Research Article

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Asymptotic behavior of global mild solutions to the Keller-Segel-Navier-Stokes system in Lorentz spaces

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Abstract: The Keller-Segel-Navier-Stokes system in $\mathbb{R}^N$ is considered, where $N \geq 3$. We show the existence and uniqueness of local mild solutions for arbitrary initial data and gravitational potential in scaling invariant Lorentz spaces. Although such a result has already been shown by Kozono, Miura, and Sugiyama (*Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid*, J. Funct. Anal. **270** (2016), no. 5, 1663–1683), we reveal the precise regularities of mild solutions by showing the smoothing estimates of the heat semigroup on Lorentz spaces. The method is based on the real interpolation. In addition, we prove that the mild solutions exist globally in time, provided that the initial data are sufficiently small. Compared with the usual result, a part of the smallness conditions is reduced. We also obtain the asymptotic behavior of the global mild solutions. In the proof of the asymptotic behavior, to overcome a lack of density for the space $L^\infty(\mathbb{R}^N)$ to which one c_0 of the initial data belongs, we show the decay of the global solutions without any approximation for c_0.

Keywords: Keller-Segel-Navier-Stokes system, asymptotic behavior, Lorentz spaces, scaling invariant

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1 Introduction

Let us consider the following initial value problem for the Keller-Segel-Navier-Stokes system in $\mathbb{R}^N$, $N \geq 3$:

$$\begin{cases} \partial_t n + \boldsymbol{u}\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c), & t > 0, \quad x \in \mathbb{R}^N, \\ \partial_t c + \boldsymbol{u}\cdot\nabla c = \Delta c - nc, & t > 0, \quad x \in \mathbb{R}^N, \\ \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \Delta\boldsymbol{u} - \nabla p + n\nabla\varphi, & t > 0, \quad x \in \mathbb{R}^N, \\ \nabla\cdot\boldsymbol{u} = 0, & t > 0, \quad x \in \mathbb{R}^N, \\ (n, c, \boldsymbol{u})(0, x) = (n_0, c_0, \boldsymbol{u}_0)(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $n = n(t,x)$, $c = c(t,x)$, $\boldsymbol{u} = \boldsymbol{u}(t,x)$, and $p = p(t,x)$ are the unknown functions denoting the density of the cell, the concentration of the oxygen, the velocity of the fluid, and the pressure, respectively. Moreover, $n_0 = n_0(x)$, $c_0 = c_0(x)$, and $\boldsymbol{u}_0 = \boldsymbol{u}_0(x)$ are the given initial data and $\varphi = \varphi(x)$ is the given gravitational potential.

In this article, we show the existence of local mild solutions of system (1.1) with arbitrary initial data $(n_0, c_0, \boldsymbol{u}_0)$ and gravitational potential φ in *scaling invariant* Lorentz spaces. By analyzing the smoothing effects of the heat semigroup on Lorentz spaces precisely, we reveal the regularities of mild solutions of (1.1) corresponding to those of the initial data $(n_0, c_0, \boldsymbol{u}_0)$. We also obtain the uniqueness result of mild solutions

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under the assumption based on the weak Lebesgue spaces. Moreover, if we take the initial data $(n_0, c_0, \boldsymbol{u}_0)$ sufficiently small, then we show that the mild solutions of (1.1) exist globally in time. The *asymptotic behavior* of the global mild solutions is also revealed.

The Keller-Segel-Navier-Stokes system given by (1.1) has been proposed in [45]. More precisely, from experimental observations, it is known that certain bacteria move in fluid drops and concentrate in special regions [19]. From the viewpoint of mathematical and theoretical biology, it is assumed that such a phenomenon occurs by the effect of the *chemotaxis*, i.e., bacteria swim due to the consumption of oxygen in fluid drops. In addition, the swimming of bacteria, a gravitational effect for bacteria, and a convective flow stir up the fluid, and hence the fluid flow also affects the diffusion of bacteria strongly. Such a hypothesis is discussed in [19,36,45]. We also mention that the original mathematical model of chemotaxis is called the Keller-Segel system [27] and given as follows:

$$\begin{cases} \partial_t n = \Delta n - \nabla\cdot(n\nabla v), t > 0, \quad x \in \mathbb{R}^N, \\ \partial_t v = \Delta v - v + n, \quad t > 0, \quad x \in \mathbb{R}^N, \\ (n, v)(0, x) = (n_0, v_0)(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

We note that the right-hand side of the second equation is not $\Delta c - nc$ but $\Delta v - v + n$: Recall that the unknown function c of system (1.1) denotes the concentration of the *oxygen*. Compared with (1.1), in the original Keller-Segel system (1.2), we usually consider the unknown function v describing the concentration of the *chemo-attractant* instead of the oxygen. This is the reason why the structure of the second equation is different. Here, we mention that Lorz [37] considers the Keller-Segel-Stokes system described by the concentration v of the chemo-attractant.

1.1 Existence of global solutions to the chemotaxis(-fluid) system

While chemotaxis is one of the significant features in the biological field, the Keller-Segel system (1.2) also has a special property in the mathematical sense, *critical mass phenomena.* One of the typical problems for evolution equations may be the construction of solutions. However, in general, we do not know whether we may construct *global* solutions of nonlinear evolution equations for arbitrarily *large* initial data. Concerning this problem, in the 2D Keller-Segel system (1.2), it is known that solutions exist globally in time if the initial density n_0 satisfies $\int_{\mathbb{R}^2} n_0(x)\mathrm{d}x < 8\pi$. Conversely, for any $M > 8\pi$, we may take some initial density n_0 such that $\int_{\mathbb{R}^2} n_0(x)\mathrm{d}x = M$ and corresponding solutions of (1.2) *blow up* in finite time. We refer to [2–5,14,16,18,39,40] for such results and related topics. In addition, similar results hold even if we consider the case of the Keller-Segel-Navier-Stokes system [23,29,34]. We also note that there have been a lot of studies on the existence and asymptotic behavior of global solutions to the 2D Keller-Segel-Navier-Stokes system in various domains. For instance, we refer to [6,8–10,15,35,43,46,47,49].

As mentioned above, mathematical models of chemotaxis in the 2D case have the critical mass phenomenon, and thus, the global existence of solutions may be characterized by the mass of the initial density n_0. However, we do not know the corresponding results in the higher dimensional case $N \geq 3$. Therefore, to construct global solutions in such a case, we have to consider *weak solutions* or we have to assume that all of the initial data are sufficiently *small.* In the former case, there have been various kinds of the existence results of global weak solutions to the 3D Keller-Segel-Navier-Stokes system. In fact, Liu and Lorz [35] considered the case where the term Δn appearing in (1.1) is replaced by $\Delta n^{4/3}$, nonlinear diffusion type. Winkler [46] studied the case where the domain is bounded and the Navier-Stokes system is replaced by the Stokes system. Chae et al. [8] considered the case where some nonlinear terms have special structures. Tao and Winkler [44] treated the case where the domain is bounded and Δn is replaced by Δn^m, $m > 8/7$. In addition, Winkler [48] considered the case of a bounded domain and achieved obtaining global weak solutions *without* any restriction, such as considering nonlinear diffusion and Stokes system and assuming structures of nonlinear terms. Kang et al. [25] also treated the whole space case.

Next, we mention another way to construct global solutions, assuming the smallness conditions. A starting point to obtain such a result may be the article by Duan et al. [20]. In fact, they considered the 3D case and constructed global classical solutions for small initial data in the Sobolev spaces $H^{3,2}(\mathbb{R}^3)$ framework. After that, several results were obtained by following the well-known method of Kato [26]. Here, let us consider the *scaling invariant* spaces to system (1.1). Suppose that $(n, c, \boldsymbol{u}, p)$ is a global classical solution of (1.1) with a gravitational potential φ. Then, by setting

$$(n_\lambda(t,x), c_\lambda(t,x), \boldsymbol{u}_\lambda(t,x), p_\lambda(t,x)) \coloneqq (\lambda^2 n(\lambda^2 t, \lambda x), c(\lambda^2 t, \lambda x), \lambda \boldsymbol{u}(\lambda^2 t, \lambda x), \lambda^2 p(\lambda^2 t, \lambda x)) \tag{1.3}$$

for $(t,x) \in (0,\infty) \times \mathbb{R}^N$ and $0 < \lambda < \infty$, we observe that $(n_\lambda, c_\lambda, \boldsymbol{u}_\lambda, p_\lambda)$ also satisfies (1.1) with φ replaced by $\varphi_\lambda(x) \coloneqq \varphi(\lambda x)$ for all $0 < \lambda < \infty$. Since it holds that

$$\|n_\lambda(0,\cdot)\|_{L^{N/2}(\mathbb{R}^N)} = \|n(0,\cdot)\|_{L^{N/2}(\mathbb{R}^N)}, \quad \|c_\lambda(0,\cdot)\|_{L^\infty(\mathbb{R}^N)} = \|c(0,\cdot)\|_{L^\infty(\mathbb{R}^N)},$$
$$\|\boldsymbol{u}_\lambda(0,\cdot)\|_{L^N(\mathbb{R}^N)} = \|\boldsymbol{u}(0,\cdot)\|_{L^N(\mathbb{R}^N)}, \quad \|\nabla\varphi_\lambda\|_{L^N(\mathbb{R}^N)} = \|\nabla\varphi\|_{L^N(\mathbb{R}^N)}$$

for all $0 < \lambda < \infty$, we see that

$$n_0 \in L^{N/2}(\mathbb{R}^N), \quad c_0 \in L^\infty(\mathbb{R}^N), \quad \boldsymbol{u}_0 \in (L^N(\mathbb{R}^N))^N, \quad \nabla\varphi \in (L^N(\mathbb{R}^N))^N$$

is one of the scaling invariant spaces to (1.1). Kozono et al. [28] considered a more involved system by adding a new unknown function v to system (1.1), where v stands for the concentration of the chemo-attractant. From the viewpoint of the scaling invariance, they obtained global mild solutions by assuming that the given data $(n_0, c_0, v_0, \boldsymbol{u}_0)$ and φ are sufficiently small in the following spaces:

$$n_0 \in L^{N/2,\infty}(\mathbb{R}^N), \quad c_0 \in L^\infty(\mathbb{R}^N), \quad \nabla c_0 \in (L^{N,\infty}(\mathbb{R}^N))^N,$$
$$\nabla v_0 \in (L^{N,\infty}(\mathbb{R}^N))^N, \quad \boldsymbol{u}_0 \in P(L^{N,\infty}(\mathbb{R}^N))^N, \quad \nabla\varphi \in (L^{N,\infty}(\mathbb{R}^N))^N,$$

where P denotes the Helmholtz projection operator. Note that their result is valid for all $N \geq 3$ and that the case of $N = 2$ is also valid if $n_0 \in L^{N/2,\infty}(\mathbb{R}^N)$ is replaced by $n_0 \in L^1(\mathbb{R}^2)$. Cao and Lankeit [6] treated the case of a bounded domain $\Omega \subset \mathbb{R}^N$ for $N \in \{2,3\}$ and assumed that the initial data satisfy either $\|n_0 - \bar{n}_0\|_{L^q(\Omega)} + \|c_0\|_{L^\infty(\Omega)} + \|\boldsymbol{u}_0\|_{L^N(\Omega)} \ll 1$ with $N/2 < q < \infty$ or $\|n_0\|_{L^q(\Omega)} + \|\nabla c_0\|_{L^N(\Omega)} + \|\boldsymbol{u}_0\|_{L^N(\Omega)} \ll 1$ with $N/2 < q < N$ and some suitable assumptions, where $\bar{n}_0 \coloneqq |\Omega|^{-1}\int_\Omega n_0(x)\mathrm{d}x$. Then, they constructed global classical solutions. Jiang [24] considered a similar system to (1.1) in a bounded domain with $N \geq 2$ by replacing c with the concentration v of the chemo-attractant. He assumed that the initial data satisfy $\|n_0 - \bar{n}_0\|_{L^{N/2}(\Omega)} + \|\nabla v_0\|_{L^N(\Omega)} + \|\boldsymbol{u}_0\|_{L^N(\Omega)} \ll 1$ with certain assumptions to show the existence of global classical solutions. Moreover, concerning more involved scaling invariant spaces, Choe et al. [13] and Choe and Lkhagvasuren [12] considered the 3D case in the homogeneous Besov spaces framework; $n_0 \in \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3)$, $c_0 \in \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$, $\boldsymbol{u}_0 \in P(\dot{B}^{-1+3/q}_{q,1}(\mathbb{R}^3))^3$, and $\varphi \in \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ for $1 \leq q < 3$. For the case of the other function spaces based on Besov-type spaces, we refer to, e.g., [17,22,42,51–55].

1.2 Our motivations in this article

In this article, we consider the case where the given data $(n_0, c_0, \boldsymbol{u}_0)$ and φ satisfy

$$\begin{cases} n_0 \in L^{N/2,\rho}(\mathbb{R}^N), & c_0 \in L^\infty(\mathbb{R}^N), \quad \nabla c_0 \in (L^{N,\rho}(\mathbb{R}^N))^N, \\ \boldsymbol{u}_0 \in P(L^{N,\rho}(\mathbb{R}^N))^N, & \nabla\varphi \in (L^{N,\infty}(\mathbb{R}^N))^N, \end{cases} \tag{1.4}$$

where $N \geq 3$ and $1 \leq \rho < \infty$. In particular, we focus on scaling invariant *Lorentz spaces*. As stated before, Kozono et al. [28] have already shown the existence of global solutions to a more involved system under the condition $\rho = \infty$. One of the reasons why we consider the scaling invariant spaces is to enlarge the spaces of the given data that ensure the existence of global solutions. Although our assumption (1.4) does not seem to provide an advantage from this viewpoint, what we want to do is to reveal the detailed properties of solutions of system (1.1) by focusing on a difference of the index $1 \leq \rho < \infty$. More precisely, we show that mild solutions of (1.1) have *higher* regularities than those of [28]. We also reduce a part of the *smallness conditions* of the given data. In addition, we improve the *asymptotic behavior* of the global mild solutions to (1.1) by relying on the condition $1 \le \rho < \infty$.

Before stating our main results, we shall mention the key ideas of the analysis in this article. The method to obtain regularities of solutions is mainly based on the *real interpolation*. We recall that the Lorentz spaces are characterized by the real interpolation spaces of the usual Lebesgue spaces. Kozono and Shimizu [33, Proposition 2.1] focused on the smoothing estimates of the heat semigroup defined on homogeneous Besov spaces, which were shown by Kozono et al. [30, Lemma 2.2 (ii)], and relied on the real interpolation technique to obtain the space-time estimates of the heat semigroup:

$$\|\Delta e^{t\Delta} f\|_{L^{\alpha,\rho}((0,\infty);\dot{B}^{s-2+2/\alpha}_{q,1}(\mathbb{R}^N))} \le C\|f\|_{\dot{B}^{s}_{q,\rho}(\mathbb{R}^N)}.$$

From this result, we may expect that a similar estimate is valid in the case of the Lorentz spaces. By showing the corresponding estimate, we will observe that mild solutions of (1.1) have higher regularities. It should be noted that Yamazaki [50, Corollary 2.3] has obtained almost the same estimate as ours. We also refer to [31,32] for related papers. Concerning the reduction of the smallness condition of the given data, we note that the term $n\nabla\varphi$ appearing in (1.1) is a *linear* term with respect to n. From the viewpoint of perturbation theory, we have to assume that $\nabla\varphi$ is small to obtain solutions. However, we may reduce such a condition by a simple *modification* of the usual method in the construction of solutions. We remark that Choe and Lkhagvasuren [12] have used such a method. Moreover, for the initial datum $c_0 \in L^\infty(\mathbb{R}^N)$ with $\nabla c_0 \in (L^{N,\rho}(\mathbb{R}^N))^N$, we only assume that $\|\nabla c_0\|_{L^{N,\rho}(\mathbb{R}^N)}$ is small to obtain global solutions, while it is *not necessary* to assume that $\|c_0\|_{L^\infty(\mathbb{R}^N)}$ is small. In other words, we may construct global solutions even if c_0 is close to a *large* constant. We shall refer to the result of Cao and Lankeit [6, Theorem 1.2], who have obtained a similar result through a different approach. Finally, let us mention the asymptotic behavior; by following the approach due to Kato [26], we may show that the global solutions decay as $t \to \infty$ with suitable rates. For instance, Kozono et al. [28] obtained the decay of global solutions n such that

$$\|n(t)\|_{L^q(\mathbb{R}^N)} = O(t^{-(N/2)(2/N-1/q)})$$

as $t \to \infty$ for some $q > N/2$. Compared with this result, we show that

$$\|n(t)\|_{L^{q,1}(\mathbb{R}^N)} = o(t^{-(N/2)(2/N-1/q)}), \quad \|n(t)\|_{L^{N/2,\rho}(\mathbb{R}^N)} = o(1)$$

as $t \to \infty$ for some $q > N/2$ and $1 \le \rho < \infty$. Such a result is obtained by the method of Kato [26, Note]. Mention that the set $C_0^\infty(\mathbb{R}^N)$ of smooth functions with compact support is dense in $L^{N/2,\rho}(\mathbb{R}^N)$ for $1 \le \rho < \infty$ but *not* dense in $L^{N/2,\infty}(\mathbb{R}^N)$. This is an advantage of considering the case of $1 \le \rho < \infty$. However, we should remark that our condition (1.4) contains $c_0 \in L^\infty(\mathbb{R}^N)$, and thus, the problem of a *lack of density* still remains. To overcome this problem, we show the decay of the global solutions *without* any approximation for the initial datum c_0. In particular, we approximate only the initial data n_0 and $\boldsymbol{u}_0$.

1.3 Main results

In what follows, we shall state our main results. To this end, we consider the integral forms of system (1.1). Let P denote the Helmholtz projection operator defined on $\mathbb{R}^N$, namely, we set $P \coloneqq I + \nabla(-\Delta)^{-1}\nabla\cdot$. Then, since $P(\nabla p) = 0$ and $P\boldsymbol{u} = \boldsymbol{u}$ under the condition $\nabla\cdot\boldsymbol{u} = 0$, we observe that the third equation of (1.1) is given by $\partial_t\boldsymbol{u} - \Delta\boldsymbol{u} = -P((\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - n\nabla\varphi)$. We also note that $\boldsymbol{u}\cdot\nabla n = \nabla\cdot(n\boldsymbol{u})$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u})$ due to $\nabla\cdot\boldsymbol{u} = 0$. As the Laplacian $-\Delta$ generates the heat semigroup $e^{t\Delta} \coloneqq G_t *$ for $0 < t < \infty$, where $G_t(x) \coloneqq (4\pi t)^{-N/2}e^{-|x|^2/(4t)}$ for $(t,x) \in (0,\infty) \times \mathbb{R}^N$, we obtain the following integral forms of system (1.1):

$$\begin{cases} n = e^{t\Delta}n_0 - \int_0^t e^{(t-\tau)\Delta}(\nabla\cdot(n\boldsymbol{u}) + \nabla\cdot(n\nabla c))\mathrm{d}\tau & \text{in } (0,\infty)\times\mathbb{R}^N, \\ c = e^{t\Delta}c_0 - \int_0^t e^{(t-\tau)\Delta}(\boldsymbol{u}\cdot\nabla c + nc)\mathrm{d}\tau & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \boldsymbol{u} = e^{t\Delta}\boldsymbol{u}_0 - \int_0^t e^{(t-\tau)\Delta}P(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) - n\nabla\varphi)\mathrm{d}\tau & \text{in } (0,\infty)\times\mathbb{R}^N. \end{cases} \tag{1.5}$$

A solution $(n, c, \boldsymbol{u})$ of the integral forms (1.5) may be called a *mild solution* of the original system (1.1). Throughout this article, we assume that $N \geq 3$.

Our first result is the existence and uniqueness theorem of local mild solutions of (1.1) with initial data in scaling invariant Lorentz spaces.

Theorem 1.1. (Local mild solution) *Let* $1 \leq \rho < \infty$ *and let* q, r, α, *and* β *satisfy*

$$N/2 < q < N, \quad N < r < \min\{2N, Nq/(2q-N), Nq/(N-q)\}, \tag{1.6}$$

$$1/\alpha = (N/2)(2/N - 1/q), \quad 1/\beta = (N/2)(1/N - 1/r). \tag{1.7}$$

Suppose that the initial data $(n_0, c_0, \boldsymbol{u}_0)$ *and the gravitational potential* φ *satisfy* (1.4). *Then, the following statements hold*:

(*i*) *There exist* $T > 0$ *and a solution* $(n, c, \boldsymbol{u})$ *on* $(0, T)\times\mathbb{R}^N$ *of* (1.5) *satisfying*

$$\begin{aligned} n &\in \mathrm{BC}([0,T);\, L^{N/2,\rho}(\mathbb{R}^N)) \cap L^{\alpha,\rho}((0,T);\, L^{q,1}(\mathbb{R}^N)), \\ t^{1/\alpha}n &\in \mathrm{BC}([0,T);\, L^{q,1}(\mathbb{R}^N)), \\ c &\in \mathrm{BC}((0,T);\, \mathrm{BUC}(\mathbb{R}^N)), \\ \nabla c &\in \mathrm{BC}([0,T);\, (L^{N,\rho}(\mathbb{R}^N))^N) \cap L^{\beta,\rho}((0,T);\, (L^{r,1}(\mathbb{R}^N))^N), \\ t^{1/\beta}\nabla c &\in \mathrm{BC}([0,T);\, (L^{r,1}(\mathbb{R}^N))^N), \\ \boldsymbol{u} &\in \mathrm{BC}([0,T);\, P(L^{N,\rho}(\mathbb{R}^N))^N) \cap L^{\beta,\rho}((0,T);\, P(L^{r,1}(\mathbb{R}^N))^N), \\ t^{1/\beta}\boldsymbol{u} &\in \mathrm{BC}([0,T);\, P(L^{r,1}(\mathbb{R}^N))^N). \end{aligned}$$

Moreover, the solution has the properties

$$\begin{aligned} \|(n,c,\boldsymbol{u})\|_{X_T} \coloneqq{}& \|n\|_{L_T^\infty(L^{N/2,\rho}(\mathbb{R}^N))} + \|c\|_{L_T^\infty(L^\infty(\mathbb{R}^N))} + \|\nabla c\|_{L_T^\infty(L^{N,\rho}(\mathbb{R}^N))} + \|\boldsymbol{u}\|_{L_T^\infty(L^{N,\rho}(\mathbb{R}^N))} \\ &+ \sup_{0<t<T}(t^{1/\alpha}\|n(t)\|_{L^{q,1}(\mathbb{R}^N)} + t^{1/\beta}\|\nabla c(t)\|_{L^{r,1}(\mathbb{R}^N)} + t^{1/\beta}\|\boldsymbol{u}(t)\|_{L^{r,1}(\mathbb{R}^N)}) \\ &+ \|n\|_{L_T^{\alpha,\rho}(L^{q,1}(\mathbb{R}^N))} + \|\nabla c\|_{L_T^{\beta,\rho}(L^{r,1}(\mathbb{R}^N))} + \|\boldsymbol{u}\|_{L_T^{\beta,\rho}(L^{r,1}(\mathbb{R}^N))} \\ \leq{}& 2\|c_0\|_{L^\infty(\mathbb{R}^N)} + C(1 + \|\nabla\varphi\|_{L^{N,\infty}(\mathbb{R}^N)})(1 + \|c_0\|_{L^\infty(\mathbb{R}^N)}) \\ &\times (\|n_0\|_{L^{N/2,\rho}(\mathbb{R}^N)} + \|\nabla c_0\|_{L^{N,\rho}(\mathbb{R}^N)} + \|\boldsymbol{u}_0\|_{L^{N,\rho}(\mathbb{R}^N)}), \end{aligned} \tag{1.8}$$

$$\|c\|_{L_T^\infty(L^\infty(\mathbb{R}^N))} \leq 2\|c_0\|_{L^\infty(\mathbb{R}^N)} \tag{1.9}$$

and

$$\lim_{t\to+0}(\|n(t) - n_0\|_{L^{N/2,\rho}(\mathbb{R}^N)} + \|\nabla c(t) - \nabla c_0\|_{L^{N,\rho}(\mathbb{R}^N)} + \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{L^{N,\rho}(\mathbb{R}^N)}) = 0, \tag{1.10}$$

$$\lim_{t\to+0}\|c(t) - e^{t\Delta}c_0\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{1.11}$$

$$\lim_{t\to+0}(t^{1/\alpha}\|n(t)\|_{L^{q,1}(\mathbb{R}^N)} + t^{1/\beta}\|\nabla c(t)\|_{L^{r,1}(\mathbb{R}^N)} + t^{1/\beta}\|\boldsymbol{u}(t)\|_{L^{r,1}(\mathbb{R}^N)}) = 0, \tag{1.12}$$

where $C > 0$ *is a constant independent of* T, n_0, c_0, $\boldsymbol{u}_0$, φ, n, c, *and* $\boldsymbol{u}$.

(*ii*) *Let* $0 < T_0 \leq \infty$. *There exists a constant* $0 < \delta < 1$ *independently of* T_0, n_0, c_0, $\boldsymbol{u}_0$, *and* φ *such that a solution* $(n, c, \boldsymbol{u})$ *on* $(0, T_0)\times\mathbb{R}^N$ *of* (1.5) *satisfying*

$$\begin{aligned}
&n \in L^{\alpha,\infty}((0,T_0); L^{q,\infty}(\mathbb{R}^N)), \quad c \in L^{\infty}((0,T_0); L^{\infty}(\mathbb{R}^N)),\\
&\nabla c \in L^{\beta,\infty}((0,T_0); (L^{r,\infty}(\mathbb{R}^N))^N), \quad \boldsymbol{u} \in L^{\beta,\infty}((0,T_0); P(L^{r,\infty}(\mathbb{R}^N))^N),\\
&\limsup_{\lambda\to\infty}\{\lambda\mu(t\in(0,T_0) \mid \|n(t)\|_{L^{q,\infty}(\mathbb{R}^N)} > \lambda)^{1/\alpha} + \lambda\mu(t\in(0,T_0) \mid \|\nabla c(t)\|_{L^{r,\infty}(\mathbb{R}^N)} > \lambda)^{1/\beta}\\
&\quad + \lambda\mu(t\in(0,T_0) \mid \|\boldsymbol{u}(t)\|_{L^{r,\infty}(\mathbb{R}^N)} > \lambda)^{1/\beta}\} \le \delta(1+\|\nabla\varphi\|_{L^{N,\infty}(\mathbb{R}^N)})^{-1}(1+\|c\|_{L^{\infty}_{T_0}(L^{\infty}(\mathbb{R}^N))})^{-1}
\end{aligned}$$

is unique, where μ denotes the usual Lebesgue measure on $(0,T_0)$.

(*iii*) *In the statement of* (i), *suppose that $(n,c,\boldsymbol{u})$ is the solution on $(0,T)\times\mathbb{R}^N$ of* (1.5) *with the initial data $(n_0,c_0,\boldsymbol{u}_0)$ and the gravitational potential φ. In addition, suppose that $(\tilde{n},\tilde{c},\tilde{\boldsymbol{u}})$ is a solution on $(0,T)\times\mathbb{R}^N$ of* (1.5) *with initial data $(\tilde{n}_0,\tilde{c}_0,\tilde{\boldsymbol{u}}_0)$ and a gravitational potential $\tilde{\varphi}$ such that*

$$\begin{aligned}
&\tilde{n}_0 \in L^{N/2,\rho}(\mathbb{R}^N), \quad \tilde{c}_0 \in L^{\infty}(\mathbb{R}^N), \quad \nabla\tilde{c}_0 \in (L^{N,\rho}(\mathbb{R}^N))^N,\\
&\tilde{\boldsymbol{u}}_0 \in P(L^{N,\rho}(\mathbb{R}^N))^N, \quad \nabla\tilde{\varphi} \in (L^{N,\infty}(\mathbb{R}^N))^N.
\end{aligned}$$

Then, it holds that

$$\begin{aligned}
&\|(n-\tilde{n}, c-\tilde{c}, \boldsymbol{u}-\tilde{\boldsymbol{u}})\|_{X_T}\\
&\le C(1+\|\nabla\varphi\|_{L^{N,\infty}(\mathbb{R}^N)})(1+\|c_0\|_{L^{\infty}(\mathbb{R}^N)})\|n_0-\tilde{n}_0\|_{L^{N/2,\rho}(\mathbb{R}^N)}\\
&\quad + C(1+\|\nabla\tilde{\varphi}\|_{L^{N,\infty}(\mathbb{R}^N)})\|c_0-\tilde{c}_0\|_{L^{\infty}(\mathbb{R}^N)} + C\|\nabla c_0-\nabla\tilde{c}_0\|_{L^{N,\rho}(\mathbb{R}^N)}\\
&\quad + C(1+\|c_0\|_{L^{\infty}(\mathbb{R}^N)})(\|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}(\mathbb{R}^N)} + \|\boldsymbol{u}_0-\tilde{\boldsymbol{u}}_0\|_{L^{N,\rho}(\mathbb{R}^N)}),
\end{aligned} \tag{1.13}$$

where $C>0$ is a constant independent of T, n_0, c_0, $\boldsymbol{u}_0$, φ, $\tilde{n}_0$, $\tilde{c}_0$, $\tilde{\boldsymbol{u}}_0$, $\tilde{\varphi}$, n, c, $\boldsymbol{u}$, $\tilde{n}$, $\tilde{c}$, and $\tilde{\boldsymbol{u}}$.

Remark 1.2. (i) The spaces of solutions of (1.1) are *scaling invariant* spaces, which correspond to those of the initial data. In particular, for the function $(n_\lambda, c_\lambda, \boldsymbol{u}_\lambda)$ given by (1.3), we observe that $\|(n_\lambda,c_\lambda,\boldsymbol{u}_\lambda)\|_{X_\infty} = \|(n,c,\boldsymbol{u})\|_{X_\infty}$ for all $0<\lambda<\infty$.

(ii) In Theorem 1.1, we focus on the local solvability. In this case, we do not have to assume the smallness conditions of the given data unlike the result of [28]. Moreover, even though $C_0^{\infty}(\mathbb{R}^N)$ is not dense in $L^{\infty}(\mathbb{R}^N)$, it is *not necessary* to assume that $\|c_0\|_{L^{\infty}(\mathbb{R}^N)}$ is small. We remark that the smallness condition of φ is also reduced by the method of [12].

(iii) A main difference between our result and that of [28] is the regularities of solutions in both space and time directions. Concerning the space direction, we see that $t^{1/\alpha}n(t)\in L^{q,1}(\mathbb{R}^N)$ for all $0<t<T$, where $L^{q,1}(\mathbb{R}^N)\subset L^q(\mathbb{R}^N)$. Moreover, we also have $n\in L^{\alpha,\rho}((0,T);L^{q,1}(\mathbb{R}^N))$, which may be regarded as a gain of regularities in the time direction. Note that ρ is the *same* as the index appearing in (1.4), and thus, this is an advantage of considering *general* $1\le\rho<\infty$. We refer to [50, Corollary 2.3] and [33, Proposition 2.1] for such results.

(iv) We also mention that properties (1.10) and (1.12) are obtained by considering the case of $1\le\rho<\infty$; this is also different from the result of [28]. Since (1.10) and (1.12) rely on the *density* argument, for the space $L^{\infty}(\mathbb{R}^N)$, we only know that $c\in \mathrm{BC}((0,T);\mathrm{BUC}(\mathbb{R}^N))$ with (1.11). Specifically, we do *not* know whether $\lim_{t\to+0}\|c(t)-c_0\|_{L^{\infty}(\mathbb{R}^N)}=0$ holds. However, by combining (1.11) and the duality argument, we see that $\lim_{t\to+0}\langle c(t)-c_0,\psi\rangle=0$ for all $\psi\in L^1(\mathbb{R}^N)$.

(v) The statement of (ii) states the *uniqueness* of mild solutions. To ensure the uniqueness in weaker assumptions, we consider the case of weak Lebesgue spaces in both space and time. Note that some smallness conditions are needed. Recall that the weak Lebesgue spaces are characterized as follows:

$$\|n\|_{L^{\alpha,\infty}_{T_0}(L^{q,\infty}(\mathbb{R}^N))} = \sup_{0<\lambda<\infty}\lambda\mu(t\in(0,T_0) \mid \|n(t)\|_{L^{q,\infty}(\mathbb{R}^N)}>\lambda)^{1/\alpha}.$$

Therefore, the smallness condition implies that a *part* of the norm is sufficiently small. We also remark that [42, Proposition 3.7] yields

$$\lim_{\lambda\to\infty}\lambda\mu(t\in(0,T_0) \mid \|n(t)\|_{L^{q,\infty}(\mathbb{R}^N)}>\lambda)^{1/\alpha}=0,$$

provided that $t^{1/\alpha}n \in L^\infty((0,T_0);L^{q,\infty}(\mathbb{R}^N))$ satisfies $\lim_{t\to+0} t^{1/\alpha}\|n(t)\|_{L^{q,\infty}(\mathbb{R}^N)} = 0$. Thus, we observe that the solution obtained in the statement of (i) satisfies the assumption of uniqueness. We refer to [33, Theorem 2 (ii)] for a similar assumption.

Our second result states that if the initial data $(n_0, c_0, \boldsymbol{u}_0)$ are sufficiently small, then we may obtain global mild solutions of (1.1). In addition, the asymptotic behavior of the global mild solutions is obtained.

Theorem 1.3. (Global existence and asymptotic behavior) *In the statement of* Theorem 1.1 (*i*), *there exists a constant* $0 < \varepsilon < 1$ *independently of* n_0, c_0, $\boldsymbol{u}_0$, *and* φ *such that if*

$$\begin{aligned}&\|n_0\|_{L^{N/2,\rho}(\mathbb{R}^N)} + \|\nabla c_0\|_{L^{N,\rho}(\mathbb{R}^N)} + \|\boldsymbol{u}_0\|_{L^{N,\rho}(\mathbb{R}^N)}\\ &\le \varepsilon(1+\|\nabla\varphi\|_{L^{N,\infty}(\mathbb{R}^N)})^{-2}(1+\|c_0\|_{L^\infty(\mathbb{R}^N)})^{-2},\end{aligned} \tag{1.14}$$

then, the solution $(n, c, \boldsymbol{u})$ *of* (1.5) *exists globally in time, namely, the existence time interval* T *in the statement of* Theorem 1.1 (*i*) *may be chosen as* $T = \infty$. *In addition, the following properties hold*:

$$\lim_{t\to\infty}(\|n(t)\|_{L^{N/2,\rho}(\mathbb{R}^N)} + \|\nabla c(t)\|_{L^{N,\rho}(\mathbb{R}^N)} + \|\boldsymbol{u}(t)\|_{L^{N,\rho}(\mathbb{R}^N)}) = 0, \tag{1.15}$$

$$\lim_{t\to\infty}\|c(t) - e^{t\Delta}c_0\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{1.16}$$

$$\lim_{t\to\infty}(t^{1/\alpha}\|n(t)\|_{L^{q,1}(\mathbb{R}^N)} + t^{1/\beta}\|\nabla c(t)\|_{L^{r,1}(\mathbb{R}^N)} + t^{1/\beta}\|\boldsymbol{u}(t)\|_{L^{r,1}(\mathbb{R}^N)}) = 0. \tag{1.17}$$

Remark 1.4. (i) As stated before, we do not have to assume that $\|c_0\|_{L^\infty(\mathbb{R}^N)}$ and $\|\nabla\varphi\|_{L^{N,\infty}(\mathbb{R}^N)}$ are small unlike the result of [28]. In particular, we may take c_0 such that c_0 is close to a *large* constant. This consequence is regarded as a whole space version of [6, Theorem 1.2]. In addition, the dependence of the smallness of the initial data with respect to $\|c_0\|_{L^\infty(\mathbb{R}^N)}$ and $\|\nabla\varphi\|_{L^{N,\infty}(\mathbb{R}^N)}$ is given by (1.14). We also remark that a similar result to Theorem 1.3 is still valid even if $\rho = \infty$. However, in this case, we have to modify the regularities of solutions and the properties (1.10), (1.11), (1.12), (1.15), (1.16), and (1.17) due to a lack of density.

(ii) Noting (1.7), we see that property (1.17) implies

$$\begin{aligned}\|n(t)\|_{L^{q,1}(\mathbb{R}^N)} &= o(t^{-(N/2)(2/N-1/q)}),\\ \|\nabla c(t)\|_{L^{r,1}(\mathbb{R}^N)} &= o(t^{-(N/2)(1/N-1/r)}),\\ \|\boldsymbol{u}(t)\|_{L^{r,1}(\mathbb{R}^N)} &= o(t^{-(N/2)(1/N-1/r)})\end{aligned}$$

as $t\to\infty$. Hence, we see that these decay rates coincide with those of the solutions to the linear heat equation. In addition, our result is regarded as an *improved* version compared with [28]. Here, we mention that our method of construction of global solutions is based on the linear analysis and perturbation theory. We also refer to [9–11] for the decay property of global solutions to (1.1).

(iii) Such a result and the remaining property (1.15) are shown by the density argument. In particular, our condition $1 \le \rho < \infty$ plays a key role in the proof of refined decay properties. More precisely, we use the method of Kato [26, Note]. To this end, we combine the *continuous dependence* of solutions given in Theorem 1.1 (iii) and the density argument. However, from estimate (1.13), we see that the term $\|c_0 - \tilde{c}_0\|_{L^\infty(\mathbb{R}^N)}$ disturbs the decay of the global solutions since $C_0^\infty(\mathbb{R}^N)$ is *not* dense in $L^\infty(\mathbb{R}^N)$. Therefore, to show the properties (1.15) and (1.17), we consider smooth initial data $n_0 \in C_0^\infty(\mathbb{R}^N)$ and $\boldsymbol{u}_0 \in (C_0^\infty(\mathbb{R}^N))^N$ and the *original* initial datum $c_0 \in L^\infty(\mathbb{R}^N)$. Even if we consider such a case, we may show that the corresponding solutions have the desired decay properties. We also mention that a similar remark to property (1.11) may be said for (1.16); although a lack of density disturbs the usual decay property, we see that $\lim_{t\to\infty}\langle c(t),\psi\rangle = 0$ for all $\psi \in L^1(\mathbb{R}^N)$ from the duality argument.

This article is organized as follows: In the next section, we recall the notation and definitions of function spaces. Some fundamental properties of the Lorentz spaces are also stated. In Section 3, we show the smoothing estimates of the heat semigroup on Lorentz spaces. We also prove the standard decay properties of the heat semigroup. Section 4 is devoted to the proof of our main results. To this end, we show the suitable estimates of the nonlinear terms. After that, we construct local solutions of (1.5), i.e., we show Theorem 1.1. Finally, we also show the existence of global solutions and asymptotic behavior, i.e., we show Theorem 1.3.

2 Preliminaries

2.1 Notation and function spaces

Here, we recall the notation and definitions of the function spaces: Let $d \in \mathbb{N}$. Assume that $\Omega \subset \mathbb{R}^d$ is a domain and E is a Banach space. Then, $\mathrm{BC}(\Omega;\, E)$ and $\mathrm{BUC}(\Omega;\, E)$ denote the spaces of all bounded continuous E-valued functions on Ω and bounded uniformly continuous E-valued functions on Ω, respectively. Note that $\mathrm{BC}(\Omega;\, E)$ and $\mathrm{BUC}(\Omega;\, E)$ are the Banach spaces equipped with the norm $\|f\|_{L^\infty(\Omega;\, E)} \coloneqq \sup_{x\in\Omega}\|f(x)\|_E$ of the Bochner-Lebesgue space $L^\infty(\Omega;\, E)$. In particular, we set $\mathrm{BUC} \coloneqq \mathrm{BUC}(\mathbb{R}^N;\, \mathbb{C})$.

For the Bochner-Lebesgue space $L^1(\Omega;\, E)$, let $L^1_{\mathrm{loc}}(\Omega;\, E)$ denote the set of all functions f such that $f \in L^1(K;\, E)$ for any compact subset $K \subset \Omega$. In addition, for the function $f \in L^1_{\mathrm{loc}}(\Omega;\, E)$, let us define the rearrangement f^* of f by setting

$$f^*(\lambda) \coloneqq \inf\{\eta \in (0,\infty) | \mu(x \in \Omega | \|f(x)\|_E > \eta) \le \lambda\}, \quad 0 < \lambda < \infty,$$

where μ denotes the usual Lebesgue measure on Ω. We also define the Lorentz spaces $L^{q,\rho}(\Omega;\, E) \coloneqq \{f \in L^1_{\mathrm{loc}}(\Omega;\, E) \mid \|f\|_{L^{q,\rho}(\Omega;\, E)} < \infty\}$ for $1 < q < \infty$ and $1 \le \rho \le \infty$, where

$$\|f\|_{L^{q,\rho}(\Omega;\, E)} \coloneqq \begin{cases} \left[\displaystyle\int_0^\infty (\lambda^{1/q} f^*(\lambda))^\rho \mathrm{d}\lambda/\lambda\right]^{1/\rho} & \text{if } 1 \le \rho < \infty, \\ \displaystyle\sup_{0<\lambda<\infty} \lambda^{1/q} f^*(\lambda) & \text{if } \rho = \infty. \end{cases}$$

We remark that $\|f\|_{L^{q,\rho}(\Omega;\, E)} = \|\|f\|_E\|_{L^{q,\rho}(\Omega;\, \mathbb{C})}$ holds from a simple calculation. In particular, we set $L^{q,\rho}(\Omega) \coloneqq L^{q,\rho}(\Omega;\, \mathbb{C})$ and $L^{q,\rho} \coloneqq L^{q,\rho}(\mathbb{R}^N;\, \mathbb{C})$. We also abbreviate as follows:

$$\|f\|_{L^\infty_T(E)} \coloneqq \|f\|_{L^\infty((0,T);\, E)}, \quad \|f\|_{L^{\alpha,\rho}_T(E)} \coloneqq \|f\|_{L^{\alpha,\rho}((0,T);\, E)},$$
$$\|f\|_{L^\infty(E)} \coloneqq \|f\|_{L^\infty((0,\infty);\, E)}, \quad \|f\|_{L^{\alpha,\rho}(E)} \coloneqq \|f\|_{L^{\alpha,\rho}((0,\infty);\, E)}.$$

For the Helmholtz projection operator P, we set $P(L^{q,\rho})^N \coloneqq \{P\boldsymbol{u} \in (L^{q,\rho})^N \mid \boldsymbol{u} \in (L^{q,\rho})^N\}$, where the boundedness of $P : (L^{q,\rho})^N \to (L^{q,\rho})^N$ is mentioned in, e.g., [50, p. 641].

2.2 Fundamental properties of the Lorentz spaces

In this article, we focus on the Lorentz spaces, and thus, we recall the fundamental properties here.

Proposition 2.1. *Let $d \in \mathbb{N}$. Assume that $\Omega \subset \mathbb{R}^d$ is a domain and E is a Banach space. Then, the following statements hold:*

(*i*) *Let $1 < q < \infty$ and $1 \le \rho \le \infty$. Then, for every $f, g \in L^{q,\rho}(\Omega;\, E)$, it holds that $f + g \in L^{q,\rho}(\Omega;\, E)$ with the estimate*

$$\|f + g\|_{L^{q,\rho}(\Omega;\, E)} \le 2^{1/q}(\|f\|_{L^{q,\rho}(\Omega;\, E)} + \|g\|_{L^{q,\rho}(\Omega;\, E)}).$$

In addition, it holds that $L^{q,q}(\Omega;\, E) = L^q(\Omega;\, E)$. Furthermore, for every $f \in L^{q,\infty}(\Omega;\, E)$, the following equality

$$\|f\|_{L^{q,\infty}(\Omega;\,E)} = \sup_{0<\lambda<\infty} \lambda\mu(x \in \Omega \mid \|f(x)\|_E > \lambda)^{1/q}$$

holds.

(*ii*) *Let* $1 < q < \infty$ *and* $1 \le \rho_0 \le \rho_1 \le \infty$. *Then, the following continuous embeddings*

$$L^{q,1}(\Omega;\,E) \subset L^{q,\rho_0}(\Omega;\,E) \subset L^{q,\rho_1}(\Omega;\,E) \subset L^{q,\infty}(\Omega;\,E)$$

hold.

(*iii*) *Let* $1 < q < \infty$ *and* $1 \le \rho \le \infty$. *Suppose that* $q_0, q_1 \in (1, \infty)$ *and* $\rho_0, \rho_1 \in [1, \infty]$ *satisfy* $1/q = 1/q_0 + 1/q_1$ *and* $1/\rho = 1/\rho_0 + 1/\rho_1$, *respectively. Then, for every* $f \in L^{q_0,\rho_0}(\Omega)$ *and* $g \in L^{q_1,\rho_1}(\Omega)$, *it holds that* $fg \in L^{q,\rho}(\Omega)$ *with the estimate*

$$\|fg\|_{L^{q,\rho}(\Omega)} \le 2^{1/q}\|f\|_{L^{q_0,\rho_0}(\Omega)}\|g\|_{L^{q_1,\rho_1}(\Omega)}.$$

(*iv*) *Let* $1 < q < \infty$ *and* $1 \le \rho \le \infty$. *Then, for every* $f \in L^{\infty}(\Omega)$ *and* $g \in L^{q,\rho}(\Omega)$, *it holds that* $fg \in L^{q,\rho}(\Omega)$ *with the estimate*

$$\|fg\|_{L^{q,\rho}(\Omega)} \le 2^{1/q}\|f\|_{L^{\infty}(\Omega)}\|g\|_{L^{q,\rho}(\Omega)}.$$

(*v*) *Let* $1 \le q_0 < q_1 \le \infty$, $1 \le \rho \le \infty$, *and* $0 < \theta < 1$. *Then, it holds that*

$$(L^{q_0}(\Omega;\,E), L^{q_1}(\Omega;\,E))_{\theta,\rho} = L^{q,\rho}(\Omega;\,E),$$

where q satisfies

$$1/q = (1-\theta)/q_0 + \theta/q_1 \tag{2.1}$$

and $(\cdot,\cdot)_{\theta,\rho}$ *denotes the real interpolation spaces. In addition, let* $1 < q_0 < q_1 < \infty$, $\rho, \rho_0, \rho_1 \in [1, \infty]$, *and* $0 < \theta < 1$. *Then, it holds that*

$$(L^{q_0,\rho_0}(\Omega;\,E), L^{q_1,\rho_1}(\Omega;\,E))_{\theta,\rho} = L^{q,\rho}(\Omega;\,E),$$

where q satisfies (2.1).

(*vi*) *Let* $1 < q < \infty$ *and* $1 \le \rho < \infty$. *Then, the set* $C_0^\infty(\Omega)$ *of smooth functions with compact support is dense in* $L^{q,\rho}(\Omega)$.

Proof. The assertions of (i), (ii), and (v) are shown by Castillo and Rafeiro [7, p. 216 and Theorems 6.3 (b), 6.4 (a), and 6.6] and Bergh and Löfström [1, Theorems 5.2.1 and 5.3.1]. Hence, it remains to verify the assertions of (iii), (iv), and (vi).

Concerning (iii), we assume that $\rho, \rho_0, \rho_1 \in [1, \infty)$. Then, since $(fg)^*(\lambda) \le f^*(\lambda/2)g^*(\lambda/2)$ for all $0 < \lambda < \infty$ from [7, Theorem 4.11], we see by the Hölder inequality that

$$\begin{aligned}\int_0^\infty (\lambda^{1/q}(fg)^*(\lambda))^\rho \frac{\mathrm{d}\lambda}{\lambda} &\le \int_0^\infty (\lambda^{1/q} f^*(\lambda/2) g^*(\lambda/2))^\rho \frac{\mathrm{d}\lambda}{\lambda} \\ &= 2^{\rho/q} \int_0^\infty (\eta^{1/q_0} f^*(\eta) \cdot \eta^{1/q_1} g^*(\eta))^\rho \frac{d\eta}{\eta} \\ &\le 2^{\rho/q}\|f\|^\rho_{L^{q_0,\rho_0}(\Omega)}\|g\|^\rho_{L^{q_1,\rho_1}(\Omega)},\end{aligned}$$

which yields the desired result. The remaining cases are shown in the same way. Moreover, we may show the assertion of (iv) as well.

Finally, we show (vi). The condition $1 \le \rho < \infty$ and [7, Theorem 6.19] imply that we may take a sequence $\{f_j\}_{j\in\mathbb{N}} \subset (L^1 \cap L^\infty)(\Omega)$ of simple functions satisfying $\lim_{j\to\infty}\|f - f_j\|_{L^{q,\rho}(\Omega)} = 0$. In addition, since $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$, for each $j \in \mathbb{N}$, there exists a sequence $\{f_{j,k}\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ of functions such that $\lim_{k\to\infty}\|f_j - f_{j,k}\|_{L^1(\Omega)} = 0$. Noting that the assertion of (v) yields $(L^1(\Omega), L^\infty(\Omega))_{1-1/q,\rho} = L^{q,\rho}(\Omega)$, we see by [38, Corollary 1.7] that

$$\begin{aligned}\|f - f_{j,k}\|_{L^{q,\rho}(\Omega)} &\le C(\|f - f_j\|_{L^{q,\rho}(\Omega)} + \|f_j - f_{j,k}\|_{L^{q,\rho}(\Omega)}) \\ &\le C(\|f - f_j\|_{L^{q,\rho}(\Omega)} + \|f_j - f_{j,k}\|_{L^1(\Omega)}^{1/q} \|f_j - f_{j,k}\|_{L^\infty(\Omega)}^{1-1/q}),\end{aligned}$$

which gives

$$\limsup_{k\to\infty} \|f - f_{j,k}\|_{L^{q,\rho}(\Omega)} \le C\|f - f_j\|_{L^{q,\rho}(\Omega)}.$$

Therefore, by letting $j \to \infty$, we have the desired result. □

Remark 2.2. (i) Proposition 2.1 (i) gives the *quasi-triangle inequality* on the Lorentz spaces $L^{q,\rho}(\Omega; E)$. More precisely, it is known that $L^{q,\rho}(\Omega; E)$ are *not* normed spaces in general [7, Theorem 6.5 (a)]. However, by introducing

$$[\![f]\!]_{L^{q,\rho}(\Omega;\,E)} := \begin{cases} \left\{ \int_0^\infty (\lambda^{1/q} f^{**}(\lambda))^\rho d\lambda/\lambda \right\}^{1/\rho} & \text{if } 1 \le \rho < \infty, \\ \sup_{0<\lambda<\infty} \lambda^{1/q} f^{**}(\lambda) & \text{if } \rho = \infty, \end{cases}$$

where $f^{**}(\lambda) := \lambda^{-1}\int_0^\lambda f^*(\eta)d\eta$ for $0 < \lambda < \infty$, we may show that $[\![\cdot]\!]_{L^{q,\rho}(\Omega;\,E)}$ are *norms* and

$$\|f\|_{L^{q,\rho}(\Omega;\,E)} \le [\![f]\!]_{L^{q,\rho}(\Omega;\,E)} \le q(q-1)^{-1}\|f\|_{L^{q,\rho}(\Omega;\,E)}$$

holds for all $f \in L^{q,\rho}(\Omega; E)$ [7, p. 228 and Theorem 6.14]. In addition, the Lorentz spaces $L^{q,\rho}(\Omega; E)$ equipped with the norms $[\![\cdot]\!]_{L^{q,\rho}(\Omega;\,E)}$ become Banach spaces [7, Theorem 6.18]. Hence, we may regard $L^{q,\rho}(\Omega; E)$ as Banach spaces even though $\|\cdot\|_{L^{q,\rho}(\Omega;\,E)}$ do not satisfy the triangle inequality.

(ii) In this article, we consider the Lorentz spaces $L^{q,\rho}$ for $1 < q < \infty$. If we consider the case of $q = 1$, then we do *not* know the equivalence of norms $[\![\cdot]\!]_{L^{1,\rho}(\Omega;\,E)}$ and $\|\cdot\|_{L^{1,\rho}(\Omega;\,E)}$ due to the coefficient $q(q-1)^{-1}$. In addition, if we consider the case of $q = \infty$, then we see by [7, p. 216] that $L^{\infty,\rho}(\Omega; E) = \{0\}$ for any $1 \le \rho < \infty$. Thus, we exclude the *marginal cases* $q = 1$ and $q = \infty$.

3 Heat semigroup on Lorentz spaces

In this section, we show the properties of the heat semigroup $\{e^{t\Delta}\}_{t>0}$ defined on Lorentz spaces $L^{q,\rho}$. We first show the smoothing estimates and space-time estimates of the heat semigroup. These estimates play a key role in the proof of our main results. We may regard the assertions of (iii) and (iv) below as the result corresponding to that given by Kozono and Shimizu [33, Proposition 2.1]; they showed the space-time estimate in the homogeneous Besov spaces framework, where the homogeneous Besov spaces are characterized by the real interpolation spaces of the homogeneous Sobolev spaces. Our method also relies on real interpolation. We have to remark that Yamazaki [50, Corollary 2.3] has shown a part of our estimates: Even so, we attach all the proofs in detail for the convenience of the readers.

Proposition 3.1. *Let $1 < q_0 < q_1 < \infty$ and $1 \le \rho \le \infty$. Set $(Hf)(t) := e^{t\Delta}f$ for $f \in L^1_{\mathrm{loc}}$ and $0 < t < \infty$. Then, the following assertions hold*:

(i) *For every $f \in L^{q_0,\rho}$, it holds that $Hf \in L^\infty((0,\infty); L^{q_0,\rho})$ with the estimate*

$$\|Hf\|_{L^\infty(L^{q_0,\rho})} \le C\|f\|_{L^{q_0,\rho}},$$

where $C > 0$ is a constant independent of f. In addition, it holds that $\nabla(Hf)(t) \in (L^{q_0,\rho})^N$ and $\nabla^2(Hf)(t) \in (L^{q_0,\rho})^{N^2}$ for all $0 < t < \infty$ with the estimate

$$\|\nabla^j(Hf)(t)\|_{L^{q_0,\rho}} \le Ct^{-j/2}\|f\|_{L^{q_0,\rho}}$$

for $j \in \{1, 2\}$, where $C > 0$ is a constant independent of t and f.

(ii) For every $f \in L^{q_0,\infty}$, *it holds that* $(Hf)(t) \in L^{q_1,1} \cap L^\infty$ *and* $\nabla(Hf)(t) \in (L^{q_1,1} \cap L^\infty)^N$ *for all* $0 < t < \infty$ *with the estimates*

$$\|\nabla^j(Hf)(t)\|_{L^{q_1,1}} \le Ct^{-(N/2)(1/q_0-1/q_1)-j/2}\|f\|_{L^{q_0,\infty}},$$
$$\|\nabla^j(Hf)(t)\|_{L^\infty} \le Ct^{-N/(2q_0)-j/2}\|f\|_{L^{q_0,\infty}}$$

for $j \in \{0,1\}$. *In addition, for every* $f \in L^1$, *it holds that* $(Hf)(t) \in L^{q_1,1}$ *and* $\nabla(Hf)(t) \in (L^{q_1,1})^N$ *for all* $0 < t < \infty$ *with the estimate*

$$\|\nabla^j(Hf)(t)\|_{L^{q_1,1}} \le Ct^{-(N/2)(1-1/q_1)-j/2}\|f\|_{L^1}$$

for $j \in \{0,1\}$.

(iii) Let $j \in \{0,1\}$ *and let* α_j *satisfy* $1/\alpha_j = (N/2)(1/q_0 - 1/q_1) + j/2$. *Suppose that* $1/q_0 - 1/q_1 < 2/N$. *Then, for every* $f \in L^{q_0,\rho}$, *it holds that* $Hf \in L^{\alpha_0,\rho}((0,\infty); L^{q_1,1})$ *with the estimate*

$$\|Hf\|_{L^{\alpha_0,\rho}(L^{q_1,1})} \le C\|f\|_{L^{q_0,\rho}}.$$

In addition, suppose that $1/q_0 - 1/q_1 < 1/N$. *Then, for every* $f \in L^{q_0,\rho}$, *it holds that* $\nabla Hf \in L^{\alpha_1,\rho}((0,\infty); (L^{q_1,1})^N)$ *with the estimate*

$$\|\nabla Hf\|_{L^{\alpha_1,\rho}(L^{q_1,1})} \le C\|f\|_{L^{q_0,\rho}}.$$

(iv) Let $j \in \{0,1\}$ *and let* β_j *satisfy* $1/\beta_j = N/(2q_0) + j/2$. *Suppose that* $N/2 < q_0 < \infty$. *Then, for every* $f \in L^{q_0,\rho}$, *it holds that* $Hf \in L^{\beta_0,\rho}((0,\infty); L^\infty)$ *with the estimate*

$$\|Hf\|_{L^{\beta_0,\rho}(L^\infty)} \le C\|f\|_{L^{q_0,\rho}}.$$

In addition, suppose that $N < q_0 < \infty$. *Then, for every* $f \in L^{q_0,\rho}$, *it holds that* $\nabla Hf \in L^{\beta_1,\rho}((0,\infty); (L^\infty)^N)$ *with the estimate*

$$\|\nabla Hf\|_{L^{\beta_1,\rho}(L^\infty)} \le C\|f\|_{L^{q_0,\rho}}.$$

Proof. (i) Since $1 < q_0 < q_1 < \infty$, we may take $q_-, q_+ \in (1,\infty)$ so that

$$1 < q_- < q_0 < q_+ < q_1. \tag{3.1}$$

In addition, we take $0 < \theta < 1$ satisfying

$$1/q_0 = (1-\theta)/q_- + \theta/q_+. \tag{3.2}$$

Then, we see by the usual estimates of the heat semigroup that

$$\|\nabla^j e^{t\Delta} f_-\|_{L^{q_-}} \le Ct^{-j/2}\|f_-\|_{L^{q_-}}, \quad \|\nabla^j e^{t\Delta} f_+\|_{L^{q_+}} \le Ct^{-j/2}\|f_+\|_{L^{q_+}}$$

for all $0 < t < \infty$, $j \in \{0,1,2\}$, $f_- \in L^{q_-}$, and $f_+ \in L^{q_+}$. Since (3.2) and Proposition 2.1 (v) yield $(L^{q_-}, L^{q_+})_{\theta,\rho} = L^{q_0,\rho}$ for all $1 \le \rho \le \infty$, we see by [38, Theorem 1.6] that the desired estimates are valid.

(ii) We consider $q_-, q_+ \in (1,\infty)$ and $0 < \theta < 1$ satisfying (3.1) and (3.2) again. The usual estimates of the heat semigroup yield

$$\|\nabla^j e^{t\Delta} f_-\|_{L^{q_1}} \le Ct^{-(N/2)(1/q_- - 1/q_1)-j/2}\|f_-\|_{L^{q_-}},$$
$$\|\nabla^j e^{t\Delta} f_+\|_{L^{q_1}} \le Ct^{-(N/2)(1/q_+ - 1/q_1)-j/2}\|f_+\|_{L^{q_+}},$$
$$\|\nabla^j e^{t\Delta} f_-\|_{L^\infty} \le Ct^{-N/(2q_-)-j/2}\|f_-\|_{L^{q_-}},$$
$$\|\nabla^j e^{t\Delta} f_+\|_{L^\infty} \le Ct^{-N/(2q_+)-j/2}\|f_+\|_{L^{q_+}}$$

for all $0 < t < \infty$, $j \in \{0,1\}$, $f_- \in L^{q_-}$, and $f_+ \in L^{q_+}$. Therefore, by applying Proposition 2.1 (v) and [38, Theorem 1.6], we deduce that

$$\|\nabla^j e^{t\Delta} f\|_{L^{q_1}} \le C(t^{-(N/2)(1/q_- - 1/q_1)-j/2})^{1-\theta}(t^{-(N/2)(1/q_+ - 1/q_1)-j/2})^{\theta}\|f\|_{L^{q_0,\infty}} = Ct^{-(N/2)(1/q_0-1/q_1)-j/2}\|f\|_{L^{q_0,\infty}},$$
$$\|\nabla^j e^{t\Delta} f\|_{L^\infty} \le C(t^{-N/(2q_-)-j/2})^{1-\theta}(t^{-N/(2q_+)-j/2})^{\theta}\|f\|_{L^{q_0,\infty}} = Ct^{-N/(2q_0)-j/2}\|f\|_{L^{q_0,\infty}}$$

for all $0 < t < \infty$ and $f \in L^{q_0,\infty}$. Since the first estimate is valid for all $q_0 < q_1 < \infty$, by taking $r_-, r_+ \in (1, \infty)$ and $0 < \sigma < 1$ such that

$$q_0 < r_- < q_1 < r_+ < \infty, \quad 1/q_1 = (1-\sigma)/r_- + \sigma/r_+,$$

we have

$$\|\nabla^j e^{t\Delta} f\|_{L^{r_-}} \le C t^{-(N/2)(1/q_0 - 1/r_-) - j/2} \|f\|_{L^{q_0,\infty}},$$
$$\|\nabla^j e^{t\Delta} f\|_{L^{r_+}} \le C t^{-(N/2)(1/q_0 - 1/r_+) - j/2} \|f\|_{L^{q_0,\infty}}$$

for all $0 < t < \infty$ and $f \in L^{q_0,\infty}$. As Proposition 2.1 (v) yields $(L^{r_-}, L^{r_+})_{\sigma,1} = L^{q_1,1}$, we see by [38, Corollary 1.7] that

$$\begin{aligned}\|\nabla^j e^{t\Delta} f\|_{L^{q_1,1}} &\le C \|\nabla^j e^{t\Delta} f\|_{L^{r_-}}^{1-\sigma} \|\nabla^j e^{t\Delta} f\|_{L^{r_+}}^{\sigma} \\ &\le C(t^{-(N/2)(1/q_0-1/r_-)-j/2} \|f\|_{L^{q_0,\infty}})^{1-\sigma} (t^{-(N/2)(1/q_0-1/r_+)-j/2} \|f\|_{L^{q_0,\infty}})^{\sigma} \\ &= C t^{-(N/2)(1/q_0-1/q_1)-j/2} \|f\|_{L^{q_0,\infty}}.\end{aligned}$$

Similarly, noting that

$$\|\nabla^j e^{t\Delta} f\|_{L^{r_-}} \le C t^{-(N/2)(1-1/r_-) - j/2} \|f\|_{L^1},$$
$$\|\nabla^j e^{t\Delta} f\|_{L^{r_+}} \le C t^{-(N/2)(1-1/r_+) - j/2} \|f\|_{L^1}$$

for all $0 < t < \infty$ and $f \in L^1$, we also see by Proposition 2.1 (v) and [38, Corollary 1.7] that

$$\begin{aligned}\|\nabla^j e^{t\Delta} f\|_{L^{q_1,1}} &\le C \|\nabla^j e^{t\Delta} f\|_{L^{r_-}}^{1-\sigma} \|\nabla^j e^{t\Delta} f\|_{L^{r_+}}^{\sigma} \\ &\le C(t^{-(N/2)(1-1/r_-)-j/2} \|f\|_{L^1})^{1-\sigma} (t^{-(N/2)(1-1/r_+)-j/2} \|f\|_{L^1})^{\sigma} \\ &= C t^{-(N/2)(1-1/q_1)-j/2} \|f\|_{L^1},\end{aligned}$$

which yields the desired estimate.

(iii) Let $1 < q < q_1$. Then, we see by the assertion of (ii) that

$$\begin{aligned}&\mu(t \in (0,\infty) \mid \|\nabla^j e^{t\Delta} f\|_{L^{q_1,1}} > \lambda)^{(N/2)(1/q-1/q_1)+j/2} \\ &\le \mu(t \in (0,\infty) \mid C t^{-(N/2)(1/q-1/q_1)-j/2} \|f\|_{L^{q,\infty}} > \lambda)^{(N/2)(1/q-1/q_1)+j/2} \\ &= \mu(t \in (0,\infty) \mid (C\lambda^{-1} \|f\|_{L^{q,\infty}})^{\{(N/2)(1/q-1/q_1)+j/2\}^{-1}} > t)^{(N/2)(1/q-1/q_1)+j/2} \\ &= C\lambda^{-1} \|f\|_{L^{q,\infty}}\end{aligned}$$

for all $0 < \lambda < \infty$ and $f \in L^{q,\infty}$. In what follows, we consider the case of $j = 0$. Suppose that $1/q_0 - 1/q_1 < 2/N$. Then, we may take $q_-, q_+ \in (1, \infty)$ so that (3.1) and $1/q_- - 1/q_1 < 2/N$. We also take $0 < \theta < 1$ such that (3.2). In addition, let α_- and α_+ satisfy

$$1/\alpha_- = (N/2)(1/q_- - 1/q_1), \quad 1/\alpha_+ = (N/2)(1/q_+ - 1/q_1).$$

Then, we obtain $\alpha_-, \alpha_+ \in (1, \infty)$ and

$$(1-\theta)/\alpha_- + \theta/\alpha_+ = 1/\alpha_0$$

from condition (3.2). Therefore, since setting $q = q_-$ and $q = q_+$ in the above estimate yields

$$\mu(t \in (0,\infty) \mid \|e^{t\Delta} f_-\|_{L^{q_1,1}} > \lambda)^{1/\alpha_-} \le C\lambda^{-1} \|f_-\|_{L^{q_-,\infty}},$$
$$\mu(t \in (0,\infty) \mid \|e^{t\Delta} f_+\|_{L^{q_1,1}} > \lambda)^{1/\alpha_+} \le C\lambda^{-1} \|f_+\|_{L^{q_+,\infty}}$$

for all $0 < \lambda < \infty$, $f_- \in L^{q_-,\infty}$, and $f_+ \in L^{q_+,\infty}$, we see by Proposition 2.1 (i) that

$$\|Hf_-\|_{L^{\alpha_-,\infty}(L^{q_1,1})} \le C \|f_-\|_{L^{q_-,\infty}}, \quad \|Hf_+\|_{L^{\alpha_+,\infty}(L^{q_1,1})} \le C \|f_+\|_{L^{q_+,\infty}}.$$

By using Proposition 2.1 (v), we have

$$(L^{\alpha_-,\infty}((0,\infty); L^{q_1,1}), L^{\alpha_+,\infty}((0,\infty); L^{q_1,1}))_{\theta,\rho} = L^{\alpha_0,\rho}((0,\infty); L^{q_1,1}),$$
$$(L^{q_-,\infty}, L^{q_+,\infty})_{\theta,\rho} = L^{q_0,\rho}.$$

Hence, it holds by [38, Theorem 1.6] that the desired estimate is valid. The case of $j = 1$ may be shown in the same way.

(iv) We may show in the same way as in the proof of the assertion of (iii). This proves Proposition 3.1. □

By combining Proposition 3.1 and the standard density argument, we may show the following properties:

Proposition 3.2. *Let $1 < q_0 < q_1 < \infty$ and $1 \le \rho < \infty$. Then, for every $f \in L^{q_0,\rho}$, it holds that*

$$\lim_{t\to+0} \|e^{t\Delta}f - f\|_{L^{q_0,\rho}} + \lim_{t\to\infty} \|e^{t\Delta}f\|_{L^{q_0,\rho}} = 0, \tag{3.3}$$

$$\lim_{\substack{t\to+0,\\ t\to\infty}} \sum_{j=0}^{1} \left(t^{(N/2)(1/q_0-1/q_1)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^{q_1,1}} + t^{N/(2q_0)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^\infty}\right) = 0, \tag{3.4}$$

where the limit of (3.4) *implies either $t \to +0$ or $t \to \infty$. In particular, the heat semigroup $e^{t\Delta} : L^{q_0,\rho} \to L^{q_0,\rho}$ is a bounded analytic C_0-semigroup.*

Proof. Since $1 \le \rho < \infty$, by applying Proposition 2.1 (vi), we may take a sequence $\{f_k\}_{k\in\mathbb{N}} \subset C_0^\infty$ of functions so that $\lim_{k\to\infty}\|f - f_k\|_{L^{q_0,\rho}} = 0$. Here, noting that Proposition 2.1 (v) yields $(L^1, L^\infty)_{1-1/q_0,\rho} = L^{q_0,\rho}$, we see by [38, Corollary 1.7] that

$$\begin{aligned}\|e^{t\Delta}f_k - f_k\|_{L^{q_0,\rho}} &\le C\|e^{t\Delta}f_k - f_k\|_{L^1}^{1/q_0}\|e^{t\Delta}f_k - f_k\|_{L^\infty}^{1-1/q_0}\\ &\le C\|e^{t\Delta}f_k - f_k\|_{L^1}^{1/q_0}\cdot(2\|f_k\|_{L^\infty})^{1-1/q_0}\end{aligned}$$

for all $0 < t < \infty$ and $k \in \mathbb{N}$. Since the heat semigroup $e^{t\Delta} : L^1 \to L^1$ is a C_0-semigroup, we have $\lim_{t\to+0}\|e^{t\Delta}f_k - f_k\|_{L^{q_0,\rho}} = 0$. Hence, by the condition $q_0 > 1$ and Proposition 3.1 (i) and (ii), we observe that

$$\begin{aligned}\|e^{t\Delta}f - f\|_{L^{q_0,\rho}} &\le C(\|e^{t\Delta}f - e^{t\Delta}f_k\|_{L^{q_0,\rho}} + \|e^{t\Delta}f_k - f_k\|_{L^{q_0,\rho}} + \|f_k - f\|_{L^{q_0,\rho}})\\ &\le C(\|f_k - f\|_{L^{q_0,\rho}} + \|e^{t\Delta}f_k - f_k\|_{L^{q_0,\rho}}),\\ \|e^{t\Delta}f\|_{L^{q_0,\rho}} &\le C(\|e^{t\Delta}f - e^{t\Delta}f_k\|_{L^{q_0,\rho}} + \|e^{t\Delta}f_k\|_{L^{q_0,\rho}})\\ &\le C(\|f - f_k\|_{L^{q_0,\rho}} + t^{-(N/2)(1-1/q_0)}\|f_k\|_{L^1}),\end{aligned}$$

which yield

$$\limsup_{t\to+0}\|e^{t\Delta}f - f\|_{L^{q_0,\rho}} + \limsup_{t\to\infty}\|e^{t\Delta}f\|_{L^{q_0,\rho}} \le C\|f_k - f\|_{L^{q_0,\rho}}.$$

By letting $k \to \infty$, we obtain (3.3). Moreover, since Proposition 3.1 (i) yields $\|\Delta e^{t\Delta}f\|_{L^{q_0,\rho}} \le Ct^{-1}\|f\|_{L^{q_0,\rho}}$ for all $0 < t < \infty$ and $f \in L^{q_0,\rho}$, by applying [21, Theorem 4.6], we observe that the heat semigroup $e^{t\Delta} : L^{q_0,\rho} \to L^{q_0,\rho}$ is a bounded analytic C_0-semigroup.

In addition, by considering the sequence $\{f_k\}_{k\in\mathbb{N}} \subset C_0^\infty$ again, we also obtain

$$\begin{aligned}&t^{(N/2)(1/q_0-1/q_1)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^{q_1,1}} + t^{N/(2q_0)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^\infty}\\ &\le Ct^{(N/2)(1/q_0-1/q_1)+j/2}(\|\nabla^j e^{t\Delta}f - \nabla^j e^{t\Delta}f_k\|_{L^{q_1,1}} + \|\nabla^j e^{t\Delta}f_k\|_{L^{q_1,1}})\\ &\quad + t^{N/(2q_0)+j/2}(\|\nabla^j e^{t\Delta}f - \nabla^j e^{t\Delta}f_k\|_{L^\infty} + \|\nabla^j e^{t\Delta}f_k\|_{L^\infty})\end{aligned}$$

for all $0 < t < \infty$, $k \in \mathbb{N}$, and $j \in \{0, 1\}$. Hence, Proposition 3.1 (i) and (ii) imply that

$$\begin{aligned}&t^{(N/2)(1/q_0-1/q_1)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^{q_1,1}} + t^{N/(2q_0)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^\infty}\\ &\le C(\|f - f_k\|_{L^{q_0,\rho}} + t^{(N/2)(1/q_0-1/q_1)+j/2}\|\nabla^j f_k\|_{L^{q_1,1}} + t^{N/(2q_0)+j/2}\|\nabla^j f_k\|_{L^\infty})\end{aligned}$$

and

$$t^{(N/2)(1/q_0-1/q_1)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^{q_1,1}} + t^{N/(2q_0)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^\infty} \le C(\|f - f_k\|_{L^{q_0,\rho}} + t^{-(N/2)(1-1/q_0)}\|f_k\|_{L^1}).$$

Noting that $1 < q_0 < q_1$, we have

$$\limsup_{\substack{t\to+0,\\ t\to\infty}}\sum_{j=0}^{1}\big(t^{(N/2)(1/q_0-1/q_1)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^{q_1,1}}+t^{N/(2q_0)+j/2}\|\nabla^j e^{t\Delta}f\|_{L^\infty}\big)\le C\|f_k-f\|_{L^{q_0,\rho}},$$

and thus, we obtain (3.4) by letting $k\to\infty$. This proves Proposition 3.2. □

4 Construction of mild solutions in scaling invariant Lorentz spaces

4.1 Nonlinear estimates

In this section, we shall show our main results (Theorems 1.1 and 1.3). To this end, we follow the approach due to Kato [26]. Particularly, we show the suitable estimates of the nonlinear terms of the integral systems (1.5) by using the estimate of the heat semigroup in Proposition 3.1. We begin with the following estimates:

Proposition 4.1. *Let $1<\alpha<\infty$, $1\le\rho\le\infty$, $1/\alpha<\sigma<1$, and $0<T\le\infty$. For the function $f\in L^1_{\mathrm{loc}}(0,T)$, define $I_\sigma[f]$ by setting*

$$(I_\sigma[f])(t):=\int_0^t(t-\tau)^{-\sigma}|f(\tau)|\,\mathrm{d}\tau,\quad 0<t<T.$$

Then, the following statements hold:

(*i*) *Suppose that $f\in L^{\alpha_0,\rho}(0,T)$, where $1<\alpha_0<\infty$ satisfies $1/\alpha_0=1/\alpha+1-\sigma$. Then, it holds that $I_\sigma[f]\in L^{\alpha,\rho}(0,T)$ with the estimate*

$$\|I_\sigma[f]\|_{L_T^{\alpha,\rho}}\le C\|f\|_{L_T^{\alpha_0,\rho}},$$

where $C>0$ is a constant independent of T and f.

(*ii*) *Suppose that $f\in L^{\alpha_0,\rho}(0,T)$ and $g\in L^{\alpha_1,\rho}(0,T)$, where $\alpha_0,\alpha_1\in(1,\infty)$ satisfy $1/\alpha_0+1/\alpha_1=1/\alpha+1-\sigma$. Then, it holds that $I_\sigma[fg]\in L^{\alpha,\rho}(0,T)$ with the estimate*

$$\|I_\sigma[fg]\|_{L_T^{\alpha,\rho}}\le C\|f\|_{L_T^{\alpha_0,\rho}}\|g\|_{L_T^{\alpha_1,\rho}},$$

where $C>0$ is a constant independent of T, f, and g.

Proof. (i) Let $1<\alpha_*<\infty$ satisfy $1/\alpha_*<\sigma$. Then, we have $0<1/\alpha_*+1-\sigma<1$. Suppose that $f\in L^{(1/\alpha_*+1-\sigma)^{-1}}(0,T)$. By setting

$$f_{\mathrm{E}}(\tau):=\begin{cases}f(\tau) & \tau\in(0,T),\\ 0 & \tau\in\mathbb{R}\setminus(0,T),\end{cases}\qquad (I_{\sigma,\mathrm{E}}[f])(t):=\begin{cases}(I_\sigma[f])(t) & t\in(0,T),\\ 0 & t\in\mathbb{R}\setminus(0,T),\end{cases}$$

we obtain $(I_{\sigma,\mathrm{E}}[f])(t)\le\int_{\mathbb{R}}|t-\tau|^{-\sigma}|f_{\mathrm{E}}(\tau)|\,\mathrm{d}\tau$ for all $t\in\mathbb{R}$. Hence, the Hardy-Littlewood-Sobolev inequality [41, V, Theorem 1] yields

$$\|I_\sigma[f]\|_{L_T^{\alpha_*}}=\|I_{\sigma,\mathrm{E}}[f]\|_{L^{\alpha_*}(\mathbb{R})}\le C\|f_{\mathrm{E}}\|_{L^{(1/\alpha_*+1-\sigma)^{-1}}(\mathbb{R})}=C\|f\|_{L_T^{(1/\alpha_*+1-\sigma)^{-1}}}.\tag{4.1}$$

Next, take $\alpha_-,\alpha_+\in(1,\infty)$ so that $1<\alpha_-<\alpha<\alpha_+<\infty$ and $1/\alpha_-<\sigma$. We also take $0<\theta<1$ satisfying $1/\alpha=(1-\theta)/\alpha_-+\theta/\alpha_+$. Then, we see by (4.1) that

$$\|I_\sigma[f_-]\|_{L_T^{\alpha_-}}\le C\|f_-\|_{L_T^{(1/\alpha_-+1-\sigma)^{-1}}},\quad \|I_\sigma[f_+]\|_{L_T^{\alpha_+}}\le C\|f_+\|_{L_T^{(1/\alpha_++1-\sigma)^{-1}}}$$

for all $f_-\in L^{(1/\alpha_-+1-\sigma)^{-1}}(0,T)$ and $f_+\in L^{(1/\alpha_++1-\sigma)^{-1}}(0,T)$. Since

$$(1-\theta)(1/\alpha_-+1-\sigma)+\theta(1/\alpha_++1-\sigma)=1/\alpha+1-\sigma=1/\alpha_0,$$

Proposition 2.1 (v) implies that

$$(L^{\alpha_-}(0,T), L^{\alpha_+}(0,T))_{\theta,\rho} = L^{\alpha,\rho}(0,T),$$
$$(L^{(1/\alpha_-+1-\sigma)^{-1}}(0,T), L^{(1/\alpha_++1-\sigma)^{-1}}(0,T))_{\theta,\rho} = L^{\alpha_0,\rho}(0,T).$$

Therefore, the result of [38, Theorem 1.6] yields the desired estimate.

(ii) In the assertion of (i), by replacing f with fg, we have

$$\|I_\sigma[fg]\|_{L_T^{\alpha,\rho}} \le C\|fg\|_{L_T^{(1/\alpha+1-\sigma)^{-1},\rho}}.$$

Noting that $1/\alpha + 1 - \sigma = 1/\alpha_0 + 1/\alpha_1$, we see by Proposition 2.1 (ii) and (iii) that the desired estimate holds. □

By combining the estimates in Propositions 3.1 and 4.1, we show the estimates of the nonlinear terms in the Lorentz spaces framework.

Lemma 4.2. *Let $N/2 < q < \infty$, $N < r < \infty$, $1 \le \rho \le \infty$, and $0 \le \theta < 1$ and let $\alpha, \beta \in (1,\infty)$ satisfy* (1.7). *In addition, let $0 < T \le \infty$ and set*

$$(\mathcal{I}[f])(t) := \int_0^t e^{(t-\tau)\Delta} f(\tau)\,\mathrm{d}\tau, \quad 0 < t < T$$

for $f \in L^1_{\mathrm{loc}}((0,T); L^1_{\mathrm{loc}})$. Then, the following assertions hold:

(*i*) *Assume that $(\theta+1)/N < 1/q + 1/r < 1$ and let $1 < q_* < \infty$ satisfy $0 < 1/q + 1/r - 1/q_* < 1/N$. Then, it holds that*

$$\|\mathcal{I}[\nabla\cdot(n\boldsymbol{u})](t)\|_{L^{q_*,1}} \le Ct^{-(N/2)(2/N-1/q_*)-\theta/2} \sup_{0<\tau<t} \tau^{1/\alpha+\theta/2}\|n(\tau)\|_{L^{q,\infty}} \sup_{0<\tau<t} \tau^{1/\beta}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}},$$
$$\|\mathcal{I}[\nabla\cdot(n\nabla c)](t)\|_{L^{q_*,1}} \le Ct^{-(N/2)(2/N-1/q_*)-\theta/2} \sup_{0<\tau<t} \tau^{1/\alpha+\theta/2}\|n(\tau)\|_{L^{q,\infty}} \sup_{0<\tau<t} \tau^{1/\beta}\|\nabla c(\tau)\|_{L^{r,\infty}},$$
$$\|\mathcal{I}[\nabla\cdot(n\boldsymbol{u})]\|_{L_t^{\alpha,\rho}(L^{q,1})} \le C\|n\|_{L_t^{\alpha,\rho}(L^{q,\infty})}\|\boldsymbol{u}\|_{L_t^{\beta,\rho}(L^{r,\infty})},$$
$$\|\mathcal{I}[\nabla\cdot(n\nabla c)]\|_{L_t^{\alpha,\rho}(L^{q,1})} \le C\|n\|_{L_t^{\alpha,\rho}(L^{q,\infty})}\|\nabla c\|_{L_t^{\beta,\rho}(L^{r,\infty})}$$

for all $0 < t < T$, where $C > 0$ is a constant independent of T, t, n, c, and $\boldsymbol{u}$.

(*ii*) *Assume that $\boldsymbol{u}(t) \in P(L^{r,\infty})^N$ for all $0 < t < T$. Then, it holds that*

$$\|\mathcal{I}[\boldsymbol{u}\cdot\nabla c](t)\|_{L^\infty} \le Ct^{-\theta/2} \sup_{0<\tau<t} \|c(\tau)\|_{L^\infty} \sup_{0<\tau<t} \tau^{1/\beta+\theta/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}},$$
$$\|\mathcal{I}[nc](t)\|_{L^\infty} \le Ct^{-\theta/2} \sup_{0<\tau<t} \tau^{1/\alpha+\theta/2}\|n(\tau)\|_{L^{q,\infty}} \sup_{0<\tau<t} \|c(\tau)\|_{L^\infty}$$

for all $0 < t < T$.

(*iii*) *Assume that $\theta < 2N/r$ and let $1 < r_* < \infty$ satisfy $0 < 2/r - 1/r_* < 1/N$. Then, it holds that*

$$\|\nabla\mathcal{I}[\boldsymbol{u}\cdot\nabla c](t)\|_{L^{r_*,1}} \le Ct^{-(N/2)(1/N-1/r_*)-\theta/2} \sup_{0<\tau<t} \tau^{1/\beta+\theta/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}} \sup_{0<\tau<t} \tau^{1/\beta}\|\nabla c(\tau)\|_{L^{r,\infty}},$$
$$\|\mathcal{I}[P\nabla\cdot(\boldsymbol{u}\otimes\tilde{\boldsymbol{u}})](t)\|_{L^{r_*,1}} \le Ct^{-(N/2)(1/N-1/r_*)-\theta/2} \sup_{0<\tau<t} \tau^{1/\beta+\theta/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}} \sup_{0<\tau<t} \tau^{1/\beta}\|\tilde{\boldsymbol{u}}(\tau)\|_{L^{r,\infty}},$$
$$\|\nabla\mathcal{I}[\boldsymbol{u}\cdot\nabla c]\|_{L_t^{\beta,\rho}(L^{r,1})} \le C\|\boldsymbol{u}\|_{L_t^{\beta,\rho}(L^{r,\infty})}\|\nabla c\|_{L_t^{\beta,\rho}(L^{r,\infty})},$$
$$\|\mathcal{I}[P\nabla\cdot(\boldsymbol{u}\otimes\tilde{\boldsymbol{u}})]\|_{L_t^{\beta,\rho}(L^{r,1})} \le C\|\boldsymbol{u}\|_{L_t^{\beta,\rho}(L^{r,\infty})}\|\tilde{\boldsymbol{u}}\|_{L_t^{\beta,\rho}(L^{r,\infty})}$$

for all $0 < t < T$, where $C > 0$ is a constant independent of T, t, c, $\boldsymbol{u}$, and $\tilde{\boldsymbol{u}}$. Here, $\otimes$ denotes the tensor product.

(*iv*) *Assume that $\theta < N/q$ and let $1 < r_* < \infty$ satisfy $0 < 1/q - 1/r_* < 1/N$. Then, it holds that*

$$\|\nabla\mathcal{I}[nc](t)\|_{L^{r_*,1}} \le Ct^{-(N/2)(1/N-1/r_*)-\theta/2} \sup_{0<\tau<t} \tau^{1/\alpha+\theta/2}\|n(\tau)\|_{L^{q,\infty}} \sup_{0<\tau<t} \|c(\tau)\|_{L^\infty}$$

for all $0 < t < T$. In addition, if q and r satisfy $0 < 1/q - 1/r < 1/N$, then, it holds that

$$\|\nabla\mathcal{I}[nc]\|_{L_t^{\beta,\rho}(L^{r,1})} \le C\|n\|_{L_t^{\alpha,\rho}(L^{q,\infty})} \sup_{0<\tau<t} \|c(\tau)\|_{L^\infty}$$

for all $0 < t < T$.

(v) *Assume that* $\theta < N/q$ *and let* $1 < r_* < \infty$ *satisfy* $-1/N < 1/q - 1/r_* < 1/N$. *Then, it holds that*

$$\|\mathcal{I}[P(n\nabla\varphi)](t)\|_{L^{r_*,1}} \le Ct^{-(N/2)(1/N-1/r_*)-\theta/2}\|\nabla\varphi\|_{L^{N,\infty}} \sup_{0<\tau<t} \tau^{1/\alpha+\theta/2}\|n(\tau)\|_{L^{q,\infty}}$$

for all $0 < t < T$, *where* $C > 0$ *is a constant independent of* T, t, φ, *and* n. *In addition, if* q *and* r *satisfy* $-1/N < 1/q - 1/r < 1/N$, *then, it holds that*

$$\|\mathcal{I}[P(n\nabla\varphi)]\|_{L_t^{\beta,\rho}(L^{r,1})} \le C\|\nabla\varphi\|_{L^{N,\infty}}\|n\|_{L_t^{\alpha,\rho}(L^{q,\infty})}$$

for all $0 < t < T$.

Proof. (i) Take q_0 so that $1/q_0 = 1/q + 1/r$. Then, the assumption yields

$$\max\{(\theta + 1)/N, 1/q_*\} < 1/q_0 < \min\{1, 1/N + 1/q_*\}.$$

Since it holds that $\int_0^t (t-\tau)^{a-1}\tau^{b-1}\mathrm{d}\tau = B(a,b)t^{a+b-1}$ for all $0 < t < \infty$ and $a, b \in (0,\infty)$, where $B(a,b)$ denotes the beta function given by $B(a,b) := \int_0^1 (1-\lambda)^{a-1}\lambda^{b-1}\mathrm{d}\lambda < \infty$, we see by Propositions 2.1 (iii) and 3.1 (ii) that

$$\begin{aligned}
\|\mathcal{I}[\nabla\cdot(n\boldsymbol{u})](t)\|_{L^{q_*,1}} &\le C\int_0^t (t-\tau)^{-(N/2)(1/q_0-1/q_*)-1/2}\|(n\boldsymbol{u})(\tau)\|_{L^{q_0,\infty}}\mathrm{d}\tau \\
&\le C\int_0^t (t-\tau)^{-(N/2)(1/q_0-1/q_*)-1/2}\|n(\tau)\|_{L^{q,\infty}}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\mathrm{d}\tau \\
&\le CB(1/2 - (N/2)(1/q_0 - 1/q_*), N/(2q_0) - 1/2 - \theta/2)t^{-1+N/(2q_*)-\theta/2} \\
&\quad \times \sup_{0<\tau<t} \tau^{1-N/(2q)+\theta/2}\|n(\tau)\|_{L^{q,\infty}} \sup_{0<\tau<t}\tau^{1/2-N/(2r)}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}
\end{aligned}$$

for all $0 < t < T$. In addition, since we may take $q_* = q$, there holds

$$\|\mathcal{I}[\nabla\cdot(n\boldsymbol{u})](t)\|_{L^{q,1}} \le C\int_0^t (t-\tau)^{-(1-1/\beta)}\|n(\tau)\|_{L^{q,\infty}}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\mathrm{d}\tau.$$

Noting that $1/\alpha + 1/\beta = 1/\alpha + 1 - (1 - 1/\beta)$, we see by Proposition 4.1 (ii) that

$$\|\|\mathcal{I}[\nabla\cdot(n\boldsymbol{u})]\|_{L^{q,1}}\|_{L_t^{\alpha,\rho}} \le C\,\|\|n\|_{L^{q,\infty}}\|_{L_t^{\alpha,\rho}}\|\|\boldsymbol{u}\|_{L^{r,\infty}}\|_{L_t^{\beta,\rho}}.$$

The remaining estimates are obtained by replacing $\boldsymbol{u}$ with ∇c.

(ii) We remark that $\boldsymbol{u}\cdot\nabla c = \nabla\cdot(c\boldsymbol{u})$ due to $\nabla\cdot\boldsymbol{u} = 0$. Since $q > N/2$ and $r > N$, it holds by Propositions 2.1 (iv) and 3.1 (ii) that

$$\begin{aligned}
\|\mathcal{I}[\boldsymbol{u}\cdot\nabla c](t)\|_{L^\infty} &\le C\int_0^t (t-\tau)^{-N/(2r)-1/2}\|(c\boldsymbol{u})(\tau)\|_{L^{r,\infty}}\mathrm{d}\tau \\
&\le C\int_0^t (t-\tau)^{-N/(2r)-1/2}\|c(\tau)\|_{L^\infty}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\mathrm{d}\tau \\
&\le CB(1/2 - N/(2r), N/(2r) + 1/2 - \theta/2)t^{-\theta/2}\sup_{0<\tau<t}\|c(\tau)\|_{L^\infty}\sup_{0<\tau<t}\tau^{1/2-N/(2r)+\theta/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{I}[nc](t)\|_{L^\infty} &\le C\int_0^t (t-\tau)^{-N/(2q)}\|(nc)(\tau)\|_{L^{q,\infty}}\mathrm{d}\tau \\
&\le C\int_0^t (t-\tau)^{-N/(2q)}\|n(\tau)\|_{L^{q,\infty}}\|c(\tau)\|_{L^\infty}\mathrm{d}\tau \\
&\le CB(1 - N/(2q), N/(2q) - \theta/2)t^{-\theta/2}\sup_{0<\tau<t}\tau^{1-N/(2q)+\theta/2}\|n(\tau)\|_{L^{q,\infty}}\sup_{0<\tau<t}\|c(\tau)\|_{L^\infty}
\end{aligned}$$

for all $0 < t < T$.

(iii) Since $r > N$, $N/r - \theta/2 > 0$, and $0 < 2/r - 1/r_* < 1/N$, we see by Propositions 2.1 (iii) and 3.1 (ii) that

$$\begin{aligned}\|\nabla\mathcal{I}[\boldsymbol{u}\cdot\nabla c](t)\|_{L^{r_*,1}} &\le C\int_0^t (t-\tau)^{-(N/2)(2/r-1/r_*)-1/2}\|(\boldsymbol{u}\cdot\nabla c)(\tau)\|_{L^{r/2,\infty}}\,\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-(N/2)(2/r-1/r_*)-1/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\|\nabla c(\tau)\|_{L^{r,\infty}}\,\mathrm{d}\tau\\ &\le CB(1/2-(N/2)(2/r-1/r_*), N/r-\theta/2)t^{-1/2+N/(2r_*)-\theta/2}\\ &\quad\times\sup_{0<\tau<t}\tau^{1/2-N/(2r)+\theta/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\sup_{0<\tau<t}\tau^{1/2-N/(2r)}\|\nabla c(\tau)\|_{L^{r,\infty}}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{I}[P\nabla\cdot(\boldsymbol{u}\otimes\tilde{\boldsymbol{u}})](t)\|_{L^{r_*,1}} &\le C\int_0^t (t-\tau)^{-(N/2)(2/r-1/r_*)-1/2}\|(\boldsymbol{u}\otimes\tilde{\boldsymbol{u}})(\tau)\|_{L^{r/2,\infty}}\,\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-(N/2)(2/r-1/r_*)-1/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\|\tilde{\boldsymbol{u}}(\tau)\|_{L^{r,\infty}}\,\mathrm{d}\tau\\ &\le CB(1/2-(N/2)(2/r-1/r_*), N/r-\theta/2)t^{-1/2+N/(2r_*)-\theta/2}\\ &\quad\times\sup_{0<\tau<t}\tau^{1/2-N/(2r)+\theta/2}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\sup_{0<\tau<t}\tau^{1/2-N/(2r)}\|\tilde{\boldsymbol{u}}(\tau)\|_{L^{r,\infty}}\end{aligned}$$

for all $0 < t < T$. Since we may take $r_* = r$, we deduce that

$$\begin{aligned}\|\nabla\mathcal{I}[\boldsymbol{u}\cdot\nabla c](t)\|_{L^{r,1}} &\le C\int_0^t (t-\tau)^{-(1-1/\beta)}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\|\nabla c(\tau)\|_{L^{r,\infty}}\,\mathrm{d}\tau,\\ \|\mathcal{I}[P\nabla\cdot(\boldsymbol{u}\otimes\tilde{\boldsymbol{u}})](t)\|_{L^{r,1}} &\le C\int_0^t (t-\tau)^{-(1-1/\beta)}\|\boldsymbol{u}(\tau)\|_{L^{r,\infty}}\|\tilde{\boldsymbol{u}}(\tau)\|_{L^{r,\infty}}\,\mathrm{d}\tau.\end{aligned}$$

Noting that $1/\beta + 1/\beta = 1/\beta + 1 - (1 - 1/\beta)$, we apply Proposition 4.1 (ii) to obtain the desired estimates.

(iv) Since $N/(2q) - \theta/2 > 0$ and $0 < 1/q - 1/r_* < 1/N$, it holds by Propositions 2.1 (iv) and 3.1 (ii) that

$$\begin{aligned}\|\nabla\mathcal{I}[nc](t)\|_{L^{r_*,1}} &\le C\int_0^t (t-\tau)^{-(N/2)(1/q-1/r_*)-1/2}\|(nc)(\tau)\|_{L^{q,\infty}}\,\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-(N/2)(1/q-1/r_*)-1/2}\|n(\tau)\|_{L^{q,\infty}}\|c(\tau)\|_{L^\infty}\,\mathrm{d}\tau\\ &\le CB(1/2-(N/2)(1/q-1/r_*), N/(2q)-\theta/2)t^{-1/2+N/(2r_*)-\theta/2}\\ &\quad\times\sup_{0<\tau<t}\tau^{1-N/(2q)+\theta/2}\|n(\tau)\|_{L^{q,\infty}}\sup_{0<\tau<t}\|c(\tau)\|_{L^\infty}\end{aligned}$$

for all $0 < t < T$. Since the condition $0 < 1/q - 1/r < 1/N$ implies that we may take $r_* = r$, it holds that

$$\|\nabla\mathcal{I}[nc](t)\|_{L^{r,1}} \le C\sup_{0<\tau<t}\|c(\tau)\|_{L^\infty}\int_0^t (t-\tau)^{-(1-1/\alpha+1/\beta)}\|n(\tau)\|_{L^{q,\infty}}\,\mathrm{d}\tau.$$

Hence, we see by $1/\alpha = 1/\beta + 1 - (1 - 1/\alpha + 1/\beta)$ and Proposition 4.1 (i) that the desired estimate holds.

(v) Take r_0 so that $1/r_0 = 1/q + 1/N$. Then, the assumption yields $1/r_* < 1/r_0 < 2/N + 1/r_*$. Thus, we see by $N/(2q) - \theta/2 > 0$ and Propositions 2.1 (iii) and 3.1 (ii) that

$$\begin{aligned}\|\mathcal{I}[P(n\nabla\varphi)](t)\|_{L^{r_*,1}} &\le C\int_0^t (t-\tau)^{-(N/2)(1/r_0-1/r_*)}\|n(\tau)\nabla\varphi\|_{L^{r_0,\infty}}\,\mathrm{d}\tau\\ &\le C\int_0^t (t-\tau)^{-(N/2)(1/r_0-1/r_*)}\|n(\tau)\|_{L^{q,\infty}}\|\nabla\varphi\|_{L^{N,\infty}}\,\mathrm{d}\tau\\ &\le CB(1-(N/2)(1/r_0-1/r_*), N/(2q)-\theta/2)t^{-1/2+N/(2r_*)-\theta/2}\\ &\quad\times\|\nabla\varphi\|_{L^{N,\infty}}\sup_{0<\tau<t}\tau^{1-N/(2q)+\theta/2}\|n(\tau)\|_{L^{q,\infty}}\end{aligned}$$

for all $0 < t < T$. Since the condition $-1/N < 1/q - 1/r < 1/N$ implies that we may take $r_* = r$, it holds that

$$\|\mathcal{I}[P(n\nabla\varphi)](t)\|_{L^{r,1}} \le C\|\nabla\varphi\|_{L^{N,\infty}}\int_0^t (t-\tau)^{-(1-1/\alpha+1/\beta)}\|n(\tau)\|_{L^{q,\infty}}\,\mathrm{d}\tau,$$

which yields the desired estimate with the aid of Proposition 4.1 (i). This completes the proof of Lemma 4.2. □

Remark 4.3. Let $N/2 < q < N$. Then, it holds that $\max\{2/N - 1/q, 1/q - 1/N\} < 1/N$, namely, $N < \min\{Nq/(2q-N), Nq/(N-q)\}$, and thus, we may take r so that (1.6). Hence, α and β given by (1.7) satisfy $2 < \alpha < \infty$ and $4 < \beta < \infty$, respectively. In addition, since q and r satisfy

$$1/(2N) < 1/r < 1/N < 1/q < 2/N, \quad 2/N < 1/q + 1/r < 3/N, \quad 0 < 1/q - 1/r < 1/N,$$

by setting $q_* = N/2$ or $q_* = q$ and $r_* = N$ or $r_* = r$, we observe that all the estimates given in Lemma 4.2 are valid for any $0 \le \theta < 1$.

4.2 Local mild solutions: Proof of Theorem 1.1

Once we show the nonlinear estimates in Lemma 4.2, we may show the existence of mild solutions of (1.1) by the Banach fixed point theorem. We mention that since the term $n\nabla\varphi$ appearing in (1.5) is regarded as a linear term, we have to choose $\nabla\varphi$ sufficiently small in general. However, by a slight modification of the definition of the mapping, we may show that the mapping is a contraction mapping *without* assuming any smallness of $\nabla\varphi$. In addition, it is *not necessary* to assume the smallness condition of the norm $\|c_0\|_{L^\infty}$ although C_0^∞ is not dense in L^∞.

Proof of Theorem 1.1. (i) Take α and β so that (1.7). Let $T > 0$ and define the function space X_T by setting

$$X_T := \left\{ (n, c, \boldsymbol{u}) \;\middle|\; \begin{array}{l} n \in BC([0,T); L^{N/2,\rho}) \cap L^{\alpha,\rho}((0,T); L^{q,1}),\\ c \in BC((0,T); BUC),\\ \nabla c \in BC([0,T); (L^{N,\rho})^N) \cap L^{\beta,\rho}((0,T); (L^{r,1})^N),\\ \boldsymbol{u} \in BC([0,T); P(L^{N,\rho})^N) \cap L^{\beta,\rho}((0,T); P(L^{r,1})^N),\\ t^{1/\alpha} n \in BC([0,T); L^{q,1}), \quad t^{1/\beta}\nabla c \in BC([0,T); (L^{r,1})^N),\\ t^{1/\beta}\boldsymbol{u} \in BC([0,T); P(L^{r,1})^N), \lim_{t\to+0} t^{1/\alpha}\|n(t)\|_{L^{q,1}} = 0, \quad \lim_{t\to+0} t^{1/\beta}\|\nabla c(t)\|_{L^{r,1}} = 0,\\ \lim_{t\to+0} t^{1/\beta}\|\boldsymbol{u}(t)\|_{L^{r,1}} = 0 \end{array} \right\} \tag{4.2}$$

and set

$$\begin{aligned}\|(n,c,\boldsymbol{u})\|_{X_t} &:= \|n\|_{L_t^\infty(L^{N/2,\rho})} + \|c\|_{L_t^\infty(L^\infty)} + \|\nabla c\|_{L_t^\infty(L^{N,\rho})} + \|\boldsymbol{u}\|_{L_t^\infty(L^{N,\rho})} + [(n,c,\boldsymbol{u})]_{X_t},\\ [(n,c,\boldsymbol{u})]_{X_t} &:= \sup_{0<\tau<t}(\tau^{1/\alpha}\|n(\tau)\|_{L^{q,1}} + \tau^{1/\beta}\|\nabla c(\tau)\|_{L^{r,1}} + \tau^{1/\beta}\|\boldsymbol{u}(\tau)\|_{L^{r,1}})\\ &\quad + \|n\|_{L_t^{\alpha,\rho}(L^{q,1})} + \|\nabla c\|_{L_t^{\beta,\rho}(L^{r,1})} + \|\boldsymbol{u}\|_{L_t^{\beta,\rho}(L^{r,1})}\end{aligned}$$

for $0 < t < T$ and $(n, c, \boldsymbol{u}) \in X_T$. First, we consider $v_0(t) := (e^{t\Delta}n_0, e^{t\Delta}c_0, e^{t\Delta}\boldsymbol{u}_0)$ for $0 < t < \infty$. By applying Propositions 3.1 (i), (ii), and (iii) and 3.2, we observe that

$$\begin{cases} [v_0]_{X_T} \le C(\|n_0\|_{L^{N/2,\rho}} + \|\nabla c_0\|_{L^{N,\rho}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}}), \\ \|v_0\|_{X_T} \le C(\|n_0\|_{L^{N/2,\rho}} + \|c_0\|_{L^\infty} + \|\nabla c_0\|_{L^{N,\rho}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}}) \end{cases} \tag{4.3}$$

and

$$\lim_{t\to+0}[v_0]_{X_t} = 0. \tag{4.4}$$

Hence, we see that $v_0 \in X_T$. Next we take $(n, c, \boldsymbol{u}) \in X_T$ and set $v := (n, c, \boldsymbol{u})$. Define the mapping $\Phi v := (\Phi_0 v, \Phi_1 v, \boldsymbol{\Phi}_2 v)$, where Φ_0, Φ_1, and $\boldsymbol{\Phi}_2$ are given by

$$(\Phi_0 v)(t) := e^{t\Delta}n_0 - \int_0^t e^{(t-\tau)\Delta}(\nabla\cdot(n\boldsymbol{u}) + \nabla\cdot(n\nabla c))(\tau)\mathrm{d}\tau, \quad 0 < t < T,$$

$$(\Phi_1 v)(t) := e^{t\Delta}c_0 - \int_0^t e^{(t-\tau)\Delta}((\boldsymbol{\Phi}_2 v)\cdot\nabla c + (\Phi_0 v)c)(\tau)\mathrm{d}\tau, \quad 0 < t < T,$$

$$(\boldsymbol{\Phi}_2 v)(t) := e^{t\Delta}\boldsymbol{u}_0 - \int_0^t e^{(t-\tau)\Delta}P(\nabla\cdot(\boldsymbol{u}\otimes \boldsymbol{u}) - (\Phi_0 v)\nabla\varphi)(\tau)\mathrm{d}\tau, \quad 0 < t < T.$$

Then, we see by Proposition 3.1 (i) and Lemma 4.2 (i) that

$$\begin{aligned} \|\Phi_0 v\|_{L^\infty_T(L^{N/2,\rho})} &\le C\|n_0\|_{L^{N/2,\rho}} + C[v]^2_{X_T}, \\ \|(\Phi_0 v)(t)\|_{L^{q,1}} &\le C\|e^{t\Delta}n_0\|_{L^{q,1}} + Ct^{-1/\alpha}[v]^2_{X_t}, \\ \|\Phi_0 v\|_{L^{\alpha,\rho}_t(L^{q,1})} &\le C[v_0]_{X_t} + C[v]^2_{X_t} \end{aligned}$$

for all $0 < t < T$. Thus, we have

$$\sup_{0<\tau<t} \tau^{1/\alpha}\|(\Phi_0 v)(\tau)\|_{L^{q,1}} + \|\Phi_0 v\|_{L^{\alpha,\rho}_t(L^{q,1})} \le C([v_0]_{X_t} + [v]^2_{X_t}). \tag{4.5}$$

By combining the estimate (4.5), Proposition 3.1 (i), and Lemma 4.2 (iii) and (v), we deduce that

$$\begin{aligned} \|\boldsymbol{\Phi}_2 v\|_{L^\infty_T(L^{N,\rho})} &\le C\|\boldsymbol{u}_0\|_{L^{N,\rho}} + C[v]^2_{X_T} + C\|\nabla\varphi\|_{L^{N,\infty}} \sup_{0<\tau<T} \tau^{1/\alpha}\|(\Phi_0 v)(\tau)\|_{L^{q,\infty}} \\ &\le C\|\boldsymbol{u}_0\|_{L^{N,\rho}} + C(1 + \|\nabla\varphi\|_{L^{N,\infty}})([v_0]_{X_T} + [v]^2_{X_T}), \\ \|(\boldsymbol{\Phi}_2 v)(t)\|_{L^{r,1}} &\le C\|e^{t\Delta}\boldsymbol{u}_0\|_{L^{r,1}} + Ct^{-1/\beta}[v]^2_{X_t} + Ct^{-1/\beta}\|\nabla\varphi\|_{L^{N,\infty}} \sup_{0<\tau<t} \tau^{1/\alpha}\|(\Phi_0 v)(\tau)\|_{L^{q,\infty}} \\ &\le C\|e^{t\Delta}\boldsymbol{u}_0\|_{L^{r,1}} + Ct^{-1/\beta}(1 + \|\nabla\varphi\|_{L^{N,\infty}})([v_0]_{X_t} + [v]^2_{X_t}), \\ \|\boldsymbol{\Phi}_2 v\|_{L^{\beta,\rho}_t(L^{r,1})} &\le C[v_0]_{X_t} + C[v]^2_{X_t} + C\|\nabla\varphi\|_{L^{N,\infty}}\|\Phi_0 v\|_{L^{\alpha,\rho}_t(L^{q,\infty})} \\ &\le C(1 + \|\nabla\varphi\|_{L^{N,\infty}})([v_0]_{X_t} + [v]^2_{X_t}) \end{aligned}$$

for all $0 < t < T$. Hence, there holds

$$\sup_{0<\tau<t} \tau^{1/\beta}\|(\boldsymbol{\Phi}_2 v)(\tau)\|_{L^{r,1}} + \|\boldsymbol{\Phi}_2 v\|_{L^{\beta,\rho}_t(L^{r,1})} \le CK_\varphi([v_0]_{X_t} + [v]^2_{X_t}), \tag{4.6}$$

where

$$K_\varphi := 1 + \|\nabla\varphi\|_{L^{N,\infty}}. \tag{4.7}$$

In addition, since (4.5) and (4.6) hold, we use Proposition 3.1 (i) and Lemma 4.2 (ii), (iii), and (iv) to obtain

$$\begin{aligned} \|\Phi_1 v\|_{L^\infty_T(L^\infty)} &\le \|c_0\|_{L^\infty} + C\|c\|_{L^\infty_T(L^\infty)} \sup_{0<\tau<T} (\tau^{1/\beta}\|(\boldsymbol{\Phi}_2 v)(\tau)\|_{L^{r,\infty}} + \tau^{1/\alpha}\|(\Phi_0 v)(\tau)\|_{L^{q,\infty}}) \\ &\le \|c_0\|_{L^\infty} + CK_\varphi\|c\|_{L^\infty_T(L^\infty)}([v_0]_{X_T} + [v]^2_{X_T}) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\|\nabla\Phi_1 v\|_{L^\infty_T(L^{N,\rho})} &\le C\|\nabla c_0\|_{L^{N,\rho}} + C[v]_{X_T}\sup_{0<\tau<T}\tau^{1/\beta}\|(\boldsymbol{\Phi}_2 v)(\tau)\|_{L^{r,\infty}} + C\|c\|_{L^\infty_T(L^\infty)}\sup_{0<\tau<T}\tau^{1/\alpha}\|(\Phi_0 v)(\tau)\|_{L^{q,\infty}}\\ &\le C\|\nabla c_0\|_{L^{N,\rho}} + CK_\varphi(\|c\|_{L^\infty_T(L^\infty)} + [v]_{X_T})([v_0]_{X_T} + [v]^2_{X_T}),\\ \|\nabla(\Phi_1 v)(t)\|_{L^{r,1}} &\le C\|\nabla e^{t\Delta}c_0\|_{L^{r,1}} + Ct^{-1/\beta}[v]_{X_t}\sup_{0<\tau<t}\tau^{1/\beta}\|(\boldsymbol{\Phi}_2 v)(\tau)\|_{L^{r,\infty}}\\ &\quad + Ct^{-1/\beta}\|c\|_{L^\infty_T(L^\infty)}\sup_{0<\tau<t}\tau^{1/\alpha}\|(\Phi_0 v)(\tau)\|_{L^{q,\infty}}\\ &\le C\|\nabla e^{t\Delta}c_0\|_{L^{r,1}} + CK_\varphi t^{-1/\beta}(\|c\|_{L^\infty_T(L^\infty)} + [v]_{X_t})([v_0]_{X_t} + [v]^2_{X_t}),\\ \|\nabla\Phi_1 v\|_{L^{\beta,\rho}_t(L^{r,1})} &\le C[v_0]_{X_t} + C([v]_{X_t}\|\boldsymbol{\Phi}_2 v\|_{L^{\beta,\rho}_t(L^{r,1})} + \|c\|_{L^\infty_T(L^\infty)}\|\Phi_0 v\|_{L^{\alpha,\rho}_t(L^{q,1})})\\ &\le C[v_0]_{X_t} + CK_\varphi(\|c\|_{L^\infty_T(L^\infty)} + [v]_{X_t})([v_0]_{X_t} + [v]^2_{X_t})\end{aligned}$$

for all $0 < t < T$. Therefore, we observe that

$$[\Phi v]_{X_t} \le CK_\varphi(1 + \|c\|_{L^\infty_T(L^\infty)} + [v]_{X_t})([v_0]_{X_t} + [v]^2_{X_t}) \tag{4.9}$$

and

$$\|\Phi v\|_{X_T} \le \|c_0\|_{L^\infty} + C(\|n_0\|_{L^{N/2,\rho}} + \|\nabla c_0\|_{L^{N,\rho}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}}) + CK_\varphi(1 + \|c\|_{L^\infty_T(L^\infty)} + [v]_{X_T})([v_0]_{X_T} + [v]^2_{X_T}).$$

Here, definition (4.2) of X_T and (4.4) imply that $\lim_{t\to+0}[\Phi v]_{X_t} = 0$. Moreover, estimate (4.3) yields

$$\|\Phi v\|_{X_T} \le \|c_0\|_{L^\infty} + CK_\varphi(1 + \|c\|_{L^\infty_T(L^\infty)} + [v]_{X_T})(\|n_0\|_{L^{N/2,\rho}} + \|\nabla c_0\|_{L^{N,\rho}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}} + [v]^2_{X_T}). \tag{4.10}$$

Thus, we observe that $\Phi v \in X_T$ for all $v \in X_T$. Let $(\tilde{n}, \tilde{c}, \tilde{\boldsymbol{u}}) \in X_T$ and set $\tilde{v} := (\tilde{n}, \tilde{c}, \tilde{\boldsymbol{u}})$. Since we have

$$\begin{aligned}&(\Phi_0 v)(t) - (\Phi_0\tilde{v})(t)\\ &\quad = -\int_0^t e^{(t-\tau)\Delta}(\nabla\cdot(n(\boldsymbol{u} - \tilde{\boldsymbol{u}})) + \nabla\cdot((n - \tilde{n})\tilde{\boldsymbol{u}}) + \nabla\cdot((n - \tilde{n})\nabla c) + \nabla\cdot(\tilde{n}\nabla(c - \tilde{c})))(\tau)\,d\tau,\\ &(\Phi_1 v)(t) - (\Phi_1\tilde{v})(t)\\ &\quad = -\int_0^t e^{(t-\tau)\Delta}((\boldsymbol{\Phi}_2 v - \boldsymbol{\Phi}_2\tilde{v})\cdot\nabla c + (\boldsymbol{\Phi}_2\tilde{v})\cdot\nabla(c - \tilde{c}) + (\Phi_0 v - \Phi_0\tilde{v})c + (\Phi_0\tilde{v})(c - \tilde{c}))(\tau)\,d\tau,\\ &(\boldsymbol{\Phi}_2 v)(t) - (\boldsymbol{\Phi}_2\tilde{v})(t)\\ &\quad = -\int_0^t e^{(t-\tau)\Delta}P(\nabla\cdot((\boldsymbol{u} - \tilde{\boldsymbol{u}})\otimes\boldsymbol{u}) + \nabla\cdot(\tilde{\boldsymbol{u}}\otimes(\boldsymbol{u} - \tilde{\boldsymbol{u}})) - (\Phi_0 v - \Phi_0\tilde{v})\nabla\varphi)(\tau)\,d\tau\end{aligned}$$

for all $0 < t < T$, in the same way as the derivation of the above estimates, we obtain

$$\begin{aligned}&[\![\Phi_0 v - \Phi_0\tilde{v}]\!]_0\\ &\quad := \|\Phi_0 v - \Phi_0\tilde{v}\|_{L^\infty_T(L^{N/2,\rho})} + \sup_{0<t<T}t^{1/\alpha}\|(\Phi_0 v)(t) - (\Phi_0\tilde{v})(t)\|_{L^{q,1}} + \|\Phi_0 v - \Phi_0\tilde{v}\|_{L^{\alpha,\rho}_T(L^{q,1})}\\ &\quad \le C([v]_{X_T} + [\tilde{v}]_{X_T})[v - \tilde{v}]_{X_T}.\end{aligned}$$

In addition, it holds that

$$\begin{aligned}&[\![\boldsymbol{\Phi}_2 v - \boldsymbol{\Phi}_2\tilde{v}]\!]_2\\ &\quad := \|\boldsymbol{\Phi}_2 v - \boldsymbol{\Phi}_2\tilde{v}\|_{L^\infty_T(L^{N,\rho})} + \sup_{0<t<T}t^{1/\beta}\|(\boldsymbol{\Phi}_2 v)(t) - (\boldsymbol{\Phi}_2\tilde{v})(t)\|_{L^{r,1}} + \|\boldsymbol{\Phi}_2 v - \boldsymbol{\Phi}_2\tilde{v}\|_{L^{\beta,\rho}_T(L^{r,1})}\\ &\quad \le C([v]_{X_T} + [\tilde{v}]_{X_T})[v - \tilde{v}]_{X_T} + C\|\nabla\varphi\|_{L^{N,\infty}}[\![\Phi_0 v - \Phi_0\tilde{v}]\!]_0\\ &\quad \le CK_\varphi([v]_{X_T} + [\tilde{v}]_{X_T})[v - \tilde{v}]_{X_T}\end{aligned}$$

and

$$\begin{aligned}
&\|\Phi_1 v - \Phi_1\tilde v\|_{L^\infty_T(L^\infty)} + \|\nabla\Phi_1 v - \nabla\Phi_1\tilde v\|_{L^\infty_T(L^{N,\rho})}\\
&\quad + \sup_{0<t<T} t^{1/\beta}\|\nabla(\Phi_1 v)(t) - \nabla(\Phi_1\tilde v)(t)\|_{L^{r,1}} + \|\nabla\Phi_1 v - \nabla\Phi_1\tilde v\|_{L^{\beta,\rho}_T(L^{r,1})}\\
&\le C(\|c\|_{L^\infty_T(L^\infty)} + [v]_{X_T})([\![\boldsymbol{\Phi}_2 v - \boldsymbol{\Phi}_2\tilde v]\!]_2 + [\![\Phi_0 v - \Phi_0\tilde v]\!]_0)\\
&\quad + C(\|c - \tilde c\|_{L^\infty_T(L^\infty)} + [v - \tilde v]_{X_T})\Bigg(\sup_{0<t<T} t^{1/\alpha}\|(\Phi_0\tilde v)(t)\|_{L^{q,\infty}} + \|\Phi_0\tilde v\|_{L^{\alpha,\rho}_T(L^{q,\infty})}\\
&\quad + \sup_{0<t<T} t^{1/\beta}\|(\boldsymbol{\Phi}_2\tilde v)(t)\|_{L^{r,\infty}} + \|\boldsymbol{\Phi}_2\tilde v\|_{L^{\beta,\rho}_T(L^{r,\infty})}\Bigg)\\
&\le CK_\varphi(\|c\|_{L^\infty_T(L^\infty)} + [v]_{X_T})([v]_{X_T} + [\tilde v]_{X_T})[v - \tilde v]_{X_T} + CK_\varphi([v_0]_{X_T} + [\tilde v]^2_{X_T})\|v - \tilde v\|_{X_T},
\end{aligned}$$

which yield

$$\|\Phi v - \Phi\tilde v\|_{X_T} \le CK_\varphi(1 + \|c\|_{L^\infty_T(L^\infty)} + [v]_{X_T} + [\tilde v]_{X_T})([v]_{X_T} + [\tilde v]_{X_T})\|v - \tilde v\|_{X_T} + CK_\varphi[v_0]_{X_T}\|v - \tilde v\|_{X_T}. \tag{4.11}$$

By property (4.4), we may assume that

$$[v_0]_{X_T} \le 2^{-5}(1 + C)^{-2}K_\varphi^{-2}(1 + \|c_0\|_{L^\infty})^{-2} \tag{4.12}$$

for sufficiently small $T > 0$. For such a $T > 0$, we define the closed subspace of X_T as follows:

$$\mathscr{B}(X_T) := \left\{(n, c, \boldsymbol{u}) \in X_T \;\middle|\; \begin{array}{l}[(n, c, \boldsymbol{u})]_{X_T} \le 2^{-3}(1 + C)^{-1}K_\varphi^{-1}(1 + \|c_0\|_{L^\infty})^{-1},\\ \|c\|_{L^\infty_T(L^\infty)} \le 2\|c_0\|_{L^\infty}\end{array}\right\}. \tag{4.13}$$

Let $(n, c, \boldsymbol{u}), (\tilde n, \tilde c, \tilde{\boldsymbol{u}}) \in \mathscr{B}(X_T)$ and set $v := (n, c, \boldsymbol{u})$ and $\tilde v := (\tilde n, \tilde c, \tilde{\boldsymbol{u}})$. Then, estimates (4.8), (4.9), (4.11), and (4.12) imply that

$$\begin{aligned}
[\Phi v]_{X_T} &\le CK_\varphi(1 + 2\|c_0\|_{L^\infty} + 1)\cdot(2^{-5} + 2^{-6})(1 + C)^{-2}K_\varphi^{-2}(1 + \|c_0\|_{L^\infty})^{-2}\\
&\le 2^{-3}(1 + C)^{-1}K_\varphi^{-1}(1 + \|c_0\|_{L^\infty})^{-1},\\
\|\Phi_1 v\|_{L^\infty_T(L^\infty)} &\le \|c_0\|_{L^\infty} + CK_\varphi \cdot 2\|c_0\|_{L^\infty}\cdot(2^{-5} + 2^{-6})(1 + C)^{-2}K_\varphi^{-2}(1 + \|c_0\|_{L^\infty})^{-2}\\
&\le 2\|c_0\|_{L^\infty},\\
\|\Phi v - \Phi\tilde v\|_{X_T} &\le CK_\varphi(1 + 2\|c_0\|_{L^\infty} + 1)\cdot 2^{-2}(1 + C)^{-1}K_\varphi^{-1}(1 + \|c_0\|_{L^\infty})^{-1}\|v - \tilde v\|_{X_T}\\
&\quad + CK_\varphi \cdot 2^{-5}(1 + C)^{-2}K_\varphi^{-2}(1 + \|c_0\|_{L^\infty})^{-2}\|v - \tilde v\|_{X_T}\\
&\le (3/4)\|v - \tilde v\|_{X_T}.
\end{aligned}$$

Therefore, we deduce that the mapping $\Phi : \mathscr{B}(X_T) \to \mathscr{B}(X_T)$ is a contraction mapping. The Banach fixed point theorem ensures the existence of a unique $v = (n, c, \boldsymbol{u}) \in \mathscr{B}(X_T)$ such that $\Phi v = v$. In particular, we have $\Phi_0 v = n$, $\Phi_1 v = c$, and $\boldsymbol{\Phi}_2 v = \boldsymbol{u}$ with estimate (1.9). In addition, since $v \in \mathscr{B}(X_T)$, it holds by (4.10) that

$$\begin{aligned}
\|v\|_{X_T} &\le \|c_0\|_{L^\infty} + CK_\varphi(1 + 2\|c_0\|_{L^\infty} + 1)(\|n_0\|_{L^{N/2,\rho}} + \|\nabla c_0\|_{L^{N,\rho}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}})\\
&\quad + CK_\varphi(1 + 2\|c_0\|_{L^\infty} + 1)\cdot 2^{-3}(1 + C)^{-1}K_\varphi^{-1}(1 + \|c_0\|_{L^\infty})^{-1}\cdot\|v\|_{X_T}\\
&\le \|c_0\|_{L^\infty} + CK_\varphi(1 + \|c_0\|_{L^\infty})(\|n_0\|_{L^{N/2,\rho}} + \|\nabla c_0\|_{L^{N,\rho}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}}) + (1/4)\|v\|_{X_T},
\end{aligned}$$

which yields (1.8). Concerning properties (1.10) and (1.11), since (1.5) yields

$$\begin{aligned}
n(t) - n_0 &= e^{t\Delta}n_0 - n_0 - \int_0^t e^{(t-\tau)\Delta}(\nabla\cdot(n\boldsymbol{u}) + \nabla\cdot(n\nabla c))(\tau)\,\mathrm{d}\tau,\\
c(t) - c_0 &= e^{t\Delta}c_0 - c_0 - \int_0^t e^{(t-\tau)\Delta}(\boldsymbol{u}\cdot\nabla c + nc)(\tau)\,\mathrm{d}\tau,\\
\boldsymbol{u}(t) - \boldsymbol{u}_0 &= e^{t\Delta}\boldsymbol{u}_0 - \boldsymbol{u}_0 - \int_0^t e^{(t-\tau)\Delta}P(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) - n\nabla\varphi)(\tau)\,\mathrm{d}\tau
\end{aligned}$$

for all $0 < t < T$, we see by Lemma 4.2 (i), (iii), (iv), and (v) that

$$\begin{aligned}\|n(t) - n_0\|_{L^{N/2,\rho}} &\le C\|e^{t\Delta}n_0 - n_0\|_{L^{N/2,\rho}} + C[(n,c,\boldsymbol{u})]_{X_t}^2,\\ \|\nabla c(t) - \nabla c_0\|_{L^{N,\rho}} &\le C\|e^{t\Delta}\nabla c_0 - \nabla c_0\|_{L^{N,\rho}} + C(\|c\|_{L^\infty_T(L^\infty)} + [(n,c,\boldsymbol{u})]_{X_t})[(n,c,\boldsymbol{u})]_{X_t},\\ \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{L^{N,\rho}} &\le C\|e^{t\Delta}\boldsymbol{u}_0 - \boldsymbol{u}_0\|_{L^{N,\rho}} + C(\|\nabla\varphi\|_{L^{N,\infty}} + [(n,c,\boldsymbol{u})]_{X_t})[(n,c,\boldsymbol{u})]_{X_t}.\end{aligned}$$

Thus, we have (1.10) by virtue of definition (4.2) of X_T and Proposition 3.2. Moreover, we also see by Lemma 4.2 (ii) that

$$\|c(t) - e^{t\Delta}c_0\|_{L^\infty} \le C[(n,c,\boldsymbol{u})]_{X_t}\|c\|_{L^\infty_T(L^\infty)},$$

which yields (1.11).

(ii) Let $0 < T_0 \le \infty$ and set

$$Y_{T_0} := \left\{(n,c,\boldsymbol{u}) \;\middle|\; \begin{array}{c} n \in L^{\alpha,\infty}((0,T_0); L^{q,\infty}), \quad c \in L^\infty((0,T_0); L^\infty),\\ \nabla c \in L^{\beta,\infty}((0,T_0); (L^{r,\infty})^N), \quad \boldsymbol{u} \in L^{\beta,\infty}((0,T_0); P(L^{r,\infty})^N),\\ M(n,c,\boldsymbol{u}) \le \delta K_\varphi^{-1}(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1}\end{array}\right\}$$

for sufficiently small $0 < \delta < 1$, where K_φ is the constant given by (4.7) and

$$\begin{aligned}M(n,c,\boldsymbol{u}) := \limsup_{\lambda\to\infty}\{&\lambda\mu(t \in (0,T_0) \mid \|n(t)\|_{L^{q,\infty}} > \lambda)^{1/\alpha}\\ &+ \lambda\mu(t \in (0,T_0) \mid \|\nabla c(t)\|_{L^{r,\infty}} > \lambda)^{1/\beta} + \lambda\mu(t \in (0,T_0) \mid \|\boldsymbol{u}(t)\|_{L^{r,\infty}} > \lambda)^{1/\beta}\}.\end{aligned}$$

We first show that there is $0 < h < 1$ satisfying

$$\|n\|_{L^{\alpha,\infty}((t,t+h);\, L^{q,\infty})} + \|\nabla c\|_{L^{\beta,\infty}((t,t+h);\, L^{r,\infty})} + \|\boldsymbol{u}\|_{L^{\beta,\infty}((t,t+h);\, L^{r,\infty})} \le C\delta K_\varphi^{-1}(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1} \tag{4.14}$$

for all $0 \le t < T_0 - h$ and $(n,c,\boldsymbol{u}) \in Y_{T_0}$, where $C > 0$ is a constant independent of T_0, δ, h, t, φ, n, c, and $\boldsymbol{u}$. By applying [42, Proposition 4.9], we may take $R_\delta > 0$ and $n_{\mathrm{B}} \in L^\infty((0,T_0); L^{q,\infty})$ so that

$$\|n_{\mathrm{B}}\|_{L^\infty_{T_0}(L^{q,\infty})} \le R_\delta, \quad \|n - n_{\mathrm{B}}\|_{L^{\alpha,\infty}_{T_0}(L^{q,\infty})} \le 2\delta K_\varphi^{-1}(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1}.$$

Hence, we observe that

$$\begin{aligned}\|n\|_{L^{\alpha,\infty}((t,t+h);\, L^{q,\infty})} &\le C(\|n_{\mathrm{B}}\|_{L^\alpha((t,t+h);\, L^{q,\infty})} + \|n - n_{\mathrm{B}}\|_{L^{\alpha,\infty}_{T_0}(L^{q,\infty})})\\ &\le C\{R_\delta h^{1/\alpha} + 2\delta K_\varphi^{-1}(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1}\}.\end{aligned}$$

Thus, we have

$$\|n\|_{L^{\alpha,\infty}((t,t+h);\, L^{q,\infty})} \le C\delta K_\varphi^{-1}(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1}$$

by taking $0 < h < 1$ sufficiently small. In the same way, we obtain (4.14). We assume that $(n,c,\boldsymbol{u}), (\tilde{n},\tilde{c},\tilde{\boldsymbol{u}}) \in Y_{T_0}$ are solutions of (1.5) and set

$$\begin{aligned}D(t) := &\|n - \tilde{n}\|_{L^{\alpha,\infty}((0,t);\, L^{q,\infty})} + \|c - \tilde{c}\|_{L^\infty((0,t);\, L^\infty)}\\ &+ \|\nabla c - \nabla\tilde{c}\|_{L^{\beta,\infty}((0,t);\, L^{r,\infty})} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^{\beta,\infty}((0,t);\, L^{r,\infty})}\end{aligned}$$

for $0 < t < T_0$. Then, since (4.14) holds and since

$$\begin{aligned}n(t) - \tilde{n}(t) &= -\int_0^t e^{(t-\tau)\Delta}(\nabla\cdot(n(\boldsymbol{u} - \tilde{\boldsymbol{u}})) + \nabla\cdot((n - \tilde{n})\tilde{\boldsymbol{u}}) + \nabla\cdot((n - \tilde{n})\nabla c) + \nabla\cdot(\tilde{n}\nabla(c - \tilde{c})))(\tau)\,\mathrm{d}\tau,\\ c(t) - \tilde{c}(t) &= -\int_0^t e^{(t-\tau)\Delta}((\boldsymbol{u} - \tilde{\boldsymbol{u}})\cdot\nabla c + \tilde{\boldsymbol{u}}\cdot\nabla(c - \tilde{c}) + (n - \tilde{n})c + \tilde{n}(c - \tilde{c}))(\tau)\,\mathrm{d}\tau,\\ \boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t) &= -\int_0^t e^{(t-\tau)\Delta}P(\nabla\cdot((\boldsymbol{u} - \tilde{\boldsymbol{u}}) \otimes \boldsymbol{u}) + \nabla\cdot(\tilde{\boldsymbol{u}} \otimes (\boldsymbol{u} - \tilde{\boldsymbol{u}})) - (n - \tilde{n})\nabla\varphi)(\tau)\,\mathrm{d}\tau\end{aligned}$$

for all $0 < t < T_0$, we see by Lemma 4.2 that

$$\begin{aligned}\|n - \tilde{n}\|_{L^{\alpha,\infty}((0,h);\, L^{q,\infty})} &\le C\delta K_\varphi^{-1}(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1}D(h),\\ \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^{\beta,\infty}((0,h);\, L^{r,\infty})} &\le C\delta(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1}D(h) + C\|\nabla\varphi\|_{L^{N,\infty}}\|n - \tilde{n}\|_{L^{\alpha,\infty}((0,h);\, L^{q,\infty})}\\ &\le C\delta(1 + \|c\|_{L^\infty_{T_0}(L^\infty)})^{-1}D(h)\end{aligned}$$

and

$$\begin{aligned}&\|c - \tilde{c}\|_{L^\infty((0,h);\, L^\infty)} + \|\nabla c - \nabla\tilde{c}\|_{L^{\beta,\infty}((0,h);\, L^{r,\infty})}\\ &\le\ C\delta D(h) + C\|c\|_{L^\infty_{T_0}(L^\infty)}(\|n - \tilde{n}\|_{L^{\alpha,\infty}((0,h);\, L^{q,\infty})} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^{\beta,\infty}((0,h);\, L^{r,\infty})})\\ &\le\ C\delta D(h),\end{aligned}$$

which yield $D(h) \le C\delta D(h)$. By taking $0 < \delta < 1$ so that $C\delta \le 1/2$, we have $D(h) = 0$. Thus, we deduce that $(n, c, \boldsymbol{u}) = (\tilde{n}, \tilde{c}, \tilde{\boldsymbol{u}})$ in $(0, h) \times \mathbb{R}^N$. In addition, since

$$\begin{aligned}D(2h) = &\|n - \tilde{n}\|_{L^{\alpha,\infty}((h,2h);\, L^{q,\infty})} + \|c - \tilde{c}\|_{L^\infty((h,2h);\, L^\infty)}\\ &+ \|\nabla c - \nabla\tilde{c}\|_{L^{\beta,\infty}((h,2h);\, L^{r,\infty})} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^{\beta,\infty}((h,2h);\, L^{r,\infty})},\end{aligned}$$

by setting $t = h$ in (4.14), we may show that $D(2h) \le C\delta D(2h)$ in a similar way to the derivation of the above estimates. Thus, we have $(n, c, \boldsymbol{u}) = (\tilde{n}, \tilde{c}, \tilde{\boldsymbol{u}})$ in $(0,2h) \times \mathbb{R}^N$. By repeating the same argument, we have the desired result.

(iii) Note that

$$\begin{aligned}n(t) - \tilde{n}(t) &= e^{t\Delta}(n_0 - \tilde{n}_0) - \int_0^t e^{(t-\tau)\Delta}(\nabla\cdot(n(\boldsymbol{u} - \tilde{\boldsymbol{u}})) + \nabla\cdot((n - \tilde{n})\tilde{\boldsymbol{u}}) + \nabla\cdot((n - \tilde{n})\nabla c) + \nabla\cdot(\tilde{n}\nabla(c - \tilde{c})))(\tau)\mathrm{d}\tau,\\ c(t) - \tilde{c}(t) &= e^{t\Delta}(c_0 - \tilde{c}_0) - \int_0^t e^{(t-\tau)\Delta}((\boldsymbol{u} - \tilde{\boldsymbol{u}})\cdot\nabla c + \tilde{\boldsymbol{u}}\cdot\nabla(c - \tilde{c}) + (n - \tilde{n})c + \tilde{n}(c - \tilde{c}))(\tau)\mathrm{d}\tau,\\ \boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t) &= e^{t\Delta}(\boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_0) - \int_0^t e^{(t-\tau)\Delta}P(\nabla\cdot((\boldsymbol{u} - \tilde{\boldsymbol{u}}) \otimes \boldsymbol{u}) + \nabla\cdot(\tilde{\boldsymbol{u}} \otimes (\boldsymbol{u} - \tilde{\boldsymbol{u}})) - (n - \tilde{n})\nabla\varphi - \tilde{n}\nabla(\varphi - \tilde{\varphi}))(\tau)\mathrm{d}\tau\end{aligned}$$

for all $0 < t < T$. We set $v \coloneqq (n, c, \boldsymbol{u})$ and $\tilde{v} \coloneqq (\tilde{n}, \tilde{c}, \tilde{\boldsymbol{u}})$. In the same way as the derivation of the estimate (4.11), we observe that

$$\begin{aligned}[\![n - \tilde{n}]\!]_0 \coloneqq &\|n - \tilde{n}\|_{L^\infty_T(L^{N/2,\rho})} + \sup_{0<t<T} t^{1/\alpha}\|n(t) - \tilde{n}(t)\|_{L^{q,1}} + \|n - \tilde{n}\|_{L^{\alpha,\rho}_T(L^{q,1})}\\ &\le C\|n_0 - \tilde{n}_0\|_{L^{N/2,\rho}} + C([v]_{X_T} + [\tilde{v}]_{X_T})[v - \tilde{v}]_{X_T}.\end{aligned}$$

In addition, it holds that

$$\begin{aligned}[\![\boldsymbol{u} - \tilde{\boldsymbol{u}}]\!]_2 \coloneqq &\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^\infty_T(L^{N,\rho})} + \sup_{0<t<T} t^{1/\beta}\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|_{L^{r,1}} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^{\beta,\rho}_T(L^{r,1})}\\ &\le C\|\boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_0\|_{L^{N,\rho}} + C([v]_{X_T} + [\tilde{v}]_{X_T})[v - \tilde{v}]_{X_T} + C\|\nabla\varphi\|_{L^{N,\infty}}[\![n - \tilde{n}]\!]_0\\ &\quad + C\|\nabla\varphi - \nabla\tilde{\varphi}\|_{L^{N,\infty}}\left(\sup_{0<t<T} t^{1/\alpha}\|\tilde{n}(t)\|_{L^{q,\infty}} + \|\tilde{n}\|_{L^{\alpha,\rho}_T(L^{q,\infty})}\right)\\ &\le C(K_\varphi\|n_0 - \tilde{n}_0\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0 - \tilde{\boldsymbol{u}}_0\|_{L^{N,\rho}}) + C([\tilde{v}_0]_{X_T} + [\tilde{v}]^2_{X_T})\|\nabla\varphi - \nabla\tilde{\varphi}\|_{L^{N,\infty}}\\ &\quad + CK_\varphi([v]_{X_T} + [\tilde{v}]_{X_T})[v - \tilde{v}]_{X_T}\end{aligned}$$

and

$$
\begin{aligned}
&\|c-\tilde{c}\|_{L_T^\infty(L^\infty)} + \|\nabla c-\nabla\tilde{c}\|_{L_T^\infty(L^{N,\rho})} + \sup_{0<t<T} t^{1/\beta}\|\nabla c(t)-\nabla\tilde{c}(t)\|_{L^{r,1}} + \|\nabla c-\nabla\tilde{c}\|_{L_T^{\beta,\rho}(L^{r,1})}\\
&\quad\le \|c_0-\tilde{c}_0\|_{L^\infty} + C\|\nabla c_0-\nabla\tilde{c}_0\|_{L^{N,\rho}} + C(\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T})([\![\boldsymbol{u}-\tilde{\boldsymbol{u}}]\!]_2 + [\![n-\tilde{n}]\!]_0)\\
&\qquad + C(\|c-\tilde{c}\|_{L_T^\infty(L^\infty)} + [v-\tilde{v}]_{X_T})\left(\sup_{0<t<T} t^{1/\alpha}\|\tilde{n}(t)\|_{L^{q,\infty}} + \|\tilde{n}\|_{L_T^{\alpha,\rho}(L^{q,\infty})} + \sup_{0<t<T} t^{1/\beta}\|\tilde{\boldsymbol{u}}(t)\|_{L^{r,\infty}} + \|\tilde{\boldsymbol{u}}\|_{L_T^{\beta,\rho}(L^{r,\infty})}\right)\\
&\quad\le \|c_0-\tilde{c}_0\|_{L^\infty} + C\|\nabla c_0-\nabla\tilde{c}_0\|_{L^{N,\rho}} + C(\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T})\\
&\qquad \times \{K_\varphi\|n_0-\tilde{n}_0\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0-\tilde{\boldsymbol{u}}_0\|_{L^{N,\rho}} + ([\tilde{v}_0]_{X_T} + [\tilde{v}]_{X_T}^2)\|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}}\}\\
&\qquad + CK_{\tilde{\varphi}}([\tilde{v}_0]_{X_T} + [\tilde{v}]_{X_T}^2)(\|c-\tilde{c}\|_{L_T^\infty(L^\infty)} + [v-\tilde{v}]_{X_T})\\
&\qquad + CK_\varphi(\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T})([v]_{X_T} + [\tilde{v}]_{X_T})[v-\tilde{v}]_{X_T},
\end{aligned}
$$

where K_φ is the constant given by (4.7). Therefore, we deduce that

$$
\begin{aligned}
\|v-\tilde{v}\|_{X_T} \le{}& \|c_0-\tilde{c}_0\|_{L^\infty} + C\|\nabla c_0-\nabla\tilde{c}_0\|_{L^{N,\rho}} + C(1+\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T})\\
&\times \{K_\varphi\|n_0-\tilde{n}_0\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0-\tilde{\boldsymbol{u}}_0\|_{L^{N,\rho}} + ([\tilde{v}_0]_{X_T} + [\tilde{v}]_{X_T}^2)\|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}}\}\\
&+ CK_\varphi(1+\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T})([v]_{X_T} + [\tilde{v}]_{X_T})[v-\tilde{v}]_{X_T}\\
&+ CK_{\tilde{\varphi}}([\tilde{v}_0]_{X_T} + [\tilde{v}]_{X_T}^2)\|v-\tilde{v}\|_{X_T}.
\end{aligned}
$$

Since v, $\tilde{v}$, and $\tilde{v}_0$ satisfy

$$
\begin{aligned}
[v]_{X_T} &\le 2^{-3}(1+C)^{-1}K_\varphi^{-1}(1+\|c_0\|_{L^\infty})^{-1},\\
[\tilde{v}]_{X_T} &\le 2^{-3}(1+C)^{-1}K_{\tilde{\varphi}}^{-1}(1+\|\tilde{c}_0\|_{L^\infty})^{-1},\\
[\tilde{v}_0]_{X_T} &\le 2^{-5}(1+C)^{-2}K_{\tilde{\varphi}}^{-2}(1+\|\tilde{c}_0\|_{L^\infty})^{-2},\\
\|c\|_{L_T^\infty(L^\infty)} &\le 2\|c_0\|_{L^\infty}, \quad \|\tilde{c}\|_{L_T^\infty(L^\infty)} \le 2\|\tilde{c}_0\|_{L^\infty}
\end{aligned}
$$

from condition (4.12) and definition (4.13) of $\mathscr{B}(X_T)$, we obtain

$$
\begin{aligned}
1+\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T} &\le 2(1+\|c_0\|_{L^\infty}),\\
[\tilde{v}_0]_{X_T} + [\tilde{v}]_{X_T}^2 &\le 1,\\
CK_\varphi(1+\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T})[v]_{X_T}[v-\tilde{v}]_{X_T} &\le 2^{-2}\|v-\tilde{v}\|_{X_T},\\
CK_{\tilde{\varphi}}([\tilde{v}_0]_{X_T} + [\tilde{v}]_{X_T}^2)\|v-\tilde{v}\|_{X_T} &\le 2^{-4}\|v-\tilde{v}\|_{X_T}
\end{aligned}
$$

and

$$
\begin{aligned}
&CK_\varphi(1+\|c\|_{L_T^\infty(L^\infty)} + [v]_{X_T})[\tilde{v}]_{X_T}[v-\tilde{v}]_{X_T}\\
&\quad\le C(\|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}} + K_{\tilde{\varphi}})(1+\|c_0\|_{L^\infty})[\tilde{v}]_{X_T}[v-\tilde{v}]_{X_T}\\
&\quad\le C(1+\|c_0\|_{L^\infty})\|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}} + CK_{\tilde{\varphi}}(\|c_0-\tilde{c}_0\|_{L^\infty} + 1 + \|\tilde{c}_0\|_{L^\infty})[\tilde{v}]_{X_T}[v-\tilde{v}]_{X_T}\\
&\quad\le C(1+\|c_0\|_{L^\infty})\|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}} + CK_{\tilde{\varphi}}\|c_0-\tilde{c}_0\|_{L^\infty} + CK_{\tilde{\varphi}}(1+\|\tilde{c}_0\|_{L^\infty})[\tilde{v}]_{X_T}\|v-\tilde{v}\|_{X_T}\\
&\quad\le C(1+\|c_0\|_{L^\infty})\|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}} + CK_{\tilde{\varphi}}\|c_0-\tilde{c}_0\|_{L^\infty} + 2^{-3}\|v-\tilde{v}\|_{X_T},
\end{aligned}
$$

which imply that

$$
\begin{aligned}
\|v-\tilde{v}\|_{X_T} \le{}& CK_\varphi(1+\|c_0\|_{L^\infty})\|n_0-\tilde{n}_0\|_{L^{N/2,\rho}} + CK_{\tilde{\varphi}}\|c_0-\tilde{c}_0\|_{L^\infty} + C\|\nabla c_0-\nabla\tilde{c}_0\|_{L^{N,\rho}}\\
&+ C(1+\|c_0\|_{L^\infty})(\|\boldsymbol{u}_0-\tilde{\boldsymbol{u}}_0\|_{L^{N,\rho}} + \|\nabla\varphi-\nabla\tilde{\varphi}\|_{L^{N,\infty}}) + (1/2)\|v-\tilde{v}\|_{X_T}.
\end{aligned}
$$

Hence, we have (1.13), which completes the proof of Theorem 1.1. □

4.3 Global mild solutions and asymptotic behavior: Proof of Theorem 1.3

Finally, we show the global existence of mild solutions to (1.1). We also show that the global solutions decay as $t \to \infty$ in the suitable norms, namely, we show Theorem 1.3. Such a result is also mentioned by Kato [26, Note], but we have to pay attention to the proof of Theorem 1.3; recall the method of [26, Note]. To show the decay of the global solutions, we use the *continuous dependence* (1.13) of solutions $(n, c, \boldsymbol{u})$ with respect to the initial data $(n_0, c_0, \boldsymbol{u}_0)$. However, since estimate (1.13) contains the term $\|c_0 - \tilde{c}_0\|_{L^\infty}$ and since C_0^∞ is *not* dense in L^∞, the initial datum $c_0 \in L^\infty$ disturbs the proof of the decay of the global solutions. Concerning this problem, we note that system (1.5) does not contain the nonlinear term given by the *product of c with itself.* Therefore, we obtain the additional regularities of the solutions *without* an additional condition of c_0.

Lemma 4.4. *Let* $1 \le \rho < \infty$ *and* $0 < \theta < 1$. *Suppose that* q, r, α, *and* β *satisfy* (1.6) *and* (1.7). *In addition, suppose that the initial data* $(n_0, c_0, \boldsymbol{u}_0)$ *and the gravitational potential* φ *satisfy* (1.4). *Then, the following statements hold*:

(*i*) *In the statement of Theorem* 1.1 (*i*), *there exists a constant* $0 < \varepsilon < 1$ *independently of* $n_0, c_0, \boldsymbol{u}_0$, *and* φ *such that if* (1.14), *then the solution* $(n, c, \boldsymbol{u})$ *of* (1.5) *exists globally in time, namely, the existence time interval T in the statement of* Theorem 1.1 (*i*) *may be chosen as* $T = \infty$.

(*ii*) *In the statement of* (*i*), *suppose that the initial data* $(n_0, c_0, \boldsymbol{u}_0)$ *satisfy the additional conditions* $n_0 \in L^{N/(\theta+2)}$ *and* $\boldsymbol{u}_0 \in P(L^{N/(\theta+1)})^N$. *Then, the global solution* $(n, c, \boldsymbol{u})$ *of* (1.5) *satisfies* (1.15), (1.16), *and* (1.17).

Remark 4.5. As $0 < \theta < 1$ is given before the assertion of (i), the constant $0 < \varepsilon < 1$ will be chosen depending on θ although it is related only to the assertion of (ii). However, choosing ε in such a way, we will not have to assume any *smallness conditions* of the norms $\|n_0\|_{L^{N/(\theta+2)}}$ and $\|\boldsymbol{u}_0\|_{L^{N/(\theta+1)}}$ even though the additional conditions $n_0 \in L^{N/(\theta+2)}$ and $\boldsymbol{u}_0 \in P(L^{N/(\theta+1)})^N$ are assumed in (ii); we emphasize that the smallness assumption of the norm only of *scaling invariant* spaces will be sufficient to obtain the desired decay properties.

Proof of Lemma 4.4. (i) We recall that in the proof of Theorem 1.1 (i), it is necessary to take $T > 0$ sufficiently small so that (4.12) holds. Since we see that (4.3), assumption (1.14) ensures that (4.12) holds for any $T > 0$. Thus, we may take $T = \infty$.

(ii) Let $X \coloneqq X_\infty$, where X_∞ is defined by (4.2). We also set

$$Z \coloneqq \left\{ (n_*, \boldsymbol{u}_*) \;\middle|\; \begin{array}{c} (n_*, 0, \boldsymbol{u}_*) \in X, \\ t^{1/\alpha+\theta/2} n_* \in \mathrm{BC}([0,\infty); L^{q,1}), \\ t^{1/\beta+\theta/2} \boldsymbol{u}_* \in \mathrm{BC}([0,\infty); P(L^{r,1})^N) \end{array} \right\}$$

and

$$\|(n_*, \boldsymbol{u}_*)\|_Z \coloneqq \|(n_*, 0, \boldsymbol{u}_*)\|_X + \sup_{0<t<\infty} (t^{1/\alpha+\theta/2} \|n_*(t)\|_{L^{q,1}} + t^{1/\beta+\theta/2} \|\boldsymbol{u}_*(t)\|_{L^{r,1}})$$

for $(n_*, \boldsymbol{u}_*) \in Z$. In addition, let $v_* \coloneqq (n_*, \boldsymbol{u}_*) \in Z$ and let $\Psi v_* \coloneqq (\Psi_0 v_*, \boldsymbol{\Psi}_2 v_*)$ be the mapping such that

$$(\Psi_0 v_*)(t) \coloneqq e^{t\Delta} n_0 - \int_0^t e^{(t-\tau)\Delta} (\nabla \cdot (n_* \boldsymbol{u}) + \nabla \cdot (n_* \nabla c))(\tau) \mathrm{d}\tau, \quad 0 < t < \infty,$$

$$(\boldsymbol{\Psi}_2 v_*)(t) \coloneqq e^{t\Delta} \boldsymbol{u}_0 - \int_0^t e^{(t-\tau)\Delta} P(\nabla \cdot (\boldsymbol{u}_* \otimes \boldsymbol{u}) + (\Psi_0 v_*) \nabla \varphi)(\tau) \mathrm{d}\tau, \quad 0 < t < \infty,$$

where $(n, c, \boldsymbol{u}) \in \mathscr{B}(X)$ denotes the global solution of (1.5). By applying Lemma 4.2 (i) and (ii), we have

$$\|(\Psi_0 v_*)(t)\|_{L^{q,1}} \le C \|e^{t\Delta} n_0\|_{L^{q,1}} + C t^{-1/\alpha-\theta/2} [(n, c, \boldsymbol{u})]_{X_t} \|v_*\|_Z$$

for all $0 < t < \infty$, which yields

$$\sup_{0<\tau<\infty} \tau^{1/\alpha+\theta/2} \|(\Psi_0 v_*)(\tau)\|_{L^{q,1}} \le C \|n_0\|_{L^{N/(\theta+2)}} + C [(n, c, \boldsymbol{u})]_X \|v_*\|_Z$$

from Proposition 3.1 (ii). Similarly, we deduce that

$$\begin{aligned}\|(\boldsymbol{\Psi}_2 v_*)(t)\|_{L^{r,1}} &\le C\|e^{t\Delta}\boldsymbol{u}_0\|_{L^{r,1}} + Ct^{-1/\beta-\theta/2}[(n,c,\boldsymbol{u})]_{X_t}\|v_*\|_Z \\ &\quad + Ct^{-1/\beta-\theta/2}\|\nabla\varphi\|_{L^{N,\infty}} \sup_{0<\tau<\infty} \tau^{1/\alpha+\theta/2}\|(\Psi_0 v_*)(\tau)\|_{L^{q,\infty}} \\ &\le C\|e^{t\Delta}\boldsymbol{u}_0\|_{L^{r,1}} + CK_\varphi t^{-1/\beta-\theta/2}(\|n_0\|_{L^{N/(\theta+2)}} + [(n,c,\boldsymbol{u})]_X\|v_*\|_Z)\end{aligned}$$

for all $0 < t < \infty$, where K_φ is the constant given by (4.7). Moreover, in a similar way to the derivation of the estimate (4.10), we obtain

$$\|(\Psi_0 v_*, 0, \boldsymbol{\Psi}_2 v_*)\|_X \le CK_\varphi(\|n_0\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}} + [(n,c,\boldsymbol{u})]_X\|v_*\|_Z).$$

Hence, it holds by Proposition 3.1 (ii) that

$$\|\Psi v_*\|_Z \le CK_\varphi(\|n_0\|_{L^{N/2,\rho}} + \|n_0\|_{L^{N/(\theta+2)}} + \|\boldsymbol{u}_0\|_{L^{N,\rho}} + \|\boldsymbol{u}_0\|_{L^{N/(\theta+1)}} + [(n,c,\boldsymbol{u})]_X\|v_*\|_Z),$$

which yields $\Psi v_* \in Z$. In a similar way to the derivation of the above estimate, we also see that

$$\begin{aligned}\|(\Psi_0 v_* - \Psi_0\tilde{v}_*, 0, \boldsymbol{\Psi}_2 v_* - \boldsymbol{\Psi}_2\tilde{v}_*)\|_X &\le CK_\varphi[(n,c,\boldsymbol{u})]_X\|(n_* - \tilde{n}_*, 0, \boldsymbol{u}_* - \tilde{\boldsymbol{u}}_*)\|_X, \\ \|\Psi v_* - \Psi\tilde{v}_*\|_Z &\le CK_\varphi[(n,c,\boldsymbol{u})]_X\|v_* - \tilde{v}_*\|_Z\end{aligned}$$

for all $v_* = (n_*, \boldsymbol{u}_*) \in Z$ and $\tilde{v}_* = (\tilde{n}_*, \tilde{\boldsymbol{u}}_*) \in Z$. Recall definition (4.13) of $\mathscr{B}(X)$ and note that the constant $C > 0$ appearing in the above estimates might be larger than that in (4.13). However, we may follow the same argument as in the proof of Theorem 1.1 even if the constant $C > 0$ in (4.13) is replaced with a larger one. Since we have assumed that (1.14), we may regard these two constants as the same ones by going back to the proof of the assertion of (i) and choosing $0 < \varepsilon < 1$ sufficiently small if necessary. Then, as the condition $(n, c, \boldsymbol{u}) \in \mathscr{B}(X)$ implies that

$$\begin{aligned}\|(\Psi_0 v_* - \Psi_0\tilde{v}_*, 0, \boldsymbol{\Psi}_2 v_* - \boldsymbol{\Psi}_2\tilde{v}_*)\|_X &\le 2^{-3}\|(n_* - \tilde{n}_*, 0, \boldsymbol{u}_* - \tilde{\boldsymbol{u}}_*)\|_X, \\ \|\Psi v_* - \Psi\tilde{v}_*\|_Z &\le 2^{-3}\|v_* - \tilde{v}_*\|_Z,\end{aligned}$$

we observe that the mapping $\Psi : Z \to Z$ is a contraction mapping. Hence, by the Banach fixed point theorem, we obtain a unique $v_* = (n_*, \boldsymbol{u}_*) \in Z$ such that $\Psi v_* = v_*$. Note that $(n_*, \boldsymbol{u}_*) \in Z$ satisfies

$$\begin{cases} n_*(t) = e^{t\Delta}n_0 - \displaystyle\int_0^t e^{(t-\tau)\Delta}(\nabla\cdot(n_*\boldsymbol{u}) + \nabla\cdot(n_*\nabla c))(\tau)\,\mathrm{d}\tau, & 0 < t < \infty, \\ \boldsymbol{u}_*(t) = e^{t\Delta}\boldsymbol{u}_0 - \displaystyle\int_0^t e^{(t-\tau)\Delta}P(\nabla\cdot(\boldsymbol{u}_* \otimes \boldsymbol{u}) + n_*\nabla\varphi)(\tau)\,\mathrm{d}\tau, & 0 < t < \infty \end{cases} \tag{4.15}$$

and the global solution $(n, c, \boldsymbol{u}) \in X$ of (1.5) also satisfies (4.15). Furthermore, we deduce that a global solution $(n_*, \boldsymbol{u}_*)$ of (4.15) is unique under the condition $(n_*, 0, \boldsymbol{u}_*) \in X$, and thus, we have $n_* = n$ and $\boldsymbol{u}_* = \boldsymbol{u}$. Therefore, we observe that the global solution $(n, c, \boldsymbol{u})$ of (1.5) satisfies $(n, \boldsymbol{u}) \in Z$.

Now, we shall verify that (1.15), (1.16), and (1.17). To this end, we estimate the original system (1.5). By applying Lemma 4.2 (i), (iii), (iv), and (v) again, we have

$$\begin{aligned}\|n(t)\|_{L^{N/2,\rho}} &\le C\|e^{t\Delta}n_0\|_{L^{N/2,\rho}} + Ct^{-\theta/2}\|(n,c,\boldsymbol{u})\|_X\|(n,\boldsymbol{u})\|_Z, \\ \|n(t)\|_{L^{q,1}} &\le C\|e^{t\Delta}n_0\|_{L^{q,1}} + Ct^{-1/\alpha-\theta/2}\|(n,c,\boldsymbol{u})\|_X\|(n,\boldsymbol{u})\|_Z, \\ \|\nabla c(t)\|_{L^{N,\rho}} &\le C\|\nabla e^{t\Delta}c_0\|_{L^{N,\rho}} + Ct^{-\theta/2}\|(n,c,\boldsymbol{u})\|_X\|(n,\boldsymbol{u})\|_Z, \\ \|\nabla c(t)\|_{L^{r,1}} &\le C\|\nabla e^{t\Delta}c_0\|_{L^{r,1}} + Ct^{-1/\beta-\theta/2}\|(n,c,\boldsymbol{u})\|_X\|(n,\boldsymbol{u})\|_Z, \\ \|\boldsymbol{u}(t)\|_{L^{N,\rho}} &\le C\|e^{t\Delta}\boldsymbol{u}_0\|_{L^{N,\rho}} + Ct^{-\theta/2}(\|\nabla\varphi\|_{L^{N,\infty}} + \|(n,c,\boldsymbol{u})\|_X)\|(n,\boldsymbol{u})\|_Z, \\ \|\boldsymbol{u}(t)\|_{L^{r,1}} &\le C\|e^{t\Delta}\boldsymbol{u}_0\|_{L^{r,1}} + Ct^{-1/\beta-\theta/2}(\|\nabla\varphi\|_{L^{N,\infty}} + \|(n,c,\boldsymbol{u})\|_X)\|(n,\boldsymbol{u})\|_Z\end{aligned}$$

for all $0 < t < \infty$. Thus, we see by Proposition 3.2 that (1.15) and (1.17). In addition, since Lemma 4.2 (ii) yields

$$\|c(t) - e^{t\Delta}c_0\|_{L^\infty} \le Ct^{-\theta/2}\|(n,c,\boldsymbol{u})\|_X\|(n,\boldsymbol{u})\|_Z,$$

we obtain (1.16). This completes the proof of Lemma 4.4. □

Proof of Theorem 1.3. Since we may construct a global solution $(n, c, \boldsymbol{u})$ of (1.5) from Lemma 4.4 (i), in the following, we show properties (1.15), (1.16), and (1.17). We take sequences $\{n_{0,j}\}_{j\in\mathbb{N}} \subset C_0^\infty$ and $\{\boldsymbol{u}_{0,j}\}_{j\in\mathbb{N}} \subset (C_0^\infty)^N$ of functions satisfying the following conditions:

$$\lim_{j\to\infty}(\|n_0 - n_{0,j}\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_{0,j}\|_{L^{N,\rho}}) = 0,$$
$$\sup_{j\in\mathbb{N}}(\|n_{0,j}\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_{0,j}\|_{L^{N,\rho}}) + \|\nabla c_0\|_{L^{N,\rho}} \le \varepsilon K_\varphi^{-2}(1 + \|c_0\|_{L^\infty})^{-2},$$

where K_φ is the constant given by (4.7). Then, we see that the initial data $(n_{0,j}, c_0, \boldsymbol{u}_{0,j})$ satisfy the conditions corresponding to (1.14), and thus, we may obtain a sequence $\{(n_j, c_j, \boldsymbol{u}_j)\}_{j\in\mathbb{N}}$ of global solutions of (1.5). In addition, Theorem 1.1 (iii) implies that

$$\begin{aligned}
&\|n - n_j\|_{L^\infty(L^{N/2,\rho})} + \|c - c_j\|_{L^\infty(L^\infty)} + \|\nabla c - \nabla c_j\|_{L^\infty(L^{N,\rho})} + \|\boldsymbol{u} - \boldsymbol{u}_j\|_{L^\infty(L^{N,\rho})} \\
&\quad + \sup_{0<t<\infty}(t^{1/\alpha}\|n(t) - n_j(t)\|_{L^{q,1}} + t^{1/\beta}\|\nabla c(t) - \nabla c_j(t)\|_{L^{r,1}} + t^{1/\beta}\|\boldsymbol{u}(t) - \boldsymbol{u}_j(t)\|_{L^{r,1}}) \\
&\le CK_\varphi(1 + \|c_0\|_{L^\infty})(\|n_0 - n_{0,j}\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_{0,j}\|_{L^{N,\rho}})
\end{aligned}$$

for all $j \in \mathbb{N}$. Therefore, by setting

$$\begin{aligned}
E(n, c, \boldsymbol{u})(t) &:= \|n(t)\|_{L^{N/2,\rho}} + t^{1/\alpha}\|n(t)\|_{L^{q,1}} + \|\nabla c(t)\|_{L^{N,\rho}} + t^{1/\beta}\|\nabla c(t)\|_{L^{r,1}} \\
&\quad + \|\boldsymbol{u}(t)\|_{L^{N,\rho}} + t^{1/\beta}\|\boldsymbol{u}(t)\|_{L^{r,1}} + \|c(t) - e^{t\Delta}c_0\|_{L^\infty}, \quad 0 < t < \infty,
\end{aligned}$$

we have

$$\begin{aligned}
E(n, c, \boldsymbol{u})(t) &\le CE(n_j, c_j, \boldsymbol{u}_j)(t) + C(\|n - n_j\|_{L^\infty(L^{N/2,\rho})} + \|c - c_j\|_{L^\infty(L^\infty)} + \|\nabla c - \nabla c_j\|_{L^\infty(L^{N,\rho})} + \|\boldsymbol{u} - \boldsymbol{u}_j\|_{L^\infty(L^{N,\rho})}) \\
&\quad + C\sup_{0<\tau<\infty}(\tau^{1/\alpha}\|n(\tau) - n_j(\tau)\|_{L^{q,1}} + \tau^{1/\beta}\|\nabla c(\tau) - \nabla c_j(\tau)\|_{L^{r,1}} + \tau^{1/\beta}\|\boldsymbol{u}(\tau) - \boldsymbol{u}_j(\tau)\|_{L^{r,1}}) \\
&\le CE(n_j, c_j, \boldsymbol{u}_j)(t) + CK_\varphi(1 + \|c_0\|_{L^\infty})(\|n_0 - n_{0,j}\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_{0,j}\|_{L^{N,\rho}})
\end{aligned}$$

for all $0 < t < \infty$ and $j \in \mathbb{N}$. Since Lemma 4.4 (ii) implies that $\lim_{t\to\infty}E(n_j, c_j, \boldsymbol{u}_j)(t) = 0$ for all $j \in \mathbb{N}$, we have

$$\limsup_{t\to\infty}E(n, c, \boldsymbol{u})(t) \le CK_\varphi(1 + \|c_0\|_{L^\infty})(\|n_0 - n_{0,j}\|_{L^{N/2,\rho}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_{0,j}\|_{L^{N,\rho}}),$$

and hence we obtain (1.15), (1.16), and (1.17) by letting $j \to \infty$. This completes the proof of Theorem 1.3. □

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