

Research Article

Hui-Sheng Ding, Quan Liu, Wei Long*, and Lan Zhong

Choquard equations with recurrent potentials

<https://doi.org/10.1515/anona-2024-0057>

received May 13, 2024; accepted November 7, 2024

Abstract: In this article, we are concerned with the existence of nontrivial solutions to the Choquard equation

$$-\Delta u + \alpha(x)u = (|x|^{-\mu} * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N \quad (N \geq 2),$$

with recurrent potential α , where $0 < \mu < N$ and $\frac{2N-\mu}{N} < q < \frac{2N-\mu}{N-2}$. Our results include some classical cases where α is constant and α is periodic, as well as some new cases, such as α being almost periodic and α being only bounded and uniformly continuous.

Keywords: Choquard equation, recurrent potential, almost periodic function

MSC 2020: 35J20, 35J60, 47G20

1 Introduction

We consider the problem

$$-\Delta u + \alpha(x)u = (|x|^{-\mu} * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N \quad (N \geq 2), \quad (1.1)$$

where $0 < \mu < N$ and $\frac{2N-\mu}{N} < q < \frac{2N-\mu}{N-2}$. Equation (1.1) is usually called the nonlinear Choquard (or Choquard-Pekar) equation, and it has a strong physical background. Equation (1.1) originates from the model of Fröhlich and Pekar for polaron (cf. [7,21]). In the approximation developed by the Hartree-Fock theory of one component plasma, Choquard also used equation (1.1) to describe an electron trapped in its own hole (cf. [13]). In addition, the Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics with nonrelativistic Newtonian gravity (cf. [11,22]). Moreover, the equation can be derived from the Einstein-Klein-Gordon and Einstein-Dirac system (cf. [9]). Such a model was also proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter (cf. [10,26]).

When the potential $\alpha \equiv 1$, problem (1.1) becomes

$$-\Delta u + u = (|x|^{-\mu} * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N \quad (N \geq 2), \quad (1.2)$$

and it is called the autonomous homogeneous Choquard equation. Some classical results about existence, symmetry and decay of ground state solutions for equation (1.2) have been established (cf. [13,15,16,18,19,28] and the references therein). For some recent relevant contributions to some variants of equation (1.2), we refer

* **Corresponding author: Wei Long**, School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, People's Republic of China, e-mail: lw hope@jxnu.edu.cn

Hui-Sheng Ding: School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, People's Republic of China, e-mail: dinghs1979@jxnu.edu.cn

Quan Liu: School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, People's Republic of China, e-mail: quan_liu@jxnu.edu.cn

Lan Zhong: School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, People's Republic of China, e-mail: 2691552681@qq.com

the reader to [3,17]. When the potential α is nonconstant, problem (1.1) becomes more tricky, and most known results focus on the cases of α being a perturbation of a constant or periodic. For example, Lions [15] studied the existence of solutions to problem (1.1) with $p = 2$ when α is a perturbation of a constant potential, and recently in [2,4], minimizers for the energy function have been constructed when α is a positive periodic function. Moreover, Ackermann and Qin et al. [1,24] considered the existence of solutions for problem (1.1) with a sign changing periodic potential.

In this article, we are interested in problem (1.1) with a more general positive potential α . As one will see, our recurrent assumptions on α include many special cases such as α being constant, α being periodic, α being almost periodic, and even α being only bounded and uniformly continuous. More specifically, we assume that the potential function α satisfies the following conditions:

- (α_1) α is bounded and continuous on $\mathbb{R}^N$.
- (α_2) $\inf_{x \in \mathbb{R}^N} \alpha(x) > 0$.
- (α_3) for any sequence $\{y_n\} \subset \mathbb{R}^N$, there exist a subsequence $\{y_{n_k}\}$ and a function $\alpha^h(x) : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$\lim_{k \rightarrow \infty} \alpha(x + y_{n_k}) = \alpha^h(x), \quad \text{for all } x \in \mathbb{R}^N.$$

Hereafter, we denote $\text{Hull}(\alpha)$ as the set of all these $\alpha^h(x)$.

- (α'_3) $\alpha(x) = \alpha(x^1, \dots, x^N)$ is T_i -periodic in x^i for $i = 2, \dots, N$ and *almost periodic* in x^1 uniformly with respect to $(x^2, x^3, \dots, x^N) \in [0, T_2] \times [0, T_3] \times \dots \times [0, T_N]$ (c.f. Definition A.1 in Appendix).

Remark 1.1. We note that (α_3) is a pretty weak assumption, which includes many special cases. In fact, (α_3) automatically holds if α is constant, continuous and periodic, almost periodic, or even bounded and uniformly continuous. Moreover, if $\alpha = \beta + \gamma$, where β satisfies (α_3) and γ is continuous on $\mathbb{R}^N$ with

$$\lim_{x \rightarrow \infty} \gamma(x) = 0,$$

then α still satisfies (α_3) . Moreover, (α_3) is weaker than the recurrent assumptions in [20], where Pankov considered a semilinear uniformly elliptic partial differential equation with nonstabilizing coefficients.

Since the function α^h in (α_3) is not necessarily a translation of α , which is the biggest difference of (α_3) from the corresponding assumptions in earlier results, we need consider the following *shell equation* of (1.1)

$$-\Delta u + \alpha^h(x)u = (|x|^{-\mu} * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \quad (1.3)$$

where $\alpha^h(x) \in \text{Hull}(\alpha)$. Denote

$$A_\alpha = \left\{ \beta \in C(\mathbb{R}^N, \mathbb{R}) \mid \inf_{x \in \mathbb{R}^N} \alpha(x) \leq \beta(x) \leq \sup_{x \in \mathbb{R}^N} \alpha(x) \quad \text{for all } x \in \mathbb{R}^N \right\}.$$

It is easy to see that $\text{Hull}(\alpha) \subset A_\alpha$. For every $\beta \in A_\alpha$ and $u \in H^1(\mathbb{R}^N)$, let

$$J(\beta, u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \beta(x)|u|^2) dx - \frac{1}{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy,$$

and

$$\langle J'(\beta, u), \varphi \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + \beta(x)u\varphi) dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^{q-2} u(y) \varphi(y)}{|x-y|^\mu} dx dy, \quad \varphi \in H^1(\mathbb{R}^N).$$

Hereafter, for simplicity, we denote $J(\alpha, u) = J(u)$, $J'(\alpha, u) = J'(u)$, $J(\alpha^h, u) = J^h(u)$, and $J'(\alpha^h, u) = J^h'(u)$ if there is no confusion.

The Hardy-Littlewood-Sobolev inequality (cf. [14]) implies that $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy$ is well defined for $u \in H^1(\mathbb{R}^N)$ if $\frac{2N-\mu}{N} \leq q \leq \frac{2N-\mu}{N-2}$. Usually, $\frac{2N-\mu}{N}$ is called the *lower critical exponent* and $\frac{2N-\mu}{N-2}$ is the *upper critical exponent* of the Choquard equation. The upper critical exponent plays a similar role as the Sobolev critical

exponent in local semilinear equations, while the lower critical exponent is related to the bubbling at infinity phenomenon. Several existence and nonexistence results of solutions have been established for various of q . For instance, in view of the Pohozaev identity, the autonomous Choquard equation (1.1) (if $\alpha \equiv 1$) does not have nontrivial solutions if either $q \leq \frac{2N-\mu}{N}$ or $q \geq \frac{2N-\mu}{N-2}$. We refer to Li and Ma [12] for details.

We know that solutions of problems (1.1) and (1.3) are critical points of the energy functionals $J(\alpha, u)$ and $J(\alpha^h, u)$, respectively. By a classical argument, the energy functional $J(\beta, u)$ is of class C^1 on $H^1(\mathbb{R}^N)$ for all $\beta \in A_\alpha$. Moreover, $J(\beta, u)$ is a C^2 -functional on $H^1(\mathbb{R}^N)$ for all $\beta \in A_\alpha$ if $q \geq 2$.

Consider the following constrained minimization problems:

$$I = \inf\{J(u) | u \in H^1(\mathbb{R}^N), \langle J'(u), u \rangle = 0, u \neq 0\}$$

and

$$I^h = \inf\{J^h(u) | u \in H^1(\mathbb{R}^N), \langle J^{h'}(u), u \rangle = 0, u \neq 0\}.$$

Theorem 1.2. Assume that $(\alpha_1) \sim (\alpha_3)$ hold and $\frac{2N-\mu}{N} < q < \frac{2N-\mu}{N-2}$. Then, problem (1.3) has a nontrivial solution for some $\alpha^h(x) \in \text{Hull}(\alpha)$. Moreover, if $I < I^h$ for every $\alpha^h(x) \in \text{Hull}(\alpha)$, then there exists a nontrivial solution of (1.1).

Remark 1.3. The assumptions $(\alpha_1) \sim (\alpha_3)$ include several classical cases such as α being constant and α being periodic. If α is constant then $\alpha \equiv \alpha^h$, and if α is periodic, then α^h is a translation of α . So in both these two cases, the original equation (1.1) admits a nontrivial solution. This means that Theorem 1.2 recover some classical results in [2] and [16].

Corollary 1.4. Let q be as in Theorem 1.2 and $\alpha(x) = \bar{\alpha}(x) + \tilde{\alpha}(x)$ with

- (i) $\bar{\alpha}$ satisfying $(\alpha_1) \sim (\alpha_3)$;
- (ii) $\tilde{\alpha} \in C(\mathbb{R}^N, \mathbb{R})$, $\tilde{\alpha}(x) \leq 0$ for every $x \in \mathbb{R}^N$ with $\tilde{\alpha}(x) \neq 0$, and $\lim_{x \rightarrow \infty} \tilde{\alpha}(x) = 0$.

If

$$-\Delta u + \bar{\alpha}(x)u = (|x|^{-\mu} * |u|^q)|u|^{q-2}u, \quad x \in \mathbb{R}^N \quad (1.4)$$

has a ground state solution, then $I < I^h$ for every $\alpha^h(x) \in \text{Hull}(\alpha)$. Moreover, equation (1.1) has a nontrivial solution.

Remark 1.5. Corollary 1.4 means that any result for the ground state solution to equation (1.4) would lead to a corresponding result for equation (1.1). Moreover, by Corollary 1.4, we know that the condition of $I < I^h$ for every $\alpha^h(x) \in \text{Hull}(\alpha)$ does occur in many cases. In particular, for the case of $\bar{\alpha}$ being a constant, then by Corollary 1.4, problem (1.1) has a nontrivial solution, which recovers a classical result in Part III of [15].

Although Theorem 1.2 establishes the existence of nontrivial solutions to equation (1.1) in many cases (Remarks 1.3 and 1.5), one cannot use Theorem 1.2 to obtain directly the existence of nontrivial solution to equation (1.1) for some cases. In fact, from the proof of Theorem 1.2 and Lemma 2.4, we will know that $I \equiv I^h$ if α is an almost periodic function, that is,

$$\limsup_{k \rightarrow \infty} \sup_{x \in \mathbb{R}^N} |\alpha(x + y_{n_k}) - \alpha^h(x)| = 0 \quad \text{in } (\alpha_3).$$

Moreover, in this case, one cannot expect that α^h is always a translation of α . Thus, for a general almost periodic potential, Theorem 1.2 can only show that some shell equation (1.3) (not necessarily the original equation) has a nontrivial solution. In fact, the problem on nontrivial solution to equation (1.1) with almost periodic potentials appears to be difficult to answer completely. Fortunately, by utilizing an idea in [27], we can solve partially this problem.

Theorem 1.6. If α satisfies (α_1) , (α_2) , (α'_3) , and $2 < q < \frac{2N-\mu}{N-2}$, then problem (1.1) has a nontrivial solution.

This article is organized as follows. In Section 2, we present some estimates for the constrained minimizers and the proof of Theorem 1.2. The proof of Theorem 1.6 is given in Section 3. In the final part of this article, we state some auxiliary results in the Appendix.

2 Constrained minimizers and proof of Theorem 1.2

In this section, we give some basic estimates of energy functionals, which are used in the sequel.

Denote

$$l(u) = J(u) - \frac{1}{2} \langle J'(u), u \rangle = \left(\frac{1}{2} - \frac{1}{2q} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy,$$

$$I_a = \inf \{ l(u) | u \in H^1(\mathbb{R}^N), \langle J'(u), u \rangle = a \}, \quad \forall a \in \mathbb{R}.$$

Let $\mathcal{N} = \{u \in H^1(\mathbb{R}^N) | \langle J'(u), u \rangle = 0, u \neq 0\}$ and $I = \inf_{u \in \mathcal{N}} J(u)$.

Lemma 2.1. *We have $I > 0$ and $I_a \geq 0$. Moreover, $I_a > I$ if $a < 0$.*

Proof. We know

$$I = \inf_{u \in \mathcal{N}} J(u) = \inf_{u \in \mathcal{N}} l(u).$$

For every $u \in \mathcal{N}$, we have

$$\begin{aligned} J(u) = l(u) &= J(u) - \frac{1}{2} \langle J'(u), u \rangle = \left(\frac{1}{2} - \frac{1}{2q} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy \\ &= \left(\frac{1}{2} - \frac{1}{2q} \right) \int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha(x)|u|^2) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2q} \right) \min\{1, \underline{\alpha}\} \|u\|^2, \end{aligned}$$

where $\underline{\alpha} = \inf_{x \in \mathbb{R}^N} \alpha(x)$. There exists $r > 0$ such that $\|u\| > r$ for every $u \in \mathcal{N}$. It follows that $I \geq \left(\frac{1}{2} - \frac{1}{2q} \right) \min\{1, \underline{\alpha}\} r^2 > 0$.

Moreover, for every $u \in \{u \in H^1(\mathbb{R}^N) | \langle J'(u), u \rangle = a\}$, we obtain $l(u) \geq 0$. So

$$I_a \geq 0.$$

We will prove that $I_a > I$ if $a < 0$, and this will be done in four steps.

Step 1: We claim that

$$I_a = \inf \{ l(u) | u \in H^1(\mathbb{R}^N), \langle J'(u), u \rangle = a, J(u) \leq c_0 \},$$

where $c_0 = I_a - \frac{1}{2}a + 1 > 0$.

Denote

$$I_{M_1} = \inf \{ l(u) | u \in M_1 \}, M_1 = \{u \in H^1(\mathbb{R}^N) | \langle J'(u), u \rangle = a\}$$

and

$$I_{M_2} = \inf \{ l(u) | u \in M \}, M = \{u \in H^1(\mathbb{R}^N) | \langle J'(u), u \rangle = a, J(u) \leq c_0\}.$$

Since $M \subset M_1$, we obtain $I_{M_1} \leq I_{M_2}$. On the other hand, for $0 < \varepsilon < 1$, there exists $v \in M_1$ such that

$$I_{M_1} + \varepsilon > J(v) - \frac{1}{2} \langle J'(v), v \rangle = J(v) - \frac{1}{2}a,$$

which yields

$$J(v) < I_{M_1} + \varepsilon + \frac{1}{2}a \leq I_{M_1} - \frac{1}{2}a + 1 = c_0.$$

Thus, $v \in M$ and $I_{M_1} + \varepsilon > J(v) - \frac{1}{2} \langle J'(v), v \rangle \geq I_{M_2}$. According to the arbitrariness of ε , we have $I_{M_1} \geq I_{M_2}$. So

$$I_{M_1} = I_{M_2}.$$

Step 2: For $u \in M$, there exists a $0 < t = t(u) < 1$ such that $t(u)u \in \mathcal{N}$.

We define

$$\begin{aligned} G(t) &= \langle J'(tu), tu \rangle \\ &= t^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha(x)|u|^2) dx - t^{2q} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy, \quad t > 0, \quad u \in M. \end{aligned}$$

By

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy \leq c \|u\|^{2q},$$

we have

$$G(t) \geq \min\{1, \underline{a}\} t^2 \|u\|^2 - ct^{2q} \|u\|^{2q} = \|tu\|^2 (\min\{1, \underline{a}\} - c \|tu\|^{(2q-2)}).$$

We can choose $r > 0$ such that $G(t) = \langle J'(tu), tu \rangle > 0$ when $0 < t < \frac{r}{\|u\|}$. In addition, $G(1) = \langle J'(u), u \rangle = a < 0$ and $G(t) \in C(\mathbb{R})$. Therefore, there exists $t = t(u) \in (0, 1)$ such that $G(t(u)) = \langle J'(t(u)u), t(u)u \rangle = 0$, that is, $t(u)u \in \mathcal{N}$.

Step 3: We claim that $t(u) \leq s < 1$, where $s \in (0, 1)$ and $u \in M$.

We prove this by contradiction. Suppose that there are $s_n = 1 - \frac{1}{n} < 1$, $\{v_n\} \subset M$ and $t_n = t(v_n) > s_n$. Since $s_n < t_n < 1$ and $\lim_{n \rightarrow \infty} s_n = 1$, we obtain $\lim_{n \rightarrow \infty} t_n = 1$.

According to $t_n v_n \in \mathcal{N}$, we obtain

$$\langle J'(t_n v_n), t_n v_n \rangle = t_n^2 \int_{\mathbb{R}^N} (|\nabla v_n|^2 + \alpha(x)|v_n|^2) dx - t_n^{2q} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x)|^q |v_n(y)|^q}{|x-y|^\mu} dx dy = 0,$$

that is,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x)|^q |v_n(y)|^q}{|x-y|^\mu} dx dy = t_n^{2-2q} \int_{\mathbb{R}^N} (|\nabla v_n|^2 + \alpha(x)|v_n|^2) dx.$$

By $v_n \in M$, one obtains

$$(1 - t_n^{2-2q}) \int_{\mathbb{R}^N} (|\nabla v_n|^2 + \alpha(x)|v_n|^2) dx = a. \quad (2.1)$$

On the one hand, according to $v_n \in M$ and choosing $\gamma = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2q} \right)$, $\frac{1}{2q} < \gamma < \frac{1}{2}$, we have

$$\begin{aligned} I_a - \frac{1}{2}a + 1 - \gamma a &\geq J(v_n) - \gamma \langle J'(v_n), v_n \rangle \\ &= \left(\frac{1}{2} - \gamma \right) \min\{1, \underline{a}\} \|v_n\|^2 + \left(\gamma - \frac{1}{2q} \right) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x)|^q |v_n(y)|^q}{|x-y|^\mu} dx dy \\ &\geq \left(\frac{1}{2} - \gamma \right) \min\{1, \underline{a}\} \|v_n\|^2 > 0. \end{aligned}$$

Thus, $\{v_n\}$ is bounded. Taking the limit on both sides of form (2.1), then $a = 0$, which contradicts with $a < 0$.

Step 4: $I_a > I$ if $a < 0$.

Let

$$g(t) = l(tu) = \left(\frac{1}{2} - \frac{1}{2q} \right) t^{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy.$$

Thus,

$$\begin{aligned} g(1) - g(t(u)) &= \left(\frac{1}{2} - \frac{1}{2q} \right) [1 - (t(u))^{2q}] \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy \\ &= \left(\frac{1}{2} - \frac{1}{2q} \right) [1 - (t(u))^{2q}] \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha(x)|u|^2) dx - a \right] \\ &\geq \left(\frac{1}{2} - \frac{1}{2q} \right) (1 - s^{2q})(-a). \end{aligned}$$

Moreover,

$$l(u) \geq \left(\frac{1}{2} - \frac{1}{2q} \right) (1 - s^{2q})(-a) + l(t(u)u) \geq \left(\frac{1}{2} - \frac{1}{2q} \right) (1 - s^{2q})(-a) + I.$$

So

$$I_a \geq \left(\frac{1}{2} - \frac{1}{2q} \right) (1 - s^{2q})(-a) + I > I,$$

which completes the proof. $\square$

Lemma 2.2. $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded. There exists a constant k such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\mu} dx dy > k > 0.$$

Then there is a subsequence $\{t_n\} \subset (a_1, a_2)$ ($a_1, a_2 > 0$), such that $\langle J'(t_n u_n), t_n u_n \rangle = 0$.

Proof. By $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\mu} dx dy > k > 0$, we know for every fixed n ,

$$\langle J'(t_n u_n), t_n u_n \rangle = t_n^2 \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x)|u_n|^2) dx - t_n^{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\mu} dx dy = 0$$

has a solution

$$t_n = \left(\frac{\int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x)|u_n|^2) dx}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\mu} dx dy} \right)^{\frac{1}{2q-2}}.$$

If $t_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\|t_n u_n\|^2 = t_n^2 \|u_n\|^2 \rightarrow 0.$$

Moreover, $t_n u_n \rightarrow 0$ and $t_n u_n \in \mathcal{N}$, which is a contradiction. Thus, $t_n \not\rightarrow 0$. On the other hand, by $\int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x)|u_n|^2) dx < c$, we have

$$0 = \langle J'(t_n u_n), t_n u_n \rangle \leq t_n^2 \cdot c - t_n^{2q} \cdot k \rightarrow -\infty.$$

This is a contradiction. Thus, the result follows. $\square$

Remark 2.3. As the aforementioned discussion, we have the following results:

- (1) By the similar way with $\mathcal{N}$, $l(u)$, $J(u)$, I and I_a , we define $\mathcal{N}^h$, $l^h(u)$, $J^h(u)$, I^h , and I_a^h , respectively. Thus, we also have $I^h > 0$, $I_a^h \geq 0$ and $I_a^h > I^h$ when $a < 0$.
- (2) Similar to Lemma 2.2, if $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded, there exists a constant k such that

$$\iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\mu} dx dy > k > 0.$$

Then there is a subsequence $\{t_n\} \subset (a_1, a_2)$ ($a_1, a_2 > 0$) and $\langle J^{h'}(t_n u_n), t_n u_n \rangle = 0$.

Lemma 2.4. For every $\alpha^h(x) \in \text{Hull}(\alpha)$, we have $I \leq I^h$.

Proof. By the definition of $\text{Hull}(\alpha)$, for every $\alpha^h(x) \in \text{Hull}(\alpha)$, there exists $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow \infty} |\alpha(x + y_n) - \alpha^h(x)| = 0, \quad \text{for } x \in \mathbb{R}^N. \quad (2.2)$$

By the definition of I^h , for all $\varepsilon > 0$, there exists $w \in \mathcal{N}^h$ such that $I^h + \varepsilon > J^h(w)$. Let $w_n(x) = w(x - y_n)$, then we have

$$\begin{aligned} \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w_n(x)|^q |w_n(y)|^q}{|x-y|^\mu} dx dy &= \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w(x - y_n)|^q |w(y - y_n)|^q}{|x-y|^\mu} dx dy \\ &= \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w(x)|^q |w(y)|^q}{|x-y|^\mu} dx dy > 0. \end{aligned}$$

According to Lemma 2.2, going if necessary to a subsequence, there exists $t_n \rightarrow t_0$ as $n \rightarrow \infty$ such that

$$\begin{aligned} 0 &= \langle J'(t_n w_n), t_n w_n \rangle \\ &= t_n^2 \int_{\mathbb{R}^N} (|\nabla w_n|^2 + \alpha(x) |w_n|^2) dx - t_n^{2q} \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w_n(x)|^q |w_n(y)|^q}{|x-y|^\mu} dx dy \\ &= t_n^2 \int_{\mathbb{R}^N} (|\nabla w|^2 + \alpha(x + y_n) |w|^2) dx - t_n^{2q} \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w(x)|^q |w(y)|^q}{|x-y|^\mu} dx dy. \end{aligned}$$

We obtain

$$\left| (t_n^2 - t_0^2) \int_{\mathbb{R}^N} |\nabla w|^2 dx \right| \leq |t_n^2 - t_0^2| \int_{\mathbb{R}^N} |\nabla w|^2 dx \rightarrow 0, \quad (2.3)$$

$$|(t_n^{2q} - t_0^{2q}) \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w(x)|^q |w(y)|^q}{|x-y|^\mu} dx dy| \leq |t_n^{2q} - t_0^{2q}| \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w(x)|^q |w(y)|^q}{|x-y|^\mu} dx dy \rightarrow 0 \quad (2.4)$$

and

$$\begin{aligned} &\left| t_n^2 \int_{\mathbb{R}^N} \alpha(x + y_n) |w|^2 dx - t_0^2 \int_{\mathbb{R}^N} \alpha^h(x) |w|^2 dx \right| \\ &\leq \left| t_n^2 \int_{\mathbb{R}^N} \alpha(x + y_n) |w|^2 dx - t_0^2 \int_{\mathbb{R}^N} \alpha(x + y_n) |w|^2 dx \right| + \left| t_0^2 \int_{\mathbb{R}^N} \alpha(x + y_n) |w|^2 dx - t_0^2 \int_{\mathbb{R}^N} \alpha^h(x) |w|^2 dx \right| \\ &\leq |t_n^2 - t_0^2| \int_{\mathbb{R}^N} |w|^2 dx + t_0^2 \int_{\mathbb{R}^N} \lim_{n \rightarrow \infty} |\alpha(x + y_n) - \alpha^h(x)| |w|^2 dx. \end{aligned} \quad (2.5)$$

By combining with (2.2)–(2.5), we have

$$t_0^2 \int_{\mathbb{R}^N} (|\nabla w|^2 + \alpha^h(x) |w|^2) dx - t_0^{2q} \iint_{\mathbb{R}^N \mathbb{R}^N} \frac{|w(x)|^q |w(y)|^q}{|x-y|^\mu} dx dy = 0.$$

That is, $\langle J^h(t_0 w), t_0 w \rangle = 0$. Since $\langle J^h(w), w \rangle = 0$, we obtain $t_0 = 1$. By the direct computations, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} J(t_n w_n) &= \lim_{n \rightarrow \infty} \left\{ \frac{t_n^2}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + \alpha(x)|w_n|^2) dx - \frac{t_n^{2q}}{2q} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x)|^q |w_n(y)|^q}{|x-y|^\mu} dx dy \right\} \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + \alpha^h(x)|w|^2) dx - \frac{1}{2q} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w(x)|^q |w(y)|^q}{|x-y|^\mu} dx dy \\ &= J^h(w). \end{aligned}$$

Thus,

$$I^h + \varepsilon > J^h(w) = \lim_{n \rightarrow \infty} J(t_n w_n),$$

that is,

$$I^h + \varepsilon \geq J(t_n w_n).$$

Since $t_n w_n \in \mathcal{N}$, $J(t_n w_n) \geq I$. Thus,

$$I^h \geq I.$$

The proof is now complete. $\square$

2.1 Proof of Theorem 1.2

Proof. We choose a minimizing sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ of I . Then, going if necessary to a subsequence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We can assume that there exists $\lambda \geq 0$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^2 dx = \lambda.$$

If $\lambda = 0$, that is, $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^2 dx = 0$, setting $\gamma = \frac{2N}{2N-\mu} > 0$, there exists $0 < \theta < 1$ such that $2 < q\gamma < 2^*$ and $q\gamma = \theta \cdot 2 + (1-\theta) \cdot 2^*$ satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q\gamma} dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{\theta \cdot 2 + (1-\theta) \cdot 2^*} dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{\theta \cdot 2} |u_n|^{(1-\theta) \cdot 2^*} dx \\ &\leq \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |u_n|^2 dx \right)^\theta \left(\int_{\mathbb{R}^N} |u_n|^{2^*} dx \right)^{1-\theta} = 0. \end{aligned} \tag{2.6}$$

By Hardy-Littlewood-Sobolev inequality, we obtain

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\mu} dx dy \leq c \|u_n\|_{L^{q\gamma}(\mathbb{R}^N)}^{2q} \rightarrow 0.$$

Since $\{u_n\} \subset \mathcal{N}$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x)|u_n|^2) dx = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} J(u_n) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x)|u_n|^2) dx - \frac{1}{2q} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x-y|^\mu} dx dy \right\} = 0,$$

that is, $I = 0$. This contradicts $I > 0$, so $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^2 dx = \lambda > 0$. The vanishing Lemma [29] implies that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = \tau \in (0, \lambda).$$

If $\tau \in (0, \lambda)$, we have

$$\tau = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx,$$

then for all $0 < \varepsilon < \min\left\{\frac{3(\lambda - \tau)}{8}, \frac{\tau}{4}\right\}$, there exists $R_0 > 0$, $n_0 > 0$ and $\{y_n\}$. When $n > n_0$, we have

$$\int_{B_{R_0}(y_n)} |u_n|^2 dx \in (\tau - \varepsilon, \tau + \varepsilon). \quad (2.7)$$

Setting $\bar{u}_n(x) = u_n(x + y_n)$, we know $\bar{u}_n(x) \neq 0$ is bounded in $H^1(\mathbb{R}^N)$. We may assume, going if necessary to a subsequence

$$\bar{u}_n \rightharpoonup \bar{u} \quad \text{in } H^1(\mathbb{R}^N),$$

$$\bar{u}_n \rightarrow \bar{u} \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N).$$

On the one hand, if $\lim_{n \rightarrow \infty} |y_n| = \infty$, we can rewrite constraint condition

$$\begin{aligned} 0 = \xi(u_n) &= \langle J'(u_n), u_n \rangle = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \alpha(x)|u_n|^2) dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^q |u_n(y)|^q}{|x - y|^\mu} dx dy \\ &= \int_{\mathbb{R}^N} (|\nabla \bar{u}_n|^2 + \alpha(x + y_n)|\bar{u}_n|^2) dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\bar{u}_n(x)|^q |\bar{u}_n(y)|^q}{|x - y|^\mu} dx dy. \end{aligned} \quad (2.8)$$

By (α_3) , there exists $\alpha^{h_1(x)} \in \text{Hull}(\alpha)$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \alpha(x + y_n) |\bar{u}|^2 dx = \int_{\mathbb{R}^N} \alpha^{h_1(x)} |\bar{u}|^2 dx.$$

Then

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla \bar{u}_n|^2 + \alpha(x + y_n) |\bar{u}_n|^2 dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(\bar{u}_n(x - y_n) - \bar{u}(x - y_n))|^2 + \alpha(x) |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^2 dx + \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 + \alpha^{h_1(x)} |\bar{u}|^2 dx. \end{aligned} \quad (2.9)$$

On the other hand,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * |\bar{u}_n|^q) |\bar{u}_n|^q dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q) |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q dx + \int_{\mathbb{R}^N} (|x|^{-\mu} * |\bar{u}|^q) |\bar{u}|^q dx. \end{aligned} \quad (2.10)$$

Combining to (2.8)–(2.10), we have

$$0 = \lim_{n \rightarrow \infty} \xi(u_n) = \lim_{n \rightarrow \infty} \xi(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) + \xi^{h_1}(\bar{u}). \quad (2.11)$$

Similar to (2.11), we have

$$\begin{aligned}
I &= \lim_{n \rightarrow \infty} J(u_n) = \lim_{n \rightarrow \infty} J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle = \lim_{n \rightarrow \infty} l(u_n) \\
&= \left(\frac{1}{2} - \frac{1}{2q} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n|^q) |u_n|^q dx \\
&= \left(\frac{1}{2} - \frac{1}{2q} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q) |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q dx \\
&\quad + \left(\frac{1}{2} - \frac{1}{2q} \right) \int_{\mathbb{R}^N} (|x|^{-\mu} * |\bar{u}|^q) |\bar{u}|^q dx \\
&= \lim_{n \rightarrow \infty} l(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) + l^{h_1}(\bar{u}).
\end{aligned} \tag{2.12}$$

Set $\lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} \xi(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) = \xi$, $\xi^{h_1}(\bar{u}) = \tilde{a}$ and $\lim_{n \rightarrow \infty} l(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) = \tilde{l}$. We obtain

$$0 = \xi + \tilde{a}, I = \tilde{l} + l^{h_1}(\bar{u}).$$

1⁰. If $\tilde{a} < 0$, for all $n \in \mathbb{N}_+$, $l(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) \geq 0$, then $\tilde{l} \geq 0$. By definition $I_a^{h_1}$, we have

$$I = \tilde{l} + l^{h_1}(\bar{u}) \geq l^{h_1}(\bar{u}) \geq I_a^{h_1} > I^{h_1}.$$

But by Lemma 2.4, we have $I \leq I^{h_1}$. This is a contradiction.

2⁰. If $\tilde{a} > 0$, then $\xi < 0$. Hence, when n enough large, we have $\xi(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) < 0$. By definition I_{ξ_n} and Lemma 2.1,

$$l(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) \geq I_{\xi_n} > I.$$

So,

$$\tilde{l} \geq I.$$

But $\bar{u} \neq 0$, we have $l^{h_1}(\bar{u}) > 0$, and thus

$$I = \tilde{l} + l^{h_1}(\bar{u}) > \tilde{l} \geq I.$$

This is a contradiction.

3⁰. If $\tilde{a} = 0$, then $\xi = 0$ and

$$\begin{aligned}
0 &= \xi = \lim_{n \rightarrow \infty} \xi(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(\bar{u}_n(x - y_n) - \bar{u}(x - y_n))|^2 + \alpha(x) |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^2 dx \\
&\quad - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q) |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\tilde{l} &= \lim_{n \rightarrow \infty} l(\bar{u}_n(x - y_n) - \bar{u}(x - y_n)) \\
&= \left(\frac{1}{2} - \frac{1}{2q} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q) |\bar{u}_n(x - y_n) - \bar{u}(x - y_n)|^q dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2q} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \alpha(x + y_n) |\bar{u}_n - \bar{u}|^2 dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2q} \right) \left[\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\bar{u}_n|^2 dx - \int_{\mathbb{R}^N} |\bar{u}|^2 dx \right].
\end{aligned}$$

By $\lim_{R \rightarrow \infty} \int_{B_R} |\bar{u}|^2 dx = \int_{\mathbb{R}^N} |\bar{u}|^2 dx$, we obtain that for any $0 < \varepsilon < \min\{\frac{3(\lambda - \tau)}{8}, \frac{\tau}{4}\}$, there exists $R_1 > R_0 > 0$ such that $\int_{\mathbb{R}^N} |\bar{u}|^2 dx < \int_{B_R} |\bar{u}|^2 dx + \varepsilon$ when $R \geq R_1$. So

$$\begin{aligned} \int_{\mathbb{R}^N} |\bar{u}|^2 dx &< \int_{B_{R_1}} |\bar{u}|^2 dx + \varepsilon = \lim_{n \rightarrow \infty} \int_{B_{R_1}} |\bar{u}_n|^2 dx + \varepsilon \\ &= \lim_{n \rightarrow \infty} \int_{B_{R_1}(y_n)} |u_n|^2 dx + \varepsilon \leq \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_{R_1}(y)} |u_n|^2 dx + \varepsilon \\ &\leq \tau + \frac{4\varepsilon}{3} \leq \tau + \frac{\lambda - \tau}{2}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\bar{u}_n|^2 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^2 dx = \lambda$, we have

$$\tilde{l} \geq \left(\frac{1}{2} - \frac{1}{2q} \right) \alpha \left[\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\bar{u}_n|^2 dx - \int_{\mathbb{R}^N} |\bar{u}|^2 dx \right] \geq \left(\frac{1}{2} - \frac{1}{2q} \right) \alpha \left(\frac{\lambda - \tau}{2} \right) > 0.$$

Moreover,

$$I = \tilde{l} + l^{h_1}(\bar{u}) > l^{h_1}(\bar{u}) \geq I^{h_1}.$$

This is a contradiction with $I \leq I^{h_1}$.

On the other hand, if $|y_n|$ is bounded, then going if necessary to a subsequence $\{y_n\}$, we have $\lim_{n \rightarrow \infty} |y_n| = y_0$, that is, it follows from (2.7), and there exists $R_2 > 0$, and we have $B_{R_0}(y_n) \subset B_{R_2}$,

$$0 < \tau - \varepsilon < \int_{B_{R_0}(y_n)} |u_n|^2 dx < \int_{B_{R_2}} |u_n|^2 dx.$$

Similar to $|y_n| \rightarrow \infty$, by the definition of I , one can obtain a contradiction. Thus, the dichotomy cannot happen. Thereby, we have

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = \lambda.$$

For any $0 < \varepsilon < \frac{\lambda}{4}$, there exists $r > 0$, $n_1 > 0$, $\{\tilde{y}_n\} \subset \mathbb{R}^N$, we obtain

$$0 < \int_{B_r(\tilde{y}_n)} |u_n|^2 dx \in (\lambda - \varepsilon, \lambda + \varepsilon) \quad (2.13)$$

when $n > n_1$. If $\lim_{n \rightarrow \infty} |\tilde{y}_n| = \infty$, set $\tilde{u}_n(x) = u_n(x + \tilde{y}_n)$. It is verified through the compactness that

$$\tilde{u}_n \rightarrow \tilde{u} \quad \text{in } L^2(\mathbb{R}^N), \quad (2.14)$$

where $\tilde{u} \neq 0$. Hence, by (2.14) and (α_3) , there exists $\alpha^{h_2} \in \text{Hull}(\alpha)$ such that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (|\nabla \tilde{u}_n|^2 + \alpha(x + \tilde{y}_n) |\tilde{u}_n|^2) dx \geq \int_{\mathbb{R}^N} (|\nabla \tilde{u}|^2 + \alpha^{h_2}(x) |\tilde{u}|^2) dx. \quad (2.15)$$

Moreover,

$$0 = \lim_{n \rightarrow \infty} \xi(u_n) \geq \xi^{h_2}(\tilde{u}),$$

and

$$I = \lim_{n \rightarrow \infty} J(u_n) = \lim_{n \rightarrow \infty} J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle = \lim_{n \rightarrow \infty} l(u_n) = l^{h_2}(\tilde{u}).$$

If $\xi^{h_2}(\tilde{u}) < 0$, by definition of $I_{\xi^{h_2}(\tilde{u})}^{h_2}$ and Remark 2.3, we have

$$I = l^{h_2}(\tilde{u}) \geq I_{\xi^{h_2}(\tilde{u})}^{h_2} > I^{h_2}.$$

This is a contradiction to Lemma 2.4. If $\xi^{h_2}(\tilde{u}) = 0$, by definition of I^{h_2} ,

$$I = I^{h_2}(\tilde{u}) \geq I^{h_2}.$$

By $I \leq I^{h_2}$, we obtain $I = I^{h_2} = I^{h_2}(\tilde{u})$, $\xi^{h_2}(\tilde{u}) = 0$, that is, $\tilde{u}$ is a minimax function of I^{h_2} . Thus, $\tilde{u}$ is a nontrivial solution of

$$-\Delta u + \alpha^{h_2}(x)u = (|x|^{-\mu} * |u|^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^N.$$

If $|\tilde{y}_n|$ is bounded, by (2.13), there exists $\{\tilde{y}_n\}$, $\lim_{n \rightarrow \infty} |\tilde{y}_n| = \tilde{y}_0$, for enough large $r_0 > 0$ such that

$$0 < \lambda - \varepsilon < \int_{B_r(\tilde{y}_n)} |u_n|^2 dx < \int_{B_{r_0}} |u_n|^2 dx,$$

when $B_r(\tilde{y}_n) \subset B_{r_0}$. By $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, there exists $\hat{u} \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup \hat{u} \quad \text{in } H^1(\mathbb{R}^N).$$

Analogous to $|\tilde{y}_n| \rightarrow \infty$, $\hat{u}$ is a nontrivial solution of (1.1).

Particularly, if $I < I^h$ for every $\alpha^{h(x)} \in \text{Hull}(\alpha)$, then $\lim_{n \rightarrow \infty} |\tilde{y}_n| = \infty$ never happens. Hence, $\lim_{n \rightarrow \infty} |\tilde{y}_n| \neq \infty$ is necessary, which means that equation (1.1) has a nontrivial solution. $\square$

2.2 Proof of Corollary 1.4

Proof. Set

$$J_1(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \bar{\alpha}(x)|u|^2) dx - \frac{1}{2q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} dx dy$$

and

$$I_1 = \inf\{J_1(u) | u \in H^1(\mathbb{R}^N), \langle J_1'(u), u \rangle = 0, \quad u \neq 0\}.$$

We have $I < I_1$. Suppose I_1 is attained by u_0 , then there holds

$$\begin{aligned} a &:= \langle J'(u_0), u_0 \rangle = \int_{\mathbb{R}^N} (|\nabla u_0|^2 + \alpha(x)|u_0|^2) dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^q |u_0(y)|^q}{|x-y|^\mu} dx dy \\ &< \int_{\mathbb{R}^N} (|\nabla u_0|^2 + \bar{\alpha}(x)|u_0|^2) dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^q |u_0(y)|^q}{|x-y|^\mu} dx dy = 0 \end{aligned}$$

and

$$\begin{aligned} I_1 &= J_1(u_0) = J_1(u_0) - \frac{1}{2} \langle J_1'(u_0), u_0 \rangle = \left(\frac{1}{2} - \frac{1}{2q} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^q |u_0(y)|^q}{|x-y|^\mu} dx dy \\ &= I(u_0) \geq \inf\{I(u) | u \in H^1(\mathbb{R}^N), \quad \langle J'(u), u \rangle = a\} = I_a. \end{aligned} \quad (2.16)$$

By Lemma 2.1, we obtain $I_a > I$. So $I < I_1$.

On the other hand, for all $\alpha^h(x) \in \text{Hull}(\alpha)$, by definition of $\text{Hull}(\alpha)$, we know there exists $|y_n| \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \alpha(x + y_n) = \alpha^h(x), \quad \text{for any } x \in \mathbb{R}^N.$$

Since $\bar{\alpha}(x)$ satisfies (α_3) , there is $\bar{\alpha}^h \in C(\mathbb{R}^N)$ such that

$$\lim_{n \rightarrow \infty} \bar{\alpha}(x + y_n) = \bar{\alpha}^h(x), \quad \text{for any } x \in \mathbb{R}^N.$$

By $\lim_{n \rightarrow \infty} |\tilde{\alpha}(x + y_n)| = 0$, we have $\alpha^h(x) = \bar{\alpha}^h(x)$, that is, $I^h = I_1^h$. Thus, $I_1 \leq I^h$ by Lemma 2.4. Hence,

$$I < I^h.$$

By Theorem 1.2, Equation (1.1) has a nontrivial solution. $\square$

3 Proof of Theorem 1.6

Set

$$K_\infty := \{v \in H^1(\mathbb{R}^N) | v \neq 0, \exists \beta \in A_\alpha, J'(\beta, v) = 0\}.$$

For every given $\bar{l} \in \mathbb{N}_+$, define

$$Q_\infty := \left\{ \Phi \in H^1(\mathbb{R}^N) | \Phi = \sum_{i=1}^{\bar{l}} v_i(x + z_i), v_i \in K_\infty, z_i \in \mathbb{R}^N, i = 1, \dots, \bar{l}, \bar{l} = 1, \dots, \bar{l} \right\}.$$

By Lemma A.1 in [5], we have

Theorem 3.1. *If $v \in K_\infty$, then $v \in C^2(\mathbb{R}^N)$ and v is a solution of the following equation:*

$$\begin{cases} -\Delta v + \beta(x)v = (|x|^{-\mu} * |v|^q)|v|^{q-2}v, \\ \lim_{|x| \rightarrow \infty} v(x) = 0. \end{cases}$$

Hence, we can assume that $v(x) > 0$ and $v \in C^2(\mathbb{R}^N)$ for all $v \in K_\infty$. So if $\Phi \in Q_\infty$, we have

- (i) $\Phi \in C^2(\mathbb{R}^N)$;
- (ii) $\lim_{|x| \rightarrow \infty} \Phi(x) = 0$;
- (iii) $\Phi(x) > 0$.

The next result contains an uniform qualitative property of the elements in Q_∞ .

Lemma 3.2. *There exists $\delta > 0$ such that $\Delta \Phi(x) > \frac{\alpha}{2} \Phi(x)$ when $\Phi(x) < \delta$.*

Proof. For $v \in K_\infty$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|v(y)|^q}{|x-y|^\mu} dy &= \int_{B_R(x)} \frac{|v(y)|^q}{|x-y|^\mu} dy + \int_{\mathbb{R}^N \setminus B_R(x)} \frac{|v(y)|^q}{|x-y|^\mu} dy \\ &\leq \max_{y \in B_R(x)} |v(y)|^q c(N) \frac{1}{N-\mu} R^{N-\mu} + \frac{c}{R^\mu} \|v\|^q, \end{aligned}$$

that is, $\int_{\mathbb{R}^N} \frac{|v(y)|^q}{|x-y|^\mu} dy$ is bounded. Set $h_i(x) = \int_{\mathbb{R}^N} \frac{|v_i(y)|^q}{|x-y|^\mu} dy$, $i = 1, \dots, \bar{l}$, $h_0 = \max\{h_i(x + z_i), i = 1, \dots, \bar{l}\}$. By $q > 2$, for every $\varepsilon > 0$, $\exists \delta > 0$, we have $\frac{h_0 \cdot |v(x)|^{q-2} v(x)}{v(x)} < \varepsilon$ when $v(x) < \delta$. Choosing $\varepsilon = \frac{\alpha}{2}$, when $\Phi(x) < \delta$, for every v_i , $i = 1, \dots, \bar{l}$, we have $v_i(x + z_i) < \delta$ and

$$\sum_{i=1}^{\bar{l}} h_i(x + z_i) |v_i(x + z_i)|^{q-2} v_i(x + z_i) < \sum_{i=1}^{\bar{l}} \frac{\alpha}{2} v_i(x + z_i) = \frac{\alpha}{2} \sum_{i=1}^{\bar{l}} v_i(x + z_i).$$

For $v_i \in K_\infty$, it follows from Theorem 3.1 that v_i satisfy

$$-\Delta v_i + \beta_i(x) v_i = (|x|^{-\mu} * |v_i|^q) |v_i|^{q-2} v_i = h_i(x) |v_i|^{q-2} v_i.$$

Thus,

$$-\Delta v_i(x + z_i) + \beta_i(x + z_i) v_i(x + z_i) = h_i(x + z_i) |v_i(x + z_i)|^{q-2} v_i(x + z_i).$$

Moreover,

$$\begin{aligned}
 \Delta\Phi &= \sum_{i=1}^l \Delta v_i(x + z_i) \\
 &= \sum_{i=1}^l \beta_i(x + z_i) v_i(x + z_i) - \sum_{i=1}^l h_i(x + z_i) |v_i(x + z_i)|^{q-2} v_i(x + z_i) \\
 &> \frac{\alpha}{2} \sum_{i=1}^l v_i(x + z_i) - \frac{\alpha}{2} \sum_{i=1}^l v_i(x + z_i) = \frac{\alpha}{2} \sum_{i=1}^l v_i(x + z_i) \\
 &= \frac{\alpha}{2} \Phi.
 \end{aligned}$$

The proof is now complete. $\square$

Similar to regularity arguments ([8] or [25]), using α_1 , we obtain that for all $v \in K_\infty$ and for every open ball $B \subset \mathbb{R}^N$, there results

$$\|v\|_{C^{2,\gamma}(B)} \leq C\|v\|,$$

where $C > 0$ depends on $N, \bar{\alpha}, \gamma \in (0, 1)$ and $\text{diam}(B)$. Then it follows that for all $\phi \in Q_\infty$, we have

$$\|\phi\|_{C^{2,\gamma}(B)} \leq C_1\|v\|,$$

where $C_1 > 0$ depends on $N, \bar{\alpha}, \gamma \in (0, 1)$ and $\text{diam}(B)$. Consider a bounded subset Q in Q_∞ . It follows from the aforementioned estimates that

$$\|\nabla\phi\|_{L^\infty(\mathbb{R}^N)} \leq C$$

for every $\phi \in Q$, where $C > 0$ does not depend on $\phi \in Q$. Therefore, we have the following:

Theorem 3.3. *If Q is a bounded set in Q_∞ . Then for every $\Phi \in Q_\infty$, there exists $k(Q) > 0$, such that*

- (i) $|\Phi(x) - \Phi(y)| \leq k|x - y|$, for all $x, y \in \mathbb{R}^N$;
- (ii) for every $\varepsilon > 0$, there exists $r = r(\varepsilon, Q) > 0$ such that $|\Phi(x) - \Phi(y)| \leq \varepsilon$ when $|x - y| < r$.

By using the properties, we can give an estimate of the L^∞ distance between elements in Q_∞ .

Lemma 3.4. *For every bounded subset Q in Q_∞ , there exists a constant $c = c(Q, N) > 0$ such that*

$$\|\Phi - \Psi\|_{L^\infty(\mathbb{R}^N)} \leq c\|\Phi - \Psi\|^{\frac{2}{N+2}}, \quad \text{for all } \Phi, \Psi \in Q.$$

Proof. For every $x_0 \in \mathbb{R}^N$, by Theorem 3.3, there exists $k = k(Q) > 0$, for all $x \in \mathbb{R}^N$, such that

$$|\Phi(x) - \Phi(x_0)| \leq k|x - x_0|; |\Psi(x) - \Psi(x_0)| \leq k|x - x_0|, \quad \forall \Phi, \Psi \in Q.$$

Without loss of generality, we suppose $\Psi(x_0) > \Phi(x_0)$ and set $r = \frac{\Psi(x_0) - \Phi(x_0)}{4k}$. Then, for every $x \in \mathbb{R}^N$, we have

$$|\Phi(x) - \Phi(x_0)| \leq k|x - x_0| \leq kr; |\Psi(x) - \Psi(x_0)| \leq k|x - x_0| \leq kr,$$

when $|x - x_0| \leq r$, that is,

$$\Phi(x) \leq \Phi(x_0) + kr; \Psi(x_0) - kr \leq \Psi(x).$$

It follows that

$$\Psi(x) - \Phi(x) \geq \Psi(x_0) - kr - \Phi(x_0) - kr = \frac{1}{2}(\Psi(x_0) - \Phi(x_0)).$$

Moreover, we have

$$\|\Psi - \Phi\|^2 \geq \int_{B_r(x_0)} |\Psi - \Phi|^2 dx \geq \frac{1}{4}(\Psi(x_0) - \Phi(x_0))^2 c(N) r^N = c(k, N)(\Psi(x_0) - \Phi(x_0))^{N+2}.$$

Hence, $\Psi(x_0) - \Phi(x_0) \leq c\|\Psi - \Phi\|^{\frac{2}{N+2}}$. By arbitrariness of x_0 , we obtain

$$\|\Phi - \Psi\|_{L^\infty(\mathbb{R}^N)} \leq c\|\Phi - \Psi\|^{\frac{2}{N+2}},$$

which completes the proof. $\square$

Define a mapping $X^1 : Q_\infty \subset H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ by

$$X^1(\Phi) = \max\{x^1 \in \mathbb{R} \mid \Phi(x^1, x') = \delta \text{ for some } x' \in \mathbb{R}^{N-1}\}, \quad \Phi \in Q_\infty,$$

where $\delta > 0$ is the real number given in Lemma 3.2.

Lemma 3.5. *The function $X^1 : Q_\infty \subset H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ is uniformly continuous on bounded subsets of Q_∞ .*

Proof. Set $\Omega = \{x \in \mathbb{R}^N \mid x^1 > X^1(\Phi)\}$. We will prove $\Phi(x) < \delta$, $x \in \Omega$. In fact, suppose $x_0 \in \Omega$ such that $\Phi(x_0) \leq \delta$. If $\Phi(x_0) = \delta$, then $x_0^1 \leq X^1(\Phi)$. This contradicts $x_0 \in \Omega$ ($x_0^1 > X^1(\Phi)$). If $\Phi(x_0) > \delta$. By $\lim_{|x| \rightarrow \infty} \Phi(x) = 0$, there exists $y_0 \in \Omega$ and $y_0^1 > x_0^1$ such that $\Phi(y_0) < \delta$. By $\Phi \in C^2(\mathbb{R}^N)$, there exists y_1 in line $x_0 y_0$ such that $\Phi(y_1) = \delta$. Then $y_1^1 \leq X^1(\Phi)$. By y_1 in line $x_0 y_0$, then $y_1^1 > x_0^1 > X^1(\Phi)$. This is a contradiction. Similarly, we can prove $\Phi(x) \leq \delta$, $x \in \partial\Omega$.

Let $v(x) = \delta e^{-\omega(x^1 - X^1(\Phi))}$, where $\omega \in (0, (\frac{\delta}{2})^{\frac{1}{2}})$ and

$$L(v(x) - \Phi(x)) = -\Delta(v(x) - \Phi(x)) + \frac{\alpha}{2}(v(x) - \Phi(x)), \quad x \in \Omega.$$

By Lemma 3.2, we know $L(v(x) - \Phi(x)) \geq 0$, $x \in \Omega$ and $v(x) - \Phi(x) \geq 0$, $x \in \partial\Omega$. By $\liminf_{|x| \rightarrow \infty} (v(x) - \Phi(x)) = \liminf_{|x| \rightarrow \infty} v(x) \geq 0$ and maximum principle applied to the unbounded domain (see [23]), we know $v(x) - \Phi(x) \geq 0$, $x \in \Omega$. So, $\forall (x^1, x') \in \mathbb{R}^N$, $x^1 \geq X^1(\Phi)$, we have

$$\Phi(x^1, x') \leq v(x^1, x') = \delta e^{-\omega(x^1 - X^1(\Phi))}. \quad (3.1)$$

We prove the continuity of X^1 by the aforementioned estimate. Taking $\Phi, \Psi \in Q$, there exists $x(\Phi) = (X^1(\Phi), x'(\Phi))$, $x(\Psi) = (X^1(\Psi), x'(\Psi)) \in \mathbb{R}^N$ such that $\Phi(X^1(\Phi), x'(\Phi)) = \delta$ and $\Psi(X^1(\Psi), x'(\Psi)) = \delta$. Without losing of generality, we assume $X^1(\Psi) > X^1(\Phi)$. By (3.1),

$$\Phi(X^1(\Psi), x'(\Psi)) \leq \delta e^{-\omega(X^1(\Psi) - X^1(\Phi))},$$

that is,

$$-\Phi(X^1(\Psi), x'(\Psi)) \geq -\delta e^{-\omega(X^1(\Psi) - X^1(\Phi))}.$$

By $\Psi(x(\Psi)) = \delta$, we have

$$\|\Phi - \Psi\|_{L^\infty(\mathbb{R}^N)} \geq \Psi(x(\Psi)) - \Phi(x(\Psi)) \geq \delta - \delta e^{-\omega(X^1(\Psi) - X^1(\Phi))}.$$

By Lemma 3.4, we obtain

$$c\|\Phi - \Psi\|^{\frac{2}{N+2}} \geq \|\Phi - \Psi\|_{L^\infty(\mathbb{R}^N)} \geq \delta - \delta e^{-\omega(X^1(\Psi) - X^1(\Phi))}.$$

Thus,

$$0 < X^1(\Psi) - X^1(\Phi) \leq \frac{\ln \delta - \ln(\delta - c\|\Phi - \Psi\|^{\frac{2}{N+2}})}{\omega} \rightarrow 0 \quad (\|\Phi - \Psi\| \rightarrow 0).$$

This completes the proof. $\square$

Theorem 3.6. Set $\{u_m\} \subset H^1(\mathbb{R}^N)$ be a (PS) sequence for J at level $\tilde{c}$, $u_m \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ and $\lim_{m \rightarrow \infty} \|u_m - u_{m-1}\| = 0$. Then there exists $\{x_m\} \subset \mathbb{R}^N$ such that

- (1⁰) $\lim_{m \rightarrow \infty} |x_m^1 - x_{m-1}^1| = 0$, where $x_m = (x_m^1, \dots, x_m^N)$;
- (2⁰) there exists $R > 0$ such that $\lim_{m \rightarrow \infty} \int_{B_R(x_m)} |u_m|^2 dx > 0$;
- (3⁰) $\lim_{m \rightarrow \infty} |x_m| = \infty$.

Proof. By Theorem A.9, going if necessary to a subsequence $\{u_m\}$, there exists $\beta_i(x) \in A_a$ and $v_i (\neq 0) \in H^1(\mathbb{R}^N)$, $i = 1, \dots, l$ such that

- (i) $u_m - \sum_{i=1}^l v_i(x + y_m^{(i)}) \rightarrow 0$ in $H^1(\mathbb{R}^N)$;
- (ii) $J'(\beta_i, v_i) = 0$, $i = 1, \dots, l$;
- (iii) $\tilde{c} = \sum_{i=1}^l J(\beta_i, v_i)$;
- (iv) $|y_m^{(i)}| \rightarrow \infty$, $i = 1, \dots, l$.

Set $\Phi_m(x) = \sum_{i=1}^l v_i(x + y_m^{(i)})$, then $\{\Phi_m(x)\} \subset Q_\infty$ and $\lim_{m \rightarrow \infty} \|u_m - \Phi_m\| = 0$. Moreover, $\lim_{m \rightarrow \infty} \|\Phi_m - \Phi_{m-1}\| = 0$. There exists $x(\Phi_m) = (X^1(\Phi_m), X'(\Phi_m)) \in \mathbb{R}^N$ such that $\Phi_m(X^1(\Phi_m), X'(\Phi_m)) = \delta$. Let $x_m = x(\Phi_m)$. By $\{u_m\}$ is bounded in $H^1(\mathbb{R}^N)$, $\{\Phi_m\}$ is bounded in Q_∞ . By Lemma 3.5, we have $\lim_{m \rightarrow \infty} |x_m^1 - x_{m-1}^1| = 0$. (1⁰) is proved.

By Theorem 3.3 (ii) and $\Phi_m(x_m) = \delta$, choosing $\varepsilon = \frac{\delta}{2}$, there exists $R > 0$ such that $\Phi_m(x) \geq \frac{\delta}{2}$ for all $x \in B_R(x_m)$. By $\lim_{m \rightarrow \infty} \|u_m - \Phi_m\| = 0$, we know

$$\lim_{m \rightarrow \infty} \int_{B_R(x_m)} |u_m|^2 dx = \lim_{m \rightarrow \infty} \int_{B_R(x_m)} |\Phi_m|^2 dx \geq \left(\frac{\delta}{2}\right)^2 |B_R(x_m)| > 0.$$

Thus, (2⁰) is proved.

Next, we will use the contradiction method to prove (3⁰). Going if necessary to a subsequence, suppose that $\{x_m\}$ and $|x_m| < \tilde{R}$. Then

$$\lim_{m \rightarrow \infty} \int_{B_{R+\tilde{R}}} |u_m|^2 dx \geq \lim_{m \rightarrow \infty} \int_{B_R(x_m)} |u_m|^2 dx > 0.$$

This contradicts with $u_m \rightarrow 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. Thus, (3⁰) is proved. $\square$

Proof of Theorem 1.6. Since J satisfies the geometric assumption of the mountain pass theorem, using Theorem 1.2 in [27], we can find a Palais-Smale sequence $\{u_m\}$ for J at some level $\tilde{c} > 0$ such that

$$\lim_{m \rightarrow \infty} \|u_m - u_{m-1}\| = 0. \quad (3.2)$$

We have $\{u_m\}$ admits a subsequence converging weakly to a critical point u . If $u \neq 0$, one find a solution of (1.1).

Next, if $u = 0$, we will find a nontrivial solution of (1.1). By (3.2) and Theorem 3.6, there exists $\{x_m\} \subset \mathbb{R}^N$ satisfying

- (1⁰) $\lim_{m \rightarrow \infty} |x_m^1 - x_{m-1}^1| = 0$, where $x_m = (x_m^1, \dots, x_m^N)$,
- (2⁰) there exists $R > 0$ such that $\lim_{m \rightarrow \infty} \int_{B_R(x_m)} |u_m|^2 dx > 0$,
- (3⁰) $\lim_{m \rightarrow \infty} |x_m| = \infty$.

Take $\tau_m = (x_m^1, k_m^2 T_2, \dots, k_m^N T_N)$, where $k_m^i = \min\{k \in \mathbb{N}_+ \mid |k T_i - x_m^i| \leq T_i\}$, $i = 2, \dots, N$. So

$$|\tau_m - x_m| = \left(\sum_{i=2}^N |k_m^i T_i - x_m^i|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=2}^N T_i^2 \right)^{\frac{1}{2}} =: T < +\infty.$$

Denote $v_m(x) = u_m(x + \tau_m)$. We know there exists $v_0 \in H^1(\mathbb{R}^N)$ such that $v_m \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$ and $v_m \rightarrow v_0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. Moreover,

$$\int_{B_{R+T}(0)} |v_m|^2 dx = \int_{B_{R+T}(0)} |u_m(x + \tau_m)|^2 dx = \int_{B_{R+T}(\tau_m)} |u_m(x)|^2 dx \geq \int_{B_R(x_m)} |u_m(x)|^2 dx.$$

By (2⁰), we have $\lim_{m \rightarrow \infty} \int_{B_{R+T}} |v_m|^2 dx > 0$. Hence, $v_0 \neq 0$.

Case I: $\lim_{m \rightarrow \infty} |x_m^1| \neq \infty$ where x_m^1 is the first component of τ_m . There exists a subsequence $\{x_m^1\}$ such that $\lim_{m \rightarrow \infty} x_m^1 = x_0^1$. By (α'_3) , we have

$$\begin{aligned} |\alpha(x + \tau_m) - \alpha(x^1 + x_0^1, x')| &= |\alpha(x^1 + x_m^1, x^2 + k_m^2 T_2, \dots, x^N + k_m^N T_N) - \alpha(x^1 + x_0^1, x')| \\ &= |\alpha(x^1 + x_m^1, x') - \alpha(x^1 + x_0^1, x')|, \end{aligned}$$

where $x' = (x^2, \dots, x^N) \subset U = [0, T_2] \times \dots \times [0, T_N]$. By Theorem A.2, we have

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} |\alpha(x + \tau_m) - \alpha(x^1 + x_0^1, x')| &= \sup_{x \in \mathbb{R}^N} |\alpha(x^1 + x_m^1, x') - \alpha(x^1 + x_0^1, x')| \\ &= \sup_{x \in \mathbb{R} \times U} |\alpha(x^1 + x_m^1, x') - \alpha(x^1 + x_0^1, x')| \rightarrow 0. \end{aligned}$$

By Lemma A.5,

$$\begin{aligned} \|J'(\alpha(x + \tau_m), v_m) - J'(\alpha(x^1 + x_0^1, x'), v_m)\|_{H^{-1}(\mathbb{R}^N)} &\leq c \|\alpha(x + \tau_m) - \alpha(x^1 + x_0^1, x')\|_{L^\infty(\mathbb{R}^N)} \\ &\leq c \sup_{x \in \mathbb{R}^N} |\alpha(x + \tau_m) - \alpha(x^1 + x_0^1, x')| \rightarrow 0. \end{aligned}$$

So for all $\varphi \in H^1(\mathbb{R}^N)$, denoting $\tilde{\varphi}(x) = \varphi(x^1 + x_0^1, x') \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \langle J'(\alpha(x^1 + x_0^1, x'), v_m), \tilde{\varphi} \rangle &= \lim_{m \rightarrow \infty} \langle J'(\alpha(x + \tau_m), v_m), \tilde{\varphi} \rangle \\ &= \lim_{m \rightarrow \infty} \langle J'(\alpha(x), v_m(x - \tau_m)), \tilde{\varphi}(x - \tau_m) \rangle \\ &= \lim_{m \rightarrow \infty} \langle J'(u_m(x)), \tilde{\varphi}(x - \tau_m) \rangle = 0. \end{aligned}$$

By $v_m \rightharpoonup v_0$, we have $\langle J'(\alpha(x^1 + x_0^1, x'), v_0), \tilde{\varphi} \rangle = \lim_{m \rightarrow \infty} \langle J'(\alpha(x^1 + x_0^1, x'), v_m), \tilde{\varphi} \rangle = 0$, we obtain

$$\int_{\mathbb{R}^N} \nabla v_0(x) \nabla \tilde{\varphi}(x) + \alpha(x^1 + x_0^1, x') v_0(x) \tilde{\varphi}(x) dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_0(x)|^q |v_0(y)|^{q-2} v_0(y) \tilde{\varphi}(y)}{|x - y|^\mu} dx dy = 0.$$

Moreover,

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla v_0(x^1 - x_0^1, x') \nabla \varphi(x) + \alpha(x) v_0(x^1 - x_0^1, x') \varphi(x) dx \\ &- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_0(x^1 - x_0^1, x')|^q |v_0(y^1 - x_0^1, y')|^{q-2} v_0(y^1 - x_0^1, y') \varphi(y)}{|x - y|^\mu} dx dy = 0. \end{aligned}$$

Letting $\bar{v}_0 = v_0(x^1 - x_0^1, x')$, we have $\bar{v}_0$ is a nontrivial solution of (1.1).

Case II: $\lim_{m \rightarrow \infty} |x_m^1| = \infty$. Because $\alpha(x)$ satisfies (α'_3) , there exists $\{\sigma_m\} \subset \mathbb{R}$ and $\lim_{m \rightarrow \infty} |\sigma_m| = \infty$, such that

$$\lim_{m \rightarrow \infty} \sup_{x \in \mathbb{R}^N} |\alpha(x^1 + \sigma_m, x') - \alpha(x^1, x')| = \lim_{m \rightarrow \infty} \sup_{x \in \mathbb{R} \times U} |\alpha(x^1 + \sigma_m, x') - \alpha(x^1, x')| = 0.$$

By $\lim_{m \rightarrow \infty} |x_m^1| = \infty$, $\lim_{m \rightarrow \infty} |x_m^1 - x_{m-1}^1| = 0$, we can choose $\{x_{m_k}^1\} \subset \{x_m^1\}$ such that

$$\lim_{k \rightarrow \infty} |x_{m_k}^1 - \sigma_k| = 0.$$

Moreover,

$$\begin{aligned}
& \sup_{x \in \mathbb{R}^N} |\alpha(x + \tau_{m_k}) - \alpha(x)| \\
&= \sup_{x \in \mathbb{R}^N} |\alpha(x^1 + x_{m_k}^1, x^2 + k_{m_k}^2 T_2, \dots, x^N + k_{m_k}^N T_N) - \alpha(x)| \\
&= \sup_{x \in \mathbb{R}^N} |\alpha(x^1 + x_{m_k}^1, x^2, \dots, x^N) - \alpha(x)| \\
&\leq \sup_{x \in \mathbb{R}^N} |\alpha(x^1 + x_{m_k}^1, x') - \alpha(x^1 + \sigma_k, x')| + \sup_{x \in \mathbb{R}^N} |\alpha(x^1 + \sigma_k, x') - \alpha(x)| \\
&= \sup_{x \in \mathbb{R} \times U} |\alpha(x^1 + x_{m_k}^1, x') - \alpha(x^1 + \sigma_k, x')| + \sup_{x \in \mathbb{R} \times U} |\alpha(x^1 + \sigma_k, x') - \alpha(x)| \rightarrow 0.
\end{aligned}$$

By Lemma A.5, we know

$$\begin{aligned}
\|J'(\alpha(x + \tau_{m_k}), v_{m_k}) - J'(\alpha(x), v_{m_k})\|_{H^{-1}(\mathbb{R}^N)} &\leq c \|\alpha(x + \tau_{m_k}) - \alpha(x)\|_{L^\infty(\mathbb{R}^N)} \\
&\leq c \sup_{x \in \mathbb{R}^N} |\alpha(x + \tau_{m_k}) - \alpha(x)| \rightarrow 0.
\end{aligned}$$

By $v_{m_k} \rightharpoonup v_0$, for all $\varphi \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned}
\langle J'(\alpha(x), v_0), \varphi \rangle &= \lim_{k \rightarrow \infty} \langle J'(\alpha(x), v_{m_k}), \varphi \rangle \\
&= \lim_{k \rightarrow \infty} \langle J'(\alpha(x + \tau_{m_k}), v_{m_k}), \varphi \rangle \\
&= \lim_{k \rightarrow \infty} \langle J'(\alpha(x), v_{m_k(x - \tau_{m_k})}), \varphi(x - \tau_{m_k}) \rangle \\
&= \lim_{k \rightarrow \infty} \langle J'(\alpha(x), u_{m_k(x)}), \varphi(x - \tau_{m_k}) \rangle = 0,
\end{aligned}$$

that is, $J'(v_0) = 0$. So v_0 is a nontrivial solution of (1.1).

By Theorem 3.1, the nontrivial weak solution of (1.1) is a classical solution. Thus, we end the proof. $\square$

Funding information: Hui-Sheng Ding acknowledges support from the NSFC (12361023), Two Thousand Talents Program of Jiangxi Province (jxsq2019201001), and the Key Project of Jiangxi Provincial NSF (20242BAB26001). Wei Long acknowledges support from the NSFC (12271223), and the Key Project of Jiangxi Provincial NSF (20212ACB201003).

Author contributions: The all authors contribute equally.

Conflict of interest: The authors state no conflict of interest.

References

- [1] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. **248** (2004), no. 2, 423–443.
- [2] C. Albanese, *Localised solutions of Hartree equations for narrow-band crystals*, Comm. Math. Phys. **120** (1988), 97–103.
- [3] D. Cassani and J. Zhang, *Choquard-type equations with Hardy-Littlewood-Sobolev upper-critical growth*, Adv. Nonlinear Anal. **8** (2019), no. 1, 1184–1212.
- [4] I. Catto, C. Le Bris, and P. L. Lions, *On some periodic Hartree-type models for crystals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 143–190.
- [5] S. Cingolani, M. Clapp, and S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys. **63** (2012), no. 2, 233–248.
- [6] C. Corduneanu, *Almost Periodic Functions*, Interscience Publishers, New York, 1968.
- [7] H. Fröhlich, *Theory of electrical breakdown in ionic crystal*, Proc. R. Soc. Edinb. Sect. A **160** (1937), no. 901, 230–241.
- [8] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer Verlag, Berlin, 1983.
- [9] D. Giulini and A. Großardt, *The Schrödinger-Newton equation as a non-relativistic limit of self-gravitating Klein-Gordon and Dirac fields*, Class. Quantum Gravity **29** (2012), no. 21, 215010.
- [10] F. S. Guzmán and L. A. Ure na-López, *Newtonian collapse of scalar field dark matter*, Phys. Rev. D **68** (2003), 024023.
- [11] K. R. W. Jones, *Gravitational self-energy as the litmus of reality*, Mod. Phys. Lett. A **10** (1995), no. 8, 657–668.

- [12] X. Li and S. Ma, *Choquard equations with critical nonlinearities*, Commun. Contemp. Math. **22** (2020), 1950023, 28pp.
- [13] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57** (1977), no. 2, 93–105.
- [14] E. H. Lieb and M. Loss, *Analysis, Graduate Studies in Mathematics*, AMS, Providence, Rhode Island, 2001.
- [15] P. L. Lions, *The concentration-compactness principle in the calculus of variations, The locally compact case*, Part 1. Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145.
- [16] L. Ma and L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2010), 455–467.
- [17] X. Mingqi, V. D. Rădulescu, and B. Zhang, *A critical fractional Choquard-Kirchhoff problem with magnetic field*, Comm. Contemp. Math. **21** (2019), no. 4, 1850.
- [18] V. Moroz and J. van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), 153–184.
- [19] V. Moroz and J. van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6557–6579.
- [20] A. Pankov, *Semilinear elliptic equations on $\mathbb{R}^n$ with nonstabilizing coefficients*, Ukr. Math. J. **41** (1989), 1075–1078, transl. from Ukr. Mat. Zh. 41 (1989), 1247–1251.
- [21] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.
- [22] R. Penrose, *Quantum computation, entanglement and state reduction*, Philos. Trans. R. Soc. Lond. A, Math. Phys. Eng. Sci. **356** (1998), no. 1743, 1927–1939.
- [23] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, New Jersey, 1967.
- [24] D. D. Qin, V. D. Rădulescu, and X. H. Tang, *Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations*, J. Differential Equations **275** (2021), 652–683.
- [25] P. H. Rabinowitz, *A note on a semilinear elliptic equation on $\mathbb{R}^N$* , in *A Tribute in Honour of Giovanni Prodi*. Quaderni Scuola Normale Superiore, Pisa, 1991.
- [26] F. E. Schunck and E. W. Mielke, *General relativistic boson stars*, Class. Quantum Gravity **20** (2003), no. 20, R301–R356.
- [27] E. Serra, M. Tarallo, and S. Terracini, *On the existence of homoclinic solutions for almost periodic second order systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 783–812.
- [28] K. P. Tod and I. M. Moroz, *An analytical approach to the Schrödinger-Newton equations*, Nonlinearity **12** (1999), no. 2, 201–216.
- [29] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser Boston, 1996.
- [30] S. Zaidman, *Almost-Periodic Functions in Abstract Spaces*, Pitman Advanced Publishing Program, Boston, 1985.

Appendix

In this section, we give some preliminaries. First, we recall some definitions and properties concerning almost periodic functions depending on a parameter (cf. [6,30]).

Definition A.1. Let $\Omega \subset \mathbb{R}^{N-1}$ be a compact subset. A continuous function $\alpha(x^1, x')$ is called almost periodic in $x^1 \in \mathbb{R}$, uniformly with respect to $x' \in \Omega$, if for any $\varepsilon > 0$, there corresponds a number $l(\varepsilon) > 0$ such that any interval of the real line of length $l(\varepsilon)$ contains at least a real σ for which

$$\sup_{(x^1, x') \in \mathbb{R} \times \Omega} |\alpha(x^1 + \sigma, x') - \alpha(x^1, x')| \leq \varepsilon.$$

The number σ is called an ε -period for α . The uniform dependence on parameters follows from the fact that $l(\varepsilon)$ and σ are independent of x' .

The following two theorems contain classical results on almost periodic functions depending on parameters.

Theorem A.2. Let $\Omega \subset \mathbb{R}^{N-1}$ be a compact subset. Then an almost periodic function $\alpha(x^1, x')$ in x^1 uniformly with respect to $x' \in \Omega$ is uniformly continuous and bounded on $\mathbb{R} \times \Omega$.

Theorem A.3. Let $\Omega \subset \mathbb{R}^{N-1}$ be a compact subset. A necessary and sufficient condition for a function $\alpha(x^1, x')$ to be almost periodic in x^1 uniformly with respect to $x' \in \Omega$, is that the family of its translates $\{\alpha(x^1 + \tau, x'), \tau \in \mathbb{R}\}$ is (uniformly) pre-compact in $C(\mathbb{R} \times \Omega, \mathbb{R})$.

Next, we will give the property of the Choquard terms. Since the proof is simple or can be found directly, we omit the proof.

Lemma A.4. For $\beta \in A_a$, let $K_\beta = \{u \in H^1(\mathbb{R}^N) | J'(\beta, u) = 0, u \neq 0\}$. Then we have

- (i) $\inf_{\beta \in A_a} \inf_{u \in K_\beta} \|u\| > 0$;
- (ii) $\inf_{\beta \in A_a} \inf_{u \in K_\beta} J(\beta, u) > 0$.

Lemma A.5. Assume that Λ is a bounded subset of $H^1(\mathbb{R}^N)$. Then, for $\beta_1, \beta_2 \in L^\infty(\mathbb{R}^N)$ and all $u \in \Lambda$, there exists a constant $c(\Lambda) > 0$ such that

- (i) $|J(\beta_1, u) - J(\beta_2, u)| \leq c\|\beta_1 - \beta_2\|_{L^\infty(\mathbb{R}^N)}$;
- (ii) $\|J'(\beta_1, u) - J'(\beta_2, u)\|_{H^{-1}(\mathbb{R}^N)} \leq c\|\beta_1 - \beta_2\|_{L^\infty(\mathbb{R}^N)}$.

Proposition A.6. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then for every $\varphi \in H^1(\mathbb{R}^N)$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n - u|^q) |u_n - u|^{q-2} (u_n - u) \varphi \, dx = 0; \quad (\text{A1})$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [(|x|^{-\mu} * |u_n|^q) |u_n|^{q-2} u_n \varphi - (|x|^{-\mu} * |u|^q) |u|^{q-2} u \varphi] \, dx = 0. \quad (\text{A2})$$

Moreover, we have

Lemma A.7. (Brezis-Lieb lemma) [18, Lemma 2.4] Let $N \geq 3$, $p \in [1, \frac{2N}{2N-\mu}]$ and $0 < \mu < N$. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2Np}{2N-\mu}}(\mathbb{R}^N)$ such that $u_n \rightarrow u$ a.e. in $\mathbb{R}^N$, then

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n|^p) |u_n|^p \, dx - \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n - u|^p) |u_n - u|^p \, dx \right) = \int_{\mathbb{R}^N} (|x|^{-\mu} * |u|^p) |u|^p \, dx,$$

as $n \rightarrow \infty$.

By the Brezis-Lieb lemma of the Choquard terms, we obtain

Lemma A.8. *If $u_m \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, then*

- (1) $\sup_{\beta \in A_a} |J(\beta, u_m - u_0) - J(\beta, u_m) + J(\beta, u_0)| \rightarrow 0$;
- (2) $\sup_{\beta \in A_a} \|J'(\beta, u_m - u_0) - J'(\beta, u_m) + J'(\beta, u_0)\|_{H^{-1}(\mathbb{R}^N)} \rightarrow 0$.

Through Lemma A.8, we can prove the following representation theorem. Although the proof is standard [15], there exist some differences, which result from the almost periodic potential. To ensure the integrity of the article, we give the proof here.

Theorem A.9. (Representation theorem) *Suppose that a satisfies $(\alpha_1) \sim (\alpha'_3)$. Let $\{u_m\}$ be Palais-Smale sequence with the level $\tilde{c}$. Then there exist $u_0 \in H^1(\mathbb{R}^N)$, $l = l(\tilde{c}) \in \mathbb{N}$, $\beta_i \in \text{Hull}(a)$, $v_i \in H^1(\mathbb{R}^N)$, $v_i \neq 0$, $\{y_m^{(i)}\} \subset \mathbb{R}^N$, $i = 1, 2, \dots, l$, such that*

- (i) $u_m - u_0 - \sum_{i=1}^l v_i(x + y_m^{(i)}) \rightarrow 0$ in $H^1(\mathbb{R}^N)$;
- (ii) $J'(u_0) = 0$ and $J'(\beta_i, v_i) = 0$, $i = 1, \dots, l$;
- (iii) $\tilde{c} = J(u_0) + \sum_{i=1}^l J(\beta_i, v_i)$;
- (iv) $|y_m^{(i)}| \rightarrow \infty$, $i = 1, \dots, l$.

Proof. By the boundedness of $\{u_m\}$, there exists $u_0 \in H^1(\mathbb{R}^N)$, such that $w_m = u_m - u_0 \rightarrow 0$ in $H^1(\mathbb{R}^N)$. By Lemma A.8, we have

$$J'(u_0) = 0, \lim_{m \rightarrow \infty} J'(w_m) = 0 \quad \text{and} \quad J(w_m) = J(u_m) - J(u_0) + o_m(1).$$

By vanishing theorem [29], we know $\lim_{m \rightarrow \infty} \int_{\mathbb{R}^N} |w_m|^2 dx = \lambda_1 > 0$. Hence, we obtain dichotomy or compactness can occur, and hence, there exists $\tilde{R} > 0$ such that $\lim_{m \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_{\tilde{R}}(y)} |w_m|^2 dx > 0$. Thus, for every $m \in \mathbb{N}_+$, there exists $y_m^{(1)} \in \mathbb{R}^N$ such that

$$\sup_{y \in \mathbb{R}^N} \int_{B_{\tilde{R}}(y)} |w_m|^2 dx - \frac{1}{m} < \int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx,$$

that is,

$$\lim_{m \rightarrow \infty} \int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx \geq \lim_{m \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_{\tilde{R}}(y)} |w_m|^2 dx > 0.$$

Hence, when m is enough large,

$$\int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx \geq \frac{1}{2} \lim_{m \rightarrow \infty} \int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx.$$

Moreover, we know $\lim_{m \rightarrow \infty} |y_m^{(1)}| = \infty$. In fact, going if necessary to subsequence, we assume $|y_m^{(1)}| < \tilde{q}_1$. Then

$$\int_{B_{\tilde{c}_1+2\tilde{R}}} |w_m|^2 dx > \int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx \geq \frac{1}{2} \lim_{m \rightarrow \infty} \int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx > 0.$$

This contradicts with $w_m \rightarrow 0$.

So, we have $\lim_{m \rightarrow \infty} |y_m^{(1)}| = \infty$ and $\lim_{m \rightarrow \infty} \int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx > 0$. We may assume that there exists $v_1 \in H^1(\mathbb{R}^N)$, such that

$$\begin{aligned} w_m(x + y_m^{(1)}) &\rightharpoonup v_1 \quad \text{in } H^1(\mathbb{R}^N), \\ w_m(x + y_m^{(1)}) &\rightarrow v_1 \quad \text{in } L_{\text{loc}}^2(\mathbb{R}^N). \end{aligned}$$

By

$$\lim_{m \rightarrow \infty} \int_{B_{\tilde{R}}} |w_m(x + y_m^{(1)})|^2 dx = \lim_{m \rightarrow \infty} \int_{B_{\tilde{R}}(y_m^{(1)})} |w_m|^2 dx > 0,$$

we have $v_1 \neq 0$.

For each $m \in \mathbb{N}$ and $\alpha(x + y_m^{(1)})$, we choose $k_m^{(1),i} \in \mathbb{N}$, $i = 2, \dots, N$, $(z_m^{(1),2}, z_m^{(1),3}, \dots, z_m^{(1),N}) \subset [0, T_2] \times \dots \times [0, T_N]$ such that

$$\alpha(x + y_m^{(1)}) = \alpha(x^1 + y_m^{(1),1}, z_m^{(1),2} + k_m^{(1),2}T_2, \dots, z_m^{(1),N} + k_m^{(1),N}T_N) = \alpha(x^1 + y_m^{(1),1}, z_m^{(1),2}, \dots, z_m^{(1),N})$$

since the aforementioned equality is from (α'_3) . By Theorem A.3, we know there exists $\beta_1 \in A_\alpha = \{\beta \in C(\mathbb{R}^N, \mathbb{R}) | \underline{\alpha} \leq \beta(x) \leq \bar{\alpha}, \forall x \in \mathbb{R}^N\}$ such that $\lim_{m \rightarrow \infty} \sup_{x \in \mathbb{R}^N} |\alpha(x + y_m^{(1)}) - \beta_1(x)| = 0$.

Since $\{w_m(x + y_m^{(1)})\}$ is bounded in $H^1(\mathbb{R}^N)$ and Lemma A.5, we have

$$\begin{aligned} |J(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)})) - J(\beta_1, w_m(x + y_m^{(1)}))| &\leq c \|\alpha(x + y_m^{(1)}) - \beta_1(x)\|_{L^\infty(\mathbb{R}^N)} \\ &\leq c \sup_{x \in \mathbb{R}^N} |\alpha(x + y_m^{(1)}) - \beta_1(x)| \end{aligned}$$

and

$$\begin{aligned} \|J'(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)})) - J'(\beta_1, w_m(x + y_m^{(1)}))\|_{H^{-1}(\mathbb{R}^N)} \\ \leq c \|\alpha(x + y_m^{(1)}) - \beta_1(x)\|_{L^\infty(\mathbb{R}^N)} \leq c \sup_{x \in \mathbb{R}^N} |\alpha(x + y_m^{(1)}) - \beta_1(x)|. \end{aligned}$$

So

$$\lim_{m \rightarrow \infty} J(\beta_1, w_m(x + y_m^{(1)})) = \lim_{m \rightarrow \infty} J(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)})) = \lim_{m \rightarrow \infty} J(w_m) = \tilde{c} > 0$$

and

$$\lim_{m \rightarrow \infty} \|J'(\beta_1, w_m(x + y_m^{(1)}))\|_{H^{-1}(\mathbb{R}^N)} = \lim_{m \rightarrow \infty} \|J'(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)}))\|_{H^{-1}(\mathbb{R}^N)} = \lim_{m \rightarrow \infty} \|J'(w_m)\|_{H^{-1}(\mathbb{R}^N)} = 0.$$

By $w_m(x + y_m^{(1)}) \rightharpoonup v_1$, we know

$$J'(\beta_1, v_1) = 0. \quad (\text{A3})$$

By Lemmas A.4–A.8, the energy function

$$\begin{aligned} &|J(w_m - v_1(x - y_m^{(1)})) - J(w_m) + J(\beta_1, v_1)| \\ &= |J(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)}) - v_1) - J(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)})) + J(\beta_1, v_1)| \\ &\leq |J(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)}) - v_1) - J(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)})) + J(\alpha(x + y_m^{(1)}), v_1)| + |J(\alpha(x + y_m^{(1)}), v_1) \\ &\quad - J(\beta_1, v_1)| \\ &\leq \sup_{\beta \in A_\alpha} |J(\beta, w_m(x + y_m^{(1)}) - v_1) - J(\beta, w_m(x + y_m^{(1)})) + J(\beta, v_1)| + c \sup_{x \in \mathbb{R}^N} |\alpha(x + y_m^{(1)}) - \beta_1|. \end{aligned}$$

So

$$\lim_{m \rightarrow \infty} J(w_m - v_1(x - y_m^{(1)})) = \tilde{c} - J(\beta_1, v_1). \quad (\text{A4})$$

By Lemma A.4, we know $\tilde{c} - J(\beta_1, v_1) \geq 0$.

For every $\varphi \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned}
& |\langle J'(w_m - v_1(x - y_m^{(1)})), \varphi \rangle| \\
& \leq |\langle J'(w_m - v_1(x - y_m^{(1)})), \varphi \rangle - \langle J'(w_m), \varphi \rangle + \langle J'(v_1(x - y_m^{(1)})), \varphi \rangle| + |\langle J'(w_m), \varphi \rangle - \langle J'(v_1(x - y_m^{(1)})), \varphi \rangle| \\
& \leq \|J'(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)})) - v_1 - J'(\alpha(x + y_m^{(1)}), w_m(x + y_m^{(1)})) + J'(\alpha(x + y_m^{(1)}), v_1)\|_{H^{-1}(\mathbb{R}^N)} \|\varphi\| \\
& \quad + |\langle J'(w_m), \varphi \rangle - \langle J'(v_1(x - y_m^{(1)})), \varphi \rangle|.
\end{aligned} \tag{A5}$$

On the other hand, we know

$$\begin{aligned}
& |\langle J'(w_m), \varphi \rangle - \langle J'(v_1(x - y_m^{(1)})), \varphi \rangle| \\
& \leq \|J'(w_m)\|_{H^{-1}(\mathbb{R}^N)} \|\varphi\| + \|J'(\alpha(x + y_m^{(1)}), v_1)\|_{H^{-1}(\mathbb{R}^N)} \|\varphi\| \\
& \leq \|J'(w_m)\|_{H^{-1}(\mathbb{R}^N)} \|\varphi\| + \|J'(\alpha(x + y_m^{(1)}), v_1) - J'(\beta_1, v_1)\|_{H^{-1}(\mathbb{R}^N)} \|\varphi\| + \|J'(\beta_1, v_1)\|_{H^{-1}(\mathbb{R}^N)} \|\varphi\|.
\end{aligned} \tag{A6}$$

By Lemmas A.5 and A.8, combining (A3) and (A5), with (A6), we have

$$\lim_{m \rightarrow \infty} \|J'(w_m - v_1(x - y_m^{(1)}))\|_{H^{-1}(\mathbb{R}^N)} = 0. \tag{A7}$$

Combining to (A3), (A4) and (A7), as $m \rightarrow \infty$, we have

$$\begin{aligned}
J'(\beta_1, v_1) &= 0 \quad \text{in } H^{-1}(\mathbb{R}^N), \\
J(w_m - v_1(x - y_m^{(1)})) &\rightarrow \tilde{c} - J(\beta_1, v_1) \geq 0, \\
J'(w_m - v_1(x - y_m^{(1)})) &\rightarrow 0.
\end{aligned}$$

(1⁰) If $\tilde{c} - J(\beta_1, v_1) = 0$, that is,

$$\begin{aligned}
J(w_m - v_1(x - y_m^{(1)})) &\rightarrow 0, \\
J'(w_m - v_1(x - y_m^{(1)})) &\rightarrow 0,
\end{aligned}$$

then

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \{J(w_m - v_1(x - y_m^{(1)})) - \frac{1}{2q} \langle J'(w_m - v_1(x - y_m^{(1)})), w_m - v_1(x - y_m^{(1)}) \rangle\} \\
& = \lim_{m \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{2q} \right) \int_{\mathbb{R}^N} (|\nabla(w_m - v_1(x - y_m^{(1)}))|^2 + \alpha(x)|w_m - v_1(x - y_m^{(1)})|^2) dx = 0.
\end{aligned}$$

So, $\lim_{m \rightarrow \infty} \|w_m - v_1(x - y_m^{(1)})\| = 0$ and $\tilde{c} = J(\beta_1, v_1)$. Thus, the theorem is proved with $l = 1$.

(2⁰) If $\tilde{c} - J(\beta_1, v_1) > 0$, that is, $J(\beta_1, v_1) = \tilde{c}_1 < \tilde{c}$, setting $w_m^{(1)}(x) = w_m(x) - v_1(x - y_m^{(1)})$, then

$$\begin{aligned}
J(w_m^{(1)}(x)) &\rightarrow \tilde{c} - \tilde{c}_1 > 0, \\
J'(w_m^{(1)}(x)) &\rightarrow 0.
\end{aligned}$$

We iterate as in the aforementioned steps.

To prove that this procedure ends, it is enough to show that for some $l \in \mathbb{N}$, we obtain $J(\beta_l, v_l) = c_l = c - c_1 - c_2 - \dots - c_{l-1}$. By Lemma A.4, we obtain $c - c_1 - c_2 - \dots - c_l = 0$ after most for some l . $\square$