Research Article

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Global stability and asymptotic profiles of a partially degenerate reaction diffusion Cholera model with asymptomatic individuals

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Abstract: Considering the prevalence of asymptomatic individuals during the spread of disease, this article develops a model of degenerate reaction diffusion Cholera with asymptomatic individuals. First, the well-posedness of model is studied, including the global existence of solutions and the existence of attractor. Second, the basic reproduction number $\mathcal{R}_0$ is defined to determine whether the disease is vanishing or persistent. In particular, we also analyze the asymptotic behavior of the endemic steady state when the diffusion rate of susceptible or asymptomatic individuals tends to 0 or infinity. Finally, by fitting the theoretical results with some numerical simulations, we find that the spatial distribution of disease and local epidemic risk are less affected by the mobility of susceptible populations, whereas the mobility of asymptomatic or symptomatic populations significantly affects the spatial and temporal distribution of infected populations. In addition, we found that the proportion of asymptomatic individuals to infected individuals is also a key factor in disease epidemics, and how to quickly diagnose asymptomatic individuals for disease control and prevention should be of a particular concern.

Keywords: asymptomatic individuals, spatial heterogeneity, basic reproduction number and stability, asymptotic profiles

MSC 2020: 35B40, 35J57, 35K57, 92D30

1 Introduction

Cholera is an acute diarrheal infection caused by eating or drinking food and water contaminated with *Vibrio cholerae*. It is typical symptoms are diarrhea and vomiting, dehydration, muscle cramps, etc. [22]. When medical care is inadequate and treatment is not timely, the mortality rate is high. Although health care is improving in many countries around the world, the highly contagious cholera still occurs after natural disasters such as earthquakes and tsunamis, or in areas with relatively poor sanitation. In recent years, cholera outbreaks have remained severe in less developed countries, including Haiti, Yemen, Zimbabwe, and the Congo. Researchers estimate that there are 1,300,000–4,000,000 cases of cholera worldwide each year, and 21,000–143,000 deaths due to infection [38].

As is well known, mathematical models play a critical role in comprehending the dynamics of disease transmission. Over the past decade or so, many scholars have studied the spread of cholera through the

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development of various mathematical models. Capasso and Paveri-Fontana [6] proposed the first model of indirect cholera transmission in response to the cholera epidemic that occurred in the Mediterranean region in the summer of 1973, and the model included only infected population and pathogen. Codeço [7] in 2001 extended this model by adding equations for susceptible populations and showed the aquatic hosts have an important influence on cholera transmission. Subsequently, many more researchers have extended the model in [7], such as, Tien and Earn [30] extended the SIR model by adding pathogen concentration compartment (W), considered the impact of multiple transmission routes, and investigated the global stability of the equilibrium points. Note that, during the spread of the disease, the fact that asymptomatic infected individuals have no clinical symptoms associated with the disease can be just as contagious as symptomatic ones, leading to a longer duration of the disease in some areas. There is strong evidence that asymptomatic carriers of cholera pathogen may be responsible for the long-distance spread of the pathogen elsewhere, and their numbers may be much higher than reported [15]. Ogola et al. [26] analyzed a cholera model with indirect transmission and asymptomatic infected individuals, and their results suggested that asymptomatic individuals are the main cause of cholera epidemics in Senegal and other sub-Saharan African countries. Other related articles can be found in [1,3,14,23,31] and the references therein.

It is worth noting that strengthening personal hygiene, improving environmental sanitation, and using safe drinking water are the most important strategies for controlling and preventing cholera. Therefore, vaccination has become an effective measure against cholera that has received widespread interest and became the subject of many clinical and theoretical studies. Many researchers as well have taken the cholera vaccine into account and have come up with a wealth of theoretical results. In [2,16], it was shown that backward branching can exist in models of all susceptible individuals vaccinated when the basic reproduction number is less than one. Cai et al. [5] modeled SAIVB infectious diseases with vaccine age and showed that not considering vaccine age underestimates the risk of transmission of cholera outbreaks.

Note that most of the aforementioned models are either ordinary differential equations (ODEs) or hybrids with partial differential equations (PDEs), which do not introduce spatial heterogeneity, resulting in an inadequate comprehension of the spatial propagation of cholera epidemics. Due to the differences in temperature, climate and geographic characteristics among different countries and regions, which in turn may contribute to the spatial distribution of the disease. Therefore, it is particularly important to integrate geographic and pathogen characteristics to develop cholera models with spatial heterogeneity. Much work has been done on the spatial dynamics of cholera transmission. Zhang et al. [41] developed a reaction diffusion model incorporating both direct and indirect transmission routes, and investigated the effects of diffusion coefficients and multiple transmission routes on disease propagation. Wang and Feng [34] proposed a model with different diffusion rates and spatial heterogeneity to study threshold dynamics and asymptotic behavior. Wang and Wu [35] investigated a degenerate cholera diffusion model, focusing on the asymptotic behavior of the positive steady state at small and large diffusion rates in susceptible individuals. Other studies on asymptotic behavior are continuing and are detailed in [4,9,36,39].

As mentioned earlier, the existing studies have mainly focused on how spatial heterogeneity and individual mobility contribute to the cholera propagation, prevention, and control. Actually, vaccine effectiveness, the prevalence of asymptomatic individuals, and spatial heterogeneity are all essential features that must be taken into account in the propagation and sustenance of cholera. Throughout this article, we establish a model of cholera with general incidence, spatial transmission, and incomplete immunity, ignoring person-to-person transmission (direct transmission) and assuming no transmission of environmental viruses, concentrating on the effects of these features on the spatial and temporal distribution of the disease and its prevention and control.

The rest of the article is organized as follows. The model is presented in Section 2. In Section 3, the global existence, ultimate boundedness, and the existence of attractors of model solution are studied. In Section 4, some relations between the principal eigenvalues and the basic reproduction number $\mathcal{R}_0$ are presented, and formulas for the local reproduction number $\tilde{\mathcal{R}}_0$ are derived. The threshould dynamics of the model are studid by $\mathcal{R}_0$ in Section 5. The asymptotic distribution of the endemic steady state, as the diffusion coefficients converge to different scenarios, is examined in Section 6. Finally, a number of numerical simulations are carried out in Section 7 to validate the theoretical results, and a brief conclusion is presented in Section 8.

2 Model formulation

Let $S(x,t)$, $V(x,t)$, $I_A(x,t)$, and $I_S(x,t)$ stand for the densities of the susceptible individuals, vaccinated individuals, asymptomatic individuals, and symptomatic individuals at position x and time t, respectively. The concentration of *Vibrio cholerae* in the environment at position x and time t is denoted by $B(x,t)$. $D_i > 0$ ($i = 1, \ldots, 4$) means the dispersal rates of S, V, I_A, and I_S, respectively. $\Lambda(x)$ denotes the recruitment rate of the susceptible individuals. $\mu(x)$ denotes the natural mortality rate of the population. $r_A(x)$ and $r_S(x)$ denote the recovery rate of asymptomatic individuals and symptomatic individual, respectively. $\delta(x)$ means the probability that an asymptomatic individual turns into a symptomatic individuals. The functions $\eta(x)$ and σ represent the rate of immune loss and the probability that the vaccine is immune protective, respectively. $\beta(x)f(S,B)$ denotes the infectivity of a susceptible individual in contact with a pathogen, where $\beta(x)$ indicates the effective contact rate betweens usceptible individuals and pathogens. We hypothesize that a fraction p ($0 < p < 1$) of the entering susceptible individuals is vaccinated. In addition, when the susceptible population is infected with the disease, a fraction of θ $(0 \leqslant \theta \leqslant 1)$ becomes asymptomatic, while $1-\theta$ develops symptoms. Compared to symptomatic individuals, asymptomatic individuals carry a lower amount of virus infectiously, and therefore, they exhibit lower shedding rates and contribute less to the pathogen density in the environment, denoted by $\gamma_A(x)$ and $\gamma_S(x)$ for asymptomatic and symptomatic individuals, respectively. $\alpha(x)$ stands for the clearance rate of the environment. Motivated by these considerations, we propose the following model

$$\begin{cases} \dfrac{\partial S}{\partial t} = D_1\Delta S + (1-p(x))\Lambda(x) - \mu(x)S - \beta(x)f(S,B) + \eta(x)V, \\ \dfrac{\partial V}{\partial t} = D_2\Delta V + p(x)\Lambda(x) - (\mu(x)+\eta(x))V - \sigma\beta(x)g(V,B), \\ \dfrac{\partial I_A}{\partial t} = D_3\Delta I_A + \theta\beta(x)f(S,B) + \sigma\beta(x)g(V,B) - (\mu(x)+r_A(x)+\delta(x))I_A, \\ \dfrac{\partial I_S}{\partial t} = D_4\Delta I_S + (1-\theta)\beta(x)f(S,B) + \delta(x)I_A - (\mu(x)+r_S(x))I_S, \\ \dfrac{\partial B}{\partial t} = \gamma_A(x)I_A + \gamma_S(x)I_S - \alpha(x)B, \end{cases} \tag{1}$$

where $(x,t) \in \mathbb{D} \times \in [0,\infty)$, with the boundary condition

$$\frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial I_A}{\partial \nu} = \frac{\partial I_S}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \quad t > 0 \tag{2}$$

and $S(x,0) = S_0(x)$, $V(x,0) = V_0(x)$, $I_A(x,0) = I_{A0}(x)$, $I_S(x,0) = I_{S0}(x)$, $B(x,0) = B_0(x)$, and $x \in \mathbb{D}$, where $\mathbb{D}$ is a general open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\mathbb{D}$. ν is the outward normal unit vector on $\partial\mathbb{D}$. The initial value $(S_0(x), V_0(x), I_{A0}(x), I_{S0}(x), B_0(x))$ is a nonegative continuous function. The propagation diagram of model eq. (1) is depicted in Figure 1.

In this article, we formulate the following general hypotheses:

(H_1) Functions $f(S,B)$, $g(V,B) \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$, $f_1'(S,B)$, $g_1'(V,B)$, $f_2'(S,B)$, and $g_2'(V,B)$ are positive for S, V, $B > 0$; $f(S,B) = 0$ if and only if $SB = 0$; $g(V,B) = 0$ if and only if $VB = 0$; $f_{22}''(S,B) \leqslant 0$, $g_{22}''(V,B) \leqslant 0$, for S,

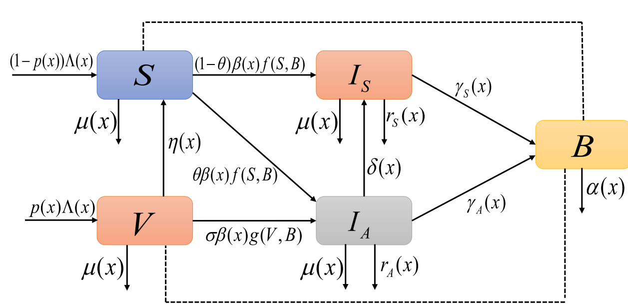

Figure 1: Schematic of the propagation of model (1).

$V, B \geqslant 0$. Here, $(\mathfrak{F})'_1$ and $(\mathfrak{F})'_2$ denote the first-order derivatives of the function $f(S, B)$ or $g(V, B)$with respect to S or V and B, respectively. $(\mathfrak{F})''_{22}$ denotes the second-order derivative of the function $f(S, B)$ or $g(V, B)$ with respect to B;

(H_2) There exists a Hölder continuous function $k_i : \mathbb{R} \to \mathbb{R}_+$ such that $f(y, B) \leqslant k_1 yB$, $g(y, B) \leqslant k_2 yB$, $y \in \{S, V\}$ for $x \in \mathbb{D}$, $S, V, B \geqslant 0$.

3 Well-posedness of the problem

Throughout this section, we shall confirm that model (1) has a unique global nonnegative solution and admits a connected global attractor. Let $X \coloneqq C(\overline{\mathbb{D}}, \mathbb{R}^4)$ be equipped with the supreme norm $\|\cdot\|_X$, and $X^+ \coloneqq C(\overline{\mathbb{D}}, \mathbb{R}^4_+)$ be the positive cone of X. For that, we introduce the notation $h^* \coloneqq \max_{x\in\overline{\mathbb{D}}}\{h(\cdot)\}$, $h_* \coloneqq \min_{x\in\overline{\mathbb{D}}}\{h(\cdot)\}$, where $h(\cdot) = \lambda(\cdot), p(\cdot), \mu(\cdot), \beta(\cdot), \eta(\cdot), r_A(\cdot), r_S(\cdot), \delta(\cdot), \gamma_A(\cdot), \alpha(\cdot), \gamma_S(\cdot)$. Let $T_k(t) : C(\overline{\mathbb{D}}, \mathbb{R}) \to C(\overline{\mathbb{D}}, \mathbb{R})$ $(k = 1, \ldots, 4)$ be the C_0-semigroups of $D_k\Delta - \pi_k(x)$ with (2), where $\pi_1(x) = \mu(x), \pi_2(x) = \mu(x) + \eta(x), \pi_3(x) = \mu(x) + r_A(x) + \delta(x)$ and $\pi_4(x) = \mu(x) + r_S(x)$. Then, for any initial value $\phi \in X^+$, we have

$$(T_k(t)\phi)(x) = \int_{\mathbb{D}} \Gamma_k(t, x, y)\mathrm{d}y, \quad \forall t > 0, \quad \phi \in C(\overline{\mathbb{D}}, \mathbb{R}), \quad k = 1, 2, 3, 4,$$

where $\Gamma_k(x, t, y)$ denotes the Green function associated with $D_k\Delta - \pi_k(x)$ subject to the Neumann boundary conditions, respectively. Let $(T_5(t)\phi)(x) = e^{-\alpha(x)t}\phi(x)$, thus, $T(t) \coloneqq (T_1(t), T_2(t), T_3(t), T_4(t), T_5(t))^T : X \to X$ forms a strongly continuous semigroup.

Furthermore, set $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5)^T : X^+ \to X$ be given by

$$\begin{cases} \mathcal{Z}_1(\phi)(\cdot) = (1 - p(x))\Lambda(x) - \beta(x)f(\phi_1, \phi_5) + \eta(x)\phi_2, \\ \mathcal{Z}_2(\phi)(\cdot) = p(x)\Lambda(x) - \sigma\beta(x)g(\phi_2, \phi_5), \\ \mathcal{Z}_3(\phi)(\cdot) = \theta\beta(x)f(\phi_1, \phi_5) + \delta(x)\beta(x)g(\phi_2, \phi_5), \\ \mathcal{Z}_4(\phi)(\cdot) = (1 - \theta)\beta(x)f(\phi_1, \phi_5) + \sigma\phi_3, \\ \mathcal{Z}_5(\phi)(\cdot) = \gamma_A(x)\phi_3 + \gamma_S(x)\phi_4. \end{cases}$$

By the aforementioned setting, we can rewrite model (1) as follows:

$$u(x, t, \phi) = T(t)\phi + \int_0^t T(t - s)\mathcal{Z}(u(x, s, \phi))\mathrm{d}s, \tag{3}$$

where $u(x, t, \phi) = (S(x, t, \phi), V(x, t, \phi), I_A(x, t, \phi), I_S(x, t, \phi), B(x, t, \phi))^T$.

The local solutions of the model (1) on X^+ is given first.

Lemma 1. *For any $\phi \in X^+$, model* (1) *has a unique mild solution $u(x, t, \phi)$ defined on $[0, \tau_{\max})$, where $\tau_{\max} \leqslant \infty$ and if $\tau_{\max} < \infty$, then $\lim_{t\to\tau_{\max}^-}\|u(x, t)\| = \infty$. Furthermore, $u(x, t, \phi)$ is a classical solution of* (1) *for all $t \in [0, \tau_{\max})$.*

Proof. For $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in X^+$ and $h \geqslant 0$, it follows that

$$\begin{aligned} \phi + h\mathcal{Z}(\phi) &= \begin{pmatrix} \phi_1 + h[(1 - p(x))\Lambda(x) - \beta(x)f(\phi_1, \phi_5) + \eta(x)\phi_2] \\ \phi_2 + h[p(x)\Lambda(x) - \sigma\beta(x)g(\phi_2, \phi_5)] \\ \phi_3 + h[\theta\beta f(\phi_1, \phi_5) + \sigma\beta(x)g(\phi_2, \phi_5)] \\ \phi_4 + h[(1 - \theta)\beta(x)f(\phi_1, \phi_5) + \sigma\phi_3] \\ \phi_5 + h[\gamma_A(x)\phi_3 + \gamma_S(x)\phi_4] \end{pmatrix} \\ &\geqslant \begin{pmatrix} \phi_1 - h\beta(x)f(\phi_1, \phi_5) \\ \phi_2 - h\sigma\beta(x)g(\phi_2, \phi_5) \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix}, \end{aligned}$$

which means

$$\lim_{h\to 0^+}\frac{1}{h}\operatorname{dist}(\phi + h\mathcal{Z}(\phi), X^+) = 0, \quad \forall \phi \in X^+.$$

It yields from the general results in [21, Corollary 4] that (1) admits positive solution $u(x,t)$ in $t \in [0, \tau_{\max})$, which is unique. The proof is completed. □

Lemma 2. *For any $\phi \in X^+$, model* (1) *has a unique nonnegative solution on* $\mathbb{D} \times [0,\infty)$, *and satisfies $u(x,0,\phi) = \phi$. Furthermore, the semiflow $\Phi(t)$ induced by the solution of* (1) *is ultimately bounded.*

Proof. From Lemma 1, we learn that model (1) has a unique solution $u(x,t)$ defined for $x \in \overline{\mathbb{D}}$ and $t \in [0, \tau_{\max})$. Then, if proves the global existence of the solution, i.e., the existence interval of the local solution can be extended to infinity. Suppose $\tau_{\max} < \infty$, then from [21, Theorem 2], we have $\|u(x,t)\|_X \to \infty$ as $t \to \tau_{\max}^-$. It follows easily from the second equation of (1) that

$$\begin{cases} \dfrac{\partial V}{\partial t} \leqslant D_2 \Delta V + p(x)\Lambda(x) - (\mu(x) + \eta(x))V, & t \in [0, \tau_{\max}), \quad x \in \mathbb{D}, \\ \dfrac{\partial V}{\partial \nu} = 0, \quad t \in [0, \tau_{\max}), \quad x \in \partial\mathbb{D}. \end{cases}$$

Combining [17, Lemma 1] and the comparison theorem leads to

$$\limsup_{t\to\tau_{\max}^-} V(x,t) \leqslant \bar{W}_2(x) \quad \text{uniformly for } x \in \overline{\mathbb{D}}, \tag{4}$$

where $\bar{W}_2(x)$ satisfies

$$\begin{cases} \dfrac{\partial \bar{V}}{\partial t} = D_2 \Delta \bar{V} + p(x)\Lambda(x) - (\mu(x) + \eta(x))\bar{V}, & t \in [0, \tau_{\max}), \quad x \in \mathbb{D}, \\ \dfrac{\partial \bar{V}}{\partial \nu} = 0, \quad t \in [0, \tau_{\max}), \quad x \in \partial\mathbb{D}. \end{cases}$$

Thus, there exists positive constant $M_1 > 0$, depending on initial value, and $t_1 = t_1(\phi) > 0$ such that $\|V(x,t)\| \leqslant M_1$, $t \geqslant t_1$. Similarly, from the S equation of model (1), it can be seen that

$$\begin{cases} \dfrac{\partial \bar{S}}{\partial t} = D_1 \Delta \bar{S} + (1 - p(x))\Lambda(x) - \mu(x)\bar{S} - \beta(x) f(\bar{S}, B) + \eta(x)\bar{W}_2(x), & t \in [t_1, \tau_{\max}) \quad x \in \mathbb{D}, \\ \dfrac{\partial \bar{S}}{\partial \nu} = 0, \quad t \in [t_1, \tau_{\max}), \quad x \in \partial\mathbb{D}, \end{cases}$$

which implies that

$$\limsup_{t\to\tau_{\max}^-} S(x,t) \leqslant \bar{W}_1(x) \quad \text{uniformly for} \quad x \in \overline{\mathbb{D}}, \tag{5}$$

where $\bar{W}_1(x)$ meets (5). Hence, there exists a positive constant M_0, depending on initial value, and $t_2 = t_2(\phi) > 0$ such that

$$\|S(x,t)\| \leqslant M_0, \quad t \geqslant t_2.$$

Choosing sufficiently large N_3, N_4 such that $\|T_3(t)\| \leqslant N_3 e^{-\kappa_3 t}$, $\|T_4(t)\| \leqslant N_4 e^{-\kappa_4 t}$, where $\kappa_3 > 0$ and $\kappa_4 > 0$ represent the principal eigenvalues of $D_3\Delta - \pi_3(x)$ and $D_4\Delta - \pi_4(x)$, respectively. Let $\tilde{N}_3 = N_3\beta^*(\theta k_1 M_0 + \sigma k_2 M_1)$ and $\tilde{N}_4 = (1-\theta)k_2\beta^* M_0 N_4$, then

$$\begin{aligned}||I_A(x,t)|| &= \left\| T_3(t)I_{A0}(x) + \int_0^t T_3(t-s)[\theta\beta(x)f(S(x,s),B(x,s)) + \sigma\beta(x)g(V(x,s),B(x,s))]\mathrm{d}s \right\| \\ &\leqslant N_3 e^{-\kappa_3 t}||I_{A0}(x)|| + \tilde{N}_3 \int_0^t e^{-\kappa_3(t-s)}||B(x,s)||\mathrm{d}s, \\ ||I_S(x,t)|| &= \left\| T_4(t)I_{S0}(x) + \int_0^t T_4(t-s)[(1-\theta)\beta(x)f(S(x,s),B(x,s)) + \delta(x)I_A(x,s)]\mathrm{d}s \right\| \\ &\leqslant N_4 e^{-\kappa_4 t}||I_{S0}(x)|| + \tilde{N}_4 \int_0^t e^{-\kappa_4(t-s)}||B(x,s)||\mathrm{d}s + N_4\delta^* \int_0^t e^{-\kappa_4(t-s)}||I_A(x,s)||\mathrm{d}s,\end{aligned}$$

for $0 \leqslant t < \tau_{\max}$. According to the fifth equation of (1), one has

$$||B(x,t)|| \leqslant e^{-\alpha_m t}||B_0|| + \gamma_A^* \int_0^t e^{-\alpha_m(t-s)}||I_A(x,s)||\mathrm{d}s + \gamma_S^* \int_0^t e^{-\alpha_m(t-s)}||I_S(x,s)||\mathrm{d}s,$$

where $\alpha_m = \min\{\kappa_3/2, \kappa_4/2, \alpha_*\}$ and $0 \leqslant t < \tau_{\max}$. Combining the aforementioned results, one has

$$\begin{aligned}||I_S(x,t)|| &\leqslant N_4 e^{-\kappa_4 t}||I_{S0}(x)|| + \tilde{N}_4 \int_0^t e^{-\kappa_4(t-s)}||B(x,s)||\mathrm{d}s \\ &\quad + \delta^* N_4 \int_0^t e^{-\kappa_4(t-s)}\left[N_3||I_{A0}(x)|| + \tilde{N}_3 \int_0^s e^{-\kappa_3(s-r)}||B(x,r)||\mathrm{d}r\right]\mathrm{d}s \\ &\leqslant N_4 e^{-\kappa_4 t}||I_{S0}(x)|| + \tilde{N}_4 \int_0^t e^{-\kappa_4(t-s)}||B(x,s)||\mathrm{d}s + \delta^* N_3 N_4 ||I_{A0}(x)||/\kappa_4 \\ &\quad + \delta^* \tilde{N}_3 N_4 \int_0^t e^{-\alpha_m(t-s)} \int_r^t e^{-\alpha_m(s-r)}||B(x,s)||\mathrm{d}s\mathrm{d}r \\ &\leqslant N_4 e^{-\kappa_4 t}||I_{S0}(x)|| + \tilde{N}_4 \int_0^t e^{-\kappa_4(t-s)}||B(x,s)||\mathrm{d}s + \delta^* N_3 N_4 ||I_{A0}(x)||/\kappa_4 \\ &\quad + \delta^* \tilde{N}_3 N_4 e^{-2\alpha_m t}(t-r) \int_0^t e^{2\alpha_m s}||B(x,s)||\mathrm{d}s,\end{aligned}$$

for $0 \leqslant t < \tau_{\max}$. Similarly, one has

$$\begin{aligned}||I_A(x,t)|| &\leqslant N_3 e^{-\kappa_3 t}||I_{A0}(x)|| + \tilde{N}_3 \int_0^t e^{-\kappa_3(t-s)}\left[e^{-\alpha_m s}||B_0|| + \gamma_A^* \int_0^s e^{-\alpha_m(s-r)}||I_A(x,r)||\mathrm{d}r + \gamma_S^* \int_0^s e^{-\alpha_m(s-r)}||I_S(x,r)||\mathrm{d}r\right]\mathrm{d}s \\ &\leqslant C_1 + C_3 e^{-\alpha_m t} \int_0^t e^{\alpha_m s}||I_A(x,s)||\mathrm{d}s + C_2 e^{-\alpha_m t} \int_0^t e^{\alpha_m s}||I_S(x,s)||\mathrm{d}s,\end{aligned}$$

where $C_1 = N_1||I_{A0}|| + \tilde{N}_1||B_0||/(\kappa_3 - \alpha_m)$ and $C_2 = \tilde{N}_3\gamma_A^*/(\kappa_3 - \alpha_m)$, $C_3 = \gamma_S^* C_2$ and $0 \leqslant t < \tau_{\max}$. By Gronwall's inequality, one has

$$||I_A(x,t)|| \leqslant \left[C_1 + C_3 e^{-\alpha_m t} \int_0^t e^{\alpha_m s}||I_S(x,s)||\mathrm{d}s\right] e^{C_2 t}, \quad 0 \leqslant t < \tau_{\max}. \tag{6}$$

Then,

$$\begin{aligned}
||I_S(x,t)|| &\leqslant N_3||I_{S0}(x)|| + \delta^* N_3 N_4 ||I_{A0}(x)||/\kappa_4 + \tilde{N}_4 \int_0^t e^{-\kappa_4(t-s)} \\
&\quad \times \left[e^{-\alpha_m s}||B_0(x)|| + \gamma_A^* \int_0^s e^{-\alpha_m(s-r)}||I_A(x,r)||\mathrm{d}r + \gamma_S^* \int_0^s e^{-\alpha_m(s-r)}||I_S(x,r)||\mathrm{d}r \right] \mathrm{d}s \\
&\quad + \delta^* \tilde{N}_3 N_4 e^{-2\alpha_m t}(t-r) \int_0^t e^{\alpha_m s} \left[e^{-\alpha_m s}||B_0(x)|| + \gamma_A^* \int_0^s e^{-\alpha_m(s-r)}||I_A(x,r)||\mathrm{d}r + \gamma_S^* \int_0^s e^{-\alpha_m(s-r)}||I_S(x,r)||\mathrm{d}r \right] \mathrm{d}s \\
&\leqslant C_4 + C_5 e^{-\alpha_m t} \int_0^t e^{\alpha_m} ||I_S(x,s)||\mathrm{d}s,
\end{aligned}$$

where $C_4 = N_4||I_{S0}(x)|| + \delta^* N_3 N_4 ||I_{A0}(x)||/\kappa_4 + \tilde{N}_4||B_0(x)||/(\kappa_4 - \alpha_m) + \delta^* \tilde{N}_3 N_4 (t-r)||B_0(x)||/\alpha_m + \tilde{N}_4 C_2 e^{(C_2+\alpha_m)t}/(C_2+\alpha_m)$, $C_5 = (\gamma_A^* + \gamma_S^*)[\tilde{N}_4 C_3 e^{C_2 t}/C_2 + \delta^* \tilde{N}_3 N_4 (t-r)]$ and $0 \leqslant t < \tau_{\max}$. Using Gronwall's inequality again yields

$$||I_S(x,t)|| \leqslant C_4 e^{C_5 t}, \quad 0 \leqslant t < \tau_{\max}. \tag{7}$$

The arrangement (6) can be derived as follows:

$$||I_A(x,t)|| \leqslant C_1 e^{C_2 t} + C_3 C_4 e^{(C_2+C_5)t}/(\alpha_m + C_5), \quad 0 \leqslant t < \tau_{\max}. \tag{8}$$

Combining (7) and (8), we obtain

$$\begin{aligned}
||B(x,t)|| &\leqslant ||B^0(x)|| + \gamma_A^* C_1 e^{C_2 t}/(\alpha_m + C_2) + \gamma_A^* C_3 C_4 e^{(C_2+C_5)t}/((\alpha_m + C_2 + C_5)(\alpha_m + C_5)) \\
&\quad + \gamma_S^* C_4 e^{C_5 t}/(\alpha_m + C_5), \quad 0 \leqslant t < \tau_{\max}.
\end{aligned}$$

These, together with Lemma 1, we realize that model (1) has a unique nonnegative global solution, i.e., $\tau_{\max} = \infty$. In the next step we prove the boundedness of the solutions of model (1). First, we claim that $(I_A(x,t), I_S(x,t), B(x,t))$ meets the following L^1-bounded estimate,

$$\limsup_{t\to\infty}(||I_A(x,t)||_{L^1} + ||I_S(x,t)||_{L^1} + ||B(x,t)||_{L^1}) \leqslant M_2. \tag{9}$$

On the basis of the model (1), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(||S(x,t)||_{L^1} + ||V(x,t)||_{L^1} + ||I_A(x,t)||_{L^1} + ||I_S(x,t)||_{L^1} + ||B(x,t)||_{L^1}) \\
&\quad \leqslant \int_{\mathbb{D}} \Lambda(x)\mathrm{d}x - \mu(x)\int_{\mathbb{D}} (S(x,t) + V(x,t) + I_A(x,t) + I_S(x,t))\mathrm{d}x \\
&\quad \leqslant \Lambda^* |\mathbb{D}| - \mu_*(||S(x,t)||_{L^1} + ||V(x,t)||_{L^1} + ||I_A(x,t)||_{L^1} + ||I_S(x,t)||_{L^1}).
\end{aligned}$$

Hence, we obtain

$$\limsup_{t\to\infty}(||S(x,t)||_{L^1} + ||V(x,t)||_{L^1} + ||I_A(x,t)||_{L^1} + ||I_S(x,t)||_{L^1}) \leqslant M_2, \tag{10}$$

where $M_2 = \Lambda^*|\mathbb{D}|/\mu_*$. Similarly, B equation satisfies

$$\frac{\partial}{\partial t}\int_{\mathbb{D}} B(x,t)\mathrm{d}x \leqslant (\gamma_A^* + \gamma_S^*)M_2 - \alpha_* \int_{\mathbb{D}} B(x,t)\mathrm{d}x,$$

it follows that $\limsup_{t\to\infty}||B(x,t)||_{L^1} \leqslant M_3$, where $M_3 = (\gamma_A^* + \gamma_S^*)M_2/\alpha_*$.

In the next step, we show that I_A, I_S and B fulfill L^{2^k}-bounded estimate for $k \geqslant 0$,

$$\limsup_{t\to\infty}(||I_A(x,t)||_{2^k}^{2^k} + ||I_S(x,t)||_{2^k}^{2^k} + ||B(x,t)||_{2^k}^{2^k}) \leqslant M_{2^k}, \quad \text{for some} \quad M_{2^k} > 0. \tag{11}$$

It is apparent that (11) holds when $k = 0$. Suppose that for $k - 1$, (11) also holds, i.e., there is constant $M_{2^{k-1}} > 0$ such that

$$\limsup_{t\to\infty}(\|I_A(x,t)\|_{2^{k-1}}^{2^{k-1}} + \|I_S(x,t)\|_{2^{k-1}}^{2^{k-1}} + \|B(x,t)\|_{2^{k-1}}^{2^{k-1}}) \leqslant M_{2^{k-1}}.$$

Multiplying the third equation of (1) by $I_A^{2^k-1}$ and integrating over $\mathbb{D}$, one has

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x \leqslant D_3\int_{\mathbb{D}} I_A^{2^k-1}\Delta I_A\mathrm{d}x + \int_{\mathbb{D}}\theta\beta(x)I_A^{2^k-1}f(S,B)\mathrm{d}x + \int_{\mathbb{D}}\sigma\beta(x)I_A^{2^k-1}g(V,B)\mathrm{d}x\\ - \int_{\mathbb{D}}(\mu(x)+r_A(x)+\delta(x))I_A^{2^k}\mathrm{d}x.\end{aligned} \tag{12}$$

Since

$$D_3\int_{\mathbb{D}} I_A^{2^k-1}\Delta I_A\mathrm{d}x \leqslant -D_3\int_{\mathbb{D}}\nabla I_A\nabla I_A^{2^k-1}\mathrm{d}x = -(2^k-1)D_3\int_{\mathbb{D}}(\nabla I_A\nabla I_A)I_A^{2^k-2}\mathrm{d}x = -\frac{2^k-1}{2^{2k-2}}D_3\int_{\mathbb{D}}|\nabla I_A^{2^{k-1}}|^2\mathrm{d}x.$$

Thus, (12) becomes

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x \leqslant -\frac{2^k-1}{2^{2k-2}}D_3\int_{\mathbb{D}}|\nabla I^{2^{k-1}}|^2\mathrm{d}x + \int_{\mathbb{D}}\theta\beta(x)I_A^{2^k-1}f(S,B)\mathrm{d}x\\ + \int_{\mathbb{D}}\sigma\beta(x)I_A^{2^k-1}g(V,B)\mathrm{d}x - \int_{\mathbb{D}}(\mu(x)+r_A(x)+\delta(x))I_A^{2^k}\mathrm{d}x.\end{aligned}$$

According to the boundedness of S and V, one can see that there exists $t_3 > 0$ such that

$$\begin{aligned}\int_{\mathbb{D}}\theta\beta(x)I_A^{2^k-1}f(S,B)\mathrm{d}x \leqslant \theta\beta^*(M_0+1)\int_{\mathbb{D}} I_A^{2^k-1}B\mathrm{d}x, \quad \text{for} \quad t \geqslant t_3,\\ \int_{\mathbb{D}}\sigma\beta(x)I_A^{2^k-1}g(V,B)\mathrm{d}x \leqslant \sigma\beta^*(M_1+1)\int_{\mathbb{D}} I_A^{2^k-1}B\mathrm{d}x, \quad \text{for} \quad t \geqslant t_3.\end{aligned}$$

By using Young's inequality, $ab \leqslant \varepsilon a^p + \varepsilon^{-\frac{q}{p}}b^q$, where $a, b, \varepsilon > 0$, $p, q > 1$ and $1/p + 1/q = 1$. One can estimate $\int_{\mathbb{D}} BI_A^{2^k-1}\mathrm{d}x$ by setting $\varepsilon_0 = \alpha_*/(4\beta^*\max\{\theta(M_0+1), \sigma(M_1+1\}))$, $p = 2^k$ and $q = 2^k/(2^k-1)$ as follows:

$$\int_{\mathbb{D}} BI_A^{2^k-1}\mathrm{d}x \leqslant \varepsilon_0\int_{\mathbb{D}} B^{2^k}\mathrm{d}x + C_{\varepsilon_0}\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x, \quad \text{where} \quad C_{\varepsilon_0} = \varepsilon_0^{-\frac{1}{2^k-1}}.$$

Hence, (12) can be estimated by

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x \leqslant -G_k\int_{\mathbb{D}}|\nabla I_A^{2^{k-1}}|^2\mathrm{d}x + \frac{\alpha_*}{4}\int_{\mathbb{D}} B^{2^k}\mathrm{d}x + C_{k_1}\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x, \tag{13}$$

where $G_k = (2^k-1)/2^{2k-2}$, $C_{k_1} = \beta^*C_{\varepsilon_0}(\theta(M_0+1)+\sigma(M_1+1))$.

Similarly, we multiply the fourth equation of (1) by $I_S^{2^k-1}$ and integrating over $\mathbb{D}$, one has

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x \leqslant D_4\int_{\mathbb{D}} I_S^{2^k-1}\Delta I_S\mathrm{d}x + \int_{\mathbb{D}}(1-\theta)\beta(x)I_S^{2^k-1}f(S,B)\mathrm{d}x\\ + \int_{\mathbb{D}}\delta(x)I_AI_S^{2^k-1}\mathrm{d}x - \int_{\mathbb{D}}(\mu(x)+r_S(x))I_S^{2^k}\mathrm{d}x.\end{aligned}$$

Since

$$D_4\int_{\mathbb{D}} I_S^{2^k-1}\Delta I_S\mathrm{d}x \leqslant -D_4\int_{\mathbb{D}}\nabla I_S\nabla I_S^{2^k-1}\mathrm{d}x = -(2^k-1)D_4\int_{\mathbb{D}}(\nabla I_S\nabla I_S)I_S^{2^k-2}\mathrm{d}x = -L_k\int_{\mathbb{D}}|\nabla I_S^{2^{k-1}}|^2\mathrm{d}x,$$

where $L_k = (2^k-1)D_3/2^{2k-2}$ and

$$\begin{aligned}\int_{\mathbb{D}}(1-\theta)\beta(x)I_S^{2^k-1}f(S,B)\mathrm{d}x \leqslant (1-\theta)\beta^*(M_0+1)\int_{\mathbb{D}} BI_S^{2^k-1}\mathrm{d}x,\\ \int_{\mathbb{D}}\delta(x)I_AI_S^{2^k-1}\mathrm{d}x \leqslant \delta^*\int_{\mathbb{D}} I_AI_S^{2^k-1}\mathrm{d}x.\end{aligned}$$

By again using Young's inequality by setting $\varepsilon_1 = \alpha_*/(4(1-\theta)\beta^*(M_0+1))$, $p = 2^k$ and $q = 2^k/(2^k-1)$, we obtain

$$\int_{\mathbb{D}} BI_S^{2^k-1}\mathrm{d}x \leqslant \varepsilon_1\int_{\mathbb{D}} B^{2^k}\mathrm{d}x + C_{\varepsilon_1}\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x, \quad \text{where} \quad C_{\varepsilon_1} = \varepsilon_1^{-\frac{1}{2^k-1}}.$$

Similarly, we also let $\varepsilon_2 = 1/\delta^*$, $p = 2^k$ and $q = 2^k/(2^k-1)$, it follows that

$$\int_{\mathbb{D}} I_A I_S^{2^k-1}\mathrm{d}x \leqslant \varepsilon_2\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x + C_{\varepsilon_2}\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x, \quad \text{where} \quad C_{\varepsilon_2} = \varepsilon_2^{-\frac{1}{2^k-1}}.$$

Hence, we can obtain

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x \leqslant -L_k\int_{\mathbb{D}} |\nabla I_S^{2^{k-1}}|^2\mathrm{d}x + \frac{\alpha_*}{4}\int_{\mathbb{D}} B^{2^k}\mathrm{d}x + \int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x + C_{k_2}\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x, \tag{14}$$

where $C_{k_2} = (1-\theta)\beta^*(M_0+1)C_{\varepsilon_1} + \delta^* C_{\varepsilon_2}$.

Multiplying the fifth equation of (1) by B^{2^k-1} and integrating over $\mathbb{D}$, one has

$$\begin{aligned}\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} B^{2^k}\mathrm{d}x &\leqslant \int_{\mathbb{D}} (\gamma_S(x)I_S + \gamma_A(x)I_A)B^{2^k-1}\mathrm{d}x - \int_{\mathbb{D}} \alpha(x)B^{2^k}\mathrm{d}x \\ &\leqslant \gamma_S^*\int_{\mathbb{D}} I_S B^{2^k-1}\mathrm{d}x + \gamma_A^*\int_{\mathbb{D}} I_A B^{2^k-1}\mathrm{d}x - \alpha_*\int_{\mathbb{D}} B^{2^k}\mathrm{d}x.\end{aligned} \tag{15}$$

By again applying Young's inequality by choosing $\varepsilon_3 = \alpha_*/8\gamma_S^*$, $\varepsilon_4 = \alpha_*/8\gamma_A^*$, $p = 2^k/(2^k-1)$, $q = 2^k$, we have

$$\gamma_S^*\int_{\mathbb{D}} I_S B^{2^k-1}\mathrm{d}x \leqslant \frac{\alpha_*}{8}\int_{\mathbb{D}} B^{2^k}\mathrm{d}x + C_{\varepsilon_3}\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x, \quad \text{for} \quad C_{\varepsilon_3} = \varepsilon_3^{-\frac{1}{2^k-1}},$$

$$r_A^*\int_{\mathbb{D}} I_A B^{2^k-1}\mathrm{d}x \leqslant \frac{\alpha_*}{8}\int_{\mathbb{D}} B^{2^k}\mathrm{d}x + C_{\varepsilon_4}\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x, \quad \text{for} \quad C_{\varepsilon_4} = \varepsilon_4^{-\frac{1}{2^k-1}}.$$

Thus, (15) becomes

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} B^{2^k}\mathrm{d}x \leqslant -\frac{3}{4}\alpha_*\int_{\mathbb{D}} B^{2^k}\mathrm{d}x + C_{\varepsilon_3}\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x + C_{\varepsilon_4}\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x. \tag{16}$$

By combining (13), (14), and (16), we have

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}} (I_A^{2^k} + I_S^{2^k} + B^{2^k})\mathrm{d}x \leqslant -G_k\int_{\mathbb{D}} |\nabla I_A^{2^{k-1}}|^2\mathrm{d}x - L_k\int_{\mathbb{D}} |\nabla I_S^{2^{k-1}}|^2\mathrm{d}x + Q_k\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x + W_k\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x - \frac{\alpha_*}{4}\int_{\mathbb{D}} B^{2^k}\mathrm{d}x,$$

where $Q_k = (C_{k_1} + 1 + C_{\varepsilon_4})$ and $W_k = C_{k_2} + C_{\varepsilon_3}$.

By applying interpolation inequality

$$\|\xi\|_2^2 \leqslant \varepsilon\|\nabla\xi\|_2^2 + C_\varepsilon\|\xi\|_1^2, \quad \text{where} \quad \xi \in W^{1,2}(\mathbb{D}).$$

Let $\varepsilon_5 = G_k/2Q_k$, $\xi = E^{2^{k-1}}$, $\varepsilon_6 = L_k/2W_k$ and $\xi = I^{2^{k-1}}$, then

$$-G_k\int_{\mathbb{D}} |\nabla I_A^{2^{k-1}}|^2\mathrm{d}x \leqslant -2Q_k\int_{\mathbb{D}} I_A^{2^k}\mathrm{d}x + 2Q_k C_{\varepsilon_5}\left(\int_{\mathbb{D}} I_A^{2^{k-1}}\mathrm{d}x\right)^2,$$

$$-L_k\int_{\mathbb{D}} |\nabla I_S^{2^{k-1}}|^2\mathrm{d}x \leqslant -2W_k\int_{\mathbb{D}} I_S^{2^k}\mathrm{d}x + 2W_k C_{\varepsilon_6}\left(\int_{\mathbb{D}} I_S^{2^{k-1}}\mathrm{d}x\right)^2.$$

Hence, we can obtain

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_{\mathbb{D}}(I_A^{2^k}+I_S^{2^k}+B^{2^k}) \leqslant -m_*\int_{\mathbb{D}}(I_A^{2^k}+I_S^{2^k}+B^{2^k})\mathrm{d}x + 2Q_kC_{\varepsilon_5}\left(\int_{\mathbb{D}} I_A^{2^{k-1}}\mathrm{d}x\right)^2 + 2W_kC_{\varepsilon_6}\left(\int_{\mathbb{D}} I_S^{2^{k-1}}\mathrm{d}x\right)^2,$$

where $m_* = \min\{Q_k, W_k, \alpha_*/4\}$. Then from (11), we can see that $\limsup_{t\to\infty}(\|I_A(x,t)\|_{2^{k-1}}^{2^{k-1}}) \leqslant M_{2^{k-1}}$ and $\limsup_{t\to\infty}(\|I_S(x,t)\|_{2^{k-1}}^{2^{k-1}}) \leqslant M_{2^{k-1}}$. Hence,

$$\limsup_{t\to\infty}(\|I_A(t,x)\|_{2^k} + \|I_S(t,x)\|_{2^k} + \|B(t,x)\|_{2^k}) \leqslant M_{2^k},$$

where $\mathcal{U} = 2Q_kC_{\varepsilon_5} + 2W_kC_{\varepsilon_6}$ and $M_{2^k} = \mathcal{U}M_{2^{k-1}}^2/m_*$. Thus, by the continuous embedding, $L^q(\mathbb{D}) \hookrightarrow L^p(\mathbb{D})$ for $q \geqslant p \geqslant 1$ yields

$$\limsup_{t\to\infty}(\|I_A(t,x)\|_{L^p} + \|I_S(t,x)\|_{L^p} + \|B(t,x)\|_{L^p}) \leqslant M_p, \quad \text{where} \quad M_p > 0.$$

Denote by Y_a $(0 \leqslant a \leqslant 1)$ the fractional power space. Similar to [39, Lemma 2.4], one obtains $Y_a \subset C(\overline{\mathbb{D}})$ by selecting $p > n/2$ and $a \geqslant n/2p$. Hence, we can obtain

$$\limsup_{t\to\infty}\|I_A(x,t)\| \leqslant M_\infty, \quad \limsup_{t\to\infty}\|I_S(x,t)\| \leqslant M_\infty, \quad \limsup_{t\to\infty}\|B(t,x)\| \leqslant \frac{\gamma_A^* + \gamma_S^*}{\alpha_*}M_\infty.$$

where $M_\infty > 0$. Thus, this completes the proof. □

As the fifth equation of model (1) has no diffusion, the weak compactness of the solution semiflow $\Phi(t)$ is hard to obtain, and we substitute the weak compactness with the asymptotic smoothness of the solution semiflow. First, we define the Kuratowski measure of noncompactness, $\tau(\cdot)$,

$$\tau(\mathbb{W}) := \inf\{r : \mathbb{W} \text{ has a finite cover of diameter } < r\},$$

for any bounded set $\mathbb{W}$.

Lemma 3. *For any $t \geqslant 0$, $\Phi(t)$ admits a global compact attractor $\mathcal{A}$ in X^+.*

Proof. Let $\mathcal{F}(I_A, I_S, B) = \gamma_A(x)I_A + \gamma_S(x)I_S - \alpha(x)B$. Taking the partial derivative of $\mathcal{F}(I_A, I_S, B)$ with respect to B yields

$$\frac{\partial\mathcal{F}(I_A, I_S, B)}{\partial B} = -\alpha(x) \leqslant -\alpha_*, \quad (I_A, I_S, B) \in X^+.$$

Let $u_0 = (S_0(x), V_0(x), I_{A0}(x), I_{S0}(x), B_0(x)) \in X^+$ and $u(x,t) = (S(x,t,u_0),\ V(x,t,u_0),\ I_A(x,t,u_0),\ I_S(x,t,u_0),\ B(x,t,u_0))$ be the solution of model (1). It is known by a similar argument as in [39, Lemma 2.5] that for any $t \geqslant 0$, the following sets

$$\mathcal{S}_1 = \left\{\int_0^t e^{-\alpha(x)(t-s)}(\gamma_A(x)I_A(x,s,u_0) + \gamma_S(x)I_S(x,s,u_0))\mathrm{d}s : u_0 \in \mathbb{W}\right\}$$

is precompact. Hence, we rewrite $\Phi(t) = \Phi_1(t) + \Phi_2(t)$, $t \geqslant 0$, where

$$\Phi_1(t)u_0 = \{S(x,t,u_0), V(x,t,u_0), I_A(x,t,u_0), I_S(x,t,u_0), \mathcal{S}_1\} \quad \text{and} \quad \Phi_2(t)u_0 = \{0, 0, 0, 0, e^{-\alpha(x)t}B_0(x)\}.$$

Obviously, $\tau(\Phi_1(t)\mathbb{W}) = 0$. We then estimate $\Phi_2(t)$ as follows:

$$\|\Phi_2(t)\| = \sup_{u_0\in\mathbb{X}}\frac{\|\Phi_2(t)u_0\|_{\mathbb{X}}}{\|u_0\|_X} \leqslant e^{-\alpha_* t}\sup_{u_0\in X}\frac{\|u_0\|_X}{\|u_0\|_X} = e^{-\alpha_* t}.$$

It then follows that for $t \geqslant 0$

$$\tau(\Phi(t)\mathbb{W}) \leqslant \|\Phi_2(t)\|\tau(\mathbb{B}) \leqslant e^{-\alpha_* t}\tau(\mathbb{B}).$$

Therefore, $\Phi(t)$ is a τ-contraction with the contraction function $e^{-\alpha_* t}$. By [13, Lemma 2.3.4], $\Phi(t)$ is asymptotically smooth. On the basis of Lemma 3 and [13, Theorem 2.4.6], one can obtain the existence of compact attarctor for model (1) in X^+. □

4 Basic reproduction number and steady states

Model (1) admits a disease free steady-state $\mathcal{E}^0 = (S^0(x), V^0(x), 0, 0, 0)$. The linearized subsystem of (1) at $\mathcal{E}^0$ is

$$\begin{cases} \dfrac{\partial I_A}{\partial t} = D_3 \Delta I_A + \theta\beta(x) f_2'(S^0(x), 0)B + \sigma\beta(x) g_2'(V^0(x), 0)B \\ \qquad - (\mu(x) + r_A(x) + \delta(x)) I_A, \quad t > 0, \quad x \in \mathbb{D}, \\ \dfrac{\partial I_S}{\partial t} = D_4 \Delta I_S + (1-\theta)\beta(x) f_2'(S^0(x), 0)B + \delta(x) I_A - (\mu(x) + r_S(x)) I_S, \quad t > 0, \quad x \in \mathbb{D}, \\ \dfrac{\partial B}{\partial t} = \gamma_S(x) I_S + \gamma_A(x) I_A - \alpha(x) B, \quad t > 0, \quad x \in \mathbb{D}, \\ \dfrac{\partial I_A}{\partial \nu} = \dfrac{\partial I_S}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial\mathbb{D}. \end{cases} \tag{17}$$

Define by $\mathcal{T}(t)$ the solution of (17), that is, $\mathcal{T}(t)\phi = (I_A(\cdot, t, \phi), I_S(\cdot, t, \phi), B(\cdot, t, \phi))$, where $\phi \in C(\overline{\mathbb{D}}, \mathbb{R}^3)$. Since (17) is cooperative, $\mathcal{T}(t)$ is a positive C_0-semigroup with generator $\mathcal{B} := \mathcal{F} + \mathcal{V}$, where

$$\mathcal{F}(x) = \begin{pmatrix} 0 & 0 & \theta\beta(x) f_2'(S^0(x), 0) + \sigma\beta(x) g_2'(V^0(x), 0) \\ 0 & 0 & (1-\theta)\beta(x) f'(S^0(x), 0) \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{V}(x) = \begin{pmatrix} D_3\Delta - (\mu(x) + r_A(x) + \delta(x)) & 0 & 0 \\ \delta(x) & D_4\Delta - (\mu(x) + r_S(x)) & 0 \\ \gamma_A(x) & \gamma_S(x) & -\alpha(x) \end{pmatrix}.$$

We denote $\tilde{\mathcal{T}}(t)$ be the positive semigroup generated $\mathcal{V}(x)$. By [29, Theorem 3.12], the next generator operator is defined by

$$\mathbf{L}(\phi)(x) = \int_0^\infty \mathcal{F}(x)\tilde{\mathcal{T}}(t)\phi(x)\mathrm{d}t = \mathcal{F}(x)\int_0^\infty \tilde{\mathcal{T}}(t)\phi(x)\mathrm{d}t, \quad \phi \in C(\overline{\mathbb{D}}, \mathbb{R}^3), \quad x \in \overline{\mathbb{D}}.$$

According to [32], we can define the spectral radius of $\mathbf{L}$ as the basic reproduction number as follows:

$$\mathcal{R}_0 := r(\mathbf{L}) = r(-\mathcal{F}\mathcal{V}^{-1}).$$

where $(\cdot)^{-1}$ denotes the inverse of $(\cdot)$. By computing the expression of $\mathcal{V}^{-1}(x)$, one finds

$$\mathcal{V}^{-1}(x) = \begin{pmatrix} (D_3\Delta - \pi_3(x))^{-1} & 0 & 0 \\ -Q_{21}(x) & (D_4 - \pi_4(x))^{-1} & 0 \\ Q_{31}(x) & \gamma_S(x)(\alpha(x))^{-1}(D_4\Delta - \pi_4(x))^{-1} & -(\alpha(x))^{-1} \end{pmatrix},$$

where

$$Q_{21}(x) = \delta(x)(D_3\Delta - \pi_3(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}, \quad \mathcal{H}(x) = \theta\beta(x) f_2'(S^0(x), 0) + \sigma\beta(x) g_2'(V^0(x), 0),$$
$$Q_{31}(x) = (\alpha(x))^{-1}(\gamma_A(x)(D_3\Delta - \pi_3(x))^{-1} - \gamma_S(x) Q_{21}(x)), \quad \mathcal{J}(x) = (1-\theta)\beta(x) f_2'(S^0(x), 0).$$

Hence, combining the expressions of $\mathcal{F}(x)$ and $\mathcal{V}^{-1}(x)$ yields

$$-\mathcal{F}(x)\mathcal{V}^{-1}(x) = -\begin{pmatrix} \mathcal{H}(x)Q_{31}(x) & \gamma_S(x)(\alpha(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}\mathcal{H}(x) & -(\alpha(x))^{-1}\mathcal{H}(x) \\ \mathcal{J}(x)Q_{31}(x) & \gamma_S(x)(\alpha(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}\mathcal{J}(x) & -(\alpha(x))^{-1}\mathcal{J}(x) \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$\mathcal{R}_0 := r(\mathbf{L}) = r(-(\mathcal{H}(x)Q_{31}(x) + \gamma_S(x)(\alpha(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}\mathcal{J}(x))).$$

This indicates that $\mathcal{R}_0$ is the principal eigenvalue of the below eigenvalue problem

$$\begin{cases} -(\mathcal{H}(x)Q_{31}(x) + \gamma_S(x)(\alpha(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}\mathcal{J}(x))\phi = \lambda\phi, & x \in \mathbb{D}, \\ \dfrac{\partial\phi(x)}{\partial\nu} = 0, & x \in \partial\mathbb{D}. \end{cases}$$

So there exists a strictly positive eigenfunction ϕ_* such that

$$\begin{cases} -(\mathcal{H}(x)Q_{31}(x) + \gamma_S(x)(\alpha(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}\mathcal{J}(x))\phi_* = \mathcal{R}_0\phi_*, & x \in \mathbb{D}, \\ \dfrac{\partial\phi_*(x)}{\partial\nu} = 0, & x \in \partial\mathbb{D}, \end{cases}$$

which is equivalent to

$$\begin{cases} -(\theta\beta(x)f_2'(S^0(x),0) + \sigma\beta(x)g_2'(V^0(x),0)(\alpha(x))^{-1}(\gamma_A(x)(D_3\Delta - \pi_3(x))^{-1} \\ \qquad - \gamma_S(x)\delta(x)(D_3\Delta - \pi_3(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}) + \gamma_S(x)(\alpha(x))^{-1} \\ \qquad \times (D_4\Delta - \pi_4(x))^{-1}(1-\theta)\beta(x)f_2'(S^0(x),0))\phi_* = \mathcal{R}_0\phi_*, & x \in \mathbb{D}, \\ \dfrac{\partial\phi_*(x)}{\partial\nu} = 0, & x \in \partial\mathbb{D}. \end{cases} \tag{18}$$

Let's further consider the following eigenvalue problem

$$\begin{cases} -(\theta\beta(x)f_2'(S^0(x),0) + \sigma\beta(x)g_2'(V^0(x),0) \times (\alpha(x))^{-1}\gamma_A(x)(D_3\Delta - \pi_3(x))^{-1})\psi = \tilde{\lambda}\psi, & x \in \mathbb{D}, \\ \dfrac{\partial\psi(x)}{\partial\nu} = 0, & x \in \partial\mathbb{D}, \end{cases} \tag{19}$$

where $\tilde{\lambda}$ is the eigenvalue and associated with the positive eigenfunction ψ. It is known from [39] that the problem (19) has a principal eigenvalue $\tilde{\lambda}^*$ and corresponds to a positive eigenfunction ψ_* satisfying

$$\begin{cases} -(\theta\beta(x)f_2'(S^0(x),0) + \sigma\beta(x)g_2'(V^0(x),0) \times (\alpha(x))^{-1}\gamma_A(x)(D_3\Delta - \pi_3(x))^{-1})\psi_* = \tilde{\lambda}^*\psi_*, & x \in \mathbb{D}, \\ \dfrac{\partial\psi_*(x)}{\partial\nu} = 0, & x \in \partial\mathbb{D}. \end{cases} \tag{20}$$

Multiplying the first equation of the problem (20) by ψ_* and integrating over $\mathbb{D}$ yields

$$-\int_{\mathbb{D}}\gamma_A(x)(\alpha(x))^{-1}(\theta\beta(x)f_2'(S^0(x),0) + \sigma\beta(x)g_2'(V^0(x),0))\psi_*^2\mathrm{d}x = -\tilde{\lambda}^*\int_{\mathbb{D}}(D_3|\nabla\psi_*|^2 + \pi_3(x)\psi_*^2)\mathrm{d}x. \tag{21}$$

This shows that

$$\tilde{\lambda}^* = \frac{\int_{\mathbb{D}}\gamma_A(x)(\theta\beta(x)f_2'(S^0(x),0) + \sigma\beta(x)g_2'(V^0(x),0))\psi_*^2\mathrm{d}x}{\int_{\mathbb{D}}\alpha(x)(D_3|\nabla\psi_*|^2 + \pi_3(x)\psi_*^2)\mathrm{d}x}. \tag{22}$$

In addition, we consider the following eigenvalue problem

$$\begin{cases} \gamma_S(x)(\alpha(x))^{-1}(D_4\Delta - \pi_4(x))^{-1}((\theta\beta(x)f_2'(S^0(x),0) + \sigma\beta(x)g_2'(V^0(x),0)) \times \delta(x)(D_3\Delta - \pi_3(x))^{-1} \\ \qquad - (1-\theta)\beta(x)f_2'(S^0(x),0))\varphi = \bar{\lambda}\varphi, & x \in \mathbb{D}, \\ \dfrac{\partial\varphi(x)}{\partial\nu} = 0, & x \in \partial\mathbb{D}, \end{cases} \tag{23}$$

which is equivalent to

$$\begin{cases} \gamma_S(x)(\alpha(x))^{-1}((\theta\beta(x)f_2'(S^0(x),0)+\sigma\beta(x)g_2'(V^0(x),0))\delta(x)(D_3\Delta-\pi_3(x))^{-1} \\ -(1-\theta)\beta(x)f_2'(S^0(x),0))\varphi=\bar{\lambda}(D_4\Delta-\pi_4(x))\varphi, \quad x\in\mathbb{D}, \\ \dfrac{\partial\varphi(x)}{\partial\nu}=0, \quad x\in\partial\mathbb{D}. \end{cases}$$

Similar to the method of (21), one derives

$$\begin{aligned} \bar{\lambda}^*=&\frac{\int_{\mathbb{D}}\gamma_S(x)(1-\theta)\beta(x)f_2'(S^0(x),0)\varphi_*^2\mathrm{d}x}{\int_{\mathbb{D}}\alpha(x)(D_4|\nabla\varphi_*|^2+\pi_4(x)\varphi_*^2)\mathrm{d}x} \\ &-\frac{\int_{\mathbb{D}}\gamma_S(x)\delta(x)(D_3\Delta-\pi_3(x))^{-1}(\theta\beta(x)f_2'(S^0(x),0)+\sigma\beta(x)g_2'(V^0(x),0))\varphi_*^2\mathrm{d}x}{\int_{\mathbb{D}}\alpha(x)(D_4|\nabla\varphi_*|^2+\pi_4(x)\varphi_*^2)\mathrm{d}x}. \end{aligned} \tag{24}$$

Next, consider the following eigenvalue problem

$$\begin{cases} -(\gamma_S(x)\delta(x)(D_3\Delta-\pi_3(x))^{-1}(\theta\beta(x)f_2'(S^0(x),0)+\sigma\beta(x)g_2'(V^0(x),0)))\chi=\hat{\lambda}\chi, \quad x\in\mathbb{D}, \\ \dfrac{\partial\chi(x)}{\partial\nu}=0, \quad x\in\partial\mathbb{D}, \end{cases}$$

where $\hat{\lambda}$ is the eigenvalue and χ is the corresponding eigenfunction. Similarly, one can also conclude that

$$\hat{\lambda}^*=\frac{\int_{\mathbb{D}}\gamma_S(x)\delta(x)(\theta\beta(x)f_2'(S^0(x),0)+\sigma\beta(x)g_2'(V^0(x),0))\chi_*^2\mathrm{d}x}{\int_{\mathbb{D}}(D_3|\nabla\chi_*|^2+\pi_3(x)\chi_*^2)\mathrm{d}x}. \tag{25}$$

From (22)–(25) and multiplying the first equation of problem (18) by $\psi_*\varphi_*\chi_*$, one obtains

$$\begin{aligned} \mathcal{R}_0=&\frac{\int_{\mathbb{D}}\gamma_S(x)(1-\theta)\beta(x)f_2'(S^0(x),0)\psi_*^2\phi_*\varphi_*\chi_*\mathrm{d}x}{\int_{\mathbb{D}}\alpha(x)(D_4|\nabla\psi_*|^2+\pi_4(x)\psi_*^2)\mathrm{d}x}+\frac{\int_{\mathbb{D}}(\theta\beta(x)f_2'(S^0(x),0)+\sigma\beta(x)g_2'(V^0(x),0))\mathrm{d}x}{\int_{\mathbb{D}}\alpha(x)(D_3|\nabla\varphi_*|^2+\pi_3(x)\varphi_*^2)\mathrm{d}x} \\ &\times\left(\int_{\mathbb{D}}\gamma_A(x)\varphi_*^2\phi_*\psi_*\chi_*\mathrm{d}x+\frac{\int_{\mathbb{D}}\gamma_S(x)\delta(x)\chi_*^2\varphi_*^2\phi_*\psi_*\mathrm{d}x}{\int_{\mathbb{D}}(D_4|\nabla\chi_*|^2+\pi_4(x)\chi_*^2)\mathrm{d}x}\right), \quad \int_{\mathbb{D}}\phi_*\psi_*\varphi_*\chi_*\mathrm{d}x=1. \end{aligned}$$

Therefore, we finally obtain

$$\begin{aligned} \mathcal{R}_0=\sup_{\phi,\psi,\varphi,\chi\in H^1(\mathbb{D}),\ \phi,\psi,\varphi,\chi\neq0}&\left\{\frac{\int_{\mathbb{D}}\gamma_S(x)(1-\theta)\beta(x)f_2'(S^0(x),0)\psi^2\phi\varphi\chi\mathrm{d}x}{\int_{\mathbb{D}}\alpha(x)(D_4|\nabla\psi|^2+\pi_4(x)\psi^2)\mathrm{d}x}\right. \\ &+\frac{\int_{\mathbb{D}}(\theta\beta(x)f_2'(S^0(x),0)+\sigma\beta(x)g_2'(V^0(x),0))\varphi^2\mathrm{d}x}{\int_{\mathbb{D}}\alpha(x)(D_3|\nabla\varphi|^2+\pi_3(x)\varphi^2)\mathrm{d}x} \\ &\left.\times\left(\int_{\mathbb{D}}\gamma_A(x)\phi\psi\chi\mathrm{d}x+\frac{\int_{\mathbb{D}}\gamma_S(x)\delta(x)\chi^2\phi\psi\mathrm{d}x}{\int_{\mathbb{D}}(D_4|\nabla\chi|^2+\pi_4(x)\chi^2)\mathrm{d}x}\right)\right\}. \end{aligned}$$

Note that in case the diffusion coefficients are all 0, i.e., $D_i=0\ (i=1,\dots,4)$, the local reproduction number $\tilde{\mathcal{R}}_0(x)$ is denoted in the following form

$$\begin{aligned}\tilde{\mathcal{R}}_0(x) &= \frac{\gamma_S(x)(1-\theta)f_2'(S^0(x),0)}{\alpha(x)(\mu(x)+r_S(x))} + \frac{\theta\beta(x)f_2'(S^0(x),0)+\sigma\beta(x)g_2'(V^0(x),0)}{\alpha(x)(\mu(x)+r_A(x)+\delta(x))}\left(\gamma_A(x)+\frac{\gamma_S(x)\delta(x)}{\mu(x)+r_S(x)}\right)\\ &= \mathcal{R}_1(x)+\mathcal{R}_2(x)+\mathcal{R}_3(x),\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_1(x) &= \frac{\gamma_S(x)\beta(x)(1-\theta)f_2'(S^0(x),0)}{\alpha(x)(\mu(x)+r_S(x))},\quad \mathcal{R}_2(x) = \frac{\gamma_A(x)\beta(x)(\theta f_2'(S^0(x),0)+\sigma g_2'(V^0(x),0))}{\alpha(x)(\mu(x)+r_A(x)+\delta(x))},\\ \mathcal{R}_3(x) &= \frac{\gamma_S(x)\delta(x)\beta(x)(\theta f_2'(S^0(x),0)+\sigma g_2'(V^0(x),0))}{\alpha(x)(\mu(x)+r_S(x))(\mu(x)+r_A(x)+\delta(x))}.\end{aligned}$$

For the relation between $\mathcal{R}_0$ and $\tilde{\mathcal{R}}_0(x)$, it holds by the following conclusions.

Remark 1. The following facts were established.
(i) If $\mathcal{R}_j(x)$ $(j=1,2,3)$ are constants, then $\mathcal{R}_0 = \tilde{\mathcal{R}}_0(x)$.
(ii) If $\min_{1\leqslant j\leqslant 3}\{\mathcal{R}_j(x)\} > 1$, then $\tilde{\mathcal{R}}_0(x) > 1$ holds.

By substituting $I_A(x,t) = e^{\lambda t}\psi_3(x)$, $I_S(x,t) = e^{\lambda t}\psi_4(x)$, and $B(x,t) = e^{\lambda t}\psi_5(x)$ into model (17), we obtain the eigenvalue problem as follows:

$$\begin{cases}\lambda\psi_3(x) = D_3\Delta\psi_3(x) + \theta\beta(x)f_2'(S^0(x),0)\psi_5(x) + \sigma\beta(x)g_2'(V^0(x),0)\psi_5(x) - (\mu(x)+r_A(x)+\delta(x))\psi_3(x),\\ x\in\mathbb{D},\\ \lambda\psi_4(x) = D_4\Delta\psi_4(x) + (1-\theta)\beta(x)f_2'(S^0(x),0)\psi_5(x) + \delta(x)\psi_3(x) - (\mu(x)+r_S(x))\psi_4(x), \quad x\in\mathbb{D},\\ \lambda\psi_5(x) = \gamma_A(x)\psi_3(x) + \gamma_S(x)\psi_4(x) - \alpha(x)\psi_5(x), \quad x\in\mathbb{D},\\ \dfrac{\partial\psi_3(x)}{\partial\nu} = \dfrac{\partial\psi_4(x)}{\partial\nu} = 0, \quad x\in\partial\mathbb{D}.\end{cases} \tag{26}$$

According to [29] and [32, Lemma 2.2], it is straightforward to derive the following result.

Lemma 4. *$\mathcal{R}_0 - 1$ has the same sign $s(\mathcal{B})$, where $s(\mathcal{B}) = \sup\{|\lambda|, \lambda\in\sigma(\mathbf{L})\}$ is the spectral bounded of $\mathcal{B}$.*

Lemma 5. *If $\mathcal{R}_0 \geqslant 1$ $(s(\mathcal{B})\geqslant 0)$, then $s(\mathcal{B})$ is the principal eigenvalue of the problem (26) associated with a strongly positive eigenfunction.*

Proof. From (17), one can derive

$$\begin{cases} I_A(x,t,\phi) = T_3(t)\phi(t) + \displaystyle\int_0^t T_3(t-s)\mathcal{P}_1(B(x,s,\phi))\mathrm{d}s,\\ I_S(x,t,\phi) = T_4(t)\phi(t) + \displaystyle\int_0^t T_4(t-s)\mathcal{P}_2(I_A(x,s,\phi),B(x,s,\phi))\mathrm{d}s,\\ B(x,t,\phi) = T_5(t)\phi(t) + \displaystyle\int_0^t T_5(t-s)(\gamma_A(x)I_A(x,s,\phi)+\gamma_S(x)I_S(x,s,\phi))\mathrm{d}s,\end{cases} \tag{27}$$

where $\mathcal{P}_1(B) = \theta\beta_1(x)f_2'(S^0(x),0)B + \sigma\beta_2(x)g_2'(V^0(x),0)B$ and $\mathcal{P}_2(I_A,B) = (1-\theta)\beta_1(x)f_2'(S^0(x),0)B + \delta(x)I_A$.

Let $\tilde{\mathcal{T}}(t)$ as $\tilde{\mathcal{T}}(t) = \tilde{\mathcal{T}}_3(t) + \tilde{\mathcal{T}}_4(t)$, where $\tilde{\mathcal{T}}_3(t)\phi = (0,0,\Gamma_5(t)\phi_5)$ and

$$\tilde{\mathcal{T}}_4(t)\phi = \left(I_A(x,t,\phi), I_S(x,t,\phi), \int_0^t T_5(t-s)(\gamma_A(x)I_A(x,s,\phi)+\gamma_S(x)I_S(x,s,\phi))\mathrm{d}s\right).$$

According to [39, Lemma 2.5], one has $\tilde{\mathcal{T}}_4(t)$ is compact. From Lemma 3, $\tilde{\mathcal{T}}(t)$ is the τ-contraction on, i.e., the essential growth bound $\omega_{\text{ess}}(\tilde{\mathcal{T}}(t)) = \lim_{t\to\infty} \ln \vartheta(\mathcal{T}(t))/t$, ϑ is the measure of non-compactness $\omega_{\text{ess}}(\tilde{\mathcal{T}}(t)) \leqslant -\alpha_*$ and the essential spectral radius $r_{\text{ess}}(\tilde{\mathcal{T}}(t)) \leqslant e^{-\alpha_* t} < 1$, $t > 0$. It is well known [11] that

$$\omega(\tilde{\mathcal{T}}(t)) = \max\{s(\mathcal{B}), \quad \omega_{\text{ess}}(\tilde{\mathcal{T}}(t))\},$$

where $\omega(\tilde{\mathcal{T}}(t))$ is the exponential growth bound of $\tilde{\mathcal{T}}(t)$ defined as $\omega(\tilde{\mathcal{T}}(t)) = \lim_{t\to\infty} \ln \|\tilde{\mathcal{T}}(t)\|/t$. Under the assumption that $s(\mathcal{B}) \geqslant 0$, the spectral radius of $\tilde{\mathcal{T}}$ satisfies

$$r(\tilde{\mathcal{T}}(t)) = e^{s(\mathcal{B})t} \geqslant 1, \quad t > 0.$$

As a result, $r_{\text{ess}}(\tilde{\mathcal{T}}(t)) < r(\tilde{\mathcal{T}}(t)), t > 0$. By the aid of the generalized Krein-Rutman theorem (see [25, lemma 2.2], we can conclude that $s(\mathcal{B})$ is the principal eigenvalue of (26), which is associated with a strong positive eigenfunction. □

5 Stability of steady states

This section focuses on the analysis of the threshold dynamics of model (1) on $\mathcal{R}_0$. We start by justifying the stability of $\mathcal{E}^0$ when $\mathcal{R}_0 < 1$.

Theorem 1. *If* $\mathcal{R}_0 < 1$ $(s(\mathcal{B}) < 0)$, *then* $\mathcal{E}^0$ *is globally asymptotically stable.*

Proof. Similar to [32, Theorem 3.1], it is clear that $\mathcal{E}^0$ is locally asymptotically stable. Therefore, we just have to prove the global attraction of $\mathcal{E}^0$. Following Lemma 4 reveals that when $\mathcal{R}_0 < 1$, $s(\mathcal{B}(S^0(x), V^0(x))) < 0$. That means there exists a sufficiently small positive number ε_0 such that $s(\mathcal{B}(S^0(x) + \varepsilon_0, V^0(x)) + \varepsilon_0) < 0$. According to Theorem 3, $S(x,t)$ and $V(x,t)$ are ultimately bounded, i.e., there exists $t_1 > 0$ such that $0 \leqslant S(x,t) \leqslant S^0(x) + \varepsilon_0$ and $0 \leqslant V(x,t) \leqslant V^0(x) + \varepsilon_0$, for $t \geqslant t_1$. From the comparison principle [21] yields $(I_A(x,t), I_S(x,t), B(x,t)) \leqslant (\hat{I}_A(x,t), \hat{I}_S(x,t), \hat{B}(x,t))$ on $\overline{\mathbb{D}} \times [0,\infty)$, where $(\hat{I}_A(x,t), \hat{I}_S(x,t), \hat{B}(x,t))$ satisfies

$$\begin{cases} \dfrac{\partial \hat{I}_A(x,t)}{\partial t} = D_3 \Delta \hat{I}_A(x,t) + \theta\beta(x) f_2'(S^0 + \varepsilon_0, 0)\hat{B}(x,t) + \sigma\beta(x) g_2'(V^0(x) + \varepsilon_0, 0)\hat{B}(x,t) \\ \qquad - (\mu(x) + r_A(x) + \delta(x))\hat{I}_A(x,t), \quad x \in \mathbb{D}, \quad t \geqslant t_1, \\ \dfrac{\partial \hat{I}_S(x,t)}{\partial t} = D_4 \Delta \hat{I}_S(x,t) + (1-\theta)\beta(x) f_2'(S^0(x) + \varepsilon_0, 0)\hat{B}(x,t) + \delta(x)\hat{I}_A(x,t) - (\mu(x) + r_S(x))\hat{I}_S(x,t), \\ x \in \mathbb{D}, \quad t \geqslant t_1, \\ \dfrac{\partial \hat{B}(x,t)}{\partial t} = \gamma_A(x)\hat{I}_A(x,t) + \gamma_S(x)\hat{I}_S(x,t) - \alpha(x)\hat{B}(x,t), \quad x \in \mathbb{D}, \quad t \geqslant t_1, \\ \dfrac{\partial \hat{I}_A(x,t)}{\partial \nu} = \dfrac{\partial \hat{I}_S(x,t)}{\partial \nu} = 0, \quad x \in \overline{\mathbb{D}}, \quad t \geqslant t_1, \\ \hat{I}_A(x,t) = I_A(x,t_1), \quad \hat{I}_S(x,t) = I_S(x,t_1), \quad x \in \mathbb{D}. \end{cases} \tag{28}$$

Choose a constant $\iota > 0$ such that $(I_A(x,t_1,\phi), I_S(x,t_1,\phi), B(x,t_1,\phi)) \leqslant \iota(\psi_3^{\varepsilon_0}(x), \psi_4^{\varepsilon_0}(x), \psi_5^{\varepsilon_0}(x))$ for $x \in \overline{\mathbb{D}}$, where $(\psi_3^{\varepsilon_0}(x), \psi_4^{\varepsilon_0}(x), \psi_5^{\varepsilon_0}(x))$ is the corresponding eigenvector of $s(\mathcal{B}(S^0(x) + \varepsilon_0, V^0(x)) + \varepsilon_0)$ and $\iota e^{s(\mathcal{B}_{\varepsilon_0})(t-t_1)}(\psi_3^{\varepsilon_0}(x), \psi_4^{\varepsilon_0}(x), \psi_5^{\varepsilon_0}(x))$ is a solution of (28). By employing the comparison principle, one has

$$(I_A(x,t_1,\phi), I_S(x,t_1,\phi), B(x,t_1,\phi)) \leqslant \iota e^{s(\mathcal{B}_{\varepsilon_0})(t-t_1)}(\psi_3^{\varepsilon_0}(x), \psi_4^{\varepsilon_0}(x), \psi_5^{\varepsilon_0}(x)), \quad t \geqslant t_1.$$

Thus, $(I_A(x,t), I_S(x,t), B(x,t)) \to (0,0,0)$ as $t \to \infty$ uniformly for $x \in \overline{\mathbb{D}}$. Moreover, using the first and second equations of (1) yields $S(x,t) \to S^0(x)$, $V(x,t) \to V^0(x)$ as $t \to \infty$ uniformly for $x \in \overline{\mathbb{D}}$. This completes the proof. □

Next, we address the uniform persistence of the disease. Before the proof, we present a few lemmas that will be utilized afterward.

Lemma 6. *Let* $(S(x,t), V(x,t), I_A(x,t), I_S(x,t), B(x,t))$ *be the solution of* (1) *satisfying* $\phi = (S_0(x), V_0(x), I_{A0}(x), I_{S0}(x), B_0(x)) \in X^+$.

(*i*) *For any* $\phi \in X^+$, *then* $S(x,t) > 0$ *and* $V(x,t) > 0$, *and there exists a positive constant* τ_0, *independent of* ϕ, *such that*

$$\liminf_{t\to\infty} S(x,t) > \tau_0, \quad \liminf_{t\to\infty} V(x,t) > \tau_0 \quad \textit{uniformly for} \quad x \in \overline{\mathbb{D}}.$$

(*ii*) *For any* $\phi \in X^+$, *if* $I_{A0}(x) \not\equiv 0$ *or* $I_{S0}(x) \not\equiv 0$ *or* $B_0(x) \not\equiv 0$, *then* $p(x,t,\phi) > 0$, *for* $\forall x \in \overline{\mathbb{D}}, t > 0$, *where* $p = I_A$, I_S, B.

Proof. (*i*) From Lemma 3, there is a constant $M_4 > 0$ such that $B(x,t) \leqslant M_4$, $\forall x \in \overline{\mathbb{D}}, t > 0$. It follows that the assumption (H_2) yields $f(S,B) \leqslant k_1 M_4 S$ and $g(V,B) \leqslant k_2 M_4 V$. In view of V equation of (1), we have

$$\frac{\partial V}{\partial t} \geqslant D_2 \Delta V - (\mu(x) + \eta(x) + \sigma k_2 \beta(x) M_4)V.$$

Hence, $V(x,t) \geqslant \tilde{V}(x,t)$, where $\tilde{V}(x,t)$ is a solution to the following problem:

$$\begin{cases} \dfrac{\partial \tilde{V}}{\partial t} = D_2 \Delta \tilde{V} + p(x)\Lambda(x) - (\mu(x) + \eta(x) + \sigma k_2 \beta(x) M_4)\tilde{V}, & x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial \tilde{V}}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \quad t > 0, \\ V(x,0) = \tilde{V}(x,0), \quad x \in \overline{\mathbb{D}}. \end{cases} \tag{29}$$

By the the maximum principle, $\tilde{V}(x,t) > 0$, $V(x,t) \geqslant \tilde{V}(x,t) > 0$ for any $x \in \overline{\mathbb{D}}$ and $t > 0$. Similarly, we can derive $S(x,t) \geqslant \tilde{S}(x,t) > 0$, where $\tilde{S}(x,t)$ is a solution to the following problem:

$$\begin{cases} \dfrac{\partial \tilde{S}}{\partial t} = D_1 \Delta \tilde{S} + (1 - p(x))\Lambda(x) - (\mu(x) + k_1 \beta(x) M_4)\tilde{S} + \eta \tilde{V}, & x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial \tilde{S}}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \\ S(x,0) = \tilde{S}(x,0), \quad x \in \overline{\mathbb{D}}. \end{cases} \tag{30}$$

Note that problem (29) and (30) possesses a unique steady state, defined by $\tilde{S}^0(x)$ and $\tilde{V}^0(x)$, from the proof of [17, Lemma 1]. Hence,

$$\liminf_{t\to\infty} S(x,t) \geqslant \inf_{x\in\overline{\mathbb{D}}} \tilde{S}^0(x) \coloneqq \tau_1, \quad \liminf_{t\to\infty} V(x,t) \geqslant \inf_{x\in\overline{\mathbb{D}}} \tilde{V}^0(x) \coloneqq \tau_2.$$

Choose $\tau \coloneqq \min\{\tau_1, \tau_2\}$. This concludes ($i$) the proof.

(ii) For $I_{A0}(x) \not\equiv 0$, from the third equation of model (1), one has

$$\begin{cases} \dfrac{\partial I_A}{\partial t} \geqslant D_3 \Delta I_A - (\mu(x) + r_A(x) + \delta(x))I_A, & x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial I_A}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \quad t > 0. \end{cases}$$

By [33, Lemma 1.26], it follows that $I_A(x,t) > 0$ for all $(x,t) \in \overline{\mathbb{D}} \times (0,\infty)$. Further, considering the third equation of model (1) yields

$$\begin{cases} \dfrac{\partial I_S}{\partial t} \geqslant D_4 \Delta I_S - (\mu(x) + r_S(x))I_S, & x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial I_S}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \quad t > 0. \end{cases}$$

Similarly, we can also derive $I_S(x,t)>0$ for $(x,t)\in\overline{\mathbb{D}}\times(0,\infty)$. Moreover, for fixed $x\in\overline{\mathbb{D}}$ and $t_0>0$, B equation of (1) yields

$$B(x,t)=e^{-\alpha(x)t}B(x,t_0)+\int_{t_0}^{t}e^{-\alpha(x)(t-s)}[\gamma_A(x)I_A(x,s,\phi)+\gamma_S(x)I_S(x,s,\phi)]\mathrm{d}s.$$

This is, $B(x,t)>0$ for $(x,t)\in\overline{\mathbb{D}}\times[t_0,\infty)$. For the cases $I_{S0}(x)\not\equiv0$ or $B_0(x)\not\equiv0$, the proof procedure is similar to $I_{A0}(x)\not\equiv0$, which we omit here. Thus, Lemma 6(ii) hold. □

Lemma 7. *If there exists $\tau_p>0$ such that*

$$\liminf_{t\to\infty}p(x,t,\phi)\geqslant\tau_p,\quad \textit{for}\quad p=I_A\quad\textit{or}\quad I_S\quad\textit{or}\quad B,\quad\textit{uniformly for}\quad x\in\mathbb{D},$$

then there exists $\sigma_p>0$ such that

$$\liminf_{t\to\infty}\tilde p(x,t,\phi)\geqslant\tau_p,\quad\textit{for}\quad\tilde p=S,V,I_A,I_S,B,\quad\textit{uniformly for}\quad x\in\mathbb{D}.\tag{31}$$

Proof. In the case of $p=I_A$, it is assumed that there exists $t_1^*>0$ such that $I_A(x,t)\geqslant1/2\tau_p$ for any $x\in\mathbb{D}$ and $t\in[t_1^*,\infty)$. Therefore, I_S equation of (1) satisfies

$$\begin{cases}\dfrac{\partial I_S}{\partial t}\geqslant D_4\Delta I_S+\dfrac{1}{2}\tau_p\delta(x)-(\mu(x)+r_S(x))I_S,\quad x\in\mathbb{D},\quad t\geqslant t_1^*,\\ \dfrac{\partial I_S}{\partial v}=0,\quad x\in\partial\mathbb{D},\quad t\geqslant t_1^*.\end{cases}\tag{32}$$

By the comparison theorem, we learn that (32) there exists positive functions $\tilde I_S^*(x)$ such that

$$\liminf_{t\to\infty}I_S(x,t)\geqslant\tilde I_S^*(x).$$

This means that there exists $t_2^*>0$ such that $I_S(x,t)\geqslant\tilde I_S^*(x)/2$, for any $x\in\mathbb{D}$, $t\geqslant t_2^*$. According to the B equation of (1), one has

$$\begin{cases}\dfrac{\partial B}{\partial t}\geqslant\dfrac12\tau_p\gamma_A(x)+\dfrac12\gamma_S(x)\tilde I_S^*(x)-\alpha(x)B,\quad x\in\mathbb{D},\quad t>t_2^*,\\ \dfrac{\partial B}{\partial v}=0,\quad x\in\partial\mathbb{D},\quad t>t_2^*.\end{cases}\tag{33}$$

According to the comparison theorem, we learn that (32) there exists positive function $\tilde B_*(x)$ such that $\liminf_{t\to\infty}B(x,t)\geqslant\tilde B_*(x)$. Therefore, the conclusion (31) holds when $p=I_A(x,t)$.

In the other $p=B$ case, from Lemma 6(ii) and (33), we learn that there exists $t_2^*>0$ such that

$$S(x,t)\geqslant\frac12\tau_p,\quad V(x,t)\geqslant\frac12\tau_p,\quad B(x,t)\geqslant\frac12\tau_p,\quad\forall x\in\mathbb{D},\quad t\geqslant t_2^*.$$

As a result, the I_A equation of (1) fulfills

$$\begin{cases}\dfrac{\partial I_A}{\partial t}\geqslant D_3\Delta I_A+\dfrac{\theta\beta_*\tau_p}{2}f_2'\left(\dfrac{\tau_p}{2},\dfrac{\tau_p}{2}\right)+\dfrac{\sigma\beta_*\tau_p}{2}g'_2\left(\dfrac{\tau_p}{2},\dfrac{\tau_p}{2}\right)-(\mu(x)+r_A(x)+\delta(x))I_A,\quad x\in\mathbb{D},\quad t\geqslant t_2^*,\\ \dfrac{\partial I_A}{\partial n}=0,\quad x\in\partial\mathbb{D},\quad t\geqslant t_2^*.\end{cases}$$

By the comparison principle, we obtain $\liminf_{t\to\infty}I_A(x,t)\geqslant\tilde I_A^*(x)$, where $\tilde I_A^*(x)$ is a positive number. Thus, there exists $t_3^*>0$ such that $I_A(x,t)\geqslant\tilde I_A^*(x)/2$, for any $x\in\mathbb{D}$ and $t>t_3^*$. By using the I_S equation from model (1), we can also derive

$$\begin{cases} \dfrac{\partial I_S}{\partial t} \geqslant D_4 \Delta I_S + \dfrac{1}{2}\delta(x)\tilde{I}_A^*(x) - (\mu(x) + r_S(x))I_S, & x \in \mathbb{D}, \quad t \geqslant t_3^*, \\ \dfrac{\partial I_S}{\partial n} = 0, \quad x \in \partial\mathbb{D}, \quad t \geqslant t_3^*, \end{cases}$$

By using the comparison principle again, one has $\liminf_{t\to\infty} I_S(x,t) \geqslant \tilde{I}_S^*(x)$. In a similar way to the demonstration above, the case $p = I_S$ can be verified. This proves (31). □

In the following, we present the uniform persistence of model (1).

Theorem 2. *If $\mathcal{R}_0 > 1$, then there exists $\sigma_* > 0$ such that for $\phi \in X^+$ with $\phi_3 \not\equiv 0$ or $\phi_4 \not\equiv 0$ or $\phi_5 \not\equiv 0$, the solution $\tilde{p} = (S(x,t,\phi), V(x,t,\phi), I_A(x,t,\phi), I_S(x,t,\phi), B(x,t,\phi))$ of (1) satisfies*

$$\liminf_{t\to\infty} \tilde{p}(x,t,\phi) \geqslant \sigma_* \quad \textit{uniformly for } x \in \overline{\mathbb{D}}.$$

In addition, the model possesses at least one positive steady state.

Proof. According to Lemmas 6 and 7 and the process of [28, Theorem 3], define the set

$$\begin{aligned} &\mathcal{W}_0 = \{\phi \in X^+ : \phi_3(x) \not\equiv 0\}, \quad \partial\mathcal{W}_0 = X^+ \backslash \mathcal{W}_0 = \{\phi \in X^+ : \phi_3(x) \equiv 0\}, \\ &\mathcal{M}_\partial = \{\phi \in \partial\mathcal{W}_0 : \Phi(t)\phi \in \partial\mathcal{W}_0, \ \forall t \geqslant 0\}, \end{aligned}$$

where $\Phi(t) : X^+ \to X^+$ is the semiflow generated by the solution of (1). For any $\phi \in \mathcal{W}_0$, it follows directly from Lemma 7 that $I_A(x,t) > 0$. This indicates that $\Phi(t)\mathcal{W}_0 \in \mathcal{W}_0$. Further, we will prove this theorem by the following claims.

- Claim I: For any $\phi \in \mathcal{M}_\partial$, the ω limit set $\omega(\phi)$ is the singleton $\{\mathcal{E}_0\}$.

For any $\phi \in \mathcal{M}_\partial$, one has $\Phi(t)\phi \in \partial\mathcal{M}_0$, i.e., $I_A(x,t,\phi) \equiv 0$, $x \in \mathbb{D}$, $\forall t \geqslant 0$. Thus, by I_A equation of model (1), one obtains

$$\theta\beta(x) f(S,B) + \sigma\beta(x) g(V,B) \equiv 0.$$

Combining the conditions (H_1) yields $B(x,t) \equiv 0$, $x \in \mathbb{D}$, $\forall t > 0$. Taking the aforementioned leads to the fourth equation of model (1) yields $I_S(x,t) \equiv 0$, $x \in \mathbb{D}$, $\forall t > 0$. Further, the first second equations of (1) yields

$$\liminf_{t\to\infty} S(x,t,\phi) = S^0(x), \quad \liminf_{t\to\infty} V(x,t,\phi) = V^0(x) \quad \text{uniformly for} \quad x \in \overline{\mathbb{D}}.$$

This proves Claim I.

Claim II: $\limsup_{t\to\infty} \|\Phi(t)\phi - \mathcal{E}^0\| \leqslant \tau_*$, $\forall \ \phi \in \mathcal{W}_0$.

By arguing by contradiction, we suppose that there exists $\phi \in \mathcal{W}_0$ such that

$$\limsup_{t\to\infty} \|\Phi(t)\phi - \mathcal{E}^0\| < \tau_*.$$

So there exists $t_1 > 0$ such that for any $x \in \overline{\mathbb{D}}$ and $t \geqslant t_1$,

$$S(x,t,\phi) > S^0(x) - \tau_*, \quad V(x,t,\phi) > V^0(x) - \tau_*, \quad I_A(x,t,\phi) < \tau_*, \quad I_S(x,t,\phi) < \tau_*, \quad B(x,t,\phi) < \tau_*.$$

Thus, one has

$$\begin{cases} \dfrac{\partial I_A}{\partial t} \geqslant D_3 \Delta I_A + \theta\beta(x) f_2'(S^0(x) - \tau_*, \tau_*)B + \sigma\beta g_2'(V^0(x) - \tau_*, \tau_*)B - (\mu(x) + r_A(x) + \delta(x))I_A, & x \in \mathbb{D}, \quad t \geqslant t_1 \\ \dfrac{\partial I_S}{\partial t} \geqslant D_4 \Delta I_S + (1-\theta)\beta(x) f_2'(S^0(x) - \tau_*, \tau_*)B + \delta(x) I_A - (\mu(x) + r_S(x))I_S, & x \in \mathbb{D}, \quad t \geqslant t_1, \\ \dfrac{\partial B}{\partial t} \geqslant \gamma_A(x) I_A + \gamma_S(x) I_S - \alpha(x) B, & x \in \mathbb{D}, \quad t \geqslant t_1, \\ \dfrac{\partial I_A}{\partial \nu} = \dfrac{\partial I_S}{\partial \nu} = 0, & x \in \partial\mathbb{D}, \quad t \geqslant t_1. \end{cases}$$

Assuming that the linear system

$$\begin{cases} \dfrac{\partial u_3}{\partial t} = D_3\Delta u_3 + \theta\beta(x)f_2'(S^0(x)-\tau_*,\tau_*)u_5 + \sigma\beta g_2'(V^0(x)-\tau_*,\tau_*)u_5 - (\mu(x)+r_A(x)+\delta(x))u_3, & x\in\mathbb{D}, \quad t\geqslant t_1 \\ \dfrac{\partial u_4}{\partial t} = D_4\Delta u_4 + (1-\theta)\beta(x)f_2'(S^0(x)-\tau_*,\tau_*)u_5 + \delta(x)u_3 - (\mu(x)+r_S(x))u_4, & x\in\mathbb{D}, \quad t\geqslant t_1, \\ \dfrac{\partial u_5}{\partial t} = \gamma_A(x)u_3 + \gamma_S(x)u_4 - \alpha(x)u_5, & x\in\mathbb{D}, \quad t\geqslant t_1, \\ \dfrac{\partial I_A}{\partial \nu} = \dfrac{\partial I_S}{\partial \nu} = 0, & x\in\partial\mathbb{D}, \quad t\geqslant t_1. \end{cases}$$

admits a solution $(u_3(x,t),u_4(x,t),u_5(x,t)) = e^{\lambda(S^0(x)-\tau_*,V^0(x)-\tau_*)(t-t_1)}$, where $\hat{\psi}(x) = (\hat{\psi}_3(x),\hat{\psi}_4(x),\hat{\psi}_5(x))$ corresponding to $\lambda_0(S^0(x)-\tau_*,V^0(x)-\tau_*)$ the corresponding eigenvector. Choosing $\xi>0$ such that $\xi(\hat{\psi}_3(x),\hat{\psi}_4(x),\hat{\psi}_5(x)) \leqslant (I_A(x,t_1),I_S(x,t_1),B(x,t_1))$ and by the comparison principle, we can obtain

$$(I_S(x,t,\phi),I_A(x,t,\phi),B(x,t,\phi)) \geqslant \xi e^{\lambda(S^0(x)-\hat{\sigma}_*,V^0(x)-\hat{\sigma}_*)(t-t_1)}\hat{\psi}(x), \quad \text{for} \quad x\in\mathbb{D}, \quad t\geqslant t_1.$$

By $\lambda(S^0(x)-\tau_*,V^0(x)-\tau_*)>0$, $\lim\limits_{t\to\infty}(I_A(x,t,\phi),I_S(x,t,\phi),B(x,t,\phi)) = (\infty,\infty,\infty)$,which contradicts Lemma 2.

Next, we define the continuous function $\varrho : X^+ \to [0,\infty)$ fulfills

$$\varrho(\phi)(x) := \min_{x\in\mathbb{D}}\{\phi_3(x)\}, \quad \phi\in X^+.$$

Evidently, $\varrho^{-1}(0,\infty)\in\mathcal{W}_0$ and ϱ possesses the property that either $\varrho(\phi)=0$ and $\phi\in\mathcal{M}_0$ or $\varrho(\phi)>0$, then $\varrho(\Phi(t)\phi)>0$. Hence, ϱ is a generalized distance function for the semiflow $\Phi(t): X^+\to X^+$ (see [28]). After the above analysis, we can derive that any forward orbit $\Phi(t)$ in $\mathcal{M}_\partial$ converges to $\mathcal{E}^0$, and $W^s(\mathcal{E}^0)\cap\mathcal{M}_0=\varnothing$, where $W^s(\mathcal{E}^0)$ is a stable subset of $\mathcal{E}^0$. Moreover, $\mathcal{E}^0$ is an isolated invariant set in X^+, and no set of $\{\mathcal{E}^0\}$ forms a circle in $\partial\mathcal{M}_0$. Based on [28, Theorem 3], there exists $\iota>0$ such that $\liminf_{t\to\infty}\varrho(\Phi(t)\phi)>\iota$, for $\phi\in\mathcal{M}_0$, which means $\liminf_{t\to\infty}I_A(x,t,\phi)\geqslant\iota$ for $\phi\in\mathcal{M}_0$. It follows from Lemma 7 that model (1) is uniformly persistent and by [20, Theorem 4.7] and Theorem 2, there exists at least one endemic steady state for model (1). This completes the proof. □

Theorem 3. *In case $\mathcal{R}_0 = 1$, then $\mathcal{E}^0$ is globally asymptotically stable.*

Proof. As a start, let us study the local stability of $\mathcal{E}^0$. Suppose that there exists $l>0$ such that $\|u_0-\mathcal{E}^0\|\leqslant l$, where $u_0=(S_0(x),V_0(x),I_{A0}(x),I_{S0}(x),B_0(x))\in X^+$. Define

$$\omega_1(x,t) = \frac{S(x,t)}{S^0(x)}-1, \quad \omega_2(x,t) = \frac{V(x,t)}{V^0(x)}-1, \quad b_1(t) = \max_{x\in\overline{\mathbb{D}}}\{\omega_1(x,t),0\}, \quad b_2(t) = \max_{x\in\overline{\mathbb{D}}}\{\omega_2(x,t),0\}.$$

By $D_1\Delta S^0(x) + (1-p(x))\Lambda(x) - \mu(x)S^0(x) + \eta(x)V^0(x) = 0$, $D_2\Delta V^0(x) + p(x)\Lambda(x) - (\mu(x)+\eta(x))V^0(x) = 0$, one has

$$\begin{aligned} &\frac{\partial\omega_1}{\partial t} - D_1\Delta\omega_1 - 2D_1\frac{\nabla S^0(x)\nabla\omega_1}{S^0(x)} + \frac{(1-p(x))\Lambda(x)+\eta(x)V^0(x)}{S^0(x)}\omega_1 = -\frac{\beta(x)f(S,B)}{S^0(x)} + \frac{\eta(x)(V-V^0(x))}{S^0(x)}, \\ &\frac{\partial\omega_2}{\partial t} - D_2\Delta\omega_2 - 2D_2\frac{\nabla V^0(x)\nabla\omega_2}{V^0(x)} + \frac{p(x)\Lambda(x)}{V^0(x)}\omega_2 = -\frac{\sigma\beta(x)g(V,B)}{V^0(x)}. \end{aligned}$$

Denote $\tilde{T}_1(t)$ and $\tilde{T}_2(t)$ be the positive semigroups generated by the below operators, respectively

$$D_1\Delta + 2D_1\frac{\nabla S^0(x)\nabla}{S^0(x)} - \frac{(1-p(x))\Lambda(x)+\eta(x)V^0(x)}{S^0(x)}, \quad D_2\Delta + 2D_2\frac{\nabla V^0(x)\nabla}{V^0(x)} - \frac{p(x)\Lambda(x)}{V^0(x)}.$$

Then for some $M_i (i=1,2)$, there has $\tilde{\rho}_i>0$ such that $\|\tilde{T}_i\|\leq M_ie^{-\tilde{\rho}_it}$. Hence,

$$\omega_1(x,t) = \tilde{T}_1(t)\omega_{10}(x) - \int_0^t \tilde{T}_1(t-s)\left[\frac{\beta(x)f(S(x,s),B(x,s))}{S^0(x)} - \frac{\eta(x)(V(x,s)-V^0(x))}{S^0(x)}\right]\mathrm{d}s,$$
$$\omega_2(x,t) = \tilde{T}_2(t)\omega_{20}(x) - \int_0^t \tilde{T}_2(t-s)\frac{\sigma\beta(x)g(V(x,s),B(x,s))}{V^0(x)}\mathrm{d}s,$$

where $\omega_{10}(x) = S_0(x)/S^0(x) - 1$ and $\omega_{20}(x) = V_0(x)/V^0(x) - 1$. Then, it follows from the positivity of $\tilde{T}_i(t)$ $(i = 1, 2)$ that

$$b_2(t) = \max_{x\in\overline{\mathbb{D}}}\{\tilde{T}_2(t)\omega_{20}(x), 0\} \leqslant \|\tilde{T}_2(t)\omega_{20}(x)\| \leqslant \frac{lM_2e^{-\tilde{\rho}_2 t}}{V_m} := H_2e^{-\tilde{\rho}_2 t},$$
$$\begin{aligned} b_1(t) &= \max_{x\in\overline{\mathbb{D}}}\left\{\tilde{T}_1(t)\omega_{10}(x) - \int_0^t \tilde{T}_1(t-s)\left[\frac{\beta(x)f(S(x,s),B(x,s))}{S^0(x)} - \frac{\eta(x)(V(x,s)-V^0(x))}{S^0(x)}\right]\mathrm{d}s, 0\right\} \\ &\leqslant \left\|\tilde{T}_1(t)\omega_{10}(x) + \int_0^t \tilde{T}_1(t-s)\frac{\eta(x)(V(x,s)-V^0(x))}{S^0(x)}\mathrm{d}s\right\| \\ &\leqslant \frac{M_1}{S_m}\left(l + \frac{\eta^* H_2}{V_m}\left|\frac{e^{-(\tilde{\rho}_1-\tilde{\rho}_2)t}}{\tilde{\rho}_1-\tilde{\rho}_2} - 1\right|\right)e^{-\tilde{\rho}_1 t} := H_1e^{-\tilde{\rho}_1 t}, \end{aligned}$$

where $S_m = \min_{x\in\overline{\mathbb{D}}}\{S^0(x)\}$ and $V_m = \min_{x\in\overline{\mathbb{D}}}\{V^0(x)\}$. Note that (I_A, I_S, B) satisfies

$$\begin{cases} \dfrac{\partial I_A}{\partial t} = D_3\Delta I_A + \theta\beta(x)f_2'(S^0(x),0)B + \sigma\beta(x)g_2'(V^0(x),0)B - (\mu(x) + r_A(x) + \delta(x))I_A + \mathcal{G}_1(x,t), \\ \dfrac{\partial I_S}{\partial t} = D_4\Delta I_S + (1-\theta)\beta(x)f_2'(S^0(x),0)B + \delta(x)I_A - (\mu(x) + r_S(x))I_S + \mathcal{G}_2(x,t), \\ \dfrac{\partial B}{\partial t} = \gamma_A(x)I_A + \gamma_S(x)I_S - \alpha(x)B, \end{cases} \tag{34}$$

for $x \in \mathbb{D}$, $t > 0$ and $\partial I_A/\partial\nu = \partial I_S/\partial\nu = 0$, $x \in \partial\mathbb{D}$, where

$$\mathcal{G}_1(x,t) = \theta\beta(x)[f(S,B) - f_2'(S^0(x),0)B] + \sigma\beta(x)[g(V,B) - g_2'(V^0(x),0)],$$
$$\mathcal{G}_2(x,t) = (1-\theta)\beta(x)[f(S,B) - f_2'(S^0(x),0)B].$$

Then we can derive

$$\begin{aligned} \begin{pmatrix} I_A \\ I_S \\ B \end{pmatrix} &\leqslant \mathcal{T}(t)\begin{pmatrix} I_{A0} \\ I_{S0} \\ B_0 \end{pmatrix} + \int_0^t \mathcal{T}(t-s)\begin{pmatrix} \theta\beta(x)(f_2'(S(x,s),0) - f_2'(S^0(x),0))B(x,s) + Q(x) \\ (1-\theta)\beta(x)(f_2'(S(x,s),0) - f_2'(S^0(x),0))B(x,s) \\ 0 \end{pmatrix} \\ &\leqslant \mathcal{T}(t)\begin{pmatrix} I_{A0} \\ I_{S0} \\ B_0 \end{pmatrix} + \int_0^t \mathcal{T}(t-s)\begin{pmatrix} \theta\beta(x)L_1|S(x,s) - S^0(x)| + \theta\beta(x)L_2|V(x,s) - V^0(x)|B \\ (1-\theta)\beta(x)L_1|S(x,s) - S^0(x)|B \\ 0 \end{pmatrix}, \end{aligned}$$

where $Q(x) = \sigma\beta(x)(g_2'(V(x,s),0) + g_2'(V^0(x),0))B(x,s)$, L_1 and L_2 are Lipschitz constants for f_2' and g_2', respectively. By taking Lemma 4 and $\mathcal{R}_0 = 1$, one derives $s(\mathcal{B}) = 0$; thus, there exists $\bar{M} > 0$ such that $\|\mathcal{T}(t)\| \leqslant \bar{M}$, $t \geqslant 0$. From $b_1(s) \leqslant H_1e^{-\tilde{\rho}_1 s}$ and $b_2(s) \leqslant H_2e^{-\tilde{\rho}_2 s}$, one has

$$\begin{aligned} &\max\{\|I_A(x,t)\|, \|I_S(x,t)\|, \|B(x,t)\|\} \\ &\leqslant \bar{M}l + \bar{M}\theta\beta^* L_1\|S^0(x)\|H_1\int_0^t e^{-\tilde{\rho}_1 s}\|B(x,s)\|\mathrm{d}s + \bar{M}\sigma\beta^* L_2\|V^0(x)\|H_2\int_0^t e^{-\tilde{\rho}_2 s}\|B(x,s)\|\mathrm{d}s \\ &\leqslant \bar{M}l + (N_1 + N_2)\int_0^t e^{-\tilde{\rho}s}\|B(x,s)\|\mathrm{d}s, \end{aligned} \tag{35}$$

where $N_1 = \bar{M}\theta\beta^* L_1\|S^0(x)\|H_1$, $N_2 = \bar{M}\sigma\beta^* L_2\|V^0(x)\|H_2$ and $\tilde{\rho} = \min\{\tilde{\rho}_1, \tilde{\rho}_2\}$. Further, by using Gronwall's inequality, one can derive

$$\|B(x,t)\| \leqslant \bar{M}le^{\int_0^t (N_1+N_2)e^{-\tilde{\rho}s}\mathrm{d}s} \leqslant \bar{M}le^{\frac{N_1+N_2}{\tilde{\rho}}}. \tag{36}$$

By combining (35) and (36), we can also derive

$$\begin{aligned} \|I_A(x,t)\| &\leqslant \bar{M}l + (N_1+N_2)\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\int_0^t e^{-\tilde{\rho}s}\mathrm{d}s \leqslant \bar{M}l\left(1 + \frac{(N_1+N_2)}{\tilde{\rho}}e^{(N_1+N_2)/\tilde{\rho}}\right), \\ \|I_S(x,t)\| &\leqslant \bar{M}l + (N_1+N_2)\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\int_0^t e^{-\tilde{\rho}s}\mathrm{d}s \leqslant \bar{M}l\left(1 + \frac{(N_1+N_2)}{\tilde{\rho}}e^{(N_1+N_2)/\tilde{\rho}}\right). \end{aligned} \tag{37}$$

As a result, by (36), $(S(x,t), V(x,t)) \geqslant (\hat{S}(x,t), \hat{V}(x,t))$, where $(\hat{S}(x,t), \hat{V}(x,t))$ satisfies

$$\begin{cases} \dfrac{\partial \hat{S}}{\partial t} = D_1\Delta\hat{S} + (1-p(x))\Lambda(x) - \mu(x)\hat{S} - \beta(x)\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\hat{S} + \eta(x)\hat{V}, \quad x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial \hat{V}}{\partial t} = D_2\Delta\hat{V} + p(x)\Lambda(x) - (\mu(x)+\eta(x))\hat{V} - \sigma\beta(x)\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\hat{V}, \quad x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial \hat{S}}{\partial \nu} = \dfrac{\partial \hat{V}}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \quad t > 0, \\ \hat{S}(x,0) = S_0(x), \quad \hat{V}(x,0) = V_0(x), \quad x \in \mathbb{D}. \end{cases} \tag{38}$$

Let $(\hat{S}_l(x,t), \hat{V}_l(x,t))$ is the positive steady state of (38) and $\bar{S}(x,t) = \hat{S}(x,t) - \hat{S}_l(x)$, $\bar{V}(x,t) = \hat{V}(x,t) - \hat{V}_l(x)$. Then $(\bar{S}(x,t), \bar{V}(x,t))$ satisfies

$$\begin{cases} \dfrac{\partial \bar{S}}{\partial t} = D_1\Delta\bar{S} - (\mu(x) + \beta(x)\bar{M}le^{(N_1+N_2)\tilde{\rho}})\bar{S} + \eta(x)\bar{V}, \quad x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial \bar{V}}{\partial t} = D_2\Delta\bar{V} - ((\mu(x)+\eta(x)) - \sigma\beta(x)\bar{M}le^{(N_1+N_2)/\tilde{\rho}})\bar{V}, \quad x \in \mathbb{D}, \quad t > 0, \\ \dfrac{\partial \bar{S}}{\partial \nu} = \dfrac{\partial \bar{V}}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \quad t > 0, \\ \bar{S}(x,0) = S(x,0) - \hat{S}_l(x), \quad \bar{V}(x,0) = V(x,0) - \hat{V}_l(x), \quad x \in \mathbb{D}. \end{cases} \tag{39}$$

Let $\tilde{T}_i(t)$ $(i = 1,2)$ be the semigroup generated by $D_1\Delta - \mu(x)$ and $D_2\Delta - (\mu(x) + \eta(x))$, respectively, and satisfying the boundary conditions. We can choose a $M_i > 0$ such that $\|\tilde{T}_1(t)\| \leqslant M_1e^{-\mu_* t}$, $\|\tilde{T}_2(t)\| \leqslant M_2e^{-(\mu_*+\eta_*)t}$. By (39), one has

$$\begin{aligned} \bar{S}(x,t) &= \tilde{T}_1(t)(S_0(x) - \hat{S}_l(x)) - \int_0^t \tilde{T}_1(t-s)(\beta(x)\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\bar{S}(x,s) - \eta(x)\bar{V}(x,s))\mathrm{d}s, \\ \bar{V}(x,t) &= \tilde{T}_2(t)(V_0(x) - \hat{V}_l(x)) - \int_0^t \tilde{T}_2(t-s)\sigma\beta(x)\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\bar{V}(x,s)\mathrm{d}s. \end{aligned}$$

Moreover, $\bar{S}(x,t)$ and $\bar{V}(x,t)$ can be estimated as follows:

$$\begin{aligned} \|\bar{S}(x,t)\| &\leqslant M_1\|S_0(x) - \hat{S}_l(x)\|e^{-\mu_* t} + \int_0^t M_1e^{-\mu_*(t-s)}(\beta^*\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\|\bar{S}(x,s)\| + \eta^*\|\bar{V}(x,s)\|)\mathrm{d}s, \\ \|\bar{V}(x,t)\| &\leqslant M_2\|V_0(x) - \hat{V}_l(x)\|e^{-(\mu_*+\eta_*)t} + \int_0^t M_2e^{-(\mu_*+\eta_*)(t-s)}\sigma\beta^*\bar{M}le^{(N_1+N_2)/\tilde{\rho}}\|\bar{V}(x,s)\|\mathrm{d}s. \end{aligned}$$

Let $K_1 = M_1\bar{M}\beta^* le^{(N_1+N_2)/\tilde{\rho}}$ and $K_2 = M_2\bar{M}l\sigma\beta^* e^{(N_1+N_2)/\tilde{\rho}}$. After that, it is possible to use Gronwall's inequality to derive the preceding equation as follows:

$$\|\hat{S}(x,t)-\hat{S}_l(x)\| = \|\bar{S}(x,t)\| \leqslant \left(M_1\|S_0(x)-\hat{S}_l(x)\| + Q_m\|V_0(x)-\hat{V}_l(x)\|\int_0^t e^{(K_2-\eta_*)s}\mathrm{d}s\right)e^{K_1t-\mu_*t},$$
$$\|\hat{V}(x,t)-\hat{V}_l(x)\| = \|\bar{V}(x,t)\| \leqslant M_2\|V_0(x)-\hat{V}_l(x)\|e^{K_2t-(\mu_*+\eta_*)t},$$

where $Q_m = \eta^* M_1 M_2$. Selecting adequately small $l > 0$ such that $K_1 < \mu_*/2$, $K_2 < \eta_*/2$, one can launch

$$\begin{aligned}\|\hat{S}(x,t)-\hat{S}_l(x)\| = \|\bar{S}(x,t)\| &\leqslant (M_1\|S_0(x)-\hat{S}_l(x)\| + \eta^*\eta_* M_1M_2\|V_0(x)-\hat{V}_l(x)\|/2)e^{-\mu_*t/2},\\ \|\hat{V}(x,t)-\hat{V}_l(x)\| = \|\bar{V}(x,t)\| &\leqslant M_2\|V_0(x)-V_l(x)\|e^{-(\mu_*+\eta_*)t/2}.\end{aligned} \tag{40}$$

Hence,

$$\begin{aligned}S(x,t)-S^0(x) &\geqslant \hat{S}(x,t)-S^0(x) \geqslant \hat{S}(x,t)-\hat{S}_l(x)+\hat{S}_l(x)-S^0(x)\\ &\geqslant -(M_1\|S_0(x)-\hat{S}_l(x)\| + \eta^*\eta_*M_1M_2\|V_0(x)-\hat{V}_l(x)\|)e^{-\mu_*t/2} + \hat{S}_l(x)-S^0(x)\\ &\geqslant M_1(\|S_0(x)-S^0(x)\| + \|S^0(x)-\hat{S}_l(x)\|) - \|\hat{S}_l(x)-S^0(x)\| - Q_M\|V_0(x)-\hat{V}_l(x)\|\\ &\geqslant -M_1l-(M_1+1)\|S^0(x)-\hat{S}_l(x)\| - Q_M\|V_0(x)-\hat{V}_l(x)\|,\\ V(x,t)-V^0(x) &\geqslant \hat{V}(x,t)-V^0(x) \geqslant \hat{V}(x,t)-\hat{V}_l(x)+\hat{V}_l(x)-V^0(x)\\ &\geqslant -M_2\|V_0(x)-\hat{V}_l(x)\|e^{-(\mu_*+\eta_*)t/2} + \hat{V}_l(x)-V^0(x)\\ &\geqslant -M_2(\|V_0(x)-V^0(x)\| + \|V^0(x)-\hat{V}_l(x)\|) - \|\hat{V}_l(x)-V^0(x)\|\\ &\geqslant -M_2l-(M_2+1)\|V^0(x)-\hat{V}_l(x)\|,\end{aligned}$$

where $Q_M = \eta^*\eta_* M_1M_2$. Since $b_1(t) \leqslant H_1e^{-\hat{\rho}_1t}$ and $b_2(t) \leqslant H_2e^{-\hat{\rho}_2t}$, one has

$$\begin{aligned}S(x,t)-S^0(x) &= S^0(x)\left(\frac{S(x,t)}{S^0(x)}-1\right) \leqslant \|S^0(x)\|H_1e^{-\hat{\rho}_1t} \leqslant \|S^0(x)\|H_1,\\ V(x,t)-V^0(x) &= V^0(x)\left(\frac{V(x,t)}{V^0(x)}-1\right) \leqslant \|V^0(x)\|H_2e^{-\hat{\rho}_2t} \leqslant \|V^0(x)\|H_2.\end{aligned} \tag{41}$$

Integrating (40) and (41) yields

$$\begin{aligned}\|S(x,t)-S^0(x)\| &\leqslant \max\{M_1l+(M_1+1)\|S^0(x)-\hat{S}_l(x)\| + Q_M\|\hat{V}^0-\hat{V}_l(x)\|, H_1\|S^0(x)\|\},\\ \|V(x,t)-V^0(x)\| &\leqslant \max\{M_2l+(M_2+1)\|V^0(x)-\hat{V}_l(x)\|, H_2\|V^0(x)\|\}.\end{aligned} \tag{42}$$

Finally, combining (36)–(37), (42) and $\lim_{l\to 0}S_l(x) = S^0(x)$, $\lim_{l\to 0}V_l(x) = V^0(x)$, choose sufficiently small l such that for any $t > 0$,

$$\|S(x,t)-S^0(x)\|, \quad \|V(x,t)-V^0(x)\|, \quad \|I_A(x,t)\|, \quad \|I_S(x,t)\|, \quad \|B(x,t)\| \leqslant \varepsilon.$$

This proves the local stability of $\mathcal{E}^0$.

Further, let us prove the global attractiveness of $\mathcal{E}^0$. Let $\Phi(t)$ and X^+ be as defined in Lemma 2. According to Lemma 3, $\Phi(t)$ has a global attractor $\mathcal{A}$. Combined with Lemmas 4 and 5, problem (26) has a positive eigenvector (ψ_3, ψ_4, ψ_5) correlated to the eigenvalue $s(\mathcal{B})$. Define

$$\partial\mathbb{X}_1 = \{(S, V, I_A, I_S, B) \in X^+ : I_A = I_S = B = 0\}.$$

Claim I: For all $\phi \in \mathcal{A}$, the ω limit set $\omega(\phi) \subset \partial\mathbb{X}_1$.

From (4) and (5), it follows that $S(x,0) \leqslant S^0(x)$ and $V(x,0) \leqslant V^0(x)$. In case $\phi_3 = \phi_4 = \phi_5 = 0$, it is easy to conclude that $\partial\mathbb{X}_1$ is invariant for $\Phi(t)$. It follows that $\phi_3 \neq 0$ or $\phi_4 \neq 0$ or $\phi_5 \neq 0$. By Lemma 7, one has $I_A(x,t)$, $I_S(x,t)$, $B(x,t) > 0$, $x \in \overline{\mathbb{D}}$, $t > 0$. Therefore, $(S(x,t), V(x,t))$ fulfills

$$\begin{cases}\dfrac{\partial S}{\partial t} < (1-p(x))\Lambda(x) - \mu(x)S + \eta(x)V, \quad x \in \mathbb{D}, \quad t > 0,\\ \dfrac{\partial V}{\partial t} < p(x)\Lambda(x) - (\mu(x)+\eta(x))V, \quad x \in \mathbb{D}, \quad t > 0,\\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0, \quad x \in \partial\mathbb{D},\\ S(x,0) \leqslant S^0(x), \quad V(x,0) \leqslant V^0(x), \quad x \in \partial\mathbb{D}.\end{cases}$$

Using the principle of comparison, it follows that $S(x,t) < S^0(x)$, $V(x,t) < V^0(x)$, $(x,t) \in \overline{\mathbb{D}} \times [0,\infty)$.

Inspired by [8], we define

$$c(t,\phi) = \inf\{\check{c} \in \mathbb{R} : I_A \leqslant \check{c}\psi_3, I_S \leqslant \check{c}\psi_4 \quad \text{and} \quad B \leqslant \check{c}\psi_5\}.$$

Clearly, $c(t,\phi) > 0$ for any $t > 0$. It follows immediately that we prove that $c(t,\phi)$ is a strictly monotonically decreasing function. For this reason, fix $\bar{t}_0 > 0$, and let $\bar{I}_A(x,t) = c(\bar{t}_0,\phi)\psi_3$, $\bar{I}_S(x,t) = c(\bar{t}_0,\phi)\psi_4$ and $\bar{B}(x,t) = c(\bar{t}_0,\phi)\psi_5$, $t \geqslant \bar{t}_0$. Combined with $S(x,t) < S^0(x)$, $V(x,t) < V^0(x)$, we have

$$\begin{cases} \dfrac{\partial \bar{I}_A}{\partial t} > D_3\Delta I_A + \theta(x)\beta(x)f(S,\bar{B}) + \sigma\beta(x)g(V,\bar{B}) - (\mu(x) + r_A(x) + \delta(x))\bar{I}_A, & x \in \mathbb{D},\ t > t_0, \\ \dfrac{\partial \bar{I}_S}{\partial t} > D_4\Delta I_S + (1-\theta)\beta(x)f(S,\bar{B}) + \delta(x)\bar{I}_A - (\mu(x) + r_S(x))\bar{I}_S, & x \in \mathbb{D},\ t > t_0, \\ \dfrac{\partial \bar{B}}{\partial t} = \gamma_A(x)\bar{I}_A + \gamma_S(x)\bar{I}_S - \alpha(x)\bar{B}, & x \in \mathbb{D},\ t > t_0, \\ \bar{I}_A(x,\bar{t}_0) \geqslant I_A(x,\bar{t}_0), \quad \bar{I}_S(x,\bar{t}_0) \geqslant I_S(x,\bar{t}_0), \quad \bar{B}(x,\bar{t}_0) \geqslant B(x,\bar{t}_0), & x \in \mathbb{D}, \end{cases} \tag{43}$$

where $\partial \bar{I}/\partial \nu = \partial \bar{I}_S/\partial \nu = 0$, $x \in \partial\mathbb{D}$, $t > t_0$. Via the comparison principle, we can acquire $(\bar{I}_A(x,t), \bar{I}_S(x,t), \bar{B}(x,t)) \geqslant (I_A(x,t), I_S(x,t), B(x,t))$ for any $x \in \overline{\mathbb{D}}$ and $t > \bar{t}_0$. From the first second equations of (43) and comparison principle, it follows that $c(\bar{t}_0,\phi)\psi_3 = \bar{I}_A(x,t) > I(x,t)$, $c(\bar{t}_0,\phi)\psi_4 = \bar{I}_S(x,t) > I_S(x,t)$ and $c(\bar{t}_0,\phi)\psi_5 = \bar{B}(x,t) > B(x,t)$, for any $x \in \overline{\mathbb{D}}$ and $t > \bar{t}_0$. Because $\bar{t}_0$ is arbitrary, $c(t,\phi)$ is strictly decreasing function.

Definition by $c_* = \lim_{t\to\infty} c(t,\phi)$, we verify that $c_* = 0$. As a matter of fact, by setting $\mathbb{F} = (S, V, I_A, I_S, B) \in \omega(\phi)$, there exists $\{t_k\} \to \infty$ satisfying $\Phi(t_k)\phi \to \mathbb{F}$. Since

$$\lim_{t_k\to\infty} \Phi(t+t_k)\phi = \Phi(t)\lim_{t\to\infty}\Phi(t_k)\phi = \Phi(t)\mathbb{F},$$

we immediately obtain that $c(t,\mathbb{F}) = c_*$, $t \geqslant 0$. If $I_A \neq 0$ or $I_S \neq 0$ or $B \neq 0$, by repeating the above processes, we know that $c(t,\mathbb{F})$ is a strictly decreasing function. This results in the contradiction that $c(t,\mathbb{F}) = c_*$. Therefore, $I_A = I_S = B = 0$.

Claim II. $\mathcal{A} = \{\mathcal{E}^0\}$.

Claim I guarantees that $\{\mathcal{E}^0\}$ is globally attractive in $\partial\mathbb{X}_1$. Moreover, $\{\mathcal{E}^0\}$ constitutes the unique invariant subset in $\partial\mathbb{X}_1$. Since the ω limit set $\omega(\phi)$ is a compact invariant and $\omega(\phi) \subset \partial\mathbb{X}_1$, we arrive at $\omega(\phi) = \{\mathcal{E}^0\}$, $\forall\phi \in \mathcal{A}$. Lemma 3 shows that $\mathcal{E}_0$ is a compact invariant in X^+. Combining the global attractivity of $\{\mathcal{E}^0\}$ as well as [39, Lemma 3.11] indicates that $\mathcal{A} = \{\mathcal{E}^0\}$.

With the aforementioned analysis, we can deduce that $\mathcal{E}^0$ is globally asymptotically stable. □

6 Asymptotic profiles of the endemic steady state

In this section, we study the asymptotic behavior of the steady state to evaluate the impact of diffusion rate on the propagation of the infection. The efficacy of vaccines is increasing with the development of medical technology, we consider ignoring the immune loss rate, i.e., taking $\eta(x) = 0$ in model (1). For the purpose of better investigating the asymptotic behavior of model (1) endemic steady state, we select $f(S,B) = SB/(1 + c_1B)$, $g(V,B) = VB/(1 + c_2B)$, where c_1 and c_2 are the saturated coefficients. In light of Theorem 2, we learn that if $\mathcal{R}_0 > 1$, then model (1) has at least one steady state (S, V, I_A, I_S, B), which satisfies

$$\begin{cases} -D_1\Delta S = (1-p(x))\Lambda(x) - \mu(x)S - \dfrac{\beta(x)SB}{1+c_1B}, \quad x \in \mathbb{D} \\ -D_2\Delta V = p(x)\Lambda(x) - (\mu(x)+\eta(x))V - \dfrac{\sigma\beta(x)VB}{1+c_2B}, \quad x \in \mathbb{D}, \\ -D_3\Delta I_A = \dfrac{\theta\beta(x)SB}{1+c_1B} + \dfrac{\sigma\beta(x)VB}{1+c_2B} - (\mu(x)+r_A(x)+\delta(x))I_A, \quad x \in \mathbb{D}, \\ -D_4\Delta I_S = \dfrac{(1-\theta)\beta(x)SB}{1+c_1B} + \delta(x)I_A - (\mu(x)+r_S(x))I_S, \quad x \in \mathbb{D}, \\ 0 = \gamma_S(x)I_S + \gamma_A(x)I_A - \alpha(x)B, \quad x \in \mathbb{D}, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = \dfrac{\partial I_A}{\partial \nu} = \dfrac{\partial I_S}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases} \tag{44}$$

It is easy to come up with

$$B = \frac{1}{\alpha(x)}(\gamma_A(x)I_A + \gamma_S(x)I_S),$$

then (S, V, I_A, I_S) satisfies

$$\begin{cases} -D_1\Delta S = (1-p(x))\Lambda(x) - \mu(x)S - \dfrac{\beta(x)SB_1}{1+c_1B_1}, \quad x \in \mathbb{D}, \\ -D_2\Delta V = p(x)\Lambda(x) - \mu(x)V - \dfrac{\sigma\beta(x)VB_1}{1+c_2B_1}, \quad x \in \mathbb{D}, \\ -D_3\Delta I_A = \dfrac{\theta\beta(x)SB_1}{1+c_1B_1} + \dfrac{\sigma\beta(x)VB_1}{1+c_2B_1} - (\mu(x)+r_A(x)+\delta(x))I_A, \quad x \in \mathbb{D}, \\ -D_4\Delta I_S = \dfrac{(1-\theta)\beta(x)SB_1}{1+c_1B_1} + \delta(x)I_A - (\mu(x)+r_S(x))I_S, \quad x \in \mathbb{D}, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = \dfrac{\partial I_A}{\partial n} = \dfrac{\partial I_S}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \end{cases} \tag{45}$$

here, B_1 is used to represent $(\gamma_A(x)I_A + \gamma_S(x)I_S)/\alpha(x)$.

Lemma 8. *Let (S, V, I_A, I_S) be any solution of (45). For any $D_1, D_2, D_3, D_4 > 0$, one has the following conclusions that hold true.*

(i) The L^1-bounded of (S, V, I_A, I_S)

$$\int_{\mathbb{D}} S(x)\mathrm{d}x \leqslant \frac{\Lambda^*|\mathbb{D}|}{\mu_*}, \quad \int_{\mathbb{D}} V(x)\mathrm{d}x \leqslant \frac{p^*\Lambda^*|\mathbb{D}|}{\mu_*}, \quad \int_{\mathbb{D}} I_A(x)\mathrm{d}x \leqslant \frac{(\theta+\sigma p^*)\beta^*\Lambda^*|\mathbb{D}|}{\mu_*(\mu_* + r_{A*} + \delta_*)},$$
$$\int_{\mathbb{D}} I_S(x)\mathrm{d}x \leqslant \frac{\delta^*(\theta+\sigma p^*)\beta^*\Lambda^*|\mathbb{D}|}{\mu_*(\mu_* + r_{S*})(\mu_* + r_{A*} + \delta_*)} + \frac{\Lambda^*|\mathbb{D}|(1-\theta)\beta^*}{\mu_*(\mu_* + r_{S*})}.$$

(ii) The lower bounds of S and V

$$S(x) \geqslant \frac{c_1(1-p^*)\Lambda_*}{c_1\mu_* + \beta^*}, \quad V(x) \geqslant \frac{c_2 p_*\Lambda_*}{c_2\mu_* + \sigma\beta^*}, \quad \forall x \in \overline{\mathbb{D}}.$$

Proof. Arranging the first four equations of (45) and integrating over $\mathbb{D}$ yields

$$\int_{\mathbb{D}} \Lambda(x)\mathrm{d}x = \int_{\mathbb{D}} (\mu(x)(S(x) + V(x) + I_A(x) + I_S(x)) + r_A(x)I_A(x) + r_S(x)I_S(x))\mathrm{d}x.$$

Therefore, one obtains $\int_{\mathbb{D}} S(x)\mathrm{d}x \leqslant \Lambda^*|\mathbb{D}|/\mu_*$. Integrating the second to fourth equations in (45) over $\mathbb{D}$, we obtain

$$\mu_*\int_{\mathbb{D}} V(x)\mathrm{d}x \leqslant \int_{\mathbb{D}} \mu(x)V(x)\mathrm{d}x \leqslant \int_{\mathbb{D}}\left(\mu(x)V(x) + \frac{\sigma\beta(x)V(x)B_1(x)}{1 + c_2B_1(x)}\right)\mathrm{d}x = \int_{\mathbb{D}} p(x)\Lambda(x)\mathrm{d}x \leqslant p^*\Lambda^*|\mathbb{D}|.$$

Similarly,

$$\begin{aligned}(\mu_* + r_{A*} + \delta_*)\int_{\mathbb{D}} I_A(x)\mathrm{d}x &\leqslant \int_{\mathbb{D}} (\mu(x) + r_A(x) + \delta(x))I_A(x)\mathrm{d}x \\ &= \int_{\mathbb{D}} \frac{\theta\beta(x)S(x)B_1(x)}{1 + c_1B_1(x)}\mathrm{d}x + \int_{\mathbb{D}} \frac{\sigma\beta(x)V(x)B_1(x)}{1 + c_2B_1(x)}\mathrm{d}x \\ &\leqslant \theta\beta^*\int_{\mathbb{D}} S(x)\mathrm{d}x + \sigma\beta^*\int_{\mathbb{D}} V(x)\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}(\mu_* + r_{S*})\int_{\mathbb{D}} I_S(x)\mathrm{d}x \leqslant \int_{\mathbb{D}} (\mu(x) + r_S(x))I_S(x)\mathrm{d}x &= \int_{\mathbb{D}}\left(\delta(x)I_A(x) + \frac{(1-\theta)\beta(x)S(x)B_1(x)}{1 + c_1B_1(x)}\right)\mathrm{d}x \\ &\leqslant \delta^*\int_{\mathbb{D}} I_A(x)\mathrm{d}x + (1-\theta)\beta^*\int_{\mathbb{D}} S(x)\mathrm{d}x.\end{aligned}$$

Thus, it follows that L^1 bound of V, I_A and I_S, i.e., $\int_{\mathbb{D}} V(x)\mathrm{d}x \leqslant p^*\Lambda^*|\mathbb{D}|/\mu_*$, $\int_{\mathbb{D}} I_A(x)\mathrm{d}x \leqslant (\theta + \sigma p^*)\beta^*\Lambda^*|\mathbb{D}|/(\mu_*(\mu_* + r_{A*} + \delta_*))$ and $\int_{\mathbb{D}} I_S(x)\mathrm{d}x \leqslant \delta^*(\theta + \sigma p^*)\beta^*\Lambda^*|\mathbb{D}|/(\mu_*(\mu_* + r_{S*})(\mu_* + r_{A*} + \delta_*)) + \Lambda^*(1-\theta)p^*|\mathbb{D}|(\mu_*(\mu_* + r_{S*}))$.

(*ii*) Set $x_0 \in \overline{\mathbb{D}}$ such that $S(x_0) = \min_{x\in\overline{\mathbb{D}}}\{S(x)\}$ and $V(x_1) = \min_{x\in\overline{\mathbb{D}}}\{V(x)\}$. By the first equation of (45), combined with [18, Proposition 2.2], one has

$$(1-p^*)\Lambda_* \leqslant (1-p(x_0))\Lambda(x_0) \leqslant \mu(x_0)S(x_0) + \frac{\beta(x_0)S(x_0)B_1(x_0)}{1 + c_1B_1(x_0)} \leqslant \left(\mu^* + \frac{\beta^*}{c_1}\right)S(x_0).$$

Thus, we obtain $S(x) \geqslant c_1(1-p^*)\Lambda^*/(c_1\mu_* + \beta^*)$. From the second equation of (45), using the same method as above, we obtain

$$p_*\Lambda_* \leqslant p(x_1)\Lambda(x_1) \leqslant \mu(x_1)V(x_1) + \frac{\sigma\beta(x_1)V(x_1)B_1(x_1)}{1 + c_2B_1(x_1)} \leqslant \left(\mu^* + \frac{\sigma\beta^*}{c_2}\right)V(x_1).$$

Therefore, it is feasible to launch $V(x) \geqslant c_2p_*\Lambda_*/(c_2\mu^* + \sigma\beta^*)$. □

6.1 The case of $D_1 \to 0$

Throughout the present section, we discuss the asymptotic behavior of $\mathcal{E}^*$ as $D_1 \to \infty$.

Theorem 4. *Suppose that* $\mathcal{R}_0 > 1$. *Fix* D_2, D_3, $D_4 > 0$, *and let* $D_1 \to 0$ (*up to a subsequences*), *then* $(S(x), V(x), I_S(x), I_A(x))$ *satisfies*

$$(S(x), V(x), I_S(x), I_A(x)) \to (\Phi_S(x), \Phi_V(x), \Phi_{I_A}(x), \Phi_{I_S}(x)) \quad \textit{uniformly on } x \in \overline{\mathbb{D}},$$

where

$$\Phi_S(x) = \frac{(1-p(x))\Lambda(x)(1 + c_1\Phi_{B_1}(x))}{\mu(x)(1 + c_1\Phi_{B_1}(x)) + \beta(x)\Phi_{B_1}(x)},$$

and $(\Phi_V(x), \Phi_{I_A}(x), \Phi_{I_S}(x))$ *is a solution of the following problem*

$$\begin{cases} -D_2\Delta\Phi_V(x) = p(x)\Lambda(x) - \mu(x)\Phi_V(x) - \dfrac{\sigma\beta(x)\Phi_V(x)\Phi_{B_1}(x)}{1+c_2\Phi_{B_1}(x)}, \quad x\in\mathbb{D}, \\ -D_3\Delta\Phi_{I_A}(x) = \dfrac{\theta\beta(x)\Phi_S(x)\Phi_{B_1(x)}}{1+c_1\Phi_{B_1}(x)} + \dfrac{\sigma\beta(x)\Phi_V(x)\Phi_{B_1}(x)}{1+c_2\Phi_{B_1}(x)} - (\mu(x)+r_A(x)+\delta(x))\Phi_{I_A}(x), \quad x\in\mathbb{D}, \\ -D_4\Delta\Phi_{I_S}(x) = \dfrac{(1-\theta)\beta(x)\Phi_S(x)\Phi_{B_1}(x)}{1+c_1\Phi_{B_1}(x)} + \delta(x)\Phi_{I_A}(x) - (\mu(x)+r_S(x))\Phi_{I_S}(x), \quad x\in\mathbb{D}, \\ \dfrac{\partial\Phi_V(x)}{\partial\nu} = \dfrac{\partial\Phi_{I_A}(x)}{\partial\nu} = \dfrac{\partial\Phi_{I_S}(x)}{\partial\nu} = 0, \quad x\in\partial\mathbb{D}. \end{cases} \tag{46}$$

Proof. In this proof, we suppose that C is a positive constant and does not depend on D_1, but allows to vary from place to place. In the interest of clarity, let's divide the proof into three steps.

Step 1: The L^p-bound of (S, V, I_A, I_S) for any $p \geqslant 1$.

It follows from Lemma 8 and the ellipse L^1-estimated [24, Lemma 2.2] that $\|I_A(\cdot)\| \leqslant C$, $\|I_S(\cdot)\| \leqslant C$ for all $1 \leqslant q \leqslant N/(N-1)$ (or if $N = 1$, $q \in [1, \infty)$). The Sobolev embedding theorem $W^{1,q}(\mathbb{D}) \hookrightarrow L^{p_1}(\mathbb{D})$ [12]. For any $q \in [1, N/(N-1))$ yields

$$\|I_A(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C, \quad \|I_S(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C, \quad p_1 \in (1, qN/(N-q)).$$

It clearly follows from the fact that q can be close to $N/(N-1)$ that

$$\|I_A(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C, \quad \|I_S(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C, \quad p_1 \in (1, qN/(N-2)).$$

Evidently, if $N \leqslant 2$, then $p_1 \in (1, \infty)$ holds. Multiplying the S equation of (45) by S^k $(k > 0)$ and integrating over $\mathbb{D}$, we obtain

$$kD_1\int_{\mathbb{D}} S^{k-1}|\nabla S|^2 \mathrm{d}x = \int_{\mathbb{D}}(1-p(x))\Lambda(x)S^k \mathrm{d}x - \int_{\mathbb{D}}\mu(x)S^{k+1}\mathrm{d}x - \int_{\mathbb{D}}\frac{\beta S^{k+1}B_1}{1+c_1B_1}\mathrm{d}x.$$

It is clear that

$$\mu_*\int_{\mathbb{D}} S^{k+1}\mathrm{d}x \leqslant (1-p_*)\Lambda^*\int_{\mathbb{D}} S^k\mathrm{d}x. \tag{47}$$

Let $k_1 = 1/q_1$ and $q_1 = 1 + 1/(p_1 - 1) = p_1/(p_1 - 1)$ (note that $1/p_1 + 1/q_1 = 1$), we conclude that

$$\mu_*\int_{\mathbb{D}} S^{k_1+1}\mathrm{d}x \leqslant (1-p_*)\Lambda^*\int_{\mathbb{D}} S^{k_1}\mathrm{d}x \leqslant (1-p_*)|\mathbb{D}|^{\frac{1}{p_1}}\left(\int_{\mathbb{D}} S\mathrm{d}x\right)^{\frac{1}{q_1}} \leqslant C.$$

Then, we derive

$$\|S(\cdot)\|_{L^{k_1+1}(\mathbb{D})} \leqslant C.$$

Combining the Hölder inequality and (47), the following result can be obtained

$$\mu_*\int_{\mathbb{D}} S^{k_2+1}\mathrm{d}x \leqslant (1-p_*)\Lambda^*\int_{\mathbb{D}} S^{k_2}\mathrm{d}x \leqslant (1-p_*)\Lambda^*|\mathbb{D}|^{\frac{1}{p_1}}\left(\int_{\mathbb{D}} S^{k_2q_1}\right)^{\frac{1}{q_1}},$$

where $k_2 = (k_1 + 1)/q_1 = 1/q_1 + 1/q_1^2$. Hence, we conclude that $\|S(\cdot)\|_{L^{k_2+1}(\mathbb{D})} \leqslant C$. The iterative process can be repeated to obtain

$$k_\infty = \frac{1}{q_1} + \frac{1}{q_1^2} + \frac{1}{q_1^3} + \cdots = p_1 - 1,$$

then one can conclude that

$$\|S(\cdot)\|_{L^{k_\infty+1}(\mathbb{D})} = \|S(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C.$$

Similarly, we can also derive $\|V(\cdot)\|_{L^{k_\infty+1}(\mathbb{D})} = \|V(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C$. Further, combining the aforementioned results and the elliptic L^p-theory for I_A equation and I_S equation of (45), one has

$$\|I_A(\cdot)\|_{W^{2,p_1}(\mathbb{D})} \leqslant C, \quad \|I_S(\cdot)\|_{W^{2,p_1}(\mathbb{D})} \leqslant C.$$

Combining Sobolev's embedding theorem $W^{2,p_1}(\mathbb{D}) \hookrightarrow L^{p_2}(\mathbb{D})$ for $1 < p_2 < Np_1/(N-2p_1)$ and $p_1 \in (1, N/(N-2))$, one has

$$\|I_A(\cdot)\|_{L^{p_2}(\mathbb{D})} \leqslant C, \quad \|I_A(\cdot)\|_{L^{p_2}(\mathbb{D})} \leqslant C, \quad \forall p_2 \in (1, N/(N-4) \quad \text{or} \quad \forall p_2 \in (1,\infty)) \quad \text{if} \quad N \leqslant 4.$$

By a similar argument as the L^{p_1}-bound of S and V, one can show $\|S(\cdot)\|_{L^{p_2}(\mathbb{D})} \leqslant C, \|V(\cdot)\|_{L^{p_2}(\mathbb{D})} \leqslant C$. Therefore, repeating the aforementioned argument concludes

$$\|S(\cdot)\|_{L^{p}(\mathbb{D})} \leqslant C, \quad \|V(\cdot)\|_{L^{p}(\mathbb{D})} \leqslant C, \quad \|I_A(\cdot)\|_{L^{p}(\mathbb{D})} \leqslant C, \quad \|I_S(\cdot)\|_{L^{p}(\mathbb{D})} \leqslant C, \quad \forall 1 \leqslant p < \infty.$$

Step 2: Convergence of I_A and I_S.

Note that I_A can be solved from

$$\begin{cases} -\Delta I_A + \left[\dfrac{\mu(x) + r_A(x) + \delta(x)}{D_3} - \dfrac{\theta\beta(x)SB_1}{D_3(1 + c_1B_1)I_A} - \dfrac{\sigma\beta(x)VB_1}{D_3(1 + c_2B_1)I_A}\right] I_A = 0, \quad x \in \mathbb{D}, \\ \dfrac{\partial I_A}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases} \tag{48}$$

Fix $D_3 > 0$, it follows from the Lemma 8 (*ii*) that

$$\begin{aligned} &\left\| \frac{\mu(x) + r_A(x) + \delta(x)}{D_3} - \frac{\theta\beta(x)SB_1}{D_3(1 + c_1B_1)I_A} - \frac{\sigma\beta(x)VB_1}{D_3(1 + c_2B_1)I_A} \right\|_{L^p(\mathbb{D})} \\ &\leqslant \left\| \frac{\mu(x) + r_A(x) + \delta(x)}{D_3} + \frac{\theta\beta(x)S}{c_1D_3I_A} + \frac{\sigma\beta(x)V}{c_2D_3I_A} \right\|_{L^p(\mathbb{D})} \leqslant C. \end{aligned}$$

Choosing p sufficiently large, by using [27, Lemma 2.2], one has

$$\max_{x\in\overline{\mathbb{D}}}\{I_A(x)\} \leqslant C \min_{x\in\overline{\mathbb{D}}}\{I_A(x)\}, \tag{49}$$

where $C > 0$ does not depend on D_1. In the light of Lemma 8(*i*) and (49), one can conclude that

$$I_A(x) \leqslant C \min_{x\in\overline{\mathbb{D}}}\{I_A(x)\} \leqslant \frac{C}{|\mathbb{D}|}\int_{\mathbb{D}} I_A \mathrm{d}x \leqslant C. \tag{50}$$

Rewrite the I_A equation in (45) as follows:

$$\begin{cases} -D_3\Delta I_A + (\mu(x) + r_A(x) + \delta(x))I_A = \dfrac{\theta\beta(x)SB_1}{1 + c_1B_1} + \dfrac{\sigma\beta(x)VB_1}{1 + c_2B_1}, \quad x \in \mathbb{D}, \\ \dfrac{\partial I_A}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases} \tag{51}$$

By (50), there holds

$$\left\| \frac{\theta\beta(x)SB_1}{1 + c_1B_1} + \frac{\sigma\beta(x)VB_1}{1 + c_2B_1} \right\|_{L^p(\mathbb{D})} \leqslant C, \quad \forall p \geqslant 1.$$

According to the standard L^p estimate for elliptic equations (see, e.g., [12]), then

$$\|I_A(\cdot)\|_{W^{2,p}(\mathbb{D})} \leqslant C, \quad \forall p > 1.$$

For sufficiently large p, the Sobolev embedding theorem [12] means that

$$\|I_A(\cdot)\|_{C^{1+\alpha}(\overline{\mathbb{D}})} \leqslant C, \quad \text{for} \quad 0 < \alpha < 1.$$

Similarly, we can also derive $||I_A(\cdot)||_{C^{1+\alpha}(\overline{\mathbb{D}})} \leqslant C$, for $0 < \alpha < 1$. Consequently one can identify a subsequence D_1(which is labeled by $D_{S,n}$), meeting $D_{S,n} \to 0$ as $n \to \infty$, and a corresponding positive solution $(S_n(x), V_n(x), I_{An}(x), I_{Sn}(x)) := (S(x, D_{S,n}), V(x, D_{S,n}), I_A(x, D_{S,n}), I_S(x, D_{S,n}))$ of (45) for $D_1 = D_{S,n}$ satisfies

$$(I_{An}(x), I_{Sn}(x)) \to (\Phi_{I_A}(x), \Phi_{I_S}(x)) \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \quad \text{as} \quad n \to \infty, \tag{52}$$

where $(\Phi_{I_A}(x), \Phi_{I_S}(x)) \in C^1(\mathbb{D}) \times C^1(\mathbb{D})$ and $\Phi_{I_A}(x), \Phi_{I_S}(x) \geqslant 0$ on $\overline{\mathbb{D}}$. From (50), it is apparent that either $\Phi_{I_A}(x) \equiv 0$ on $\overline{\mathbb{D}}$ or $\Phi_{I_A}(x) > 0$ on $\overline{\mathbb{D}}$.

Assuming that $\Phi_{I_A}(x) \equiv 0$ on $\overline{\mathbb{D}}$, that is,

$$I_{An}(x) \to 0 \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \quad \text{as} \quad n \to \infty.$$

Bringing in the third equation of (45) yields

$$\frac{\theta\beta(x)\gamma_S(x)S_n(x)I_{Sn}(x)}{\alpha(x) + c_1\gamma_S(x)I_{Sn}(x)} + \frac{\sigma\beta(x)\gamma_S(x)V_n(x)I_{Sn}(x)}{\alpha(x) + c_2\gamma_S(x)I_{Sn}(x)} = 0, \quad \text{as} \quad n \to \infty. \tag{53}$$

Combining this with the Lemma 8(*ii*) yields

$$I_{Sn}(x) \to 0 \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \quad \text{as} \quad n \to \infty.$$

Then for any small ε, there exists $n_1 > 0$, one has

$$0 \leqslant I_{An}(x) \leqslant \varepsilon, \quad 0 \leqslant I_{Sn}(x) \leqslant \varepsilon, \quad \forall x \in \overline{\mathbb{D}}, \quad \text{for} \quad n \geqslant n_1.$$

Along with the first equation of (45), this implies that $S_n(x)$ is satisfied for all large n

$$-D_{S,n}\Delta S_n \leqslant (1 - p(x))\Lambda(x) - \mu(x)S_n, \quad x \in \mathbb{D}, \quad \frac{\partial S_n}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}$$

and

$$-D_{S,n}\Delta S_n \geqslant (1 - p(x))\Lambda(x) - \varepsilon\beta^* S_n - \mu(x)S_n, \quad x \in \mathbb{D}, \quad \frac{\partial S_n}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}.$$

Taking an arbitrarily large n, there are two auxiliary systems to consider as follows:

$$-D_{S,n}\Delta S_n = (1 - p(x))\Lambda(x) - \mu(x)S_n, \quad x \in \mathbb{D}, \quad \frac{\partial S_n}{\partial n} = 0, \quad x \in \partial\mathbb{D} \tag{54}$$

and

$$-D_{S,n}\Delta S_n = (1 - p(x))\Lambda(x) - \varepsilon\beta^* S_n - \mu(x)S_n, \quad x \in \mathbb{D}, \quad \frac{\partial S_n}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \tag{55}$$

Clearly, (54) and (55) contain unique positive solutions, denoted by u_n and v_n, respectively. On the basis of the sub-superior solution argument in [40], we can conclude that

$$v_n \leqslant S_n \leqslant u_n \quad \text{on} \quad \overline{\mathbb{D}} \quad \text{for all large} \quad n.$$

Further by a similar proof as in [10, Lemma 2.4], it follows that

$$u_n \to \frac{(1 - p(x)\Lambda(x))}{\mu(x)} \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \quad \text{as} \quad n \to \infty$$

and

$$v_n \to \frac{(1 - p(x)\Lambda(x))}{\mu(x) + \varepsilon\beta^*} \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \quad \text{as} \quad n \to \infty.$$

Thus, as $n \to \infty$, one has

$$\frac{(1 - p(x)\Lambda(x))}{\mu(x) + \varepsilon\beta^*} \leqslant \liminf_{n\to\infty} S_n \leqslant \limsup_{n\to\infty} S_n \leqslant \frac{(1 - p(x)\Lambda(x))}{\mu(x)}.$$

By the arbitrariness of ε yields

$$S_n \to \frac{(1-p(x))\Lambda(x)}{\mu(x)} \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \quad \text{as} \quad n \to \infty.$$

Similarly, one can conclude that

$$V(x) \to \frac{p(x)\Lambda(x)}{\mu(x)} \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \quad \text{as} \quad n \to \infty.$$

Consider the I_A equation of (45) satisfies

$$\begin{cases} -D_3\Delta I_{An} = \dfrac{\theta\beta(x)S_nB_{1n}}{1+c_1B_{1n}} + \dfrac{\sigma\beta(x)V_nB_{1n}}{1+c_2B_{1n}} - (\mu(x)+r_A(x)+\delta(x))I_{An}, & x \in \mathbb{D}, \\ \dfrac{\partial I_{An}}{\partial \nu} = 0, & x \in \partial\mathbb{D}. \end{cases} \tag{56}$$

Define $\bar{I}_{An} = I_{An}/\|I_{An}\|_{L^\infty(\mathbb{D})}$, it is obvious that $\|\bar{I}_{An}\|_{L^\infty(\mathbb{D})} = 1$ for all $n \geqslant 1$, and $\bar{I}_{An}$ solves

$$\begin{cases} -D_3\Delta\bar{I}_{An} = \left[\dfrac{\theta\gamma_A(x)\beta(x)S_n}{\alpha(x)(1+c_1B_{1n})} + \dfrac{\sigma\gamma_A(x)\beta(x)V_n}{\alpha(x)(1+c_2B_{1n})} - (\mu(x)+r_A(x)+\delta(x))\right]\bar{I}_{An} + \dfrac{\theta\gamma_S(x)\beta(x)S_nI_{Sn}}{\alpha(x)(1+c_1B_{1n})\|I_{An}\|_{L^\infty(\mathbb{D})}} \\ \qquad + \dfrac{\sigma\gamma_S(x)\beta(x)V_nI_{Sn}}{\alpha(x)(1+c_2B_{1n})\|I_{An}\|_{L^\infty(\mathbb{D})}}, \quad x \in \mathbb{D}, \\ \dfrac{\partial \bar{I}_{An}}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases} \tag{57}$$

Starting from the standard compactness argument for elliptic equations, as $n \to \infty$, one has $\bar{I}_{An} \to \bar{I}_A$ in $C^1(\mathbb{D})$, as $n \to \infty$, where $\bar{I}_A \geqslant 0$ belongs to $C^1(\mathbb{D})$ and satisfies $\|\bar{I}_A\|_{L^\infty(\mathbb{D})} = 1$. Thus, as $n \to \infty$, (57) becomes

$$\begin{cases} -D_3\Delta\bar{I}_A = \left[\dfrac{\beta(x)\gamma_A(x)\Lambda(x)}{\alpha(x)\mu(x)}(\theta(1-p(x))+\sigma p(x)) - (\mu(x)+r_A(x)+\delta(x))\right]\bar{I}_A, & x \in \mathbb{D}, \\ \dfrac{\partial \bar{I}_A}{\partial \nu} = 0, & x \in \partial\mathbb{D}. \end{cases}$$

By the Harnack-type inequality, one has $\bar{I}_A > 0$. Further, we consider the I_S equation for (45)

$$\begin{cases} -D_4I_{Sn} = \dfrac{(1-\theta)\beta(x)S_nB_{1n}}{1+c_1B_{1n}} - (\mu(x)+r_S(x))I_{Sn} + \delta(x)\left[\dfrac{\theta\beta(x)S_nB_{1n}}{1+c_1B_{1n}} + \dfrac{\sigma\beta(x)V_nB_{1n}}{1+c_2B_{1n}}\right] \\ \qquad \times (-D_3\Delta + (\mu(x)+r_A(x)+\delta(x)))^{-1}, \quad x \in \mathbb{D}, \\ \dfrac{\partial I_{Sn}}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases}$$

Define $\bar{I}_{Sn} = I_{Sn}/\|I_{Sn}\|_{L^\infty(\mathbb{D})}$, taking a similar approach yields $\bar{I}_{Sn} \to \bar{I}_S > 0$ in $C^1(\mathbb{D})$, as $n \to \infty$. where, I_S satisfies

$$\begin{cases} -D_4\Delta\bar{I}_S = \left(\dfrac{(1-\theta)\beta(x)\gamma_S(x)(1-p(x))\Lambda(x)}{\alpha(x)\mu(x)} - (\mu(x)+r_S(x)) \right. \\ \qquad \left. + \dfrac{\Lambda(x)\gamma_S(x)\beta(x)}{\alpha(x)\mu(x)} \times \dfrac{\delta(x)(\theta(1-p(x))+\sigma p(x))}{-D_3\Delta + (\mu(x)+r_A(x)+\delta(x))}\right)\bar{I}_S, \quad x \in \mathbb{D}, \\ \dfrac{\partial \bar{I}_S}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases}$$

Moreover, due to the uniqueness of the principal eigenvalue, 0 is the principal eigenvalue of (19) and (23). Combined with (18) and Lemma 4, this contradicts $\mathcal{R}_0 > 1$. Thus, $\Phi_{I_A} > 0$ and $\Phi_{I_S} > 0$ on $\overline{\mathbb{D}}$, which means that $I_{An} \to \Phi_{I_A} > 0$ and $I_{Sn} \to \Phi_{I_S} > 0$ uniformly on $\overline{\mathbb{D}}$, as $n \to \infty$.

Step 3: Convergence of S.

From (45), we know that

$$-D_1\Delta S_n = (1-p(x))\Lambda(x) - \mu(x)S_n - \frac{\beta(x)S_nI_n}{1+c_1B_{1n}}, \quad x \in \mathbb{D}.$$

By (52), for any small $\varepsilon > 0$, it follows that we have a sufficiently large n such that

$$0 < \Phi_{I_A}(x) - \varepsilon \leqslant I_{An}(x) \leqslant \Phi_{I_A}(x) + \varepsilon, \quad 0 < \Phi_{I_S}(x) - \varepsilon \leqslant I_{Sn}(x) \leqslant \Phi_{I_S}(x) + \varepsilon, \quad x \in \overline{\mathbb{D}}.$$

Thus,

$$\begin{aligned} 0 &< \frac{1}{\alpha(x)}[(\Phi_{I_A}(x) - \varepsilon) + (\Phi_{I_S}(x) - \varepsilon)] = \Phi_{B_1}(x) - 2\varepsilon \leqslant \frac{1}{\alpha(x)}(I_{An}(x) + I_{Sn}(x)) \\ &\leqslant \frac{1}{\alpha(x)}[(\Phi_{I_A}(x) + \varepsilon) + (\Phi_{I_S}(x) + \varepsilon)] = \Phi_{B_1}(x) + 2\varepsilon. \end{aligned} \tag{58}$$

So, for sufficiently large n, one has

$$\begin{aligned} (1-p(x))\Lambda(x) - \mu(x)S_n - \frac{\beta(x)S_nB_{1n}}{1+c_1B_{1n}} &\leqslant (1-p(x))\Lambda(x) - \mu(x)S_n - \frac{\beta(x)S_n(\Phi_{B_{1n}} - 2\varepsilon)}{1+c_1(\Phi_{B_{1n}} + 2\varepsilon)} \\ &= \frac{(g^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) - S_n)\varphi^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S})}{1+c_1(\Phi_{B_1} + 2\varepsilon)}, \end{aligned}$$

where

$$\begin{aligned} g^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) &= \frac{(1-p(x))\Lambda(x)(1+c_1(\Phi_{B_1} + 2\varepsilon))}{\mu(x)(1+c_1(\Phi_{B_1} + 2\varepsilon)) + \beta(x)(\Phi_{B_1} - 2\varepsilon)}, \\ \varphi^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) &= \mu(x)(1+c_1(\Phi_{B_1} + 2\varepsilon)) + \beta(x)(\Phi_{B_1} - 2\varepsilon). \end{aligned}$$

For adequately large n, let us study the below auxiliary problem

$$-D_n\Delta z = \frac{(g^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) - z)\varphi^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S})}{1+c_1(\Phi_{B_1} + 2\varepsilon)}, \quad x \in \mathbb{D}, \tag{59}$$

with $\partial z/\partial\nu = 0$, $x \in \partial\mathbb{D}$. Note that (S_n, C) is a a couple of sub-supersolutions of (59), where $C > 0$ is a sufficiently large positive constant C meeting $S_n \leqslant C$. Therefore, (59) admits at least a positive solution $\bar{z}_n$ satisfying $S_n \leqslant \bar{z}_n \leqslant C$ on $\overline{\mathbb{D}}$. Then, similar arguments applied to [10, Lemma 2.4] can be shown that any solution $\bar{z}_n$ of (59) satisfies

$$\bar{z}_n \to g^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) \quad \text{uniformly on} \quad \overline{\mathbb{D}}.$$

By $S_n \leqslant \bar{z}_n \leqslant C$, it indicate that

$$\limsup_{n\to\infty} S_n \leqslant g^{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) \quad \text{uniformly on} \quad \overline{\mathbb{D}}. \tag{60}$$

Furthermore, by (58), one has

$$\begin{aligned} (1-p(x))\Lambda(x) - \mu(x)S_n - \frac{\beta(x)S_nB_{1n}}{1+c_1B_{1n}} &\geqslant (1-p(x))\Lambda(x) - \mu(x)S_n - \frac{\beta(x)S_n(\Phi_{B_{1n}} + 2\varepsilon)}{1+c_1(\Phi_{B_{1n}} - 2\varepsilon)} \\ &= \frac{(g_{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) - S_n)\varphi_{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S})}{1+c_1(\Phi_{B_1} - 2\varepsilon)}, \end{aligned}$$

where

$$\begin{aligned} g_{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) &= \frac{(1-p(x))\Lambda(x)(1+c_1(\Phi_{B_1} - 2\varepsilon))}{\mu(x)(1+c_1(\Phi_{B_1} - 2\varepsilon)) + \beta(x)(\Phi_{B_1} + 2\varepsilon)}, \\ \varphi_{\varepsilon}(x, \Phi_{I_A}, \Phi_{I_S}) &= \mu(x)(1+c_1(\Phi_{B_1} - 2\varepsilon)) + \beta(x)(\Phi_{B_1} - 2\varepsilon). \end{aligned}$$

Arguing similarly as before, this leads to $\liminf_{t\to\infty} S_n \geqslant g_\varepsilon(x, \Phi_{I_A}, \Phi_{I_S})$. We further obtain

$$\lim_{\varepsilon\to 0} g^\varepsilon(x, \Phi_{I_A}(x), \Phi_{I_S}(x)) = \lim_{\varepsilon\to 0} g_\varepsilon(x, \Phi_{I_A}(x), \Phi_{I_S}(x)) = \frac{(1-p(x))\Lambda(x)(1+c_1\Phi_{B_1}(x))}{\mu(x)(1+c_1\Phi_{B_1}(x)) + \beta(x)\Phi_{B_1}(x)}$$
$$= G(x, \Phi_{I_A}(x), \Phi_{I_S}(x)).$$

Combining the above results, it follows that

$$S_n(x) \to G(x, \Phi_{I_A}(x), \Phi_{I_S}(x)) \quad \text{uniformly on} \quad \overline{\mathbb{D}}, \ \text{as} \quad n\to\infty.$$

Therefore, it can be conveniently derived that $\Phi_V(x)$, $\Phi_{I_A}(x)$ and $\Phi_{I_S}(x)$ meet (46). The proof is complete. □

6.2 The case of $D_3 \to 0$

We investigate the asymptotic behavior of $\mathcal{E}^*$ with respect to $D_3 \to 0$ in this subsection.

Theorem 5. *Suppose that $\mathcal{R}_0 > 1$. Fix D_1, D_2, and D_4 are positive constants and $D_3 \to 0$ (up to a subsequnece), then every positive $(S(x), V(x), I_A(x), I_S(x))$ meets*

$$(S(x), V(x), I_A(x), I_S(x)) \to (\Psi_S(x), \Psi_V(x), \Psi_{I_A}(x), \Psi_{I_S}(x)) \quad \textit{uniformly on} \quad \overline{\mathbb{D}},$$

where

$$I_A(x) = \frac{\beta\Phi_{B_1}(\theta\Phi_S(1+c_1\Phi_{B_1}) + \sigma\Phi_V(1+c_1\Phi_{B_1}))}{(1+c_2\Phi_{B_1})(1+c_1B_1)}, \tag{61}$$

and $(\Psi_S(x), \Psi_V(x), \Psi_{I_S}(x))$ satisfies

$$\begin{cases} -D_1\Delta\Psi_S = (1-p(x))\Lambda(x) - \mu(x)\Psi_S - \dfrac{\beta(x)\Psi_S\Psi_{B_1}}{1+c_1\Psi_{B_1}}, & x\in\mathbb{D}, \\ -D_2\Delta\Psi_V = p(x)\Lambda(x) - \mu(x)\Psi_V - \dfrac{\sigma\beta(x)\Psi_V\Phi_{B_1}}{1+c_2\Psi_{B_1}}, & x\in\mathbb{D}, \\ -D_4\Delta\Psi_{I_S} = \dfrac{(1-\theta)\beta(x)\Psi_S\Psi_{B_1}}{1+c_1\Psi_{B_1}} + \delta(x)\Psi_{I_A} - (\mu(x)+r_S(x))\Psi_{I_S}, & x\in\mathbb{D}, \\ \dfrac{\partial\Psi_S}{\partial\nu} = \dfrac{\partial\Psi_V}{\partial\nu} = \dfrac{\partial\Psi_{I_S}}{\partial\nu} = 0, & x\in\partial\mathbb{D}. \end{cases} \tag{62}$$

Proof. We will deal with proof by means of belowing three claims.

Step 1: The endemic steady state (S, V, I_A, I_S) is L^p-bound for $p \geqslant 1$.

Arguing in a similar way to Step 1 of the proof of Theorem 4, the elliptic L^1-theory of he first to second equations of (45) and the Sobolev embedding theorem lead to

$$\|S(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C, \quad \|V(\cdot)\|_{L^{p_1}(\mathbb{D})} \leqslant C, \quad \forall p_1 \in (1, N/(N-2)) \ (\text{or } \forall p_1 \in (0,\infty) \quad \text{if} \quad N \leqslant 2), \tag{63}$$

where the positive constant C is independent of small D_3. For $k > 0$, integrating over $\mathbb{D}$ by multiplying the third equation in (45) by I_A^k yields

$$(\mu_* + r_{A*} + \delta_*)\int_{\mathbb{D}} I_A^{k+1}\mathrm{d}x \leqslant \int_{\mathbb{D}} (\mu(x) + r_A(x) + \delta(x)) I_A^{k+1}\mathrm{d}x \leqslant \frac{\theta\beta^*}{c_1}\int_{\mathbb{D}} SI_A^k\mathrm{d}x + \frac{\sigma\beta^*}{c_2}\int_{\mathbb{D}} VI_A^k\mathrm{d}x.$$

Using the approach of Step 1 in Theorem 4 yields the L^p-bound of I_A. Similarly, multiplying I_S equation for (45) by I_S^k and repeating the above process yields L^p-bound of I_S. Thus, we can conclude that

$$\|S(\cdot)\|_{L^p(\mathbb{D})} \leqslant C, \quad \|V(\cdot)\|_{L^p(\mathbb{D})} \leqslant C, \quad \|I_A(\cdot)\|_{L^p(\mathbb{D})} \leqslant C, \quad \|I_S(\cdot)\|_{L^p(\mathbb{D})} \leqslant C, \quad \forall 1 \leqslant q < \infty. \tag{64}$$

Step 2: Convergence of S, V and I_S.

By (63), the elliptic L^p-theory and Sobolev embedding theorem assure that

$$\|S(\cdot)\|_{C^{1+\alpha}(\overline{\mathbb{D}})} \leqslant C, \quad \|V(\cdot)\|_{C^{1+\alpha}(\overline{\mathbb{D}})} \leqslant C, \quad \|I_A(\cdot)\|_{C^{1+\alpha}(\overline{\mathbb{D}})} \leqslant C, \quad \|I_S(\cdot)\|_{C^{1+\alpha}(\overline{\mathbb{D}})} \leqslant C, \quad \alpha \in (0,1).$$

Then $D_3 \to 0$ contains a subsequence that converges to zero as $j \to \infty$, denoted $D_3 = D_{3,j} \to 0$, such that the corresponding positive solution $(S_j(x), V_j(x), I_{A,j}(x), I_{S,j}(x)) \to (\Psi_S(x), \Psi_V(x), \Psi_{I_A}(x), \Psi_{I_S}(x)) = (S(x, D_{3,j}), V(x, D_{3,j}), I_A(x, D_{3,j}), I_S(x, D_{3,j}))$ of (45) for $D_3 = D_{3,j}$ fulfills

$$(S_j(x), V_j(x), I_{S,j}(x)) \to (\Psi_S(x), \Psi_V(x), \Psi_{I_S}(x)) \quad \text{in} \quad C^1(\overline{\mathbb{D}}) \times C^1(\mathbb{D}) \times C^1(\mathbb{D}) \quad \text{as} \quad j \to \infty,$$

where $\Psi_{I_S}(x) \geqslant 0$ and $\Psi_S(x), \Psi_V(x) > 0$ due to Lemma 8(*ii*).

Step 3: Convergence of I_A.

It is noteworthy that $I_{A,j}$ satisfies

$$\begin{cases} -D_3 \Delta I_{A,j} = \dfrac{\theta\beta(x)S_j B_{1,j}}{1 + c_1 B_{1,j}} + \dfrac{\sigma\beta(x)V_j B_{1,j}}{1 + c_2 B_{1,j}} - (\mu(x) + r(x) + \delta(x))I_{A,j}, \quad x \in \mathbb{D}, \\ \dfrac{\partial I_{A,j}}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases} \tag{65}$$

From (64) and (65), It is possible to employ the sub-supersolution comparison argument as in Claim III of Theorem 4 (see [19, Theorem 1.1]) to conclude that $I_{A,j}(x) \to \Psi_{I_A}(x)$ in $C^1(\overline{\mathbb{D}})$ as $j \to \infty$, where $\Psi_{I_A}(x)$ fulfills (61). Moreover, the formulation of $\Psi_{I_A}(x)$, it is evidence that $(\Psi_S(x), \Psi_V(x), \Psi_{I_A}(x), \Psi_{I_S}(x))$ is the unique positive solution of (62). This completes the proof. □

6.3 The case of $D_1 \to \infty$ or $D_3 \to \infty$

In the following, we analyze the asymptotic behavior of $\mathcal{E}^*$ when $D_1 \to \infty$ or $D_3 \to \infty$. The results are presented below.

Theorem 6. *Suppose that* $\mathcal{R}_0 > 1$. *Fix* $D_2, D_3, D_4 > 0$ *and let* $D_1 \to \infty$, *then every solution* $(S(x), V(x), I_A(x), I_S(x))$ *of* (45) *meets*

$$(S(x), V(x), I_A(x), I_S(x)) \to (S^\infty(x), V^\infty(x), I_A^\infty(x), I_S^\infty(x)) \quad \textit{uniformly on} \quad \overline{\mathbb{D}},$$

where $S^\infty(x) > 0$ *is a constant,* $V^\infty(x), I_A^\infty(x), I_S^\infty(x) > 0$ *on* $\overline{\mathbb{D}}$, *and* $(S^\infty(x), V^\infty(x), I_A^\infty(x), I_S^\infty(x))$ *solves*

$$\begin{cases} -D_2 \Delta V^\infty = p(x)\Lambda(x) - (\mu(x) + \eta(x))V^\infty - \dfrac{\sigma\beta(x)V^\infty B_1^\infty}{1 + c_1 B_1^\infty}, \quad x \in \mathbb{D}, \\ -D_3 \Delta I_A^\infty = \dfrac{\theta\beta(x)S^\infty B_1^\infty}{1 + c_1 B_1^\infty} + \dfrac{\sigma\beta(x)V^\infty B_1^\infty}{1 + c_1 B_1^\infty} - (\mu(x) + r(x) + \delta(x))I_A^\infty, \quad x \in \mathbb{D}, \\ -D_4 \Delta I_S^\infty = \dfrac{(1-\theta)\beta(x)S^\infty B^\infty}{1 + c_1 B^\infty} + \delta(x)I_A^\infty - (\mu(x) + r_S(x))I_S^\infty, \quad x \in \mathbb{D}, \\ \dfrac{\partial V^\infty}{\partial \nu} = \dfrac{\partial I_A^\infty}{\partial \nu} = \dfrac{\partial I_S^\infty}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}, \\ \displaystyle\int_{\mathbb{D}} \left(\mu(x)S^\infty + \dfrac{\beta(x)S^\infty B_1^\infty}{1 + c_1 B_1^\infty}\right) \mathrm{d}x = \int_{\mathbb{D}} (1 - p(x))\Lambda(x)\mathrm{d}x. \end{cases} \tag{66}$$

Proof. By Theorem 4, the L^p-bound of (S, V, I_A, I_S) is valid for all $D_1 \geqslant 1$. The first equation of (45) can be rewritten as follows:

$$-D_1 \Delta S = (1 - p(x))\Lambda(x) - \mu(x)S - \frac{\beta(x)SB_1}{1 + c_1 B_1}, \quad x \in \mathbb{D}, \quad \frac{\partial S}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}.$$

Then, the elliptic L^p-theory Sobolev theorem and standard compactness argument guarantee that there exists a subsequence of $D_{S,k}$, labeled by D_k, with $D_k \to \infty$ as $k \to \infty$ and a corresponding positive solution $(S_k(x), V_k(x), I_{A,k}(x), I_{S,k}(x))$ with $D_1 = D_k$ satisfies $S_k \to S^\infty > 0$ in $C^1(\overline{\mathbb{D}})$ due to Lemma 8(*ii*). Further, S^∞ solves

$$-\Delta S^\infty = 0, \quad x \in \mathbb{D}, \quad \frac{\partial S^\infty}{\partial \nu} = 0, \quad x \in \partial\mathbb{D}.$$

Therefore, $S^\infty > 0$ must be a constant. Similar to the previous, with the help of the I_A and I_S equations of (45), passed to another subsequence if necessary, one has

$$I_{A,k} \to I_A^\infty, \quad I_{S,k} \to I_S^\infty \quad \text{on} \quad C^1(\overline{\mathbb{D}}), \ \text{as} \quad k \to \infty,$$

where $I_A^\infty \geqslant 0$ and $I_S^\infty \geqslant 0$ on $\overline{\mathbb{D}}$. As the Harnack inequality of I_A also holds, either $I_A^\infty > 0$ on $\overline{\mathbb{D}}$ or $I_A^\infty \equiv 0$ on $\overline{\mathbb{D}}$. Analyzing similarly to Step 2 in the proof of Theorem 4, let's proceed with the argument by contradiction and suppose that $I_A^\infty \equiv 0$ on $\overline{\mathbb{D}}$. Then from (53), one can derive that $I_S^\infty \equiv 0$ on $\overline{\mathbb{D}}$. Define $\tilde{I}_{A,k} := I_{A,k}/\|I_{A,k}\|_{L^\infty}(\mathbb{D})$. Thus, $\|\tilde{I}_{A,k}\|_{L^\infty(\mathbb{D})} = 1$ for $k \geqslant 1$ and $\tilde{I}_{A,k}$ satisfies

$$\begin{cases} -D_3\Delta\tilde{I}_{A,k} = \left[\dfrac{\theta\gamma_A(x)\beta(x)S_k}{\alpha(x)(1 + c_1B_{1,k})} + \dfrac{\sigma\gamma_A(x)\beta(x)V_k}{\alpha(x)(1 + c_2B_{1,k})} - (\mu(x) + r_A(x) + \delta(x))\right]\tilde{I}_{A,k} + \dfrac{\theta\gamma_S(x)\beta(x)S_k}{\alpha(x)(1 + c_1B_{1,k})\|I_{A,k}\|_{L^\infty(\mathbb{D})}} \\ \qquad + \dfrac{\sigma\gamma_S(x)\beta(x)V_k}{\alpha(x)(1 + c_2B_{1,k})\|I_{A,k}\|_{L^\infty(\mathbb{D})}}, \quad x \in \mathbb{D}, \\ \dfrac{\partial\tilde{I}_{A,k}}{\partial\nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases}$$

The proof that is similar to the Theorem 4 leads to $\tilde{I}_{A,k} \to \hat{I}_A^\infty$ in $C^1(\overline{\mathbb{D}})$ as $k \to \infty$, where $\hat{I}_A^\infty \geqslant 0$ on $\overline{\mathbb{D}}$ and $\|\hat{I}_A^\infty\|_{L^\infty(\mathbb{D})} = 1$. As $k \to \infty$, $I_{A,k} \to 0$ and $I_{S,k} \to \infty$, by sub-supersolution argument that $S^\infty = \int_{\mathbb{D}}(1 - p(x))\Lambda(x)\mathrm{d}x/\int_{\mathbb{D}}\mu(x)\mathrm{d}x$, $V^\infty = \int_{\mathbb{D}}p(x)\Lambda(x)\mathrm{d}x/\int_{\mathbb{D}}\mu(x)\mathrm{d}x$. Thus, $\hat{I}_A^\infty$ solves

$$\begin{cases} -D_3\Delta\hat{I}_A^\infty = \left[\dfrac{\theta\gamma_A(x)\beta(x)\int_{\mathbb{D}}(1 - p(x))\Lambda(x)\mathrm{d}x}{\alpha(x)\int_{\mathbb{D}}\mu(x)\mathrm{d}x} + \dfrac{\sigma\gamma_A(x)\beta(x)\int_{\mathbb{D}}p(x)\Lambda(x)\mathrm{d}x}{\alpha(x)\mathbb{D}\,\mu(x)\mathrm{d}x} - (\mu(x) + r_A(x) + \delta(x))\right]\hat{I}_A^\infty, \quad x \in \mathbb{D}, \\ \dfrac{\partial\hat{I}_A^\infty}{\partial\nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases}$$

It is evident that the Harnack-type inequality also holds for $\hat{I}_A^\infty$, then $\hat{I}_A^\infty > 0$ on $\overline{\mathbb{D}}$. Define $\tilde{I}_{S,k} := I_{S,k}/\|I_{S,k}\|_{L^\infty(\mathbb{D})}$, it is easy to derive $\tilde{I}_{S,k} \to \hat{I}_S^\infty$ in $C^1(\overline{\mathbb{D}})$ as $k \to \infty$, where $\hat{I}_S^\infty \geqslant 0$ on $\overline{\mathbb{D}}$ and satisfies

$$\begin{cases} -D_4\Delta\hat{I}_S = \left(\dfrac{(1 - \theta)\beta(x)\gamma_S(x)\int_{\mathbb{D}}(1 - p(x))\Lambda(x)\mathrm{d}x}{\alpha(x)\int_{\mathbb{D}}\mu(x)\mathrm{d}x} - (\mu(x) + r_S(x)) \right. \\ \qquad \left. + \dfrac{\gamma_S(x)\beta(x)}{\alpha(x)} \times \dfrac{\delta(x)\left(\theta\int_{\mathbb{D}}\Lambda(x)(1 - p(x))\mathrm{d}x + \sigma\mathrm{d}x\Lambda(x)p(x)\right)}{(-D_3\Delta + (\mu(x) + r_A(x) + \delta(x)))\int_{\mathbb{D}}\mu(x)\mathrm{d}x}\right)\hat{I}_S, \quad x \in \mathbb{D}, \\ \dfrac{\partial\hat{I}_S}{\partial\nu} = 0, \quad x \in \partial\mathbb{D}. \end{cases}$$

Therefore, 0 is the principal eigenvalue of (19) and (23). Combined with (18) and Lemma 4. This contradiction with $\mathcal{R}_0 > 1$. As a result, $I_A^\infty > 0$ satisfies (66). This completes the proof. □

As far as Theorem 7 is concerned, the method of proof is very similar to Theorem 6, so that we simply revise it slightly and then omit the details. It turns out to be as follows.

Theorem 7. *Suppose that* $\mathcal{R}_0 > 1$. *Fix* D_1, D_2, D_4 *are positive constants and let* $D_3 \to \infty$ (*up to a subsequence*), *then every solution* $(S(x), V(x), I_A(x), I_S(x))$ *of* (45) *meets*

$$(S(x), V(x), I_A(x), I_S(x)) \to (S_\infty(x), V_\infty(x), I_{A,\infty}(x), I_{S,\infty}(x)) \quad \text{uniformly on} \quad \overline{\mathbb{D}},$$

where I_∞ *is the normal number,* $S_\infty, V_\infty, I_{A,\infty}, I_{S,\infty} > 0$ *on* $\overline{\mathbb{D}}$, *and* $(S_\infty, V_\infty, I_{A,\infty}, I_{S,\infty})$ *satisfies*

$$\begin{cases} -D_1\Delta S_\infty = (1-p(x))\Lambda(x) - \mu(x)S_\infty - \dfrac{\beta(x)S_\infty B_{1,\infty}}{1+c_1B_{1,\infty}}, & x \in \mathbb{D}, \\ -D_2\Delta V_\infty = p(x)\Lambda(x) - (\mu(x)+\eta(x))V_\infty - \dfrac{\sigma\beta(x)V_\infty B_{1,\infty}}{1+c_1B_{1,\infty}}, & x \in \mathbb{D}, \\ -D_4\Delta I_{S,\infty} = \dfrac{(1-\theta)\beta(x)S_\infty B_{1,\infty}}{1+c_1B_{1,\infty}} + \delta(x)I_{A,\infty} - (\mu(x)+r_S(x))I_{S,\infty}, & x \in \mathbb{D}, \\ \dfrac{\partial S_\infty}{\partial \nu} = \dfrac{\partial V_\infty}{\partial \nu} = \dfrac{\partial I_{S,\infty}}{\partial \nu} = 0, & x \in \partial\mathbb{D}, \\ \displaystyle\int_{\mathbb{D}} \left(\dfrac{\theta\beta(x)S_\infty B_{1,\infty}}{1+c_1B_{1,\infty}} + \dfrac{\sigma\beta(x)V_\infty B_{1,\infty}}{1+c_1B_{1,\infty}} - (\mu(x)+r(x)+\delta(x))I_{A,\infty}\right)\mathrm{d}x = 0. \end{cases} \tag{67}$$

7 Numerical simulations

With this section, we use numerical examples to validate our theoretical findings and explore the impacts of individuals movement and spatial heterogeneity on $\mathcal{R}_0$ and disease dynamics. Suppose that the spatial domain $\mathbb{D}$ is one dimensional and select $\mathbb{D} = [0, \pi]$. Specifically, we consider the general incidence functions $f(S,B) = SB/(1+c_1B)$, $g(V,B) = VB/(1+c_2B)$, where c_1 and c_2 denote the saturation factors. Let's fix the parameters $D_1 = 0.02$, $D_2 = 0.05$, $D_3 = 0.008$, $D_4 = 0.005$, $\sigma = 0.1$, $\theta = 0.6$, $c_1 = 0.000002$, $c_2 = 0.000004$, $\mu(x) = 4.5 \times 10^{-5}(1+0.5\sin 3x)$, $p(x) = 0.7(1+0.2\sin 3x)$, $\Lambda(x) = 10(1+0.5\sin 3x)$, $\eta(x) = 0.0011(1+0.5\sin 3x)$, $\delta(x) = 0.23(1+0.1\sin 3x)$, $\gamma_A(x) = 20(1+0.1\sin 3x)$, $\gamma_S(x) = 40(1+0.1\sin 3x)$, and initial data

$$\mathcal{U}(x) = \begin{pmatrix} 2.24 \times 10^5 - 20{,}000\cos 2x \\ 6{,}700 - 250\cos 2x \\ 80 + 20\cos 2x \\ 260 + 80\cos 2x \\ 2.22 \times 10^4 + 5{,}000\cos 2x \end{pmatrix}, \quad \forall x \in [0, \pi], \quad \mathcal{U} = (S_0, V_0, I_{A0}, I_{S0}, B_0)^T.$$

First, we select $r_A(x) = 0.44(1+0.1\sin 3x)$, $r_S(x) = 0.22(1+0.1\sin 3x)$, $\alpha(x) = 0.35(1+0.1\sin 3x)$ and $\beta(x) = 9.9 \times 10^{-9}(1+0.5\sin 3x)$, the remaining parameters are shown earlier. The approach of calculation from [32], it can be derived that $\mathcal{R}_0 \approx 0.9824 < 1$. It follows by the Theorem 1 disease-free steady state $\mathcal{E}_0$ is

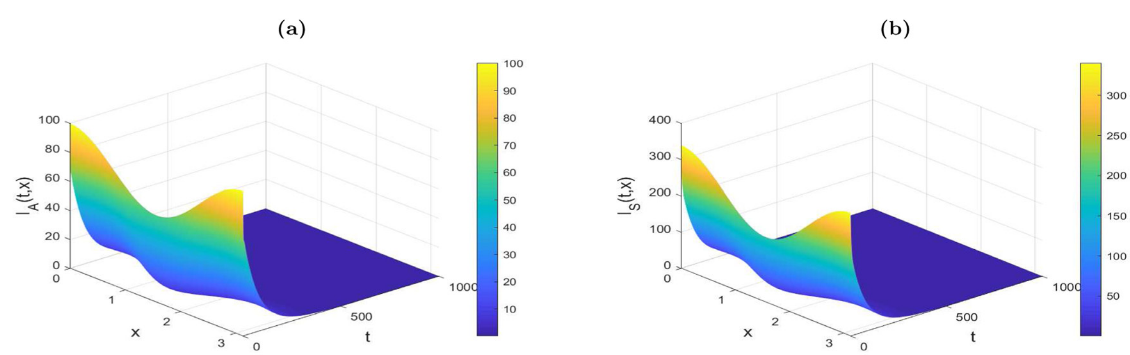

Figure 2: The spatio-temporal distribution of $I_A(x,t)$ and $I_S(x,t)$ with $\mathcal{R}_0 \approx 0.9824$: (a) $I_A(x,t)$ and (b) $I_S(x,t)$.

globally asymptotically stable. Indeed, we can observe in Figure 2(a) and (b) that the distributions of asymptomatic individuals $I_A(x,t)$ and symptomatic individuals $I_S(x,t)$ eventually converge to 0 as time t increases.

If the parameters $r_A(x) = 0.35(1 + 0.9\sin 3x)$, $r_S(x) = 0.18(1 + 0.9\sin 3x)$, $\alpha(x) = 0.3(1 + 0.8\sin 3x)$ and $\beta(x) = 1.6 \times 10^{-8}(1 + 0.5\sin 3x)$ are varied, other parameters are fixed as Figure 2. The direct calculation in

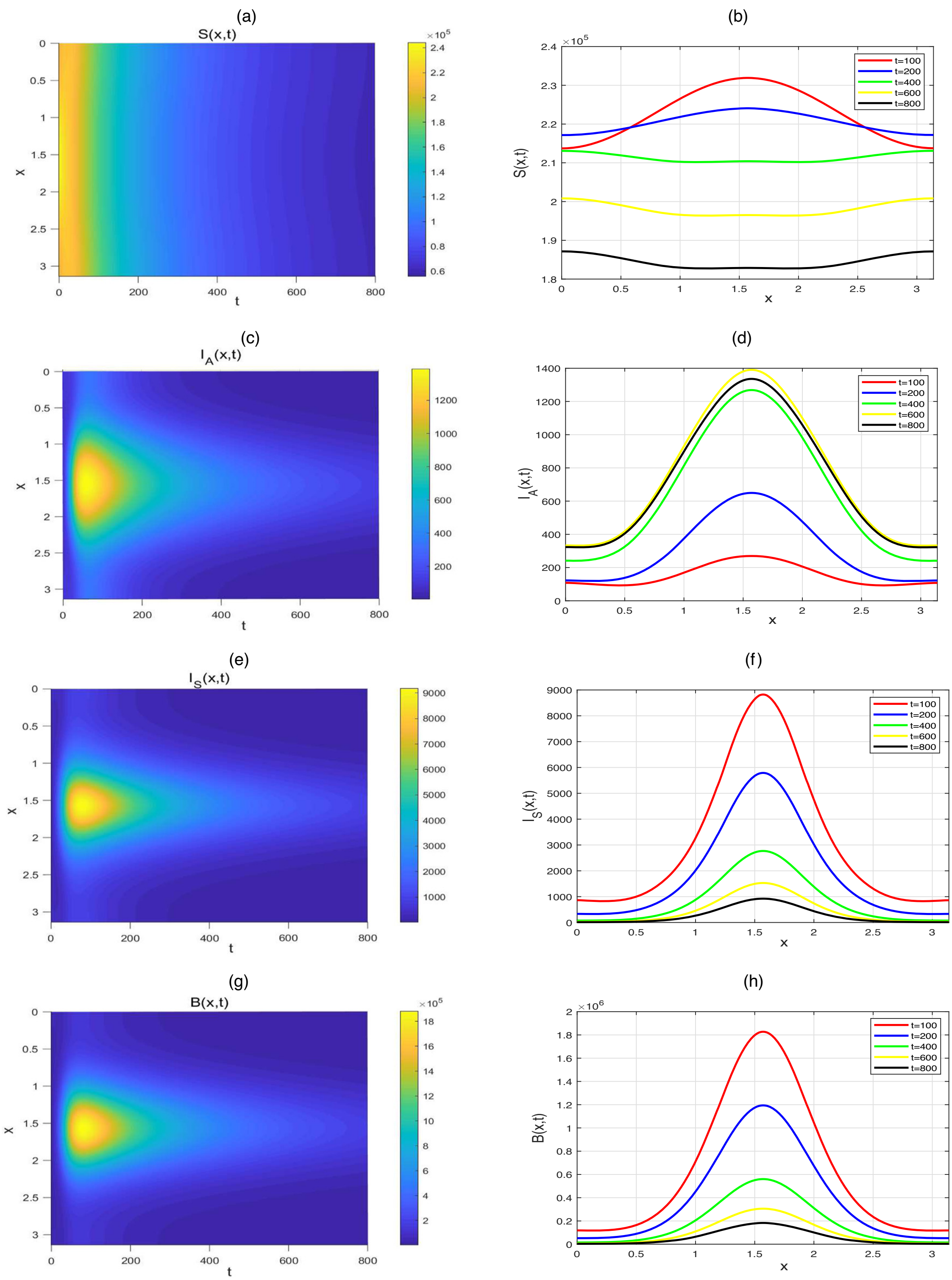

Figure 3: The persistence of disease for model (1) with $\mathcal{R}_0 \approx 1.1069 > 1$: (a), (c), (e), (g): Spatio-temporal evolution of $S(x,t)$, $I_A(x,t)$, $I_S(x,t)$ and $B(x,t)$; (b), (d), (f), (j): $S(x,t)$, $I_A(x,t)$, $I_S(x,t)$, and $B(x,t)$ cross-section curves at various times.

this case yields $\mathcal{R}_0 \approx 1.1069 > 1$. This is as shown by the Theorem 2 that the epidemic is persistent, and there is at least one epidemic steady state for model (1). It is apparent from Figure 3(a)–(h) that $S(x,t)$, $I_A(x,t)$, $I_S(x,t)$ and $B(x,t)$ converge to a positive steady state over the entire spatial region. It is also shown in Figure 3(b), (d),

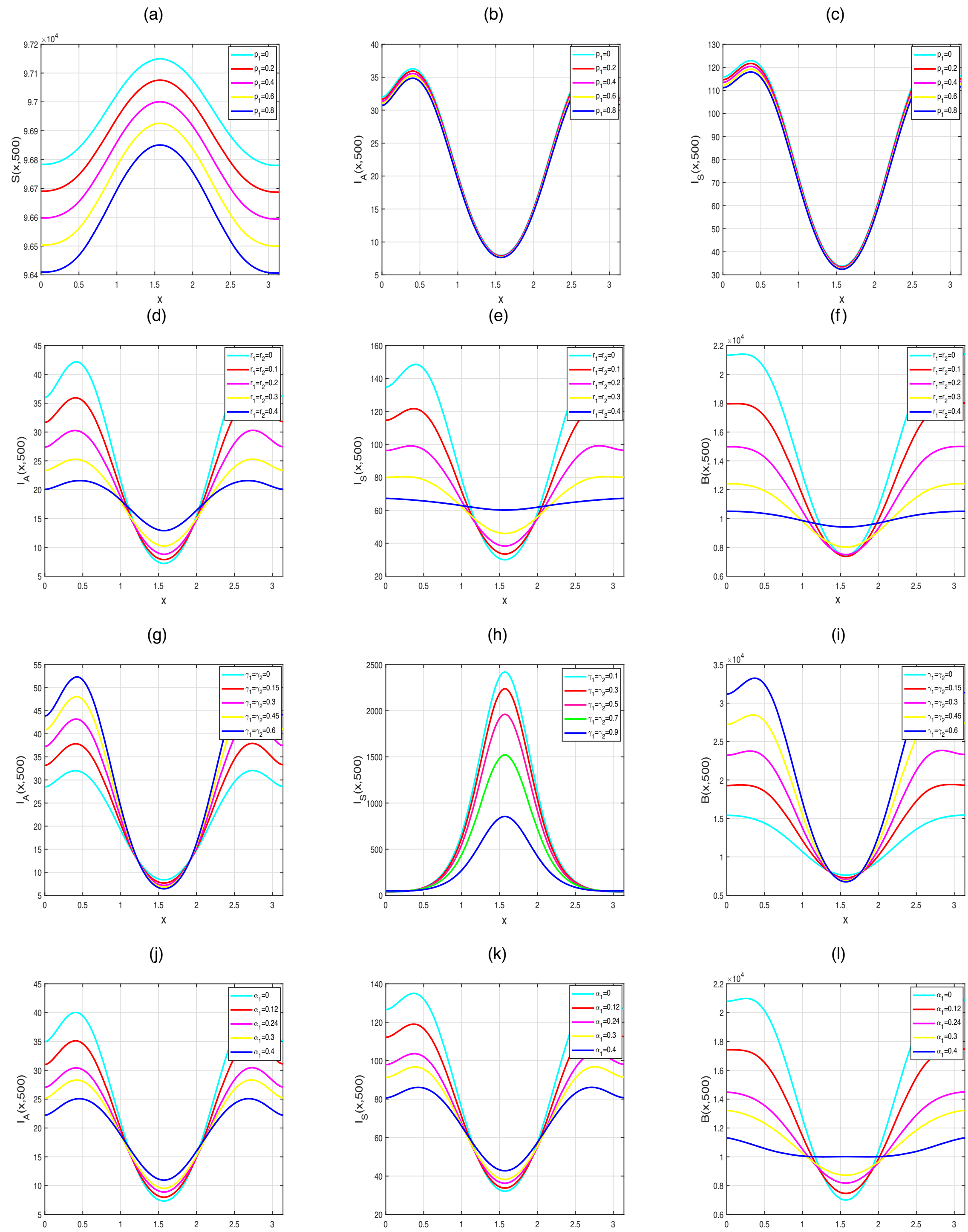

Figure 4: Effect of spatial heterogeneity parameters on the disease distribution of model (1): (a)–(c) effect of $p(x)(x)$ on $S(x,500)$, $I_A(x,500)$ and $I_S(x,500)$; (d)–(f) effect of $r_A(x)$ and $r_S(x)$ on $I_A(x,500)$, $I_S(x,500)$ and $B(x,500)$; (g)–(i) effect of $\gamma_A(x)$ and $\gamma_S(x)$ on $I_A(x,500)$, $I_S(x,500)$ and $B(x,500)$; (j)–(k) effect of $\alpha(x)$ on $I_A(x,500)$, $I_S(x,500)$ and $B(x,500)$.

(e), (j) that owing to spatial heterogeneity, there are area differences in the distributions of $S(x,t)$, $I_A(x,t)$, $I_S(x,t)$, and $B(x,t)$ across distinct time scales.

Subsequently, we consider the effects of the spatial heterogeneity parameters on disease propagation. In Figure 4(a)–(c) describe the effect of the vaccination rate $p(x)$ on the spatial distribution of susceptible individuals $S(x,t)$, asymptomatic individuals $I_A(x,t)$ and symptomatic individuals $I_S(x,t)$ at time $t = 500$. At this point, one selects $p(x) = 0.35(1 + p_1 \sin 3x)$, and p_1 is gradually increased from 0, 0.2, 0.4 to 0.6. With increasing spatial heterogeneity in vaccination rates p_1, $S(x,500)$ shows greater regional diversity, whereas $I_A(x,500)$ and $I_S(x,500)$ are not significantly fluctuating. This is due to the fact that vaccinated people are to remain at a high danger of contracting the infection. Thus, mass vaccination should be complemented by a focus on the effectiveness of the vaccine. Moreover, from Figure 4(d)–(f), it is convenient to notice that when the spatial heterogeneity of the recovery rate $(r_A(x), r_S(x)) = (0.35(1 + r_1 \sin 3x), 0.18(1 + r_2 \sin 3x))$ changes, i.e., r_1 and r_2 increase from 0, 0.15, 0.3, 0.45 to 0.6, the heterogeneity of $I_A(x,500)$, $I_S(x,500)$, and $B(x,500)$ decreases. In Figure 4(d) and (e), it can be seen that at the same point, e.g., $[0,1]$, the peaks of $I_A(x,500)(I_S(x,500))$ decrease with the enhancement of $r_1(r_2)$, which indicates that peaks in disease outbreaks can be reduced to some extent by increasing the heterogeneous strength of recovery rates. It can be found from Figure 4(g)–(i) that when the

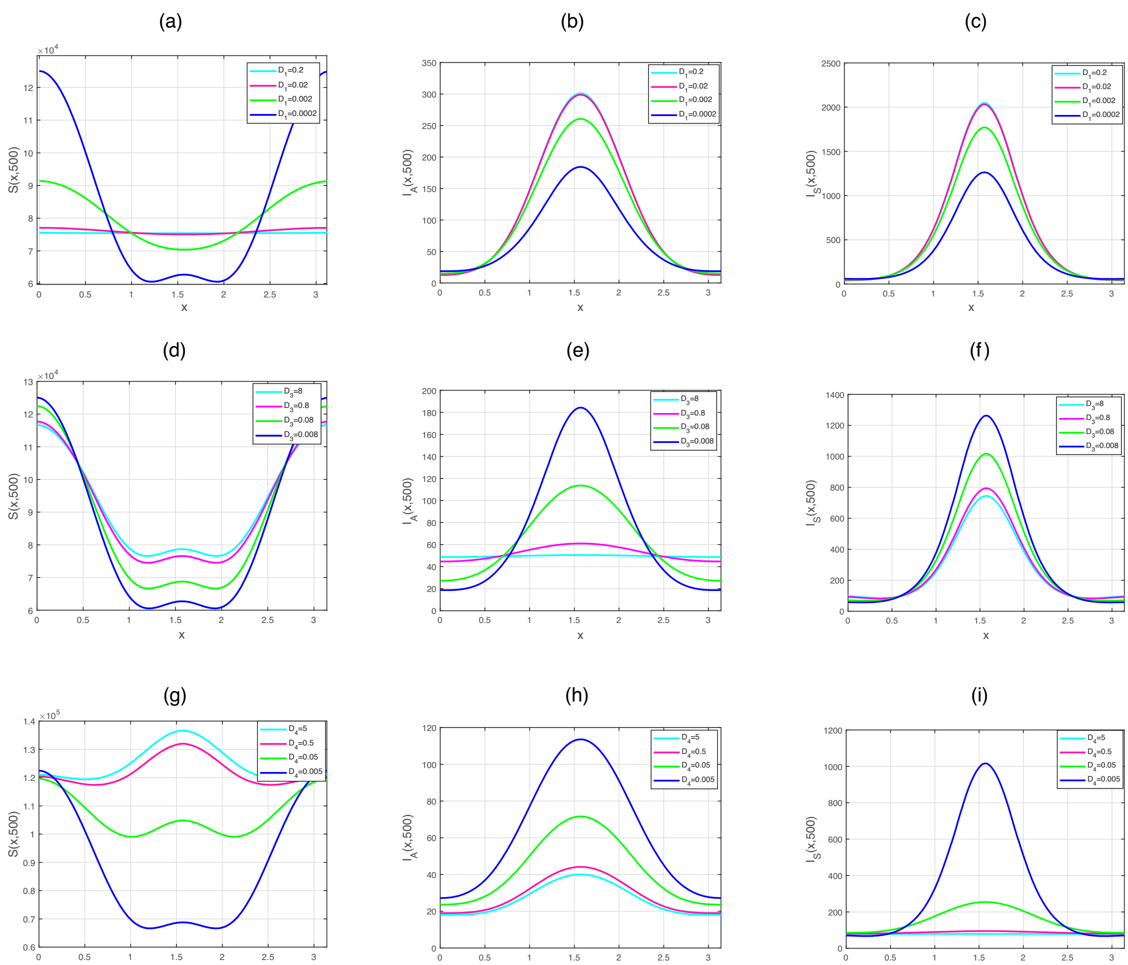

Figure 5: Distribution of different diffusion coefficients for $S(x,t)$, $I_A(x,t)$, and $B(x,t)$ prevalence at $t = 500$: (a)–(c) D_1; (d)–(f) D_3; (g)–(i) D_4.

spatial heterogeneity intensity $(\gamma_A(x), \gamma_S(x)) = (20(1 + \gamma_1 \sin 3x), 40(1 + \gamma_2 \sin 3x))$ of $\gamma_1(\gamma_2)$ increases from 0, 0.15, 0.3, 0.45 to 0.6, the regional differences in $I_A(x, 500)$ and $B(x, 500)$ become more apparent, while the regional differences in $I_S(x, 500)$ weaken, which is due to the fact that most of the infected individuals are asymptomatic, and asymptomatic individuals take some time to become symptomatic individuals. Meanwhile, it can be noticed from Figure 4(j)–(l) that when the intensity of spatial heterogeneity of $\alpha(x) = 0.3(1 + \alpha_1 \sin 3x)$ is varied, i.e., α_1 is increased from 0, 0.12, 0.24 and 0.3 to 0.4, $I_A(x, 500)$, $I_S(x, 500)$ and $B(x, 500)$ are all decreasing in spatial heterogeneity, which further suggests that improving local water sanitation and personal hygiene practices is also one way to control diseases.

Next, let's explore how diffusion coefficients affect disease propagation. Figure 5(a)–(c) displays the spatial distribution of $S(x, t)$, $I_A(x, t)$ and $I_S(x, t)$ at $t = 500$ as the diffusion rate of the susceptible individuals d_1 varies from 0.0002, 0.002, 0.02 to 0.2. It can be seen that $S(x, 500)$ becomes progressively homogeneous throughout the region as D_1 increases, due to the low other diffusion coefficients (D_2, D_3, D_4), the distributions of $I_A(x, 500)$ and $I_S(x, 500)$ are reliant on the features of local illness propagation. The Figure 5(d)–(f) reveal that the spatial distribution of $I_A(x, 500)$ gradually changes from heterogeneous to homogeneous as the diffusion rate of asymptomatic infected individuals D_3 increases from 0.0008, 0.008, 0.08, 0.8 to 8, $I_A(x, 500)$ and $I_S(x, 500)$ also show decreasing spatial variability and peaks in disease outbreaks, which suggests that restricting the movement of asymptomatic individuals could decrease the level of danger in high-risk areas to some extent, while increasing the level of danger in low-risk areas, this result is similar to that of Figure 5(g)–(i). In addition, we also found that the movement of susceptible individuals has a small effect on the spatial heterogeneity and peak value of the disease distribution in the spreading process of infection in diverse areas, whereas the spreading of infected individuals I_A and I_S contributes significantly to the spatial heterogeneity of infection. Therefore, it is possible to appropriately liberalize the mobility of susceptible individuals during the spread of the disease, and focusing on restricting the mobility of infected individuals is a necessary means of controlling the disease.

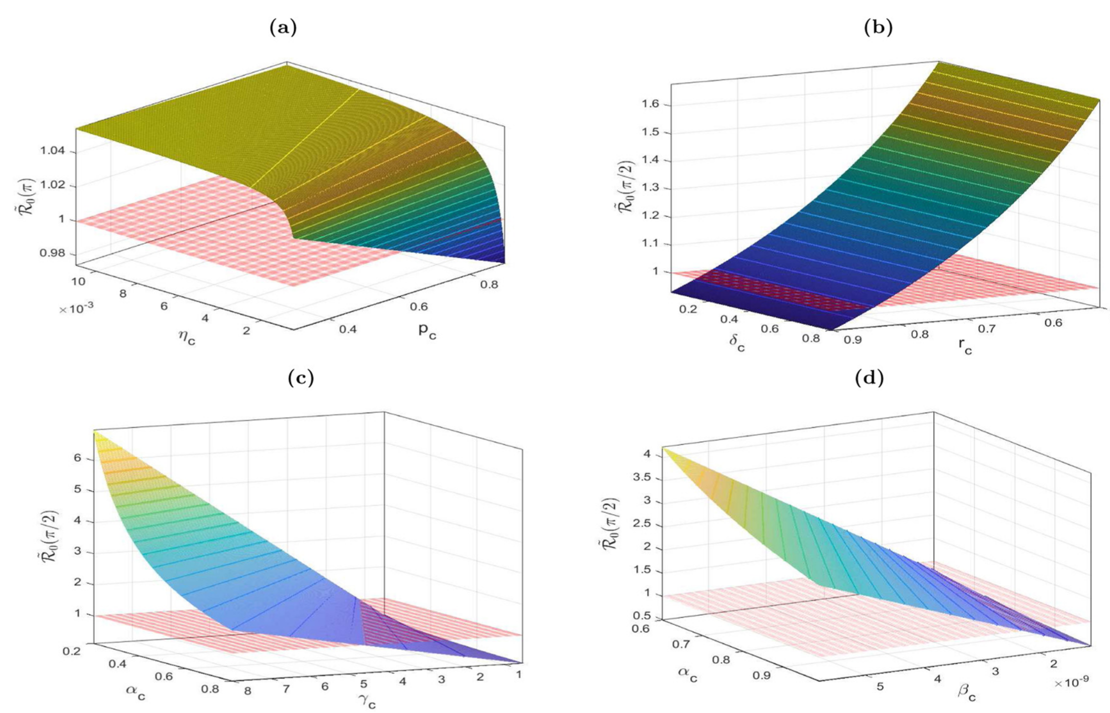

Figure 6: The correlation between the local reproduction number $\tilde{\mathcal{R}}_0(x)$ and some parameters: (a) $\tilde{\mathcal{R}}_0(x)$ with $p(x) = p_c(1 + 0.2 \sin 3x)$ and $\eta(x) = \eta_c(1 + 0.5 \sin 3x)$ at $x = \pi$; (b) $\tilde{\mathcal{R}}_0(x)$ with $(r_A(x), r_S(x)) = r_c(1 + 0.5 \sin 3x, 1 + 0.4 \sin 3x)$, and $\delta(x) = \delta_c(1 + 0.1 \sin 3x)$ at $x = \pi/2$; (c) $\tilde{\mathcal{R}}_0(x)$ with $(\gamma_A(x), \gamma_S(x)) = \gamma_c(1 + 0.1 \sin 3x, 1 + 0.2 \sin 3x)$ and $\alpha(x) = \alpha_c(1 + 0.8 \sin 3x)$ at $x = \pi/2$; (d) $\tilde{\mathcal{R}}_0(x)$ with $\beta(x) = \beta_c(1 + 0.5 \sin 3x)$ and $\alpha(x) = \alpha_c(1 + 0.8 \sin 3x)$.

In addition, let's be interested in the correlation between the primary parameters and the local reproduction number $\tilde{\mathcal{R}}_0(x)$. Here we choose $p(x) = p_c(1 + 0.2\sin 3x)$, $\eta(x) = \eta_c(1 + 0.5\sin 3x)$, $r_A(x) = r_c(1 + 0.5\sin 3x)$, $r_S(x) = r_c(1 + 0.4\sin 3x)$, $\delta(x) = \delta_c(1 + 0.1\sin 3x)$, $\gamma_A(x) = \gamma_c(1 + 0.1\sin 3x)$, $\gamma_S(x) = \gamma_c(1 + 0.2\sin 3x)$, $\alpha(x) = \alpha_c(1 + 0.8\sin 3x)$, $\beta(x) = \beta_c(1 + 0.5\sin 3x)$, and $\alpha(x) = \alpha_c(1 + 0.8\sin 3x)$. As shown in Figure 6(a), it is not uncommon to find that $\tilde{\mathcal{R}}_0(\pi)$ decreases with the increase of p_c and increases with the increase of η_c, and when $\eta_c \in [1 \times 10^{-4}, 2 \times 10^3]$, the $\tilde{\mathcal{R}}_0(\pi)$ grows faster, while $\tilde{\mathcal{R}}_0(\pi)$ grows more slowly when $\eta_c > 2 \times 10^3$. This also suggests that simply increasing the effectiveness of immunization is not an effective way to control the disease. Furthermore, in Figure 6(b)–(d) also show that at $x = \pi/2$, $\delta(x)$, $\gamma_A(x)$, $\gamma_S(x)$ and $\beta(x)$ are positively related to $\tilde{\mathcal{R}}_0(x)$, whereas $r_A(x)$, $r_S(x)$ and $\alpha(x)$ is a negative correlation. It also demonstrates that infection prevention can be attained by reducing the transmission rate between individuals and pathogens, the transfer rate from asymptomatic to infected individuals, increasing the cure rate of patients, and thoroughly disinfecting the excreta of patients and carriers. In addition, there is some risk of infection transmission due to pathogens in

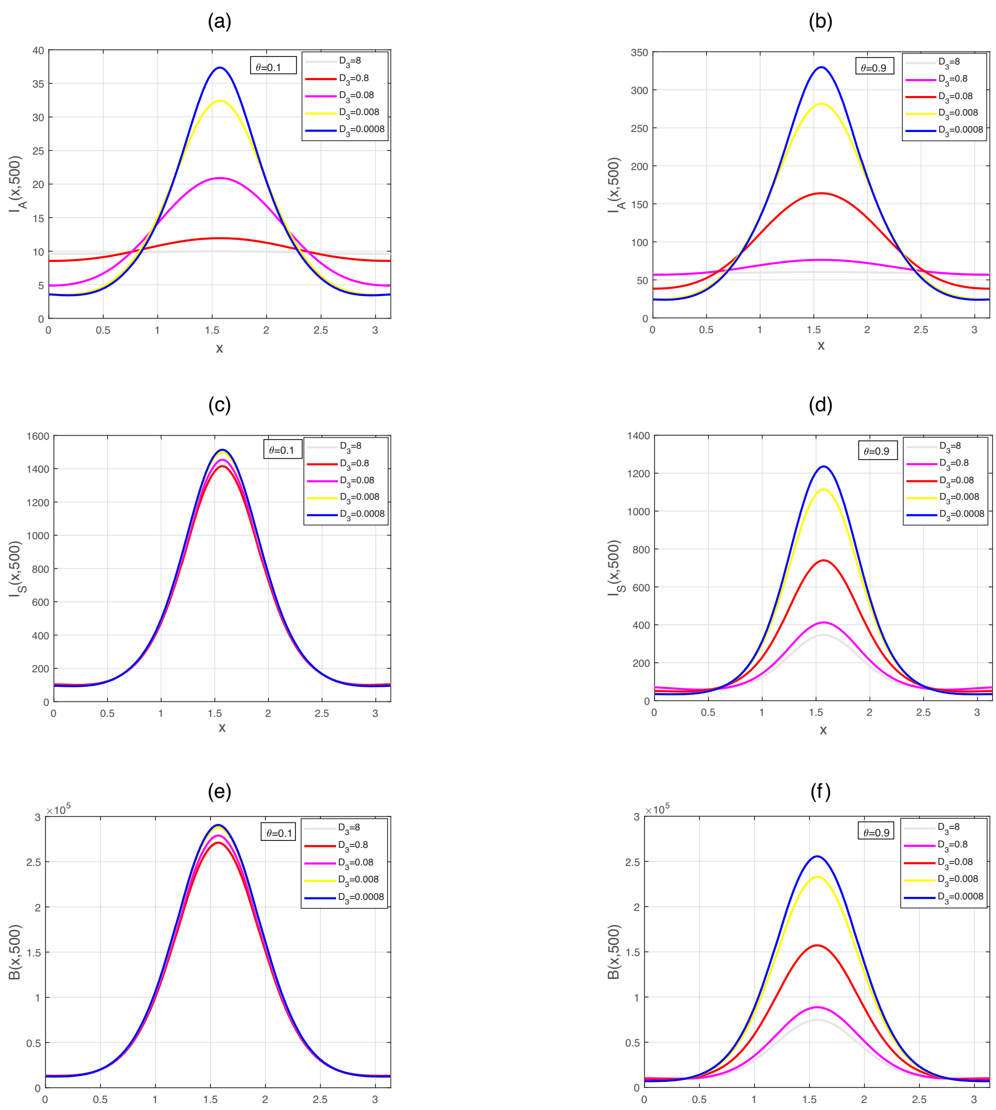

Figure 7: The effects of σ and D_3 on the spatial distribution of disease at $t = 500$: (a), (c), (e) the effects on $I_A(x, 500)$, $I_S(x, 500)$ and $B(x, 500)$ for different values of D_3 with $\sigma = 0.1$; (b), (d), (f) the effects on $I_A(x, 500)$, $I_S(x, 500)$ and $B(x, 500)$ for different values of D_3 with $\sigma = 0.9$.

the environment, i.e., $\gamma_A(x)$ and $\gamma_S(x)$ are positively correlated with $\tilde{\mathcal{R}}_0(x)$, there is a need to lessen $\gamma_A(x)$ and $\gamma_S(x)$ to reduce the number of pathogens in the environment and thus eradicate the risk of environmental spread.

Finally, we analyze in depth the effect of the proportion of asymptomatic infected individuals as a percentage of the whole infected individuals θ and D_3 on the disease dynamics. In Figure 7(a)–(f), it can be observed that as D_3 increases from 0.0008, 0.008, 0.08, 0.8 to 8, $I_A(x, 500)$, $I_S(x, 500)$, and $B(x, 500)$ will have different spatial distributions at $\theta = 0.1$ and $\theta = 0.9$, i.e., when $\theta = 0.1$, the diffusion rate D_3 has only a more significant effect on $I_A(x, 500)$ only, and very little effect on the other individuals. In contrast, when $\theta = 0.9$, D_3 has a large effect on every individuals. It also suggests that an increase in the percentage of asymptomatic individuals intensifies the spread of the disease, making it more difficult to control. Therefore, special attention should be devoted to the proportion of hidden infections to those who contract the disease. When θ is small, a policy of slow liberalization and gradual return to normal life can be adopted for groups in low-risk areas, while for high-risk areas, rapid screening of asymptomatic infected individuals is a priority.

8 Conclusion

To this end, we proposed a model of degenerate cholera with asymptomatic individuals in spatially heterogeneous environments that takes into account the rate of immune loss and general morbidity, and assumes that environmental viruses do not disperse. We studied the existence of global positive solutions to the model and proved that $\Phi(t)$ is ultimately bounded (Lemmas 1 and 2). Since model (1) lacks tightness, the τ-contraction method is utilized to justify the asymptotic smoothness of $\Phi(t)$, which proves the existence of a globally tight attractor for $\Phi(t)$ (see Lemma 3). Furthermore, we identify the basic reproduction number $\mathcal{R}_0$, give its exact expression, which is a major innovation of this article. By using $\mathcal{R}_0$ as the threshold parameter, the dynamical behavior of the model is studied. More precisely, the disease-free steady state is globally asymptotically stable if $\mathcal{R}_0 \leqslant 1$ (Theorems 1 and 3); the disease is uniformly persistent if $\mathcal{R}_0 > 1$ (see Theorem 2). Finally, it is also discussed that the asymptotic behavior of the endemic steady state as d_1 and d_3 tends to 0 or ∞, respectively (see Theorems 4 and 5, Theorems 6 and 7).

In the numerical simulation section, except for simulating the extinction and persistence of disease (Figures 2 and 3), the impacts of some crucial parameters on the spatial and temporal distributions of the disease are also explored. To be more specific, the spatial heterogeneity of model will lead to a regional variability in the spatial distribution of disease (Figure 4). Furthermore, numerical simulations show that the spread of susceptible individuals does not modify the risk level of local outbreaks when the spread of asymptomatic and infected individuals is relatively small (Figure 5(a)–(c)). This is consistent with the conclusions of [37], which stated that cholera cannot be controlled by limiting the movement of susceptible individuals. We also found the spread of asymptomatic and symptomatic individuals suppresses spatial heterogeneity so that each individual is less differentiated across the region (Figure 5(d)–(i)). Therefore, it is not advisable to entirely limit the mobility of people during epidemic periods of the disease. Further, we also analyze how the proportion of asymptomatic infected individuals to the whole infected individuals θ and d_3 affects the spread of disease (Figure 7(a)–(f)), and the results show that the proportion of asymptomatic individuals θ can change the dynamics of the epidemic and performs a critical role in the process of the spread of the disease. In real life, asymptomatic individuals are asymptomatic, which makes them unaware of whether they are infected, which can eventually lead to large-scale outbreaks of the disease, a finding similar to that in [36]. In Figure 6, we also find that thorough disinfection of the excreta of patients and carriers, as well as the surrounding area are important ways tocontrol and eradicate the risk of environmental spread.

Unfortunately, in this article, we only considered the effects of environment spread on the disease and ignored host-to-host spread (direct transmission), which exists in real situations. However, we know that cholera outbreaks are also related to seasons and temperatures, therefore, a response diffusion cholera model with cyclical and climatic variations would be more meaningful and the results would be more effective in preventing and controlling the disease.

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