 3.2. *Under the assumption that conditions* (H_1) *and* (H_2) *are satisfied, then for the initial data* $u_0\in H, u_1\in L^2(\Omega)$, *there exists a unique weak solution* $u(x,t)$ *to Problem* (1.1).

Proof. For sufficiently large positive constants $M>0$ and $T>0$, define the set $M_T=\{u\in\mathcal{H}:\|u\|_{\mathcal{H}}\le M\}$. Given any $v\in\mathcal{H}$, a solution $u(x,t)$ to Problem (3.1) exists, allowing us to introduce a mapping $S:M_T\to\mathcal{H}$ such that $S(v)=u$. Our objective is to demonstrate that S acts as a contraction mapping on M_T. Through a computation analogous to that in (3.7), combined with applications of Hölder's inequality and Young's inequality, we deduce

$$\begin{aligned}
&\|u_t\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+(g\circ\Delta u)(t)+2\int_0^t\|u_t\|_*^2\mathrm{d}t\\
&\le\|u_1\|_2^2+\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2+2\int_0^t\int_\Omega|v|^{p-2}v\ln|v|u_t\mathrm{d}x\mathrm{d}t\\
&\le\|u_1\|_2^2+\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2+2\int_0^t\||v|^{p-2}v\ln|v|\|_2\|u_t\|_2\mathrm{d}t\\
&\le\|u_1\|_2^2+\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2+2\int_0^t((e(p-1))^{-2}|\Omega|+(e\mu)^{-2}B_{2(p-1+\mu_1)}^{2(p-1+\mu_1)}\|\Delta v\|_2^{2(p-1+\mu_1)})^{\frac{1}{2}}\|u_t\|_2\mathrm{d}t\\
&\le\|u_1\|_2^2+\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2+2\int_0^t\|u_t\|_*^2\mathrm{d}t+C_4T(1+M^{2(p-1+\mu_1)}),
\end{aligned}$$

where

$$C_4=\frac{C^*}{2}\max\left\{(e(p-1))^{-2}|\Omega|,\frac{(e\mu)^{-2}B_{2(p-1+\mu_1)}^{2(p-1+\mu_1)}}{l^{p-1+\mu_1}}\right\}.$$

Then,

$$\|u_t\|_2^2+l\|\Delta u\|_2^2+\|\nabla u\|_2^2+(g\circ\Delta u)(t)\le\|u_1\|_2^2+\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2+C_4T(1+M^{2(p-1+\mu_1)}). \tag{3.20}$$

We take $M>0$ large enough so that

$$\|u_1\|_2^2+\|\Delta u_0\|_2^2+\|\nabla u_0\|_2^2\le\frac{M^2}{2}, \tag{3.21}$$

we select $T>0$ to be sufficiently small in such a way that

$$C_4T(1+M^{2(p-1+\mu_1)})\le\frac{M^2}{2}. \tag{3.22}$$

By (3.20)–(3.22), we deduce that $\|u\|_{\mathcal{H}}\le M$, i.e., $S(M_T)\subset M_T$. The subsequent task is to prove that S functions as a contraction mapping. Consider arbitrary elements $v_1,v_2\in M_T$, denote $u_1=S(v_1)$, $u_2=S(v_2)$, and define $z=u_1-u_2$, then z satisfies

$$\begin{cases} z_{tt} + \Delta^2 z - \Delta z - \omega\Delta z_t + \mu z_t - \int\limits_0^t g(t-s)\Delta^2 z(s)\mathrm{d}s \\ = |v_1|^{p-2}v_1\ln|v_1| - |v_2|^{p-2}v_2\ln|v_2|, & (x,t)\in\Omega\times(0,T), \\ z(x,t)=0, \Delta z(x,t)=0, & (x,t)\in\partial\Omega\times(0,T), \\ z(0)=0, z_t(0)=0, & x\in\Omega. \end{cases} \tag{3.23}$$

Taking the first equation in (3.23), we multiply both sides by z_t and then carry out an integration over the domain $\Omega\times(0,T)$, resulting in

$$\begin{aligned} &\|z_t\|_2^2 + l\,\|\Delta z\|_2^2 + \|\nabla z\|_2^2 + (g\circ\Delta z)(t) + 2\int\limits_0^t \|z_t\|_*^2\mathrm{d}t \\ \le\ & 2\int\limits_0^t\int\limits_\Omega (|v_1|^{p-2}v_1\ln|v_1| - |v_2|^{p-2}v_2\ln|v_2|)z_t\mathrm{d}x\mathrm{d}t + \int\limits_0^t (g'\circ\Delta z)(t)\mathrm{d}t - \int\limits_0^t g(t)\|\Delta z\|_2^2\mathrm{d}t \\ \le\ & 2\int\limits_0^t\int\limits_\Omega (|v_1|^{p-2}v_1\ln|v_1| - |v_2|^{p-2}v_2\ln|v_2|)z_t\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Furthermore, by the mean value theorem, we acquire

$$\begin{aligned} &\|z_t\|_2^2 + l\,\|\Delta z\|_2^2 + \|\nabla z\|_2^2 + (g\circ\Delta z)(t) + 2\int\limits_0^t \|z_t\|_*^2\mathrm{d}t \\ \le\ & 2\int\limits_0^t\int\limits_\Omega ((p-1)|\xi|^{p-2}\ln|\xi| + |\xi|^{p-2})(v_1-v_2)z_t\mathrm{d}x\mathrm{d}t \\ \le\ & 2\int\limits_0^t\int\limits_\Omega |\xi|^{p-2}(v_1-v_2)z_t\mathrm{d}x\mathrm{d}t + 2(p-1)\int\limits_0^t\int\limits_\Omega |\xi|^{p-2}\ln|\xi|(v_1-v_2)z_t\mathrm{d}x\mathrm{d}t \\ =\ & I_1 + I_2, \end{aligned} \tag{3.24}$$

where

$$\xi = \theta v_1 + (1-\theta)v_2,\ 0<\theta<1.$$

Through the application of Hölder's inequality and Young's inequality, we can derive

$$\begin{aligned} I_1 &\le 2\int\limits_0^t \|\xi\|_{n(p-2)}^{p-2}\|v_1-v_2\|_{\frac{2n}{n-2}}\|z_t\|_2\mathrm{d}t \\ &\le 2\int\limits_0^t B_{n(p-2)}^{p-2}\,\|\Delta\xi\|_2^{p-2}B_{\frac{2n}{n-2}}\,\|\Delta v_1-\Delta v_2\|_2\|z_t\|_2\mathrm{d}t \\ &\le C_5\varepsilon\int\limits_0^t \|\Delta\xi\|_2^{2(p-2)}\|\Delta v_1-\Delta v_2\|_2^2\mathrm{d}t + \frac{1}{\varepsilon}C^*\int\limits_0^t \|z_t\|_*^2\mathrm{d}t, \end{aligned} \tag{3.25}$$

where

$$C_5 = B_{n(p-2)}^{2(p-2)}B_{\frac{2n}{n-2}}^2 > 0,\ \varepsilon>0.$$

Let $\varepsilon = C^*$, we obtain

$$\begin{aligned} I_1 &\le C_5C^*\int\limits_0^t \|\Delta\xi\|_2^{2(p-2)}\|\Delta v_1-\Delta v_2\|_2^2\mathrm{d}t + \int\limits_0^t \|z_t\|_*^2\mathrm{d}t \\ &\le C_6M^{2(p-2)}T\,\|v_1-v_2\|_{\mathcal{H}}^2 + \int\limits_0^t \|z_t\|_*^2\mathrm{d}t, \end{aligned} \tag{3.26}$$

where $C_6 = C_5C^*l^{1-p} > 0$, the symbols $B_{n(p-2)}$ and $B_{\frac{2n}{n-2}}$ denote the optimal embedding constants for $H_0^2(\Omega) \hookrightarrow L^{n(p-2)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, respectively. We make the following estimate for I_2. We can derive from Hölder's inequality that

$$\begin{aligned} I_2 &\le 2(p-1)\int_0^t \||\xi|^{p-2}\ln|\xi|\|_n \|v_1 - v_2\|_{\frac{2n}{n-2}} \|z_t\|_2 \,\mathrm{d}x\mathrm{d}t \\ &\le 2(p-1)\int_0^t \||\xi|^{p-2}\ln|\xi|\|_n B_{\frac{2n}{n-2}} \|\Delta v_1 - \Delta v_2\|_2 \|z_t\|_2 \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.27}$$

By similar argument to (3.9), we can obtain

$$\begin{aligned} \||\xi|^{p-2}\ln|\xi|\|_n^n &\le (e(p-2))^{-n}|\Omega| + \int_{\{x\in\Omega:|\xi|\ge 1\}} ||\xi|^{-\mu_1}\ln|\xi||^n |\xi|^{n(p-2+\mu_1)}\mathrm{d}x \\ &\le (e(p-2))^{-n}|\Omega| + B_{n(p-2+\mu_1)}^{n(p-2+\mu_1)}(e\mu_1)^{-n}\|\Delta\xi\|_2^{n(p-2+\mu_1)} \\ &\le C_7(1 + M^{n(p-2+\mu_1)}), \end{aligned} \tag{3.28}$$

where

$$C_7 = \max\left\{(e(p-2))^{-n}|\Omega|, B_{n(p-2+\mu_1)}^{n(p-2+\mu_1)}(e\mu_1)^{-n}l^{-\frac{n(p-2+\mu_1)}{2}}\right\},$$

the notation $B_{n(p-2+\mu_1)}$ denotes the optimal embedding constant corresponding to the embedding $H_0^2(\Omega) \hookrightarrow L^{n(p-2+\mu_1)}(\Omega)$. Applying this to (3.27), we can derive

$$\begin{aligned} I_2 &\le 2(p-1)\int_0^t C_7^{\frac{1}{n}}(1 + M^{n(p-2+\mu_1)})^{\frac{1}{n}} B_{\frac{2n}{n-2}} \|\Delta v_1 - \Delta v_2\|_2 \|z_t\|_2 \,\mathrm{d}x\mathrm{d}t \\ &\le l^{-1}(p-1)^2 C_7^{\frac{2}{n}}(1 + M^{p-2+\mu_1})^2 B_{\frac{2n}{n-2}}^2 T\varepsilon \|v_1 - v_2\|_{\mathcal{H}}^2 + \frac{1}{\varepsilon}C^*\int_0^t \|z_t\|_*^2 \,\mathrm{d}t. \end{aligned} \tag{3.29}$$

Let $\varepsilon = C^*$, we have

$$I_2 \le C_8T(1 + M^{p-2+\mu_1})^2\|v_1 - v_2\|_{\mathcal{H}}^2 + \int_0^t \|z_t\|_*^2 \,\mathrm{d}t, \tag{3.30}$$

where

$$C_8 = l^{-1}(p-1)^2 C_7^{\frac{2}{n}} B_{\frac{2n}{n-2}}^2 C^* > 0.$$

Combining (3.24), (3.26), and (3.30), we have

$$\|z_t\|_2^2 + l\,\|\Delta z\|_2^2 + \|\nabla z\|_2^2 + (g \circ \Delta z)(t) \le T(C_8(1 + M^{p-2+\mu_1})^2 + C_6M^{2(p-2)})\|v_1 - v_2\|_H^2. \tag{3.31}$$

By selecting $T > 0$ to be sufficiently small such that the constant

$$C_R = T(C_8(1 + M^{p-2+\mu_1})^2 + C_6M^{2(p-2)}) < 1,$$

we conclude

$$\|z\|_{\mathcal{H}} = \|S(v_1) - S(v_2)\|_{\mathcal{H}} \le C_R\,\|v_1 - v_2\|_{\mathcal{H}}.$$

Thus, the contraction mapping principe ensures the uniqueness and the existence of weak solution for Problem (1.1). The proof is complete. □

4 Global solution and decay estimation

Within this part, we analyze the global existence and decay estimations for the weak solutions of Problem (1.1). Initially, we examine the invariance characteristic of the stable set W. In other words,

Lemma 4.1. *Suppose $u_0 \in W$, $u_1 \in L^2(\Omega)$. When $E(0) < d$, for every t within the interval $[0, T)$, the function $u(x, t)$ remains in the set W. Additionally, we also obtain $\|u\|_p^p < p^2 d$.*

Proof. Let $u(x, t)$ denote the weak solution of Problem (1.1) with $u_0 \in W$. This implies that $I(u_0) > 0$. Also, let T represent the maximum time for which the weak solution $u(x, t)$ exists. Based on Lemma 2.5 and given that the $E(0) < d$, we obtain the following:

$$E(t) \le E(0) < d, \forall t \in [0, T). \tag{4.1}$$

Next, we assert if $u_0 \in W$, then $u(t) \in W$. Alternatively, considering the continuity of $I(u)$, there must exist a time point t_0 within the interval $[0, T)$, where $I(u(t))$ remains positive for all t in $[0, t_0)$ and becomes zero exactly at t_0, $u(t_0) \ne 0$. This situation implies that the function $u(t_0) \in \mathcal{N}$. When taking the definition of d into account, it follows that $d \le J(u(t_0))$. Based on how $E(t)$ is defined, we can then derive that

$$E(t_0) = \frac{1}{2}\|u_t(t_0)\|_2^2 + J(u(t_0)) \ge d.$$

Then, we derive a result that contradicts equation (4.1). Consequently, $u(t) \in W$ for all $t \in [0, T)$. Now, we prove that $\|u\|_p^p < p^2 d$. From $u(t) \in W$, we have $I(u(t)) > 0$, then

$$\begin{aligned} d &> E(t) \\ &= \frac{1}{2}\|u_t\|_2^2 + J(u) \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}I(u) + \frac{p-2}{2p}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p^2}\|u\|_p^p + \frac{p-2}{2p}(g \circ \Delta u)(t) \\ &> \frac{1}{p^2}\|u\|_p^p. \end{aligned} \tag{4.2}$$

Therefore,

$$\|u(t)\|_p^p < p^2 d.$$ □

In this section, our focus is to demonstrate the global existence of weak solutions for Problem (1.1), under the scenario where the initial energy condition $E(0) < d$ holds in a subcritical regime.

Theorem 4.1. *Assume (H_1) and (H_2) holds, $u_0 \in W$, $u_1 \in L^2(\Omega)$, if $E(0) < d$, then Problem (1.1) has a global weak solution $u(x, t) \in C(0, \infty; H) \cap C^1(0, \infty; H_0^1(\Omega)) \cap C^2(0, \infty; H^{-1})$.*

Proof. By integrating the results from (2.9) and (4.1), we derive

$$\begin{aligned} d > \frac{1}{2}\|u_t\|_2^2 &+ \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) - \frac{1}{p}\int_\Omega |u|^p \ln|u|\mathrm{d}x + \frac{1}{p^2}\|u\|_p^p \\ &+ \omega\int_0^t \|\nabla u_t\|_2^2\mathrm{d}s - \frac{1}{2}\int_0^t g' \circ \Delta u(s)\mathrm{d}s + \frac{1}{2}\int_0^t g(s)\|\Delta u\|_2^2\mathrm{d}s + \mu\int_0^t \|u_t\|_2^2\mathrm{d}s. \end{aligned} \tag{4.3}$$

Through a derivation analogous to that in (3.9), it can be deduced that

$$\begin{aligned} \int_\Omega |u|^p \ln|u|\mathrm{d}x &= \int_{\Omega_1=\{x\in\Omega:|u(x)|<1\}} |u|^p \ln|u|\mathrm{d}x + \int_{\Omega_2=\{x\in\Omega:|u(x)|\ge1\}} |u|^p \ln|u|\mathrm{d}x \\ &\le \int_{\Omega_2=\{x\in\Omega:|u(x)|\ge1\}} u^{-\sigma}\,|u|^{p+\sigma} \ln|u|\mathrm{d}x \\ &\le (e\sigma)^{-1}\|u\|_{p+\sigma}^{p+\sigma}. \end{aligned} \tag{4.4}$$

By applying Lemma 2.6 together with Young's inequality, for $\varepsilon \in (0,1)$, we acquire

$$\begin{aligned} \|u\|_{p+\sigma}^{p+\sigma} &\le C\,\|\Delta u\|_2^{(p+\sigma)\theta} \cdot \|u\|_p^{(p+\sigma)(1-\theta)} \\ &\le \varepsilon(\|\Delta u\|_2^{(p+\sigma)\theta})^{p^*} + C(\varepsilon)(\|u\|_p^{(p+\sigma)(1-\theta)})^{q^*} \\ &\le \varepsilon\,\|\Delta u\|_2^2 + C(\varepsilon)\|u\|_p^{\alpha}, \end{aligned} \tag{4.5}$$

where

$$\theta = \left(\frac{1}{p} - \frac{1}{p+\sigma}\right)\left(\frac{2}{n} - \frac{1}{2} + \frac{1}{p}\right)^{-1}.$$

We choose

$$0 < \sigma < 2\left(1 + \frac{2p}{n}\right) - p,$$

then

$$p^* = \frac{2}{(p+\sigma)\theta} > 1,\ q^* = \frac{2}{2-(p+\sigma)\theta} > 1$$

and

$$\alpha = \frac{2(p+\sigma)(1-\theta)}{2-(p+\sigma)\theta} > p.$$

We combine (4.3)–(4.5) to obtain

$$\begin{aligned} &\frac{1}{2}\,\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{1}{2}\,\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) + \frac{1}{p^2}\,\|u\|_p^p + \omega\int_0^t \|\nabla u_t\|_2^2 \mathrm{d}s \\ &\quad + \mu\int_0^t \|u_t\|_2^2 \mathrm{d}s + \frac{1}{2}\int_0^t g(s)\|\Delta u\|_2^2 \mathrm{d}s \\ &\quad < d + \frac{1}{p}\int_\Omega |u|^p \ln|u| \mathrm{d}x \\ &\quad < d + \frac{\varepsilon}{pe\sigma}\,\|\Delta u\|_2^2 + \frac{C(\varepsilon)}{pe\sigma}\,\|u\|_p^{\alpha}. \end{aligned}$$

The aforementioned inequality becomes

$$\begin{aligned} &\frac{1}{2}\,\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s - \frac{\varepsilon}{pe\sigma}\right)\|\Delta u\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t) \\ &\quad + \omega\int_0^t \|\nabla u_t\|_2^2 \mathrm{d}s + \mu\int_0^t \|u_t\|_2^2 \mathrm{d}s + \frac{1}{2}\int_0^t g(s)\|\Delta u\|_2^2 \mathrm{d}s \\ &< d + \frac{C(\varepsilon)}{pe\sigma}\,\|u\|_p^{\alpha}. \end{aligned} \tag{4.6}$$

From Lemma 4.1, we obtain

$$\|u\|_p^{\alpha} = (\|u\|_p^p)^{\frac{\alpha}{p}} < (p^2 d)^{\frac{\alpha}{p}}. \tag{4.7}$$

Substituting (4.7) into (4.6), we have

$$
\begin{aligned}
&\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s-\frac{\varepsilon}{pe\sigma}\right)\|\Delta u\|_2^2+\frac{1}{2}(g\circ\Delta u)(t)\\
&\quad+\omega\int_0^t\|\nabla u_t\|_2^2\mathrm{d}s+\mu\int_0^t\|u_t\|_2^2\mathrm{d}s+\frac{1}{2}\int_0^t g(s)\|\Delta u\|_2^2\mathrm{d}s\\
&<\ C_d,
\end{aligned}
$$

where C_d denotes a positive constant relying exclusively on the parameter d. We take ε to be sufficiently small so that $l-\frac{\varepsilon}{pe\sigma}>0$. This choice leads to the following bound:

$$
\|u_t\|_2^2+\|\Delta u\|_2^2+(g\circ\Delta u)(t)+\int_0^t\|\nabla u_t\|_2^2\mathrm{d}s+\frac{1}{2}\int_0^t g(s)\|\Delta u\|_2^2\mathrm{d}s<C_d. \tag{4.8}
$$

Using an estimation similar to (3.9), we can obtain

$$
\||u|^{p-2}u\ln|u|\|_2<C_d. \tag{4.9}
$$

By the definition of the dual norm and the first equation of Problem (1.1), we can derive

$$
\begin{aligned}
\|u_{tt}\|_{H^{-1}}&=\sup_{\varphi\in H}\frac{|\langle u_{tt},\varphi\rangle|}{\|\varphi\|_H}\\
&\le\sup_{\varphi\in H}\frac{\int_\Omega|\varphi\cdot|u|^{p-2}u\ln|u||\mathrm{d}x}{\|\varphi\|_H}+\sup_{\varphi\in H}\frac{\int_\Omega|\Delta\varphi\cdot\Delta u|\mathrm{d}x}{\|\varphi\|_H}\\
&\quad+\sup_{\varphi\in H}\frac{\int_\Omega|u\cdot\Delta\varphi|\mathrm{d}x}{\|\varphi\|_H}+\sup_{\varphi\in H}\frac{\omega\int_\Omega|u_t\cdot\Delta\varphi|\mathrm{d}x}{\|\varphi\|_H}\\
&\quad+\sup_{\varphi\in H}\frac{|\mu|\int_\Omega|u_t\cdot\varphi|\mathrm{d}x}{\|\varphi\|_H}+\sup_{\varphi\in H}\frac{\int_0^t g(t-s)\int_\Omega|\Delta u(s)\cdot\Delta\varphi|\mathrm{d}x\mathrm{d}s}{\|\varphi\|_H}.
\end{aligned}
$$

In view of Hölder's inequality, hypothesis (H_2), (4.8), and (4.9), we obtain

$$
\begin{aligned}
\|u_{tt}\|_{H^{-1}}&\le\||u|^{p-2}u\ln|u|\|_2+\|\Delta u\|_2+\|u\|_2+(\omega+|\mu|)\|u_t\|_2+\|\Delta u\|_2\int_0^t g(t-s)\mathrm{d}s\\
&\le C_d.
\end{aligned}
$$

This completes the proof for $u\in C^2(0,T;H^{-1})$. The subsequent proof follows an approach entirely analogous to the demonstration outlined in Theorem 3.1. By leveraging these estimations, we are justified in extending the maximal existence time t to $+\infty$. Consequently, it can be concluded that Problem (1.1) possesses a global weak solution $u(t)\in W$. □

In the subsequent discussion, we demonstrate the decay characteristics of the weak solution as outlined in the following.

Define function

$$
H(t)\coloneqq E(t)+\varepsilon\int_\Omega u_t u\mathrm{d}x+\frac{\varepsilon\omega}{2}\|u\|^2_{H_0^1(\Omega)},
$$

where ε stands for a positive number.

Lemma 4.2. *Let $u_0\in W$, $u_1\in L^2(\Omega)$, then for a sufficiently small $\varepsilon>0$, there are two positive numbers α_1 and α_2 such that*

$$
\alpha_1E(t)\le H(t)\le\alpha_2E(t).
$$

Proof. According to Hölder's, Young's, and Poincaré's inequality and (2.2), we obtain

$$\begin{aligned}|H(t)-E(t)| &= \left|\varepsilon\int_\Omega u_t u\mathrm{d}x + \frac{\varepsilon\omega}{2}\,\|u\|^2_{H_0^1(\Omega)}\right| \\ &\le \varepsilon\left(\varepsilon\,\|u\|_2^2 + \frac{1}{4\varepsilon}\,\|u_t\|_2^2\right) + \frac{\varepsilon\omega}{2}\,\|\nabla u\|_2^2 \\ &= \frac{1}{4}\,\|u_t\|_2^2 + \varepsilon^2\,\|u\|_2^2 + \frac{\varepsilon\omega}{2}\,\|\nabla u\|_2^2 \\ &\le \frac{1}{2}E(t) + \varepsilon^2 C_P\,\|\nabla u\|_2^2 + \frac{\varepsilon\omega}{2}\,\|\nabla u\|_2^2,\end{aligned}$$

where C_P is Poincaré's constant. We choose

$$\varepsilon < \min\left\{\sqrt{\frac{1}{6C_P}}, \frac{1}{3\omega}\right\},$$

then

$$|H(t)-E(t)| \le \frac{5}{6}E(t),$$

so we acquire

$$\alpha_1 E(t) \le H(t) \le \alpha_2 E(t),$$

where $\alpha_1 = \frac{1}{6}$ and $\alpha_2 = \frac{11}{6}$. □

Theorem 4.2. *Assume* (H_1)–(H_3) *is true,* $u_0 \in W$, $u_1 \in L^2(\Omega)$, *if* $E(0)$ *satisfies*

$$E(0) < \min\left\{\left(\frac{(l-\delta)e\sigma}{B_{p+\sigma}^{p+\sigma}}\right)^{\frac{p+\sigma-2}{2}}\frac{(p-2)l}{2p}, d\right\},$$

then there are two positive numbers K_1 *and* K_2 *in such a way that* $E(t)$ *satisfies the following decay property*:

$$E(t) \le K_1\exp\left(-\int_0^t K_2\lambda(t)\mathrm{d}t\right),$$

where $0 < \delta < l$, $\lambda(t)$ *will be given later.*

Proof. Differentiating the function $H(t)$ with respect to t yields

$$\begin{aligned}H'(t) &= E'(t) + \varepsilon\int_\Omega u_{tt}u\mathrm{d}x + \varepsilon\int_\Omega |u_t|^2\mathrm{d}x + \varepsilon\omega\int_\Omega \nabla u\nabla u_t\mathrm{d}x \\ &= E'(t) + \varepsilon\int_\Omega |u|^p\ln|u|\mathrm{d}x + \varepsilon\omega\int_\Omega \Delta u_t u\mathrm{d}x \\ &\quad - \varepsilon\mu\int_\Omega u_t u\mathrm{d}x + \varepsilon\,\|u_t\|_2^2 - \varepsilon\,\|\Delta u\|_2^2 - \varepsilon\,\|\nabla u\|_2^2 \\ &\quad + \varepsilon\omega\int_\Omega \nabla u\nabla u_t\mathrm{d}x + \varepsilon\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{4.10}$$

By making use of Young's and Hölder's inequality, and referring to Lemma 2.3, we are able to infer that

$$\begin{aligned}
&\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\mathrm{d}s\mathrm{d}x\\
&= \int_\Omega \Delta u(t)\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))\mathrm{d}s\mathrm{d}x\\
&\quad + \int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(t)\mathrm{d}s\mathrm{d}x\\
&= \delta\, \|\Delta u\|_2^2 + \frac{1}{4\delta}\left\|\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))\mathrm{d}s\right\|_2^2 + (1-l)\|\Delta u\|_2^2\\
&\le (\delta + 1 - l)\|\Delta u\|_2^2 + \frac{1-l}{4\delta}(g\circ \Delta u)(t)
\end{aligned} \tag{4.11}$$

and

$$\int_\Omega u_t u\mathrm{d}x \le \delta\, \|u\|_2^2 + \frac{1}{4\delta}\,\|u_t\|_2^2, \tag{4.12}$$

where $0<\delta<l$ is a fixed constant. Combining (2.9), (4.10), (4.11), and (4.12), we derive

$$\begin{aligned}
H'(t) \le{}& \frac{1}{2}(\mathrm{g}'\circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|_2^2 - \omega\, \|\nabla u_t\|_2^2 - \varepsilon\, \|\nabla u\|_2^2\\
&+ \varepsilon\frac{1-l}{4\delta}(\mathrm{g}\circ \Delta u)(t) + \left(-\mu + \frac{\varepsilon\mu}{4\delta} + \varepsilon\right)\|u_t\|_2^2\\
&+ \varepsilon\int_\Omega |u|^p \ln|u|\mathrm{d}x + \varepsilon(\delta - l)\|\Delta u\|_2^2 + \varepsilon\mu\delta\, \|u\|_2^2.
\end{aligned} \tag{4.13}$$

From (2.1), we can obtain

$$\begin{aligned}
\varepsilon m E(t) ={}& -\frac{\varepsilon m}{2}\,\|u_t\|_2^2 + \frac{\varepsilon m}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{\varepsilon m}{2}\,\|\nabla u\|_2^2\\
&+ \frac{\varepsilon m}{2}(\mathrm{g}\circ \Delta u)(t) - \frac{\varepsilon m}{p}\int_\Omega |u|^p \ln|u|\mathrm{d}x + \frac{\varepsilon m}{p^2}\,\|u\|_p^p,
\end{aligned} \tag{4.14}$$

where m denotes a positive constant. Hence, we derive that

$$\begin{aligned}
H'(t) \le{}& -\varepsilon m E(t) + \frac{1}{2}(\mathrm{g}'\circ \Delta u)(t) + \varepsilon\left(\frac{m}{2} - 1\right)\|\nabla u\|_2^2\\
&- \frac{1}{2}g(t)\|\Delta u\|_2^2 - \omega\, \|\nabla u_t\|_2^2 - \left(\mu - \frac{\varepsilon\mu}{4\delta} - \varepsilon - \frac{\varepsilon m}{2}\right)\|u_t\|_2^2\\
&+ \varepsilon\left(\frac{m}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right) + \delta - l\right)\|\Delta u\|_2^2 + \frac{\varepsilon m}{p^2}\,\|u\|_p^p\\
&+ \varepsilon\left(1-\frac{m}{p}\right)\int_\Omega |u|^p \ln|u|\mathrm{d}x + \varepsilon\left(\frac{m}{2} + \frac{1-l}{4\delta}\right)(\mathrm{g}\circ \Delta u)(t).
\end{aligned} \tag{4.15}$$

By (2.9) and (4.2), we can obtain

$$\|\Delta u\|_2^2 \le \frac{2p}{(p-2)l}E(0). \tag{4.16}$$

So we obtain

$$\begin{aligned}\|u\|_p^p &\le B_p^p\ \|\Delta u\|_2^p\\ &\le B_p^p(\|\Delta u\|_2^2)^{\frac{p-2}{2}}\|\Delta u\|_2^2\\ &\le B_p^p\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}}\ \|\Delta u\|_2^2,\end{aligned}\tag{4.17}$$

in which B_p denotes the best embedding constant for $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$. By (4.4) and (4.16), we derive

$$\begin{aligned}\int_\Omega |u|^p \ln|u|\mathrm{d}x &\le (e\sigma)^{-1}\|u\|_{p+\sigma}^{p+\sigma}\\ &\le (e\sigma)^{-1}B_{p+\sigma}^{p+\sigma}\ \|\Delta u\|_2^{p+\sigma}\\ &\le (e\sigma)^{-1}B_{p+\sigma}^{p+\sigma}(\|\Delta u\|_2^2)^{\frac{p+\sigma-2}{2}}\|\Delta u\|_2^2\\ &\le (e\sigma)^{-1}B_{p+\sigma}^{p+\sigma}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p+\sigma-2}{2}}\ \|\Delta u\|_2^2,\end{aligned}\tag{4.18}$$

in which $B_{p+\sigma}$ represents the optimal embedding constant for $H_0^2(\Omega) \hookrightarrow L^{p+\sigma}(\Omega)$. From a combination of (4.15), (4.17), and (4.18), we derive

$$\begin{aligned}H'(t) \le &-\varepsilon mE(t) + \frac{1}{2}(g' \circ \Delta u)(t) + \varepsilon\left(\frac{m}{2} - 1\right)\ \|\nabla u\|_2^2\\ &- \frac{1}{2}g(t)\|\Delta u\|_2^2 - \omega\ \|\nabla u_t\|_2^2 - \left(\mu - \frac{\varepsilon\mu}{4\delta} - \varepsilon - \frac{\varepsilon m}{2}\right)\ \|u_t\|_2^2\\ &+ \varepsilon\left(\frac{m}{2} + \delta - l + (e\sigma)^{-1}B_{p+\sigma}^{p+\sigma}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p+\sigma-2}{2}}\right.\\ &- \frac{m}{p}(e\sigma)^{-1}B_{p+\sigma}^{p+\sigma}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p+\sigma-2}{2}}\\ &\left.+ \frac{m}{p^2}B_p^p\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}}\right)\ \|\Delta u\|_2^2 + \varepsilon\left(\frac{m}{2} + \frac{1-l}{4\delta}\right)(g \circ \Delta u)(t).\end{aligned}\tag{4.19}$$

Since

$$E(0) < \left(\frac{(l-\delta)e\sigma}{B_{p+\sigma}^{p+\sigma}}\right)^{\frac{2}{p+\sigma-2}}\frac{(p-2)l}{2p},$$

we have

$$\delta - l + (e\sigma)^{-1}B_{p+\sigma}^{p+\sigma}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p+\sigma-2}{2}} < 0.$$

By choosing $0 < m < 2$ small enough such that

$$\frac{m}{2} + \frac{m}{p^2}B_p^p\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} + \delta - l + (e\sigma)^{-1}B_{p+\sigma}^{p+\sigma}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p+\sigma-2}{2}} < 0,$$

and ε small enough to satisfy

$$\mu - \frac{\varepsilon\mu}{4\delta} - \varepsilon - \frac{\varepsilon m}{2} > 0.$$

Thus, we arrive

$$H'(t) \le -\Lambda_1 E(t) + \Lambda_2 (g \circ \Delta u)(t), \tag{4.20}$$

where

$$\Lambda_1 = \varepsilon m > 0, \Lambda_2 = \varepsilon\left(\frac{m}{2} + \frac{1-l}{4\delta}\right) > 0.$$

By taking each side of (4.20) and multiplying by $\xi(t) > 0$, then integrating condition (H_3) in conjunction with (2.9), we obtain

$$\begin{aligned} H'(t)\xi(t) &\le -\Lambda_1 \xi(t)E(t) + \Lambda_2 \xi(t)(g \circ \Delta u)(t) \\ &\le -\Lambda_1 \xi(t)E(t) + \Lambda_2 (g\xi \circ \Delta u)(t) \\ &\le -\Lambda_1 \xi(t)E(t) + \Lambda_2 (-g' \circ \Delta u)(t) \\ &\le -\Lambda_1 \xi(t)E(t) - 2\Lambda_2 E'(t). \end{aligned}$$

Define

$$Z(t) \coloneqq \xi(t)H(t) + CE(t).$$

Based on the definition of $\xi(t)$, it is straightforward to deduce the existence of positive constants α_3 and α_4 satisfying

$$\alpha_3 E(t) \le Z(t) \le \alpha_4 E(t),$$

then

$$\begin{aligned} Z'(t) &= \xi'(t)H(t) + \xi(t)H'(t) + CE'(t) \\ &\le \xi'(t)H(t) - \Lambda_1 \xi(t)E(t) - 2\Lambda_2 E'(t) + CE'(t) \\ &\le -(\Lambda_1 \xi(t) - \alpha_1 \xi'(t))E(t) + (C - 2\Lambda_2)E'(t). \end{aligned}$$

Let

$$\lambda(t) = \Lambda_1 \xi(t) - \alpha_1 \xi'(t), C = 2\Lambda_2,$$

so

$$Z'(t) \le -\lambda(t)E(t) \le -\frac{1}{\alpha_4}\lambda(t)Z(t).$$

Through simple calculation, we obtain

$$Z(t) \le Z(t_0)\exp\left(-\int_{t_0}^{t} \frac{1}{\alpha_4}\lambda(t)\mathrm{d}t\right),$$

which means

$$E(t) \le K_1 \exp\left(-\int_{t_0}^{t} K_2 \lambda(t)\mathrm{d}t\right),$$

where $K_1 = \frac{Z(t_0)}{\alpha_3}$, $K_2 = \frac{1}{\alpha_4}$. □

5 Blow-up of weak solutions

In this section, we utilize the convexity method to establish the finite-time blow-up of weak solutions to Problem (1.1) under varying energy-level conditions and derive an upper bound estimation for the blow-up time.

Theorem 5.1. *Under the assumptions that conditions (H_1) and (H_2) are satisfied, with $u_0 \in V$ and $u_1 \in L^2(\Omega)$, if*

(1) *the initial energy* $E(0)$ *takes the form* $E(0) = \alpha d$, *where the parameter* α *satisfies* $\alpha < 1$,
(2) *the kernel function* $g(s)$ *fulfills the integral inequality*

$$\int_0^t g(s)\mathrm{d}s \le \frac{p-2}{p-2+\frac{1}{(1-\alpha^-)^2 p + 2\alpha^-(1-\alpha^-)}},$$

where $\alpha^- = \max\{0, \alpha\}$, *then the solution* $u(x,t)$ *to* Problem (1.1) *exhibits finite-time blow-up behavior; this phenomenon is characterized by*

$$\lim_{t\to T^{*-}} L(t) = \infty,$$

with the function $L(t)$ *to be defined subsequently.*

Proof. We define the function

$$L(t) \coloneqq \|u\|_2^2 + \int_0^t \|u\|_*^2 \mathrm{d}t + (T-t)\|u_0\|_*^2 + b(t+T_0)^2, \tag{5.1}$$

where $T_0 > 0$, $b \ge 0$. Then, we have

$$L(t) > 0,\ t \in [0, T]. \tag{5.2}$$

Through straightforward computation, one can derive

$$L'(t) = 2\int_\Omega u u_t \mathrm{d}x + 2\int_0^t (u, u_t)_* \mathrm{d}t + 2b(t+T_0) \tag{5.3}$$

and

$$\begin{aligned} L''(t) &= 2\,\|u_t\|_2^2 + 2\int_\Omega u u_{tt}\mathrm{d}x + 2(u,u_t)_* + 2b \\ &= 2\,\|u_t\|_2^2 + 2\int_\Omega |u|^p \ln|u| \mathrm{d}x - 2\,\|\Delta u\|_2^2 - 2\,\|\nabla u\|_2^2 \\ &\quad + 2\int_0^t g(t-s)(\Delta u(s), \Delta u(t))\mathrm{d}s + 2b \\ &= 2\,\|u_t\|_2^2 + 2\int_\Omega |u|^p \ln|u| \mathrm{d}x - 2\,\|\nabla u\|_2^2 \\ &\quad - 2\int_0^t g(t-s)(\Delta u(t), \Delta u(t) - \Delta u(s))\mathrm{d}s \\ &\quad - 2\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + 2b. \end{aligned} \tag{5.4}$$

We can derive the following result from the Cauchy-Schwarz inequality and (5.3)

$$\begin{aligned} \frac{(L'(t))^2}{4} &= \left(\int_\Omega u u_t \mathrm{d}x + \int_0^t (u,u_t)_* \mathrm{d}t + b(t+T_0)\right)^2 \\ &\le \left(\|u\|_2^2 + \int_0^t \|u\|_*^2 \mathrm{d}t + b(t+T_0)^2\right)\left(\|u_t\|_2^2 + \int_0^t \|u_t\|_*^2 \mathrm{d}t + b\right) \\ &= (L(t) - (T-t)\|u_0\|_*^2)\left(\|u_t\|_2^2 + \int_0^t \|u_t\|_*^2 \mathrm{d}t + b\right) \\ &\le L(t)\left(\|u_t\|_2^2 + \int_0^t \|u_t\|_*^2 \mathrm{d}t + b\right). \end{aligned} \tag{5.5}$$

Through an analysis of (5.4) and (5.5), it can be deduced that

$$
\begin{aligned}
&L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 \\
&\geq \; L(t)L''(t) - (p+2)L(t)\left(\|u_t\|_2^2 + \int_0^t \|u_t\|_*^2 \mathrm{d}t + b\right) \\
&\geq \; L(t)\Bigg(-p\,\|u_t\|_2^2 + 2\int_\Omega |u|^p \ln|u| \mathrm{d}x - 2\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 \\
&\quad - 2\,\|\nabla u\|_2^2 - 2\int_0^t g(t-s)(\Delta u(t), \Delta u(t) - \Delta u(s))\mathrm{d}s - (p+2)\int_0^t \|u_t\|_*^2 \mathrm{d}t - pb\Bigg).
\end{aligned}
\tag{5.6}
$$

Let

$$
\begin{aligned}
F(t) = -p\,\|u_t\|_2^2 + 2\int_\Omega |u|^p \ln|u| \mathrm{d}x - 2\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 \\
- 2\,\|\nabla u\|_2^2 - 2\int_0^t g(t-s)(\Delta u(t), \Delta u(t) - \Delta u(s))\mathrm{d}s \\
- (p+2)\int_0^t \|u_t\|_*^2 \mathrm{d}t - pb,
\end{aligned}
\tag{5.7}
$$

then (5.6) can be rewritten as

$$
L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 \geq L(t)F(t).
$$

By combining (2.1), (2.9), and (5.7) and then applying Young's inequality, given an arbitrary positive number $\varepsilon > 0$, we arrive at

$$
\begin{aligned}
F(t) &= -2pE(t) + p(g \circ \Delta u)(t) - (p+2)\int_0^t \|u_t\|_*^2 \mathrm{d}t + \frac{2}{p}\,\|u\|_p^p \\
&\quad - 2\int_0^t g(t-s)(\Delta u(t), \Delta u(t) - \Delta u(s))\mathrm{d}s + (p-2)\|\nabla u\|_2^2 \\
&\quad + (p-2)\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 - pb \\
&\geq -2pE(0) - 2\int_0^t g(t-s)(\Delta u(t), \Delta u(t) - \Delta u(s))\mathrm{d}s \\
&\quad + (p-2)\int_0^t \|u_t\|_*^2 \mathrm{d}t + (p-2)\|\nabla u\|_2^2 + p(g \circ \Delta u)(t) \\
&\quad + (p-2)\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{2}{p}\,\|u\|_p^p - pb.
\end{aligned}
$$

After further simplification, we acquire

$$
\begin{aligned}
F(t) \geq & -2pE(0) + \left((p-2) - \left(p-2+\frac{1}{\varepsilon}\right)\int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 \\
& + \frac{2}{p}\|u\|_p^p + (p-\varepsilon)(g \circ \Delta u)(t) \\
& + (p-2)\|\nabla u\|_2^2 + (p-2)\int_0^t \|u_t\|_*^2 \mathrm{d}t - pb.
\end{aligned} \tag{5.8}
$$

Let us now examine the classification of the initial energy $E(0)$ into two distinct scenarios. One case is that $E(0) < 0$, and the other case is that $0 < E(0) < d$.

For the first case, let $\alpha < 0$, then $E(0) < 0$. We substitute $\varepsilon = p$ into inequality (5.8). Then, by selecting a value of b such that $0 < b \leq -2E(0)$, we obtain the following result:

$$
\begin{aligned}
F(t) \geq & \, p(-2E(0) - b) + (p-2)\int_0^t \|u_t\|_*^2 \mathrm{d}t + \frac{2}{p}\|u\|_p^p + (p-2)\|\nabla u\|_2^2 \\
& + \left((p-2) - \left(p-2+\frac{1}{p}\right)\int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 \\
\geq & \, 0.
\end{aligned} \tag{5.9}
$$

For the second case, let $0 < \alpha < 1$, then $0 < E(0) < d$. When we substitute $\varepsilon = (1-\alpha)p + 2\alpha$ into inequality (5.8), we discover that

$$
\begin{aligned}
F(t) \geq & -2pE(0) + \frac{2}{p}\|u\|_p^p + \alpha(p-2)(g \circ \Delta u) \\
& + \left((p-2) - \left(p-2+\frac{1}{(1-\alpha)p+2\alpha}\right)\int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 \\
& + (p-2)\|\nabla u\|_2^2 + (p-2)\int_0^t \|u_t\|_*^2 \mathrm{d}t - pb.
\end{aligned} \tag{5.10}
$$

Alternatively, when $I(u_0) < 0$, and $E(0) < d$, it can be shown that $I(u(t)) < 0, E(t) < d$ for all t. To prove this, suppose for contradiction that there exists some $t_0 \in (0, T)$, where $I(u(t_0)) = 0$ or $E(t_0) = d$. From equation (2.9), we know $E(t_0) \leq E(0)$, and since $E(0) < d$ by assumption, the equality $E(t_0) = d$ is immediately ruled out. Now, consider the case where $I(u(t_0)) = 0$. This implies $u(t_0) \in \mathcal{N}$ by the definition of the Nehari manifold $\mathcal{N}$. Recalling the definition of d as $d = \inf_{u\in\mathcal{N}} J(u)$, it follows that $d \leq J(u(t_0)) \leq E(t_0) \leq E(0)$, which is also contradictive with $E(0) < d$. Thus, we have $I(u) < 0$. By Lemma 2.1, there exists a parameter $\lambda_* \in (0,1)$ such that $I(\lambda_* u) = 0$, then we immediately obtain from (2.4) that

$$
\begin{aligned}
d & \leq J(\lambda_* u) \\
& = \frac{(p-2)\lambda_*{}^2}{2p}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{(p-2)\lambda_*{}^2}{2p}\|\nabla u\|_2^2 \\
& \quad + \frac{(p-2)\lambda_*{}^2}{2p}(g \circ \Delta u)(t) + \frac{\lambda_*{}^p}{p^2}\|u\|_p^p \\
& < \frac{p-2}{2p}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 \\
& \quad + \frac{p-2}{2p}(g \circ \Delta u)(t) + \frac{1}{p^2}\|u\|_p^p.
\end{aligned} \tag{5.11}
$$

Since $u(x,t)$ is continuous, a positive number $k > 0$ must exist such that

$$d + k \le \frac{p-2}{2p}\left(1 - \int_0^t g(s)\mathrm{d}s\right) \|\Delta u\|_2^2 + \frac{p-2}{2p} \|\nabla u\|_2^2 + \frac{p-2}{2p}(g \circ \Delta u)(t) + \frac{1}{p^2} \|u\|_p^p, \tag{5.12}$$

then

$$2p\alpha(d + k) \le \alpha(p-2)\left(\left(1 - \int_0^t g(s)\mathrm{d}s\right) \|\Delta u\|_2^2 + \|\nabla u\|_2^2 + (g \circ \Delta u)(t)\right) + \frac{2\alpha}{p} \|u\|_p^p. \tag{5.13}$$

By combining (5.10) and (5.13), we acquire

$$F(t) \ge -2p\alpha d + 2p\alpha(d + k) - pb = 2p\alpha k - pb. \tag{5.14}$$

By selecting a sufficiently small positive number b such that the inequality $2p\alpha k - pb \ge 0$ holds, one is able to derive

$$F(t) \ge 0. \tag{5.15}$$

By integrating the information from (5.9) and (5.15), it can be deduced that

$$L(t)L''(t) - \frac{p+2}{4}(L'(t))^2 \ge 0.$$

In both scenarios where $E(0) < 0$ and $0 < E(0) < d$, we select T_0 sufficiently large to ensure that

$$L'(0) = 2(u_0, u_1) + 2bT_0 > 0. \tag{5.16}$$

Thus, by choosing $\alpha = \frac{p-2}{4} > 0$ in Lemma 2.2, we immediately deduce that

$$\lim_{t \to T^{*-}} L(t) = \infty,$$

where

$$T^* \le \frac{4L(0)}{(p-2)L'(0)} = \frac{2\|u_0\|_2^2 + 2T\|u_0\|^2 + 2bT_0{}^2}{(p-2)bT_0 + (p-2)\int_\Omega u_0 u_1 \mathrm{d}x}.$$

So

$$T^* \le \frac{2\|u_0\|_2^2 + 2bT_0{}^2}{(p-2)bT_0 + (p-2)\int_\Omega u_0 u_1 \mathrm{d}x - 2\|u_0\|^2}.$$

The proof is complete. □

Finally, we state the conclusion on the finite-time blow-up of weak solutions with null initial energy in the form of a remark.

Remark 5.1. When $\alpha = 0$, i.e., $E(0) = 0$, we supplement the condition $(u_0, u_1) > 0$, then the weak solution $u(x, t)$ blows up in finite time.

Proof. Taking $\varepsilon = p$ and $b = 0$ in (5.8), we receive

$$\begin{aligned} F(t) &\ge \left((p-2) - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(t-s)\mathrm{d}s\right) \|\Delta u\|_2^2 \\ &\quad + \frac{2}{p} \|u\|_p^p + (p-2)\|\nabla u\|_2^2 + (p-2)\int_0^t \|u_t\|_2^2 \mathrm{d}t \\ &\ge 0. \end{aligned} \tag{5.17}$$

The condition $(u_0, u_1) > 0$ gives

$$L'(0) = 2(u_0, u_1) > 0. \tag{5.18}$$

Consequently, we arrive at a similar result that

$$\lim_{t \to T^{*-}} L(t) = \infty. \qquad \square$$

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