Research Article

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Temporal periodic solutions of non-isentropic compressible Euler equations with geometric effects

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Abstract: In this article, we investigate the general qusi-one-dimensional nozzle flows governed by non-isentropic compressible Euler system. First, the steady states of the subsonic and supersonic flows are analyzed. Then, the existence, stability, and uniqueness of the subsonic temporal periodic solutions around the steady states are proved by constructing a new iterative format technically. Besides, further regularity and stability of the obtained temporal periodic solutions are obtained, too. The main difficulty in the proof is coming from derivative loss, which is caused by the diagonalization. Observing that the entropy is conserved along the second characteristic curve, we overcome this difficulty by transforming the derivative of entropy with respect to x into a derivative along the direction of first or third characteristic. The results demonstrate that dissipative boundary feedback control can stabilize the non-isentropic compressible Euler equations in qusi-one-dimensional nozzles.

Keywords: non-isentropic compressible Euler equations, geometric effects, temporal periodic solutions, subsonic flow

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1 Introduction

This article investigates the qusi-one-dimensional non-isentropic compressible Euler equations:

$$\begin{cases} n_t + (nu)_x = -a(x)nu, \\ (nu)_t + (nu^2 + p(n,S))_x = -a(x)nu^2, \\ S_t + uS_x = 0, \end{cases} \tag{1.1}$$

for $(t,x) \in \mathbb{R}_+ \times [0,L]$. (1.1) can describe the motion of flows in a nozzle with generally varied cross-sectional area, where $L > 0$ is a constant denoting the considered nozzle length.

The unknown functions $u(t,x)$, $n(t,x)$, and $S(t,x)$ denote velocity, density, and entropy of the considered fluids, respectively. The pressure function is $p(n,S) = e^S n^\gamma$, with the adiabatic gas index $\gamma \in (1,3)$. And the function $a(x) = \frac{A'(x)}{A(x)}$, where $A(x) \in C^2([0,L])$ represents the cross-section of the duct. When $A(x)$ are $1, x, x^2$, system (1.1) denotes the classical $1D$, $2D$ rotationally, and $3D$ spherically symmetric Euler equations, respectively. We further assume there exist positive constants D_0, D_1, and D_2 such that

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$$\|a(x)\|_{C^1} \le D_0 \tag{1.2}$$

and

$$D_1 < A(x) < D_2, \tag{1.3}$$

for any $x \in [0, L]$.

There are many literature about the compressible Euler flows moving through nozzles. For example, by a numerical scheme, Liu [14] constructed the global solutions for general quasilinear hyperbolic systems, which include the model of flows in variable area pipelines. The stability and instability of transonic flows through a nozzle were discussed in [15]. Later on, an efficient shock-capturing scheme for solving the Euler equations with geometric structure was proposed in [4,5]. We can refer to [2,6,7,11,12,16,17,24,27,28] for some other existing results of global solutions. Meanwhile, many works have been done on the multidimensional non-isentropic steady Euler flows. In [3], Chen considered the non-isentropic inviscid subsonic steady flows with large vorticity in a two-dimensional infinitely long nozzle and built the existence and uniqueness of the subsonic problem. Duan and Luo[9] investigated the problem in three-dimensional axisymmetric nozzles and removed the constraint on the smallness of the vorticity in [8]. Then [20] gained the optimal convergence rates of the subsonic non-isentropic steady Euler equations through a three-dimensional axisymmetric infinitely long nozzle.

At the same time, time-periodic problems have attracted much attention for its applicability in industry. For the temporal periodic solutions excited by the time-periodic external force, we can see [1,18,19,23]. However, the time-periodic solutions motivated by boundary conditions are significant in the feedback boundary control. In [32], Yuan first gained the existence and stability of isentropic supersonic time-periodic Euler flows which are motivated by periodic boundary conditions. Inspired by this result, [31] and [21] further studied the time-periodic supersonic solutions with the source term $\beta\rho\,|u|^{\alpha}u$ for isentropic and non-isentropic cases. It is worth noting that the supersonic assumption, which means all characteristics propagate forward in both space and time, plays an important role in [21,31,32]. Therefore, after a specific start-up time $T^* > 0$, the boundary condition can control the entire region completely, thus we gain the existence of time-periodic solutions. However, it becomes more complicated for subsonic case since the characteristics propagate both forward and backward, which leads to wave interactions between different families. For one-dimensional subsonic isentropic compressible Euler system, [26,33] investigated the time-periodic solutions for linear and nonlinear damping, respectively. There are also some important researches on the existence of space and time periodic problem for non-isentropic compressible Euler systems, see [29,30]. As for more general quasi-linear hyperbolic systems, we can refer to [10,25] for details.

To our knowledge, in general ducts, few studies have been made on the subsonic temporal periodic flows driven by periodic boundary conditions so far. However, considering this problem is one of the basic steps in studying time-periodic transonic shock solutions. In this article, motivated by [10,25,26,33], we study the non-isentropic subsonic temporal periodic solutions to compressible Euler system with geometric effects. By using an iterative method, we prove that there exists an initial data such that the periodic perturbation on the dissipative boundaries can trigger a time-periodic subsonic solution. Owing to the appearance of the geometric source term, it is important to make full use of the dissipative effects brought by boundaries and construct appropriate iterative formats. Unfortunately, a new challenge, *derivative loss*, emerges from the iterative format after diagonalization. We overcome this difficulty by noticing a significant property of entropy, which enables us to switch the derivative of entropy with respect to x into a derivative along the direction of the first or third characteristic.

The remaining parts of this article are organized as follows. In Section 2, we presented some elementary properties of the C^1 subsonic and supersonic steady solutions. Section 3 was devoted to reformulate (1.1) and the steady states system by Riemann invariants and present main results. Then in Section 4–7, we gave proofs of Theorems 3.1–3.4, respectively.

2 Steady solutions

In this section, we study the solution $\tilde{Q}(x) = (\tilde{n}(x), \tilde{u}(x), \tilde{S}(x))^{\top}$ of the time-independent system

$$\begin{cases}(\tilde{n}\tilde{u})_x = -a(x)\tilde{n}\tilde{u},\\ (\tilde{n}\tilde{u}^2 + p(\tilde{n}, \tilde{S}))_x = -a(x)\tilde{n}\tilde{u}^2,\\ \tilde{u}\tilde{S}_x = 0,\end{cases} \tag{2.1}$$

or equivalently

$$\begin{cases}\tilde{n}_x\tilde{u} + \tilde{n}\tilde{u}_x = -a(x)\tilde{n}\tilde{u},\\ \tilde{u}\tilde{u}_x + \gamma e^{\tilde{S}}\tilde{n}^{\gamma-2}\tilde{n}_x + e^{\tilde{S}}\tilde{n}^{\gamma-1}\tilde{S}_x = 0,\\ \tilde{u}\tilde{S}_x = 0.\end{cases} \tag{2.2}$$

We give the inflow data at $x = 0$:

$$(\tilde{n}(x), \tilde{u}(x), \tilde{S}(x))^{\top}|_{x=0} = (n_-, u_-, S_-)^{\top} \tag{2.3}$$

with $u_- > 0$ and $n_- > 0$. By $(2.2)_1$, $(2.2)_3$, and (2.3), we know that the cross-sectional flux and entropy are two constants, i.e.,

$$A(x)\tilde{n}(x)\tilde{u}(x) = A(0)n_-u_-, \quad \tilde{S}(x) = S_-. \tag{2.4}$$

One critical parameter in fluid dynamics is the Mach number, which can be defined as follows:

$$M = \frac{\tilde{u}}{\tilde{c}},$$

where the sound speed $\tilde{c}(\tilde{n}, \tilde{S}) = \sqrt{\frac{\partial p}{\partial \tilde{n}}} = \sqrt{\gamma}\, e^{\frac{\tilde{S}}{2}}\tilde{n}^{\frac{\gamma-1}{2}}$. The fluid is supersonic for $M > 1$ and subsonic while $M < 1$. From (2.2), we could solve the continuous flows with Mach number $M \neq 1$, i.e.,

$$\begin{cases}\dfrac{\mathrm{d}\tilde{n}}{\mathrm{d}x} = -\dfrac{a(x)\tilde{n}}{1 - \frac{1}{M^2}},\\ \dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}x} = \dfrac{a(x)\tilde{u}\frac{1}{M^2}}{1 - \frac{1}{M^2}},\\ \dfrac{\mathrm{d}\tilde{S}}{\mathrm{d}x} = 0.\end{cases} \tag{2.5}$$

Then by the definition of M, it can be deduced that

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \frac{a(x)M\left(\frac{1}{M^2} + \frac{\gamma-1}{2}\right)}{1 - \frac{1}{M^2}}. \tag{2.6}$$

Therefore, if we assume the fluid velocity is supersonic (i.e., $c_- < u_-$) or subsonic (i.e., $c_- > u_-$) at the entrance $x = 0$, then problem (2.2) admits a C^1 supersonic or subsonic solution on $[0, L_*)$, where L_* is the life span of the corresponding supersonic or subsonic smooth solution. In this article, we expect the steady flow to keep its entrance supersonic or subsonic situation in the whole duct. Therefore, the length of the duct L should be smaller than L_*. While the maximum duct length L_* can be determined by (2.6), which denotes L_* only depends on γ, the initial data (n_-, u_-, S_-) and the given function $a(x)$. We will consider some other cases, such as the transonic smooth solutions and transonic shocks in the following article.

From $(2.5)_1$ and $(2.5)_2$, we directly derive

$$\tilde{n}(x) = n_-e^{-\int_0^x \frac{a(\varsigma)M^2(\varsigma)}{M^2(\varsigma)-1}\mathrm{d}\varsigma}, \quad \tilde{u}(x) = u_-e^{\int_0^x \frac{a(\varsigma)}{M^2(\varsigma)-1}\mathrm{d}\varsigma}. \tag{2.7}$$

Then, when the incoming flow is subsonic, namely, $c_- > u_- > 0$, it follows from $(2.5)_1$, $(2.5)_2$, and (2.6) that

$$\frac{\mathrm{d}\tilde{n}}{\mathrm{d}x} > 0, \quad \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} < 0, \quad \frac{\mathrm{d}M}{\mathrm{d}x} < 0, \quad \text{for } a(x) > 0; \tag{2.8}$$

$$\frac{\mathrm{d}\tilde{n}}{\mathrm{d}x} < 0, \quad \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} > 0, \quad \frac{\mathrm{d}M}{\mathrm{d}x} > 0, \quad \text{for } a(x) < 0. \tag{2.9}$$

Similarly, when the incoming flow is supersonic, i.e., $u_- > c_- > 0$, one has

$$\frac{\mathrm{d}\tilde{n}}{\mathrm{d}x} < 0, \quad \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} > 0, \quad \frac{\mathrm{d}M}{\mathrm{d}x} > 0, \quad \text{for } a(x) > 0; \tag{2.10}$$

$$\frac{\mathrm{d}\tilde{n}}{\mathrm{d}x} > 0, \quad \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} < 0, \quad \frac{\mathrm{d}M}{\mathrm{d}x} < 0, \quad \text{for } a(x) < 0. \tag{2.11}$$

Formulas (2.8)–(2.11) imply that if the duct is divergent, the flow always keeps its state of entrance for any duct length $L > 0$, regardless subsonic or supersonic. While, if the duct is convergent, to keep the upstream subsonic or supersonic state, there must exist a unique positive constant $L_* < +\infty$, such that the flow is subsonic or supersonic for $x \in [0, L_*)$. In addition, after some straightforward computations, we obtain that:

(a) If $a(x) > 0$, according to (1.3) and (2.4), for $x \in [0, +\infty)$, we have

$$n_- < \tilde{n}(x) < n_- \left(\frac{A(x)}{A(0)}\right)^{\frac{u_-^2}{c_-^2 - u_-^2}}, \quad u_- \left(\frac{A(0)}{A(x)}\right)^{\frac{c_-^2}{c_-^2 - u_-^2}} < \tilde{u}(x) < u_-, \quad \text{for } c_- > u_- > 0, \tag{2.12}$$

and

$$n_- \left(\frac{A(0)}{A(x)}\right)^{\frac{u_-^2}{u_-^2 - c_-^2}} < \tilde{n}(x) < n_-, \quad u_- < \tilde{u}(x) < u_- \left(\frac{A(x)}{A(0)}\right)^{\frac{c_-^2}{u_-^2 - c_-^2}}, \quad \text{for } u_- > c_- > 0. \tag{2.13}$$

(b) If $a(x) < 0$, for $x \in [0, L]$ and $L < L_*$, we have

$$n_- e^{\int_0^L \frac{a(\xi) M^2(\xi)}{1 - M^2(\xi)} \mathrm{d}\xi} < \tilde{n}(x) < n_-, \quad u_- < \tilde{u}(x) < \frac{u_-}{e^{\int_0^L \frac{a(\xi)}{1 - M^2(\xi)} \mathrm{d}\xi}}, \quad \text{for } c_- > u_- > 0, \tag{2.14}$$

and

$$n_- < \tilde{n}(x) < \frac{n_-}{e^{\int_0^L \frac{a(\xi) M^2(\xi)}{M^2(\xi) - 1} \mathrm{d}\xi}}, \quad u_- e^{\int_0^L \frac{a(\xi)}{M^2(\xi) - 1} \mathrm{d}\xi} < \tilde{u}(x) < u_-, \quad \text{for } u_- > c_- > 0. \tag{2.15}$$

To sum up, we conclude the properties of the steady states by the following theorem.

Theorem 2.1. *For any subsonic/supersonic smooth incoming flow* (n_-, u_-, S_-), *there exists a critical duct length* $L_* = L_*(\gamma, a(x), \tilde{Q}(0))$, *if* $L < L_*$, *system* (2.2)–(2.3) *admits a unique* C^1 *subsonic/supersonic solution* $\tilde{Q}(x) = (\tilde{n}(x), \tilde{u}(x), \tilde{S}(x))^{\mathrm{T}}$ *on* $[0, L]$. *Furthermore, it holds*:

(*i*) *The moving trends of the velocity u, density n and Mach number M can be judged by the relative slope* $a(x)$ *of the nozzle and the upstream states. See* Table 1.

(*ii*) $\tilde{n}(x)$ *and* $\tilde{u}(x)$ *are bounded, to be specific, there exist positive smooth functions* $K_i(x)$ $(i = 1, 2)$, *such that*

$$K_1(x) n_- < \tilde{n}(x) < K_2(x) n_-, \quad K_1(x) u_- < \tilde{u}(x) < K_2(x) u_-,$$

here, $K_i(x)$, $(i = 1, 2)$ *can be deduced from* (2.12)–(2.15).

(*iii*) *When* $a(x) > 0$, L_* *is infinite, while when* $a(x) < 0$, L_* *is finite.*

Table 1: Moving trend of flow

	$a(x) > 0$	$a(x) < 0$
Subsonic incoming flow	$\tilde{n} \uparrow, \tilde{u} \downarrow, M \downarrow$	$\tilde{n} \downarrow, \tilde{u} \uparrow, M \uparrow$
Supersonic incoming flow	$\tilde{n} \downarrow, \tilde{u} \uparrow, M \uparrow$	$\tilde{n} \uparrow, \tilde{u} \downarrow, M \downarrow$

3 Reformulation and main results

In this section, we rewrite systems (1.1) by Riemann invariants. Besides, we state our main results of this article. First, by simple calculations, the Riemann invariants of system (1.1) are as follows:

$$\varsigma = u - \frac{2}{\gamma - 1}\sqrt{\gamma}\, e^{\frac{S}{2}} n^{\frac{\gamma-1}{2}}, \quad v = S, \quad \eta = u + \frac{2}{\gamma - 1}\sqrt{\gamma}\, e^{\frac{S}{2}} n^{\frac{\gamma-1}{2}}. \tag{3.1}$$

Then system (1.1) changes into

$$\begin{cases} \varsigma_t + \lambda_1(\varsigma, \eta)\varsigma_x = \dfrac{\gamma - 1}{8} a(x)(\eta^2 - \varsigma^2) + \dfrac{\gamma - 1}{16\gamma}(\eta - \varsigma)^2 v_x, \\ v_t + \lambda_2(\varsigma, \eta) v_x = 0, \\ \eta_t + \lambda_3(\varsigma, \eta)\eta_x = -\dfrac{\gamma - 1}{8} a(x)(\eta^2 - \varsigma^2) + \dfrac{\gamma - 1}{16\gamma}(\eta - \varsigma)^2 v_x, \end{cases} \tag{3.2}$$

where

$$\lambda_1 = \frac{\gamma + 1}{4}\varsigma + \frac{3 - \gamma}{4}\eta, \quad \lambda_2 = \frac{1}{2}(\varsigma + \eta), \quad \lambda_3 = \frac{3 - \gamma}{4}\varsigma + \frac{\gamma + 1}{4}\eta. \tag{3.3}$$

Similarly, the steady system (2.2) can be rewritten as follows:

$$\begin{cases} \tilde{\lambda}_1 \tilde{\varsigma}_x = \dfrac{\gamma - 1}{8} a(x)(\tilde{\eta}^2 - \tilde{\varsigma}^2) + \dfrac{\gamma - 1}{16\gamma}(\tilde{\eta} - \tilde{\varsigma})^2 \tilde{v}_x, \\ \tilde{\lambda}_2 \tilde{v}_x = 0, \\ \tilde{\lambda}_3 \tilde{\eta}_x = -\dfrac{\gamma - 1}{8} a(x)(\tilde{\eta}^2 - \tilde{\varsigma}^2) + \dfrac{\gamma - 1}{16\gamma}(\tilde{\eta} - \tilde{\varsigma})^2 \tilde{v}_x, \end{cases} \tag{3.4}$$

where

$$\tilde{\varsigma} = \tilde{u} - \frac{2}{\gamma - 1}\sqrt{\gamma}\, e^{\frac{\tilde{S}}{2}} \tilde{n}^{\frac{\gamma-1}{2}}, \quad \tilde{v} = \tilde{S}, \quad \tilde{\eta} = \tilde{u} + \frac{2}{\gamma - 1}\sqrt{\gamma}\, e^{\frac{\tilde{S}}{2}} \tilde{n}^{\frac{\gamma-1}{2}},$$

and

$$\tilde{\lambda}_1 = \frac{\gamma + 1}{4}\tilde{\varsigma} + \frac{3 - \gamma}{2}\tilde{\eta}, \quad \tilde{\lambda}_2 = \frac{1}{2}\tilde{\varsigma} + \frac{1}{2}\tilde{\eta}, \quad \tilde{\lambda}_3 = \frac{3 - \gamma}{4}\tilde{\varsigma} + \frac{\gamma + 1}{4}\tilde{\eta}.$$

As noted in Section 1, the supersonic case is much elementary as it can be directly solved by wave decomposition method. More details can be seen in [21,22], here we omit the supersonic case.

To formulate the subsonic problem, we now specify the initial data and boundary conditions. Assuming that at $t = 0$:

$$\varsigma(0, x) = \varsigma_0(x), \quad \nu(0, x) = \nu_0(x), \quad \eta(0, x) = \eta_0(x), \tag{3.5}$$

and at $x = L$:

$$\varsigma(t, L) = \varsigma_p(t) + s_1(v(t, L) - \tilde{v}(L)) + s_1^*(\eta(t, L) - \tilde{\eta}(L)), \tag{3.6}$$

while at $x = 0$:

$$v(t, 0) = v_p(t) + s_2(\varsigma(t, 0) - \tilde{\varsigma}(0)), \tag{3.7}$$

$$\eta(t, 0) = \eta_p(t) + s_3(\varsigma(t, 0) - \tilde{\varsigma}(0)), \tag{3.8}$$

where the boundary conditions (3.6)–(3.8) are dissipative and time-periodic. That is, $\max\{|s_1| + |s_1^*|, |s_2|, |s_3|\} = s < 1$, and $\varsigma_p(t), \eta_p(t), v_p(t)$ are temporal periodic functions with a constant time period T. The solid boundaries (3.6)–(3.8) possess the dissipative structure in the sense of Li [13], who pointed out that without this boundary dissipation, the initial-boundary value problem of hyperbolic equations may blow up in finite time, even when the initial data is small.

Taking the steady solution $\tilde{Q}(x) = (\tilde{n}(x), \tilde{u}(x), \tilde{S}(x))^\top$ as a background solution, now we define the perturbation variables

$$\Gamma(t,x) = (\Gamma_1(t,x), \Gamma_2(t,x), \Gamma_3(t,x))^\top \overset{\text{def.}}{=} (\varsigma(t,x) - \tilde{\varsigma}(x), \upsilon(t,x) - \tilde{\upsilon}(x), \eta(t,x) - \tilde{\eta}(x))^\top, \tag{3.9}$$

and rename the steady solution as follows:

$$\tilde{\Gamma}(x) = (\tilde{\Gamma}_1(x), \tilde{\Gamma}_2(x), \tilde{\Gamma}_3(x))^\top = (\tilde{\varsigma}(x), \tilde{\upsilon}(x), \tilde{\eta}(x))^\top. \tag{3.10}$$

Then combining (3.2) and (3.4) yields

$$\begin{aligned}\Gamma_{1t} + \lambda_1(\Gamma + \tilde{\Gamma})\Gamma_{1x} = &-\frac{\gamma-1}{8}a(x)\frac{(\gamma+1)(\tilde{\Gamma}_1^2 + \tilde{\Gamma}_3^2) + 2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1 + (3-\gamma)\tilde{\Gamma}_3}\Gamma_1 + \frac{\gamma-1}{16\gamma}(\Gamma_3 - \Gamma_1 + \tilde{\Gamma}_3 - \tilde{\Gamma}_1)^2\partial_x\Gamma_2 \\ &+ \frac{\gamma-1}{8}a(x)\frac{(3-\gamma)(\tilde{\Gamma}_1^2 + \tilde{\Gamma}_3^2) + 2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1 + (3-\gamma)\tilde{\Gamma}_3}\Gamma_3 + \frac{\gamma-1}{8}a(x)(\Gamma_3^2 - \Gamma_1^2),\end{aligned} \tag{3.11}$$

$$\Gamma_{2t} + \lambda_2(\Gamma + \tilde{\Gamma})\Gamma_{2x} = 0, \tag{3.12}$$

$$\begin{aligned}\Gamma_{3t} + \lambda_3(\Gamma + \tilde{\Gamma})\Gamma_{3x} = &-\frac{\gamma-1}{8}a(x)\frac{(\gamma+1)(\tilde{\Gamma}_1^2 + \tilde{\Gamma}_3^2) + 2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1 + (\gamma+1)\tilde{\Gamma}_3}\Gamma_3 + \frac{\gamma-1}{16\gamma}(\Gamma_3 - \Gamma_1 + \tilde{\Gamma}_3 - \tilde{\Gamma}_1)^2\partial_x\Gamma_2 \\ &+ \frac{\gamma-1}{8}a(x)\frac{(3-\gamma)(\tilde{\Gamma}_1^2 + \tilde{\Gamma}_3^2) + 2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1 + (\gamma+1)\tilde{\Gamma}_3}\Gamma_1 - \frac{\gamma-1}{8}a(x)(\Gamma_3^2 - \Gamma_1^2).\end{aligned} \tag{3.13}$$

Correspondingly, the initial boundary conditions are as follows:

$$\begin{aligned} t = 0 : \Gamma(0,x) = \Gamma_0(x) &= (\Gamma_{10}(x), \Gamma_{20}(x), \Gamma_{30}(x))^T \\ &= (\varsigma_0(x) - \tilde{\varsigma}(x), \upsilon_0(x) - \tilde{\upsilon}(x), \eta_0(x) - \tilde{\eta}(x))^\top,\end{aligned} \tag{3.14}$$

$$x = L : \Gamma_1(t,L) = \Gamma_{1p}(t) + s_1\Gamma_2(t,L) + s_1^*\Gamma_3(t,L), \tag{3.15}$$

$$x = 0 : \Gamma_2(t,0) = \Gamma_{2p}(t) + s_2\Gamma_1(t,0), \tag{3.16}$$

$$\Gamma_3(t,0) = \Gamma_{3p}(t) + s_3\Gamma_1(t,0), \tag{3.17}$$

where $\Gamma_{1p}(t) = \varsigma_p(t) - \tilde{\varsigma}(L)$, $\Gamma_{2p}(t) = \upsilon_p(t) - \tilde{\upsilon}(0)$, $\Gamma_{3p}(t) = \eta_p(t) - \tilde{\eta}(0)$, and $\Gamma_{ip}(t)$, $(i = 1, 2, 3)$ satisfy

$$\Gamma_{ip}(t + T) = \Gamma_{ip}(t). \tag{3.18}$$

To gain the global existence of time-periodic solutions, we require that the incoming sound speed is a small quantity, namely, there is a small positive constant ε_0, such that

$$c_- < \varepsilon_0. \tag{3.19}$$

In fact, we can acquire the upper bound for the small constant ε_0, which depends on the relative slope $a(x)$, the duct length L, the dissipation coefficient s, and the inflow data (2.3). However, to specify this upper bound would cause a tedious primary calculation which deviates from our main purpose in this article. Concerning the slope of nozzles, it is worth pointing out that we only require the cross-section $A(x)$ be C^2 continuous, without any other restrictions.

Now we are ready to state our main results in this article.

Theorem 3.1. (Existence of the temporal periodic solution) *There exists a constant $J_E > 0$ and a small constant $\varepsilon_1 > 0$, such that for any given $\varepsilon \in (0, \varepsilon_1)$, any given $T \in \mathbb{R}_+$, and any given functions $\Gamma_{ip}(t)$ $(i = 1, 2, 3)$ satisfying* (3.18) *and*

$$\|\Gamma_{ip}(t)\|_{C^1} \le \varepsilon, \tag{3.20}$$

there exists a C^1 smooth function $\Gamma_0(x)$ with

$$\|\Gamma_0(x)\|_{C^1([0,L])} \le J_E\varepsilon, \tag{3.21}$$

such that the disturbance systems (3.11)–(3.17) *admits a* C^1 *temporal periodic classical solution* $\Gamma = \Gamma^{(T)}(t,x)$ *in the region* $\{(t,x) \in \mathbb{R}_+ \times [0,L]\}$, *satisfying*

$$\Gamma^{(T)}(t+T,x) = \Gamma^{(T)}(t,x), \quad \forall (t,x) \in \{(t,x) \in \mathbb{R}_+ \times [0,L]\}, \tag{3.22}$$

and

$$\|\Gamma^{(T)}\|_{C^1(\mathbb{R}_+\times[0,L])} \le J_E \varepsilon. \tag{3.23}$$

Theorem 3.2. (Stability of the temporal periodic solution) *There exists a constant* $J_S > 0$ *and a small constant* $\varepsilon_2 \in (0,\varepsilon_1)$, *such that for any given* $\varepsilon \in (0,\varepsilon_2)$, *any given* $\Gamma_0(x)$ *satisfying* (3.21), *and any given* $\Gamma_{ip}(t)$, $(i = 1,2,3)$ *satisfying* (3.18) *and* (3.20), *the disturbance systems* (3.11)–(3.17) *admits a unique* C^1 *classical solution* $\Gamma = \Gamma(t,x)$ *in the region* $\{(t,x) \in \mathbb{R}_+ \times [0,L]\}$, *satisfying*

$$\|\Gamma(t,\cdot) - \Gamma^{(T)}(t,\cdot)\|_{C^0} \le J_S \varepsilon \beta^{[t/T_0]}, \quad \forall t \in \mathbb{R}_+, \tag{3.24}$$

with the temporal periodic solution $\Gamma^{(T)}$ *obtained in* Theorem 3.1. *Here,* $\beta \in (0,1)$ *and* $[t/T_0]$ *denotes the largest integer smaller than* t/T_0, *with*

$$T_0 = \max_{i=1,2,3} \sup_{\Gamma} \frac{L}{|\lambda_i(u)|}. \tag{3.25}$$

Take $t \to +\infty$ in (3.24), so we obtain the uniqueness result.

Corollary 3.1. (Uniqueness of the temporal periodic solution) *There exists a small constant* $\varepsilon_3 \in (0,\varepsilon_2)$, *such that for any given* $\varepsilon \in (0,\varepsilon_3)$ *and any given* $\Gamma_{ip}(t)$ $(i = 1,2,3)$ *satisfying* (3.18) *and* (3.20), *the temporal periodic solution* $\Gamma = \Gamma^{(T)}(t,x)$ *given in* Theorem 3.1 *is unique.*

Theorem 3.3. (Regularity of the temporal periodic solution) *Under the same assumptions in* Theorem 3.1, *for all functions* $\Gamma_{ip}(t)$, $(i = 1,2,3)$ *satisfying* (3.18) *and* (3.20) *with further* $W^{2,\infty}$ *regularity as follows*:

$$\max_{i=1,2,3} \|\Gamma''_{ip}(t)\|_{L^\infty} \le H_0 < +\infty, \tag{3.26}$$

there exist a constant $H_R > 0$ *and a small constant* $\varepsilon_4 \in (0,\varepsilon_1)$, *such that for any given* $\varepsilon \in (0,\varepsilon_4)$, *the temporal periodic solution* $\Gamma = \Gamma^{(T)}(t,x)$ *given in* Theorem 3.1 *in the region* $\{(t,x) \in \mathbb{R}_+ \times [0,L]\}$, *also complies with the* $W^{2,\infty}$ *regularity*

$$\max_{i=1,2,3}\{\|\partial_t^2\Gamma^{(T)}\|_{L^\infty}, \|\partial_t\partial_x\Gamma^{(T)}\|_{L^\infty}, \|\partial_x^2\Gamma^{(T)}\|_{L^\infty}\} \le (1+B_0^2)H_R < +\infty, \tag{3.27}$$

where B_0 *is a constant defined in* (4.3).

Theorem 3.4. (Stabilization around the regular temporal periodic solution) *Under the same assumptions in* Theorem 3.3, *there exist a constant* $J_S^* > 0$ *and a small constant* $\varepsilon_5 \in (0, \min\{\varepsilon_2,\varepsilon_4\})$, *such that for any given* $\varepsilon \in (0,\varepsilon_5)$, *it holds the* C^1 *exponential stability as follows*:

$$\max\{\|\partial_t\Gamma(t,\cdot) - \partial_t\Gamma^{(T)}(t,\cdot)\|_{C^0}, \|\partial_x\Gamma(t,\cdot) - \partial_x\Gamma^{(T)}(t,\cdot)\|_{C^0}\} \le (1+B_0)J_S^*\varepsilon\beta^{[t/T_0]}, \quad \forall t \in \mathbb{R}_+. \tag{3.28}$$

4 Existence of the temporal periodic solution

In this section, we will prove the existence of subsonic time-periodic solutions to the non-isentropic compressible Euler equations by constructing iterative formats in details. Now we focus on the time-dependent system (3.11)–(3.17) and consider its periodic solutions driven by boundary conditions.

In this process, we will encounter a difficulty, which raises from the appearance of $\partial_x\Gamma_2$ in (3.11) and (3.13). This term can lead to the derivative loss in C^1 prior estimates. Fortunately, we notice Γ_2 is conserved along the second characteristic curve and overcome this difficulty by transforming $\partial_x\Gamma_2$ into the first and third characteristic directions. That is, by using (3.12), we can transform (3.11) and (3.13) into

$$\begin{aligned}\Gamma_{1t} + \lambda_1(\Gamma + \tilde{\Gamma})\Gamma_{1x} = &-\frac{\gamma-1}{8}a(x)\frac{(\gamma+1)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1+(3-\gamma)\tilde{\Gamma}_3}\Gamma_1\\ &-\frac{1}{4\gamma}(\Gamma_3-\Gamma_1+\tilde{\Gamma}_3-\tilde{\Gamma}_1)(\partial_t\Gamma_2+\lambda_1(\Gamma+\tilde{\Gamma})\partial_x\Gamma_2)\\ &+\frac{\gamma-1}{8}a(x)\frac{(3-\gamma)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1+(3-\gamma)\tilde{\Gamma}_3}\Gamma_3+\frac{\gamma-1}{8}a(x)(\Gamma_3^2-\Gamma_1^2),\end{aligned}\tag{4.1}$$

$$\begin{aligned}\Gamma_{3t} + \lambda_3(\Gamma + \tilde{\Gamma})\Gamma_{3x} = &-\frac{\gamma-1}{8}a(x)\frac{(\gamma+1)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1+(\gamma+1)\tilde{\Gamma}_3}\Gamma_3\\ &+\frac{1}{4\gamma}(\Gamma_3-\Gamma_1+\tilde{\Gamma}_3-\tilde{\Gamma}_1)(\partial_t\Gamma_2+\lambda_3(\Gamma+\tilde{\Gamma})\partial_x\Gamma_2)\\ &+\frac{\gamma-1}{8}a(x)\frac{(3-\gamma)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1+(\gamma+1)\tilde{\Gamma}_3}\Gamma_1-\frac{\gamma-1}{8}a(x)(\Gamma_3^2-\Gamma_1^2).\end{aligned}\tag{4.2}$$

Then, by integrating (4.1) and (4.2) along the first and third characteristic lines, respectively, the difficulty coming from derivative loss can be overcome.

Now, define $\mu_i(\Gamma+\tilde{\Gamma}) = \frac{1}{\lambda_i(\Gamma+\tilde{\Gamma})} (i = 1, 2, 3)$, there exists a positive constant B_0 such that

$$\max_{i=1,2,3}\sup\{|\mu_i(\Gamma+\tilde{\Gamma})|, |\mu_i(\tilde{\Gamma})|\} \le B_0.\tag{4.3}$$

Multiplying both sides of (4.1), (3.12), and (4.2) by $\mu_1(\Gamma+\tilde{\Gamma})$, $\mu_2(\Gamma+\tilde{\Gamma})$ and $\mu_3(\Gamma+\tilde{\Gamma})$, respectively, we obtain

$$\begin{aligned}\Gamma_{1x} + \mu_1(\Gamma + \tilde{\Gamma})\Gamma_{1t} = &-\frac{\gamma-1}{8}a(x)\frac{(\gamma+1)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1+(3-\gamma)\tilde{\Gamma}_3}\mu_1(\Gamma+\tilde{\Gamma})\Gamma_1\\ &-\frac{1}{4\gamma}(\Gamma_3-\Gamma_1+\tilde{\Gamma}_3-\tilde{\Gamma}_1)(\partial_x\Gamma_2+\mu_1(\Gamma+\tilde{\Gamma})\partial_t\Gamma_2)\\ &+\frac{\gamma-1}{8}a(x)\frac{(3-\gamma)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1+(3-\gamma)\tilde{\Gamma}_3}\mu_1(\Gamma+\tilde{\Gamma})\Gamma_3\\ &+\frac{\gamma-1}{8}a(x)\mu_1(\Gamma+\tilde{\Gamma})(\Gamma_3^2-\Gamma_1^2),\end{aligned}\tag{4.4}$$

$$\Gamma_{2x} + \mu_2(\Gamma+\tilde{\Gamma})\Gamma_{2t} = 0,\tag{4.5}$$

$$\begin{aligned}\Gamma_{3x} + \mu_3(\Gamma + \tilde{\Gamma})\Gamma_{3t} = &-\frac{\gamma-1}{8}a(x)\frac{(\gamma+1)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1+(\gamma+1)\tilde{\Gamma}_3}\mu_3(\Gamma+\tilde{\Gamma})\Gamma_3\\ &+\frac{1}{4\gamma}(\Gamma_3-\Gamma_1+\tilde{\Gamma}_3-\tilde{\Gamma}_1)(\partial_x\Gamma_2+\mu_3(\Gamma+\tilde{\Gamma})\partial_t\Gamma_2)\\ &+\frac{\gamma-1}{8}a(x)\frac{(3-\gamma)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2)+2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1+(\gamma+1)\tilde{\Gamma}_3}\mu_3(\Gamma+\tilde{\Gamma})\Gamma_1\\ &-\frac{\gamma-1}{8}a(x)\mu_3(\Gamma+\tilde{\Gamma})(\Gamma_3^2-\Gamma_1^2).\end{aligned}\tag{4.6}$$

For the convenience of handwriting, in the following, we denote

$$\Psi_1(x) = \frac{(\gamma+1)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2) + 2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1 + (3-\gamma)\tilde{\Gamma}_3} = \frac{2\left(\tilde{u}^2 + \frac{2}{\gamma-1}\tilde{c}^2\right)}{\tilde{c}-\tilde{u}},$$

$$\Psi_2(x) = \frac{(3-\gamma)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2) + 2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(\gamma+1)\tilde{\Gamma}_1 + (3-\gamma)\tilde{\Gamma}_3} = \frac{2\left(\frac{2}{\gamma-1}\tilde{c}^2 - \tilde{u}^2\right)}{\tilde{c}-\tilde{u}},$$

$$\Psi_3(x) = \frac{(\gamma+1)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2) + 2(3-\gamma)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1 + (\gamma+1)\tilde{\Gamma}_3} = \frac{2\left(\tilde{u}^2 + \frac{2}{\gamma-1}\tilde{c}^2\right)}{\tilde{c}+\tilde{u}},$$

$$\Psi_4(x) = \frac{(3-\gamma)(\tilde{\Gamma}_1^2+\tilde{\Gamma}_3^2) + 2(\gamma+1)\tilde{\Gamma}_1\tilde{\Gamma}_3}{(3-\gamma)\tilde{\Gamma}_1 + (\gamma+1)\tilde{\Gamma}_3} = \frac{2\left(\frac{2}{\gamma-1}\tilde{c}^2 - \tilde{u}^2\right)}{\tilde{c}+\tilde{u}},$$

$$\Psi_5(x) = \tilde{\Gamma}_3 - \tilde{\Gamma}_1 = \frac{4}{\gamma-1}\tilde{c}.$$

Since the duct length $L < L_*$, from Section 2 and (3.19), there exists positive constant C_0, depending on L, such that

$$\max_{i=1,\dots,5} |\Psi_i(x)| \le C_0\varepsilon_0. \tag{4.7}$$

Noticing the form of $\mu_1(\tilde{\Gamma})$ and $\mu_3(\tilde{\Gamma})$, we recursively construct the iterative format of system (4.4)–(4.6) as follows:

$$\begin{aligned}\Gamma_{1x}^{(k)} + \mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_{1t}^{(k)} = {} & -\frac{\gamma-1}{8}a(x)\Psi_1(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_1^{(k-1)} + \frac{\gamma-1}{8}a(x)\Psi_2(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_3^{(k-1)}\\ & - \frac{1}{4\gamma}(\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x))(\partial_x\Gamma_2^{(k-1)} + \mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\partial_t\Gamma_2^{(k-1)})\\ & + \frac{\gamma-1}{8}a(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2),\end{aligned} \tag{4.8}$$

$$\Gamma_{2x}^{(k)} + \mu_2(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_{2t}^{(k)} = 0, \tag{4.9}$$

$$\begin{aligned}\Gamma_{3x}^{(k)} + \mu_3(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_{3t}^{(k)} = {} & -\frac{\gamma-1}{8}a(x)\Psi_3(x)\mu_3(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_3^{(k-1)} + \frac{\gamma-1}{8}a(x)\Psi_4(x)\mu_3(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_1^{(k-1)}\\ & + \frac{1}{4\gamma}(\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x))(\partial_x\Gamma_2^{(k-1)} + \mu_3(\Gamma^{(k-1)}+\tilde{\Gamma})\partial_t\Gamma_2^{(k-1)})\\ & - \frac{\gamma-1}{8}a(x)\mu_3(\Gamma+\tilde{\Gamma})((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2),\end{aligned} \tag{4.10}$$

with the iterative boundary conditions as

$$x = L : \Gamma_1^{(k)}(t, L) = \Gamma_{1pr}(t) + s_1\Gamma_2^{(k-1)}(t, L) + s_1^*\Gamma_3^{(k-1)}(t, L), \tag{4.11}$$

$$x = 0 : \Gamma_2^{(k)}(t, 0) = \Gamma_{2pl}(t) + s_2\Gamma_1^{(k-1)}(t, 0), \tag{4.12}$$

$$\Gamma_3^{(k)}(t, 0) = \Gamma_{3pl}(t) + s_3\Gamma_1^{(k-1)}(t, 0), \tag{4.13}$$

where $\Gamma_{1pr}(t), \Gamma_{2pl}(t), \Gamma_{3pl}(t)$ are temporal periodic extensions in $t \in \mathbb{R}$ of $\Gamma_{ip}(t)$, $(i = 1, 2, 3)$. For system (4.8)–(4.13), assume that

$$\Gamma^{(0)}(t, x) = (\Gamma_1^{(0)}(x), \Gamma_2^{(0)}(x), \Gamma_3^{(0)}(x))^\top = \mathbf{0}. \tag{4.14}$$

Then, we define

$$\Gamma_i^{(k)}(t, x)(i = 1, 2, 3;\ k \in \mathbb{Z}_+)$$

to be the sequence of C^1 solution of the linearized iteration system (4.8)–(4.13), it holds:

Proposition 4.1. *There exist two constants $J_1, J_2 > 0$ and a small constant $\varepsilon_1 > 0$, such that for any given $\varepsilon \in (0, \varepsilon_1)$, the sequence of C^1 solutions $\Gamma_i^{(k)}(t, x)$ $(i = 1, 2, 3;\ k \in \mathbb{Z}_+)$ starting from* (4.14) *satisfying*

$$\Gamma^{(k)}(t + P, x) = \Gamma^{(k)}(t, x), \quad \forall (t, x) \in \mathbb{R} \times [0, L], k \in \mathbb{Z}_+, \tag{4.15}$$

$$\|\Gamma^{(k)}\|_{C^1} \le (J_1 + J_2)\varepsilon, \quad \forall k \in \mathbb{Z}_+, \tag{4.16}$$

$$\|\Gamma^{(k)} - \Gamma^{(k-1)}\|_{C^0} \le J_1 \theta^k \varepsilon, \quad \forall k \in \mathbb{Z}_+, \tag{4.17}$$

$$\max_{i=1,2,3}\{\varpi(\delta|\partial_t \Gamma_i^{(k)}) + \varpi(\delta|\partial_x \Gamma_i^{(k)})\} \le \left(\frac{1}{3} + \frac{1}{2}B_0\right)\Omega(\delta), \quad \forall k \in \mathbb{Z}_+, \tag{4.18}$$

where $\theta \in (0, 1)$, $\varpi(\delta|f)$ represents modulus of continuity in some senses, which is defined by

$$\varpi(\delta|f) = \sup_{\substack{|t_1 - t_2| \le \delta \\ |x_1 - x_2| \le \delta}} |f(t_1, x_1) - f(t_2, x_2)|, \tag{4.19}$$

and $\Omega(\delta)$, independent of k, is a continuous function of $\delta \in (0, 1)$ with

$$\lim_{\delta \to 0^+} \Omega(\delta) = 0. \tag{4.20}$$

Proof. We will use an inductive method to prove priori estimates (4.15)–(4.18). When $k = 0$, from (4.14), $\Gamma_i^{(0)}$, $(i = 1, 2, 3)$ satisfy (4.15)–(4.16) and (4.18) obviously. Then, for each $k \in \mathbb{Z}_+$, $\forall i = 1, 2, 3$, and $\forall (t, x) \in \mathbb{R} \times [0, L]$, we demonstrate

$$\Gamma_i^{(k)}(t + P, x) = \Gamma_i^{(k)}(t, x), \tag{4.21}$$

$$\|\Gamma_i^{(k)}\|_{C^0} \le J_1 \varepsilon, \quad \|\partial_t \Gamma_i^{(k)}\|_{C^0} \le J_1 \varepsilon, \tag{4.22}$$

$$\|\partial_x \Gamma_i^{(k)}\|_{C^0} \le J_2 \varepsilon, \tag{4.23}$$

$$\|\Gamma_i^{(k)} - \Gamma_i^{(k-1)}\|_{C^0} \le J_1 \varepsilon \theta^k, \tag{4.24}$$

$$\varpi(\delta|\partial_t \Gamma_i^{(k)}(\cdot, x)) \le \frac{1}{8[B_0 + 1]}\Omega(\delta), \tag{4.25}$$

and

$$\max_i\{\varpi(\delta|\partial_t \Gamma_i^{(k)}) + \varpi(\delta|\partial_x \Gamma_i^{(k)})\} \le \left(\frac{1}{3} + \frac{1}{2}B_0\right)\Omega(\delta) \tag{4.26}$$

under the following assumptions

$$\Gamma_i^{(k-1)}(t + P, x) = \Gamma_i^{(k-1)}(t, x), \tag{4.27}$$

$$\|\Gamma_i^{(k-1)}\|_{C^0} \le J_1 \varepsilon, \quad \|\partial_t \Gamma_i^{(k-1)}\|_{C^0} \le J_1 \varepsilon, \tag{4.28}$$

$$\|\partial_x \Gamma_i^{(k-1)}\|_{C^0} \le J_2 \varepsilon, \tag{4.29}$$

$$\|\Gamma_i^{(k-1)} - \Gamma_i^{(k-2)}\|_{C^0} \le J_1 \varepsilon \theta^{k-1}, \quad \forall k \ge 2, \tag{4.30}$$

$$\varpi(\delta|\partial_t \Gamma_i^{(k-1)}(\cdot, x)) \le \frac{1}{8[B_0 + 1]}\Omega(\delta), \tag{4.31}$$

and

$$\max_i\{\varpi(\delta|\partial_t \Gamma_i^{(k-1)}) + \varpi(\delta|\partial_x \Gamma_i^{(k-1)})\} \le \left(\frac{1}{3} + \frac{1}{2}B_0\right)\Omega(\delta). \tag{4.32}$$

Here, $[B_0 + 1]$ is the maximum integral no more than $B_0 + 1$, and

$$\varpi(\delta|f(\cdot,x)) = \sup_{|t_1-t_2|\le\delta} |f(t_1,x) - f(t_2,x)|.$$

First, the time-periodic property (4.21) can be seen directly. By changing the roles of t and x, system (4.8)–(4.13) can be treated as decoupled nonhomogeneous linear transport equations of $\Gamma_i^{(k)}$. From (4.27), we know, once $\Gamma_i^{(k)}(t,x)$, $(i = 1,2,3)$ solves problem (4.8)–(4.13), so does $\Gamma_i^{(k)}(t+P,x)$, $(i = 1,2,3)$. By the uniqueness of the solution, (4.21) is proved. □

Then, we show the C^0 estimates in (4.22). On the boundary $x = L$, from (3.20), (4.11), and (4.28), we can easily obtain

$$\|\Gamma_1^{(k)}(\cdot,L)\|_{C^0} = \|\Gamma_{1pr}(\cdot) + s_1\Gamma_2^{(k-1)}(\cdot,L) + s_1^*\Gamma_3^{(k-1)}(\cdot,L)\|_{C^0} \le \varepsilon + sJ_1\varepsilon = J_1\varepsilon - 99\varepsilon, \tag{4.33}$$

here we have used the fact $s = \max\{|s_1| + |s_1^*|, |s_2|, |s_3|\} < 1$ and take $J_1 = \frac{100}{1-s}$. Similarly, on the boundary $x = 0$, we have

$$\|\Gamma_2^{(k)}(\cdot,0)\|_{C^0} \le J_1\varepsilon - 99\varepsilon, \tag{4.34}$$

$$\|\Gamma_3^{(k)}(\cdot,0)\|_{C^0} \le J_1\varepsilon - 99\varepsilon. \tag{4.35}$$

In the interior of the region, i.e., $(t,x) \in \mathbb{R}\times(0,L)$, we define $t = t_i^{(k)}(x;\, t_0, x_0)$ to be the i-characteristic curve passing through the point (t_0,x_0), namely,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x} t_i^{(k)}(x;\, t_0,x_0) = \mu_i(\Gamma^{(k-1)} + \tilde{\Gamma})(t_i^{(k)}(x;\, t_0,x_0),x), \\ t_i^{(k)}(x_0;\, t_0,x_0) = t_0. \end{cases} \tag{4.36}$$

Integrating (4.8) from L to x along the first characteristic curve $t = t_1^{(k)}(x;\, t_0,x_0)$ to show

$$\begin{aligned} &\Gamma_1^{(k)}(t,x) - \Gamma_1^{(k)}(t_1^{(k)}(L;\, t,x),L) \\ &= \int_x^L \Bigg(-\frac{\gamma-1}{8}a(z)\Psi_1(z)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\Gamma_1^{(k-1)} + \frac{\gamma-1}{8}a(z)\Psi_2(z)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\Gamma_3^{(k-1)} \\ &\quad - \frac{1}{4\gamma}(\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(z))(\partial_x\Gamma_2^{(k-1)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t\Gamma_2^{(k-1)}) \\ &\quad + \frac{\gamma-1}{8}a(z)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2)\Bigg)\mathrm{d}z, \end{aligned}$$

then combining (1.2), (4.3), (4.7), (4.28), and (4.33), it holds

$$\|\Gamma_1^{(k)}(t,x)\|_{C^0} \le J_1\varepsilon - 99\varepsilon + \left(\frac{\gamma-1}{4}D_0C_0B_0L + \frac{1}{2\gamma}C_0L\right)J_1\varepsilon_0\varepsilon + C\varepsilon^2 \le J_1\varepsilon, \tag{4.37}$$

where we have used the fact $\gamma \in (1,3)$ and the smallness of ε_0. Similarly, we obtain

$$\|\Gamma_3^{(k)}(t,x)\|_{C^0} \le J_1\varepsilon. \tag{4.38}$$

For $\Gamma_2^{(k)}(t,x)$, since (4.9), $\Gamma_2^{(k)}(t,x)$ is conserved along the second characteristic curve $t = t_2^{(k)}(x;\, t_0,x_0)$, then with the aid of (4.34), we have

$$\|\Gamma_2^{(k)}(t,x)\|_{C^0} = \|\Gamma_2^{(k)}(\cdot,0)\|_{C^0} \le J_1\varepsilon. \tag{4.39}$$

Next, we will show the proof of C^1 estimates in (4.22) and (4.23). Denote

$$h_i^{(k)} = \partial_t\Gamma_i^{(k)}, \quad g_i^{(k)} = \partial_x\Gamma_i^{(k)}, \quad i = 1,2,3,\ \forall k \in \mathbb{Z}_+.$$

From boundary conditions (4.11)–(4.13), we have

$$x = L : h_1^{(k)}(t, L) = \Gamma'_{1pr}(t) + s_1 h_2^{(k-1)}(t, L) + s_1^* h_3^{(k-1)}(t, L), \tag{4.40}$$

$$x = 0 : h_2^{(k)}(t, 0) = \Gamma'_{2pl}(t) + s_2 h_1^{(k-1)}(t, 0), \tag{4.41}$$

$$h_3^{(k)}(t, 0) = \Gamma'_{3pl}(t) + s_3 h_1^{(k-1)}(t, 0). \tag{4.42}$$

Then by the aids of (3.20) and (4.28), we obtain

$$\begin{aligned} ||h_1^{(k)}(\cdot, L)||_{C^0} &= ||\Gamma'_{1pr}(t) + s_1 h_2^{(k-1)}(\cdot, L) + s_1^* h_3^{(k-1)}(\cdot, L)||_{C^0} \\ &\leq \varepsilon + (|s_1| + |s_1^*|) J_1 \varepsilon \leq J_1 \varepsilon - 99\varepsilon. \end{aligned} \tag{4.43}$$

Similarly calculations yield

$$||h_2^{(k)}(\cdot, 0)||_{C^0} \leq \varepsilon + |s_2| J_1 \varepsilon \leq J_1 \varepsilon - 99\varepsilon, \tag{4.44}$$

$$||h_3^{(k)}(\cdot, 0)||_{C^0} \leq \varepsilon + |s_3| J_1 \varepsilon \leq J_1 \varepsilon - 99\varepsilon. \tag{4.45}$$

According to (4.8)–(4.9), $h_i^{(k)}$, $(i = 1, 2)$ satisfy the following form:

$$\begin{aligned} &\partial_x h_1^{(k)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t h_1^{(k)} \\ &= -\sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} h_j^{(k-1)} h_1^{(k)} - \frac{\gamma - 1}{8} a(x)\Psi_1(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\cdot h_1^{(k-1)} \\ &\quad - \frac{\gamma - 1}{8} a(x)\Psi_1(x)\cdot \sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} h_j^{(k-1)} \Gamma_1^{(k-1)} \\ &\quad - \frac{1}{4\gamma}(h_3^{(k-1)} - h_1^{(k-1)})(\partial_x \Gamma_2^{(k-1)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t \Gamma_2^{(k-1)}) \\ &\quad - \frac{1}{4\gamma}(\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x))\cdot \sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} h_j^{(k-1)} h_2^{(k-1)} \\ &\quad - \frac{1}{4\gamma}(\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x))(\partial_x h_2^{(k-1)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t h_2^{(k-1)}) \\ &\quad + \frac{\gamma - 1}{8} a(x)\Psi_2(x)\sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} h_j^{(k-1)} \Gamma_3^{(k-1)} \\ &\quad + \frac{\gamma - 1}{8} a(x)\Psi_2(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma}) h_3^{(k-1)} \\ &\quad + \frac{\gamma - 1}{8} a(x)\sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} h_j^{(k-1)}((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2) \\ &\quad + \frac{\gamma - 1}{4} a(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})(\Gamma_3^{(k-1)} h_3^{(k-1)} - \Gamma_1^{(k-1)} h_1^{(k-1)}), \end{aligned} \tag{4.46}$$

$$\partial_x h_2^{(k)} + \mu_2(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t h_2^{(k)} = -\sum_{j=1}^{3} \frac{\partial \mu_2(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} h_j^{(k-1)} h_2^{(k)}. \tag{4.47}$$

It can be seen that the first item on the right-hand side of (4.46) is relevant to $h_1^{(k)}$, the estimate of which has not be gained. To overcome this, we introduce a method of *twice integral estimation*. First, we need a rough estimate of $h_1^{(k)}$.

Multiply both sides of (4.46) by $\mathrm{sgn}(h_1^{(k)})$ simultaneously to obtain

$$
\begin{aligned}
&\partial_x|h_1^{(k)}| + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t|h_1^{(k)}| \\
&= -\sum_{j=1}^{3}\frac{\partial\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)}|h_1^{(k)}| - \frac{\gamma-1}{8}\mathrm{sgn}(h_1^{(k)})a(x)\Psi_1(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\cdot h_1^{(k-1)} \\
&\quad - \frac{\gamma-1}{8}\mathrm{sgn}(h_1^{(k)})a(x)\Psi_1(x)\cdot\sum_{j=1}^{3}\frac{\partial\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)}\Gamma_1^{(k-1)} \\
&\quad - \frac{1}{4\gamma}\mathrm{sgn}(h_1^{(k)})(h_3^{(k-1)} - h_1^{(k-1)})(\partial_x\Gamma_2^{(k-1)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t\Gamma_2^{(k-1)}) \\
&\quad - \frac{1}{4\gamma}\mathrm{sgn}(h_1^{(k)})(\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x))\cdot\sum_{j=1}^{3}\frac{\partial\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)}h_2^{(k-1)} \\
&\quad - \frac{1}{4\gamma}\mathrm{sgn}(h_1^{(k)})(\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x))(\partial_x h_2^{(k-1)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t h_2^{(k-1)}) \\
&\quad + \frac{\gamma-1}{8}\mathrm{sgn}(h_1^{(k)})a(x)\Psi_2(x)\sum_{j=1}^{3}\frac{\partial\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)}\Gamma_3^{(k-1)} \\
&\quad + \frac{\gamma-1}{8}\mathrm{sgn}(h_1^{(k)})a(x)\Psi_2(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})h_3^{(k-1)} \\
&\quad + \frac{\gamma-1}{8}\mathrm{sgn}(h_1^{(k)})a(x)\sum_{j=1}^{3}\frac{\partial\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)}((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2) \\
&\quad + \frac{\gamma-1}{4}\mathrm{sgn}(h_1^{(k)})a(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})(\Gamma_3^{(k-1)}h_3^{(k-1)} - \Gamma_1^{(k-1)}h_1^{(k-1)}).
\end{aligned}
\tag{4.48}
$$

By (4.3) and (4.28), there exists a constant $K_0 > 0$, such that

$$-\sum_{j=1}^{3}\frac{\partial\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)} > -K_0\varepsilon.$$

Thus, integrating (4.48) from L to x along the first characteristic curve $t = t_1^{(k)}(x;\ t_0, x_0)$, and using (1.2), (4.3), (4.7), (4.28), (4.43), and the Gronwall's inequality, it follows:

$$
\begin{aligned}
|h_1^{(k)}(t,x)| &\le \left(J_1\varepsilon - 99\varepsilon + \left(\frac{\gamma-1}{4}B_0D_0 + \frac{1}{2\gamma}\right)C_0LJ_1\varepsilon_0\varepsilon + C\varepsilon^2\right)e^{-K_0\varepsilon L} \\
&\le J_1\varepsilon e^{-K_0\varepsilon L} \le C\varepsilon.
\end{aligned}
\tag{4.49}
$$

Integrating (4.46) from L to x along the second characteristic curve $t = t_1^{(k)}(x;\ t_0, x_0)$, using (1.2), (4.3), (4.7), (4.28), (4.43), the smallness of ε_0, and the rough estimate (4.49), we have

$$\|h_1^{(k)}(t,x)\|_{C^0} \le J_1\varepsilon - 99\varepsilon + \left(\frac{\gamma-1}{4}B_0D_0 + \frac{1}{2\gamma}\right)C_0LJ_1\varepsilon_0\varepsilon + C\varepsilon^2 \le \hbar_1J_1\varepsilon \tag{4.50}$$

with a constant $\hbar_1 \in (0,1)$. Repeating similar procedures on (4.10), one obtains

$$\|h_3^{(k)}(t,x)\|_{C^0} \le J_1\varepsilon. \tag{4.51}$$

For $h_2^{(k)}$, we integrate (4.47) along the second characteristic curve $t = t_2^{(k)}(x;\ t_0, x_0)$ from 0 to x and obtain

$$h_2^{(k)}(t,x) = h_2^{(k)}(t_2^{(k)}(0;\ t_0, x_0), 0) - \int_0^x\sum_{j=1}^{3}\frac{\partial\mu_2(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)}h_2^{(k)}\mathrm{d}s.$$

By (4.44) and the Gronwall's inequality, one obtains

$$\|h_2^{(k)}(t,x)\|_{C^0} \le \|h_2^{(k)}(\cdot, 0)\|_{C^0}e^{-C\varepsilon L} \le (\varepsilon + |s_2|J_1\varepsilon)e^{-C\varepsilon L} \le J_1\varepsilon. \tag{4.52}$$

For the spatial derivatives, it follows from (4.8) that

$$\begin{aligned}
g_1^{(k)} = &-\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_{1t}^{(k)} - \frac{\gamma-1}{8}a(x)\Psi_1(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_1^{(k-1)}\\
&+\frac{\gamma-1}{8}a(x)\Psi_2(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\Gamma_3^{(k-1)}\\
&-\frac{1}{4\gamma}(\Gamma_3^{(k-1)}-\Gamma_1^{(k-1)}+\Psi_5(x))(\partial_x\Gamma_2^{(k-1)}+\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\partial_t\Gamma_2^{(k-1)})\\
&+\frac{\gamma-1}{8}a(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})((\Gamma_3^{(k-1)})^2-(\Gamma_1^{(k-1)})^2).
\end{aligned}$$

Thus, together with (1.2), (4.3), (4.7), (4.22), (4.50), and taking J_2 large enough, one obtains

$$\|g_1^{(k)}\|_{C^0} \le B_0J_1\varepsilon + \left(\frac{\gamma-1}{4}D_0B_0C_0\varepsilon_0LJ_1 + \frac{1}{2\gamma}C_0\varepsilon_0B_0J_1\right)\varepsilon + C\varepsilon^2 \le J_2\varepsilon. \tag{4.53}$$

Then from (4.9) and (4.10), by similar arguments, we have

$$\|g_2^{(k)}\|_{C^0} \le B_0J_1\varepsilon \le J_2\varepsilon, \tag{4.54}$$

$$\|g_3^{(k)}\|_{C^0} \le J_2\varepsilon. \tag{4.55}$$

Hence, we finish the proof of (4.22) and (4.23).

Now we begin to show the proof of the C^0 Cauchy sequence property, i.e., (4.24). First, for $k=1$, from the initial iteration (4.14) and the C^0 estimates (4.22), (4.24) holds naturally. Then for $k\ge 2$, on the boundaries, since (4.11)–(4.13), we have

$$x=L:\Gamma_1^{(k)}(t,L)-\Gamma_1^{(k-1)}(t,L) = s_1(\Gamma_2^{(k-1)}(t,L)-\Gamma_2^{(k-2)}(t,L)) + s_1^*(\Gamma_3^{(k-1)}(t,L)-\Gamma_3^{(k-2)}(t,L)), \tag{4.56}$$

$$x=0:\Gamma_2^{(k)}(t,0)-\Gamma_2^{(k-1)}(t,0) = s_2(\Gamma_1^{(k-1)}(t,0)-\Gamma_1^{(k-2)}(t,0)), \tag{4.57}$$

$$\Gamma_3^{(k)}(t,0)-\Gamma_3^{(k-1)}(t,0) = s_3(\Gamma_1^{(k-1)}(t,0)-\Gamma_1^{(k-2)}(t,0)). \tag{4.58}$$

Thus,

$$\|\Gamma_1^{(k)}(t,L)-\Gamma_1^{(k-1)}(t,L)\|_{C^0} \le (|s_1|+|s_1^*|)J_1\theta^{k-1}\varepsilon \le sJ_1\theta^{k-1}\varepsilon, \tag{4.59}$$

$$\|\Gamma_2^{(k)}(t,0)-\Gamma_2^{(k-1)}(t,0)\|_{C^0} \le sJ_1\theta^{k-1}\varepsilon, \tag{4.60}$$

$$\|\Gamma_3^{(k)}(t,0)-\Gamma_3^{(k-1)}(t,0)\|_{C^0} \le sJ_1\theta^{k-1}\varepsilon. \tag{4.61}$$

Within the region $\{(t,x)\in\mathbb{R}\times(0,L)\}$, from (4.8) and (4.9), we have

$$\begin{aligned}
&\partial_x(\Gamma_1^{(k)}-\Gamma_1^{(k-1)}) + \mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})\partial_t(\Gamma_1^{(k)}-\Gamma_1^{(k-1)})\\
&\quad= -(\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})-\mu_1(\Gamma^{(k-2)}+\tilde{\Gamma}))h_1^{(k-1)} - \frac{\gamma-1}{8}a(x)\Psi_1(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})(\Gamma_1^{(k-1)}-\Gamma_1^{(k-2)})\\
&\qquad-\frac{\gamma-1}{8}a(x)\Psi_1(x)(\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})-\mu_1(\Gamma^{(k-2)}+\tilde{\Gamma}))\Gamma_1^{(k-2)}\\
&\qquad-\frac{1}{4\gamma}(\Gamma_3^{(k-1)}-\Gamma_1^{(k-1)}+\Psi_5(x))(g_2^{(k-1)}-g_2^{(k-2)}+\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})(h_2^{(k-1)}-h_2^{(k-2)}))\\
&\qquad-\frac{1}{4\gamma}(\Gamma_3^{(k-1)}-\Gamma_1^{(k-1)}+\Psi_5(x))(\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})-\mu_1(\Gamma^{(k-2)}+\tilde{\Gamma}))\cdot h_2^{(k-2)}\\
&\qquad-\frac{1}{4\gamma}(\Gamma_3^{(k-1)}-\Gamma_1^{(k-1)}-\Gamma_3^{(k-2)}+\Gamma_1^{(k-2)})(g_2^{(k-2)}+\mu_1(\Gamma^{(k-2)}+\tilde{\Gamma})h_2^{(k-2)})\\
&\qquad+\frac{\gamma-1}{8}a(x)\Psi_2(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})(\Gamma_3^{(k-1)}-\Gamma_3^{(k-2)})\\
&\qquad+\frac{\gamma-1}{8}a(x)\Psi_2(x)(\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})-\mu_1(\Gamma^{(k-2)}+\tilde{\Gamma}))\Gamma_3^{(k-2)}\\
&\qquad+\frac{\gamma-1}{8}a(x)\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})((\Gamma_3^{(k-1)})^2-(\Gamma_1^{(k-1)})^2-(\Gamma_3^{(k-2)})^2+(\Gamma_1^{(k-2)})^2)\\
&\qquad+\frac{\gamma-1}{8}a(x)(\mu_1(\Gamma^{(k-1)}+\tilde{\Gamma})-\mu_1(\Gamma^{(k-2)}+\tilde{\Gamma}))((\Gamma_3^{(k-2)})^2-(\Gamma_1^{(k-2)})^2),
\end{aligned} \tag{4.62}$$

and

$$\partial_x(\Gamma_2^{(k)} - \Gamma_2^{(k-1)}) + \mu_2(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t(\Gamma_2^{(k)} - \Gamma_2^{(k-1)}) = -(\mu_2(\Gamma^{(k-1)} + \tilde{\Gamma}) - \mu_2(\Gamma^{(k-2)} + \tilde{\Gamma}))h_2^{(k-1)}. \tag{4.63}$$

Integrating (4.62) along the first characteristic curve $t = t_1^{(k)}(x; t_0, x_0)$ from L to x, and combining (1.2), (4.3), (4.7), (4.30), and (4.59), we obtain

$$\|\Gamma_1^{(k)}(t, x) - \Gamma_1^{(k-1)}(t, x)\|_{C^0} \leq sJ_1\theta^{k-1}\varepsilon + C(\varepsilon + \varepsilon_0)J_1\theta^{k-1}\varepsilon \leq J_1\theta^k\varepsilon, \tag{4.64}$$

where we take the constant $\theta \in (s, 1)$.

Then, integrating (4.63) from 0 to x along the second characteristic curve $t = t_2^{(k)}(x; t_0, x_0)$, we have

$$\|\Gamma_2^{(k)}(t, x) - \Gamma_2^{(k-1)}(t, x)\|_{C^0} \leq sJ_1\theta^{k-1}\varepsilon + C\varepsilon J_1\theta^{k-1}\varepsilon \leq J_1\theta^k\varepsilon. \tag{4.65}$$

Similarly to (4.64), one obtains

$$\|\Gamma_3^{(k)}(t, x) - \Gamma_3^{(k-1)}(t, x)\|_{C^0} \leq J_1\theta^k\varepsilon. \tag{4.66}$$

Thus, the C^0 Cauchy sequence property (4.24) is completed.

Now we give the proof of the modulus of continuity estimates for $\Gamma_i^{(k)}$ on the temporal directions, i.e., (4.25), which is significant to prove (4.26). On the boundaries, for any fixed $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| \leq \delta$, from (4.40)–(4.42), we obtain

$$\begin{aligned} x = L : &|h_1^{(k)}(t_1, L) - h_1^{(k)}(t_2, L)| \\ &\leq |\Gamma'_{1pr}(t_1) - \Gamma'_{1pr}(t_2)| + |s_1||h_2^{(k-1)}(t_1, L) - h_2^{(k-1)}(t_2, L)| \\ &+ |s_1^*||h_3^{(k-1)}(t_1, L) - h_3^{(k-1)}(t_2, L)|, \end{aligned} \tag{4.67}$$

$$\begin{aligned} x = 0 : &|h_2^{(k)}(t_1, 0) - h_1^{(k)}(t_2, 0)| \\ &\leq |\Gamma'_{2pl}(t_1) - \Gamma'_{2pl}(t_2)| + |s_2||h_1^{(k-1)}(t_1, 0) - h_1^{(k-1)}(t_2, 0)|, \end{aligned} \tag{4.68}$$

$$\begin{aligned} &|h_3^{(k)}(t_1, 0) - h_3^{(k)}(t_1, 0)| \\ &\leq |\Gamma'_{3pl}(t_1) - \Gamma'_{3pl}(t_2)| + |s_3||h_1^{(k-1)}(t_1, 0) - h_1^{(k-1)}(t_2, 0)|. \end{aligned} \tag{4.69}$$

Then we have

$$\varpi(\delta|h_1^{(k)}(\cdot, L)) \leq \varpi(\delta|\Gamma'_{1pr}) + |s_1|\varpi(\delta|h_2^{(k-1)}(\cdot, L)) + |s_1^*|\varpi(\delta|h_3^{(k-1)}(\cdot, L))$$

and

$$\begin{aligned} \varpi(\delta|h_2^{(k)}(\cdot, 0)) &\leq \varpi(\delta|\Gamma'_{2pl}) + |s_2|\varpi(\delta|h_1^{(k-1)}(\cdot, 0)), \\ \varpi(\delta|h_3^{(k)}(\cdot, 0)) &\leq \varpi(\delta|\Gamma'_{3pl}) + |s_3|\varpi(\delta|h_1^{(k-1)}(\cdot, 0)). \end{aligned}$$

Taking

$$\Omega(\delta) = \frac{24}{1 - s}[B_0 + 1](\varpi(\delta|\Gamma'_{1pr}) + \varpi(\delta|\Gamma'_{2pl}) + \varpi(\delta|\Gamma'_{3pl}) + \varepsilon\delta), \tag{4.70}$$

we obtain

$$\varpi(\delta|h_1^{(k)}(\cdot, L)) \leqslant \frac{1 - s}{24[B_0 + 1]}\Omega(\delta) + \frac{s}{8[B_0 + 1]}\Omega(\delta) = \frac{1 + 2s}{24[B_0 + 1]}\Omega(\delta) < \frac{1}{8[B_0 + 1]}\Omega(\delta), \tag{4.71}$$

and similarly to obtain

$$\varpi(\delta|h_2^{(k)}(\cdot, 0)) < \frac{1}{8[B_0 + 1]}\Omega(\delta), \tag{4.72}$$

$$\varpi(\delta|h_3^{(k)}(\cdot, 0)) < \frac{1}{8[B_0 + 1]}\Omega(\delta). \tag{4.73}$$

To obtain the modulus of continuity for $\Gamma_i^{(k)}$ on the temporal directions within the region $\{(t,x)\in\mathbb{R}\times(0,L)\}$, we will make some preparations at first. Since (4.36), we have

$$\begin{aligned}
&|t_i^{(k)}(y;\, t_2, x) - t_i^{(k)}(y;\, t_1, x)|\\
&\le |t_2 - t_1| + \int_x^y |\mu_i(\Gamma^{(k-1)}(t_i^{(k)}(s;\, t_2, x), y) + \tilde{\Gamma}) - \mu_i(\Gamma^{(k-1)}(t_i^{(k)}(s;\, t_1, x), y) + \tilde{\Gamma})|\mathrm{d}s\\
&\le |t_2 - t_1| + \int_x^y \sum_{j=1}^{3}\left|\frac{\partial\mu_i}{\partial\Gamma_j}\right| \|h_j^{(k-1)}\|_{C^0}|t_i^{(k)}(s;\, t_2, x) - t_i^{(k)}(s;\, t_1, x)|\mathrm{d}s, \quad i = 1,2,3,
\end{aligned}$$

then by the Gronwall's inequality, it follows:

$$|t_i^{(k)}(y;\, t_2, x) - t_i^{(k)}(y;\, t_1, x)| \le (1 + C\varepsilon)|t_2 - t_1| \le (1 + C\varepsilon)\delta. \tag{4.74}$$

According to (4.70), $\Omega(\delta)$ is a continuous, bounded and concave function of δ, with

$$\lim_{\delta\to 0^+}\Omega(\delta) = 0.$$

Due to the concavity, it follows:

$$\frac{1}{1 + C\varepsilon}\Omega((1 + C\varepsilon)\delta) + \frac{C\varepsilon}{1 + C\varepsilon}\Omega(0) \le \Omega(\delta),$$

i.e.,

$$\Omega((1 + C\varepsilon)\delta) \le (1 + C\varepsilon)\Omega(\delta). \tag{4.75}$$

Then, from (4.31), we have

$$\begin{aligned}
|h_i^{(k-1)}(t_j^{(k)}(x;\, t_2, x_0)) - h_i^{(k-1)}(t_j^{(k)}(x;\, t_1, x_0))| &\le \frac{1}{8[B_0 + 1]}\Omega((1 + C\varepsilon)\delta)\\
&\le \frac{1}{8[B_0 + 1]}(1 + C\varepsilon)\Omega(\delta), \quad i, j = 1,2,3.
\end{aligned} \tag{4.76}$$

Integrating (4.46) along the first characteristic curve $t = t_1^{(k)}(y;\, t_2, x)$ and $t = t_1^{(k)}(y;\, t_1, x)$, respectively, and using (1.2), (4.3), (4.7), (4.28)–(4.29), (4.71), (4.74)–(4.76), and the Gronwall's inequality to obtain

$$|h_i^{(k)}(t_1, x) - h_i^{(k)}(t_2, x)| \le \frac{C}{8[B_0 + 1]}\Omega(\delta). \tag{4.77}$$

Then reusing the integral expression of (4.46) and with the aids of (1.2), (4.3), (4.7), (4.28)–(4.29), (4.71), (4.74)–(4.76), we obtain

$$\varpi(\delta|h_1^{(k)}(\cdot, x)) \le \frac{1 + 2s}{24[B_0 + 1]}(1 + C\varepsilon)\Omega(\delta) + C(\varepsilon + \varepsilon_0)\Omega(\delta) + C\varepsilon^2\Omega(\delta) \le \frac{1}{8[B_0 + 1]}\Omega(\delta). \tag{4.78}$$

Similarly, it holds

$$\varpi(\delta|h_3^{(k)}(\cdot, x)) \le \frac{1}{8[B_0 + 1]}\Omega(\delta). \tag{4.79}$$

Now we integrate (4.47) along the second characteristic curve $t = t_2^{(k)}(y;\, t_2, x)$ and $t = t_2^{(k)}(y;\, t_1, x)$, respectively, and have

$$\begin{aligned}
h_2^{(k)}(t_2, x) - h_2^{(k)}(t_1, x) = {}& h_2^{(k)}(t_2^{(k)}(0;\, t_2, x), 0) - h_2^{(k)}(t_1^{(k)}(0;\, t_1, x), 0)\\
&- \int_0^x \left(\sum_{j=1}^{3}\frac{\partial\mu_2(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial\Gamma_j}h_j^{(k-1)}h_2^{(k)}\right)\Bigg|_{(t_1^{(k)}(y;\, t_1,x),y)}^{(t_1^{(k)}(y;\, t_2,x),y)}\mathrm{d}y,
\end{aligned}$$

thus using the Gronwall's inequality and combining (4.3), (4.22), (4.72), it shows

$$\varpi(\delta|h_2^{(k)}(\cdot, x)) \le \left(\frac{1+2s}{24[B_0+1]}(1+C\varepsilon)\Omega(\delta) + C\varepsilon\Omega(\delta)\right)e^{C\varepsilon L} \le \frac{1}{8[B_0+1]}\Omega(\delta). \tag{4.80}$$

Based on (4.25), we could prove (4.26) finally. First, we look to a simple case for two given points (t_1, x_1) and (t_2, x_2) lie on one characteristic curve, i.e., $t_2 = t_1^{(k)}(x_2; t_1, x_1)$ with

$$|t_1 - t_2| < \delta \quad \text{and} \quad |x_1 - x_2| < \delta.$$

By the same method of (4.50), we can obtain the modulus of continuity estimation

$$|h_1^{(k)}(t_1, x_1) - h_1^{(k)}(t_2, x_2)| \le C\varepsilon^2\delta + C\varepsilon\Omega(\delta) \le \frac{1}{12}\Omega(\delta). \tag{4.81}$$

Then, for general case, i.e., the two given points (t_1, x_1) and (t_2, x_2) satisfying $|t_1 - t_2| < \delta$ and $|x_1 - x_2| < \delta$, we can take another point (t_*, x_1) lying on the first characteristic curve passing through (t_1, x_2), i.e.,

$$t_* = t_1^{(k)}(x_2; t_1, x_1).$$

According to (4.3) and (4.36),

$$|t_* - t_2| \le |\mu_1||x_1 - x_2| \le B_0\delta,$$

thus we have

$$|t_* - t_1| \le |t_* - t_2| + |t_2 - t_1| \le (B_0 + 1)\delta.$$

Reusing formulas (4.78) and (4.81), we obtain

$$\begin{aligned}
&|h_1^{(k)}(t_1, x_1) - h_1^{(k)}(t_2, x_2)| \\
&\le \left|h_1^{(k)}(t_1, x_1) - h_1^{(k)}\left(\frac{[B_0+1]t_1 + t_*}{B_0+1}, x_1\right)\right| + \left|h_1^{(k)}\left(\frac{[B_0+1]t_1 + t_*}{B_0+1}, x_1\right)\right. \\
&\quad \left. - h_1^{(k)}\left(\frac{([B_0+1]-1)t_1 + 2t_*}{B_0+1}, x_1\right)\right| + \cdots + \left|h_1^{(k)}\left(\frac{t_1 + [B_0+1]t_*}{B_0+1}, x_1\right) - h_1^{(k)}(t_*, x_1)\right| \\
&\quad + |h_1^{(k)}(t_*, x_1) - h_1^{(k)}(t_2, x_2)| \\
&\le \frac{[B_0+1]+1}{8[B_0+1]}\Omega(\delta) + \frac{1}{12}\Omega(\delta) \\
&\le \frac{1}{3}\Omega(\delta).
\end{aligned} \tag{4.82}$$

Similarly argument to have

$$|h_2^{(k)}(t_1, x_1) - h_2^{(k)}(t_2, x_2)| \le \frac{1}{3}\Omega(\delta), \tag{4.83}$$

$$|h_3^{(k)}(t_1, x_1) - h_3^{(k)}(t_2, x_2)| \le \frac{1}{3}\Omega(\delta). \tag{4.84}$$

Combining (1.2), (4.3), (4.7)–(4.10), (4.16), and (4.82)–(4.84), we finally obtain

$$\varpi(\delta|g_i^{(k)}) \le \frac{1}{3}B_0\Omega(\delta) + C\varepsilon^2\delta + C\varepsilon\delta \le \frac{1}{2}B_0\Omega(\delta), \quad i = 1, 2, 3. \tag{4.85}$$

With the help of (4.82)–(4.85), (4.26) has been proved. Therefore, we complete the proof of Proposition 4.1.

Resemble [25], once we finish the proof of Proposition 4.1,Theorem 3.1 can be derived directly. Utilizing the Arzelà-Ascoli theorem and Cauchy convergence criterion, $\{\Gamma_i^{(k)}\}$, $(i = 1, 2, 3)$ converges to $\Gamma^{(T)}$ in C^1 norm. Taking $\Gamma_0^{(T)}(x) = \Gamma^{(T)}(0, x)$, then from (4.16), we can obtain (3.21).

5 Stability of the temporal periodic solution

In this section, we consider the stability of the temporal periodic solution captured in Theorem 3.1. To begin with, we need to give the following Lemma 5.1, which presents the global existence and uniqueness of C^1 solution for the initial-boundary value problem (4.1)–(3.17). By using the same way in [13], we can complete a proof of the following Lemma 5.1. Here, we omit the details.

Lemma 5.1. *There exists a small constant* $\varepsilon_6 > 0$, *such that for any* $\varepsilon \in (0, \varepsilon_6 > 0)$, *there exists* $\sigma > 0$, *if*

$$||\Gamma_{i0}||_{C^1[0,L]} + ||\Gamma_{ip}(t)||_{C^1(0,+\infty)} \le \sigma, \quad i = 1, 2, 3, \tag{5.1}$$

then the initial-boundary problems (3.11)–(3.17) *admits a unique* C^1 *solution satisfying*

$$||\Gamma_i(t, x)||_{C^1([0,L]\times(0,+\infty))} \le J_E \varepsilon. \tag{5.2}$$

Now we will prove Theorem 3.2 by an induction. Assume that for certain $t_0 \ge 0$ and $N \in \mathbb{N}$, it holds

$$\max_{i=1,2,3} ||\Gamma_i(t, \cdot) - \Gamma_i^{(T)}(t, \cdot)||_{C^0} \le J_S \varepsilon \beta^N, \quad \forall t \in [t_0, t_0 + T_0], \tag{5.3}$$

where $N = [t/T_0]$, $J_S = 2J_E$, $\beta \in (0, 1)$ is a constant to be determined later. Then we will show that

$$\max_{i=1,2,3} ||\Gamma_i(t, \cdot) - \Gamma_i^{(T)}(t, \cdot)||_{C^0} \le J_S \varepsilon \beta^{N+1}, \quad \forall t \in [t_0 + T_0, t_0 + 2T_0]. \tag{5.4}$$

For simplicity, we denote the continuous function

$$\Lambda(t) = \max_{i=1,2,3} \sup_{x\in[0,L]} |\Gamma_i(t, x) - \Gamma_i^{(T)}(t, x)|.$$

Obviously,

$$\Lambda(t_0 + T_0) \le J_S \varepsilon \beta^N. \tag{5.5}$$

To show (3.24), we just need to prove

$$\Lambda(t) \le J_S \varepsilon \beta^{N+1}, \quad \forall t \in [t_0 + T_0, \tau] \tag{5.6}$$

under the hypothesis

$$\Lambda(t) \le J_S \varepsilon \beta^N, \quad \forall t \in [t_0, \tau], \tau \in [t_0 + T_0, t_0 + 2T_0]. \tag{5.7}$$

For this purpose, we need to specify the formula of $\Gamma^{(T)}(t, x)$. On the boundaries, similarly to (3.15)–(3.17), we have

$$x = L : \Gamma_1^{(T)}(t, L) = \Gamma_{1p}(t) + s_1\Gamma_2^{(T)}(t, L) + s_1^*\Gamma_3^{(T)}(t, L), \tag{5.8}$$

$$x = 0 : \Gamma_2^{(T)}(t, 0) = \Gamma_{2p}(t) + s_2\Gamma_1^{(T)}(t, 0), \tag{5.9}$$

$$\Gamma_3^{(T)}(t, 0) = \Gamma_{3p}(t) + s_3\Gamma_1^{(T)}(t, 0). \tag{5.10}$$

Therefore, on $x = L$, it is easy to see

$$||\Gamma_1(t, L) - \Gamma_1^{(T)}(t, L)||_{C^0} \le |s_1|||\Gamma_2(t, L) - \Gamma_2^{(T)}(t, L)||_{C^0} + |s_1^*|||\Gamma_3(t, L) - \Gamma_3^{(T)}(t, L)||_{C^0} \le sJ_S \varepsilon \beta^N. \tag{5.11}$$

Similarly, on $x = 0$, we have

$$||\Gamma_2(t, 0) - \Gamma_2^{(T)}(t, 0)||_{C^0} \le sJ_S \varepsilon \beta^N, \tag{5.12}$$

$$||\Gamma_3(t, 0) - \Gamma_3^{(T)}(t, 0)||_{C^0} \le sJ_S \varepsilon \beta^N. \tag{5.13}$$

Within the region $\{(t, x) \in \mathbb{R}_+ \times (0, L)\}$, similarly to (4.4), we have

$$
\begin{aligned}
&\Gamma_{1x}^{(T)} + \mu_1(\Gamma^{(T)} + \tilde{\Gamma})\Gamma_{1t}^{(T)} \\
&\quad = -\frac{\gamma - 1}{8} a(x)\Psi_1(x)\mu_1(\Gamma^{(T)} + \tilde{\Gamma})\Gamma_1^{(T)} + \frac{\gamma - 1}{8} a(x)\Psi_2(x)\mu_1(\Gamma^{(T)} + \tilde{\Gamma})\Gamma_3^{(T)} \\
&\qquad - \frac{1}{4\gamma}(\Gamma_3^{(T)} - \Gamma_1^{(T)} + \Psi_5(x))(\partial_x \Gamma_2^{(T)} + \mu_1(\Gamma^{(T)} + \tilde{\Gamma})\partial_t \Gamma_2^{(T)}) \\
&\qquad + \frac{\gamma - 1}{8} a(x)\mu_1(\Gamma^{(T)} + \tilde{\Gamma})((\Gamma_3^{(T)})^2 - (\Gamma_1^{(T)})^2).
\end{aligned}
\tag{5.14}
$$

Combine (4.4) and (5.14) to obtain

$$
\begin{aligned}
&\partial_x(\Gamma_1 - \Gamma_1^{(T)}) + \mu_1(\Gamma + \tilde{\Gamma})\partial_t(\Gamma_1 - \Gamma_1^{(T)}) \\
&\quad = -(\mu_1(\Gamma + \tilde{\Gamma}) - \mu_1(\Gamma^{(T)} + \tilde{\Gamma}))\partial_t \Gamma_1^{(T)} - \frac{\gamma - 1}{8} a(x)\Psi_1(x)\mu_1(\Gamma + \tilde{\Gamma})(\Gamma_1 - \Gamma_1^{(T)}) \\
&\qquad - \frac{\gamma - 1}{8} a(x)\Psi_1(x)(\mu_1(\Gamma + \tilde{\Gamma}) - \mu_1(\Gamma^{(T)} + \tilde{\Gamma}))\Gamma_1^{(T)} \\
&\qquad - \frac{1}{4\gamma}(\Gamma_3 - \Gamma_1 + \Psi_5(x))((g_2 - g_2^{(T)}) + \mu_1(\Gamma + \tilde{\Gamma})(h_2 - h_2^{(T)})) \\
&\qquad - \frac{1}{4\gamma}(\Gamma_3 - \Gamma_1 + \Psi_5(x))(\mu_1(\Gamma + \tilde{\Gamma}) - \mu_1(\Gamma^{(T)} + \tilde{\Gamma}))\cdot h_2^{(T)} \\
&\qquad - \frac{1}{4\gamma}(\Gamma_3 - \Gamma_1 - \Gamma_3^{(T)} + \Gamma_1^{(T)})(g_2^{(T)} + \mu_1(\Gamma^{(T)} + \tilde{\Gamma})h_2^{(T)}) \\
&\qquad + \frac{\gamma - 1}{8} a(x)\Psi_2(x)\mu_1(\Gamma + \tilde{\Gamma})(\Gamma_3 - \Gamma_3^{(T)}) \\
&\qquad + \frac{\gamma - 1}{8} a(x)\Psi_2(x)(\mu_1(\Gamma + \tilde{\Gamma}) - \mu_1(\Gamma^{(T)} + \tilde{\Gamma}))\Gamma_3^{(T)} \\
&\qquad + \frac{\gamma - 1}{8} a(x)\mu_1(\Gamma + \tilde{\Gamma})((\Gamma_3)^2 - (\Gamma_1)^2 - (\Gamma_3^{(T)})^2 + (\Gamma_1^{(T)})^2) \\
&\qquad + \frac{\gamma - 1}{8} a(x)(\mu_1(\Gamma + \tilde{\Gamma}) - \mu_1(\Gamma^{(T)} + \tilde{\Gamma}))((\Gamma_3^{(T)})^2 - (\Gamma_1^{(T)})^2).
\end{aligned}
\tag{5.15}
$$

Now we denote the ith, $(i = 1, 2, 3)$ characteristic curve $t = t_i(x;\, t, x)$ as follows:

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}x} t_i(x;\, \hat{t}, \hat{x}) = \mu_i(\Gamma + \tilde{\Gamma})(t_i(x;\, \hat{t}, \hat{x}), x), \\
t_i(\hat{x};\, \hat{t}, \hat{x}) = \hat{t},
\end{cases}
\tag{5.16}
$$

and integrate (5.15) along the first characteristic curve $t = t_1(x;\, \hat{t}, \hat{x})$. In view of the definition (3.25), then for any point $(\hat{t}, \hat{x}) \in [t_0 + T_0, \tau] \times [0, L]$, the backward characteristic curve $t = t_1(x;\, \hat{t}, \hat{x})$ must intersect the boundary $x = 0$ or $x = L$ in a finite time $t < T_0$. Thus, we obtain

$$
\|\Gamma_1(\hat{t}, \hat{x}) - \Gamma_1^{(T)}(\hat{t}, \hat{x})\|_{C^0} \le s J_S \beta^N \varepsilon + C(\varepsilon + \varepsilon_0) J_S \beta^N \varepsilon \le J_S \beta^{N+1} \varepsilon,
\tag{5.17}
$$

where we take the constant $\beta \in (s, 1)$. By using similarly method, one has

$$
\|\Gamma_2(\hat{t}, \hat{x}) - \Gamma_2^{(T)}(\hat{t}, \hat{x})\|_{C^0} \le s J_S \beta^N \varepsilon + C\varepsilon J_S \beta^N \varepsilon \le J_S \beta^{N+1} \varepsilon,
\tag{5.18}
$$

$$
\|\Gamma_3(\hat{t}, \hat{x}) - \Gamma_3^{(T)}(\hat{t}, \hat{x})\|_{C^0} \le J_S \beta^{N+1} \varepsilon.
\tag{5.19}
$$

From (5.17)–(5.19), it holds

$$
\Lambda(\hat{t}) \le J_S \varepsilon \beta^{N+1}, \quad \forall \hat{t} \in [t_0 + T_0, \tau].
\tag{5.20}
$$

Since τ is arbitrary, we complete the proof of Theorem 3.2.

6 Regularity of the temporal periodic solution

This section aims to prove the regularity of the temporal periodic solution $\Gamma^{(T)}$. When the boundary functions $\Gamma_{ip}(t)$, $(i = 1, 2, 3)$ are $W^{2,\infty}$ regular, we will show the corresponding temporal periodic solution has the same $W^{2,\infty}$ regularity. First, we give the following estimates.

Proposition 6.1. *For the iteration systems* (4.8)–(4.14), *if the boundary conditions meet the* $W^{2,\infty}$ *regularity* (3.26), *then there exists a large enough constant* $H_R > 0$, *such that for each* $k \in \mathbb{N}^+$, *the solutions of the iteration system* (4.8)–(4.14) *satisfy*

$$\|\partial_t^2 \Gamma^{(k)}\|_{L^\infty} \le H_R, \tag{6.1}$$

$$\|\partial_t \partial_x \Gamma^{(k)}\|_{L^\infty} \le B_0 H_R, \tag{6.2}$$

$$\|\partial_x^2 \Gamma^{(k)}\|_{L^\infty} \le B_0^2 H_R, \tag{6.3}$$

under the hypotheses

$$\|\partial_t^2 \Gamma^{(k-1)}\|_{L^\infty} \le H_R, \tag{6.4}$$

$$\|\partial_t \partial_x \Gamma^{(k-1)}\|_{L^\infty} \le B_0 H_R, \tag{6.5}$$

$$\|\partial_x^2 \Gamma^{(k-1)}\|_{L^\infty} \le B_0^2 H_R. \tag{6.6}$$

As long as Proposition 6.1 is proved, we can obtain the uniform $W^{2,\infty}$ boundedness of $\{\Gamma^{(k)}\}_{k=1}^{\infty}$. Consequently, the weak * convergence emerges.

Proof. According to Theorem 3.1, $\{\Gamma^{(k)}\}_{k=1}^{\infty}$ has strong convergence. On the basis of that, we will show the $W^{2,\infty}$ regularity of $\Gamma^{(T)}$. In fact, from Proposition 4.1, we already have (4.15)–(4.18), and especially

$$\|\Gamma^{(k)}\|_{C^1} \le (J_1 + J_2)\varepsilon, \quad \|\Gamma^{(k-1)}\|_{C^1} \le (J_1 + J_2)\varepsilon, \tag{6.7}$$

for each $k \in \mathbb{Z}_+$. Denote

$$\chi_i^{(k)} = \partial_t h_i^{(k)} = \partial_t^2 \Gamma_i^{(k)}, \quad i = 1, 2, 3, \tag{6.8}$$

then from the boundary conditions (4.40)–(4.42), we have

$$x = L : \chi_1^{(k)}(t, L) = \Gamma''_{1pr}(t) + s_1 \chi_2^{(k-1)}(t, L) + s_1^* \chi_3^{(k-1)}(t, L) \tag{6.9}$$

$$x = 0 : \chi_2^{(k)}(t, 0) = \Gamma''_{2pl}(t) + s_2 \chi_1^{(k-1)}(t, 0), \tag{6.10}$$

$$\chi_3^{(k)}(t, 0) = \Gamma''_{3pl}(t) + s_3 \chi_1^{(k-1)}(t, 0). \tag{6.11}$$

Thus, we obtain

$$|\chi_1^{(k)}(t, L)| \le H_0 + sH_R. \tag{6.12}$$

Similarly,

$$|\chi_2^{(k)}(t, 0)| \le H_0 + sH_R, \tag{6.13}$$

$$|\chi_3^{(k)}(t, 0)| \le H_0 + sH_R. \tag{6.14}$$

Within the region $\{(t, x) \in \mathbb{R} \times (0, L)\}$, we take ∂_t on (4.46). Then by using a similar way to (4.50) to have

$$\|\chi_1^{(k)}\|_{L^\infty} \le \hbar_2 H_R \tag{6.15}$$

for a constant $\hbar_2 \in (0, 1)$ and lager enough constant H_R. Similarly, we obtain

$$\|\chi_2^{(k)}\|_{L^\infty} \leq \hbar_2 H_R, \tag{6.16}$$

$$\|\chi_3^{(k)}\|_{L^\infty} \leq \hbar_2 H_R. \tag{6.17}$$

Then utilizing (4.46), combining (4.50)–(4.52) and (6.15)–(6.17), we acquire

$$\|\partial_t \partial_x \Gamma_i^{(k)}\|_{L^\infty} \leq B_0 \hbar_2 H_R + C(\varepsilon + \varepsilon_0) + C\varepsilon^2 \leq B_0 \hbar_3 H_R, \quad i = 1, 2, 3, \tag{6.18}$$

where $\hbar_3 \in (\hbar_2, 1)$ is a constant and independent of k.

Furthermore, taking the spatial derivative on (4.8), we have

$$\begin{aligned}
&\partial_x^2 \Gamma_1^{(k)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_x \partial_t \Gamma_1^{(k)} \\
&= -\sum_{j=1}^{3} \left(\frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} \partial_x \Gamma_j^{(k-1)} + \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \tilde{\Gamma}_j} \partial_x \tilde{\Gamma}_j \right) \partial_t \Gamma_1^{(k)} \\
&\quad - \frac{\gamma - 1}{8} a(x)\Psi_1(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_x \Gamma_1^{(k-1)} \\
&\quad - \frac{\gamma - 1}{8} (a'(x)\Psi_1(x) + a(x)\Psi_1'(x))\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\Gamma_1^{(k-1)} \\
&\quad - \frac{\gamma - 1}{8} a(x)\Psi_1(x) \sum_{j=1}^{3} \left(\frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} \partial_x \Gamma_j^{(k-1)} + \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \tilde{\Gamma}_j} \partial_x \tilde{\Gamma}_j \right) \Gamma_1^{(k-1)} \\
&\quad - \frac{1}{4\gamma} (\partial_x \Gamma_3^{(k-1)} - \partial_x \Gamma_1^{(k-1)} + \Psi_5'(x))(\partial_x \Gamma_2^{(k-1)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_t \Gamma_2^{(k-1)}) \\
&\quad - \frac{1}{4\gamma} (\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x))(\partial_x^2 \Gamma_2^{(k-1)} + \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_x \partial_t \Gamma_2^{(k-1)}) \\
&\quad - \frac{1}{4\gamma} (\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x)) \sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} \partial_x \Gamma_j^{(k-1)} \cdot \partial_t \Gamma_2^{(k-1)} \\
&\quad - \frac{1}{4\gamma} (\Gamma_3^{(k-1)} - \Gamma_1^{(k-1)} + \Psi_5(x)) \sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \tilde{\Gamma}_j} \partial_x \tilde{\Gamma}_j \cdot \partial_t \Gamma_2^{(k-1)} \\
&\quad + \frac{\gamma - 1}{8} a(x)\Psi_2(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\partial_x \Gamma_3^{(k-1)} \\
&\quad + \frac{\gamma - 1}{8} (a'(x)\Psi_1(x) + a(x)\Psi_2'(x))\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})\Gamma_3^{(k-1)} \\
&\quad + \frac{\gamma - 1}{8} a(x)\Psi_2'(x) \sum_{j=1}^{3} \left(\frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} \partial_x \Gamma_j^{(k-1)} + \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \tilde{\Gamma}_j} \partial_x \tilde{\Gamma}_j \right) \Gamma_3^{(k-1)} \\
&\quad + \frac{\gamma - 1}{4} a(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})(\Gamma_3^{(k-1)}\partial_x \Gamma_3^{(k-1)} - \Gamma_1^{(k-1)}\partial_x \Gamma_1^{(k-1)}) \\
&\quad + \frac{\gamma - 1}{8} a'(x)\mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2) \\
&\quad + \frac{\gamma - 1}{8} a(x) \sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \Gamma_j} \partial_x \Gamma_j^{(k-1)}((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2) \\
&\quad + \frac{\gamma - 1}{8} a(x) \sum_{j=1}^{3} \frac{\partial \mu_1(\Gamma^{(k-1)} + \tilde{\Gamma})}{\partial \tilde{\Gamma}_j} \partial_x \tilde{\Gamma}_j((\Gamma_3^{(k-1)})^2 - (\Gamma_1^{(k-1)})^2).
\end{aligned} \tag{6.19}$$

From (1.2), (4.3), (4.7), (4.16), (6.6), and (6.19), we finally obtain

$$\|\partial_x^2 \Gamma_1^{(k)}\|_{L^\infty} \leq B_0^2 \hbar_3 H_R + C(\varepsilon + \varepsilon_0) B_0^2 \hbar_3 H_R + C\varepsilon + C\varepsilon^2 \leq B_0^2 H_R. \tag{6.20}$$

Similarly,

$$\|\partial_x^2 \Gamma_2^{(k)}\|_{L^\infty} \leq B_0^2 H_R, \tag{6.21}$$

$$\|\partial_x^2\Gamma_3^{(k)}\|_{L^\infty} \le B_0^2 H_R. \tag{6.22}$$

Hence, we complete the proof of Proposition 6.1, and thus Theorem 3.3. □

7 Stabilization around the regular temporal periodic solution

In this section, we will give the stabilization result for the system around the corresponding temporal periodic solution $\Gamma^{(T)}$. Actually, from Section 5, we have the C^0 convergence results:

$$\max_{i=1,2,3} \|\Gamma_i(t,\cdot) - \Gamma_i^{(T)}(t,\cdot)\|_{C^0} \le J_S \varepsilon\beta^N, \quad \forall t \in [NT_0, (N+1)T_0), \tag{7.1}$$

and further, we have

$$\max_{i=1,2,3} \|\Gamma_i(t,\cdot) - \Gamma_i^{(T)}(t,\cdot)\|_{C^0} \le J_S \varepsilon\beta^{N+1}, \quad \forall t \in [(N+1)T_0, (N+2)T_0) \tag{7.2}$$

for each $N \in \mathbb{Z}_+$. Moreover, since Theorem 3.1,Lemma 5.1, and Theorem 3.3, we obtain

$$\|\Gamma\|_{C^1} \le J_E\varepsilon, \quad \|\Gamma^{(T)}\|_{C^1} \le J_E\varepsilon, \quad \|\Gamma^{(T)}\|_{W^{2,\infty}} \le (1+B_0)^2 H_R. \tag{7.3}$$

Due to the continuity, we will prove the C^0 estimates of the first derivatives iteratively, namely, for each $N \in \mathbb{Z}_+$ and $\tau \in [(N+1)T_0, (N+2)T_0)$, suppose that

$$\|\partial_t\Gamma_i(t,\cdot) - \partial_t\Gamma_i^{(T)}(t,\cdot)\|_{C^0} \le J_S^*\beta^N\varepsilon, \quad \forall t \in [NT_0, \tau], \tag{7.4}$$

$$\|\partial_x\Gamma_i(t,\cdot) - \partial_x\Gamma_i^{(T)}(t,\cdot)\|_{C^0} \le B_0 J_S^*\beta^N\varepsilon, \quad \forall t \in [NT_0, \tau], \tag{7.5}$$

then we will show

$$\|\partial_t\Gamma_i(t,\cdot) - \partial_t\Gamma_i^{(T)}(t,\cdot)\|_{C^0} \le J_S^*\beta^{N+1}\varepsilon, \quad \forall t \in [(N+1)T_0, \tau], \tag{7.6}$$

$$\|\partial_x\Gamma_i(t,\cdot) - \partial_x\Gamma_i^{(T)}(t,\cdot)\|_{C^0} \le B_0 J_S^*\beta^N\varepsilon, \quad \forall t \in [(N+1)T_0, \tau]. \tag{7.7}$$

Denote

$$h_i = \partial_t\Gamma_i, \quad g_i = \partial_x\Gamma_i, \quad i = 1,2,3$$

and

$$h_i^{(T)} = \partial_t\Gamma_i^{(T)}, \quad g_i^{(T)} = \partial_x\Gamma_i^{(T)}, \quad i = 1,2,3.$$

Similarly to other sections, we focus on boundary estimates at first. Taking the temporal derivative on (3.15)–(3.17), we have

$$x = L : h_1(t,L) = \Gamma_{1p}'(t) + s_1 h_2(t,L) + s_1^* h_3(t,L), \tag{7.8}$$

$$x = 0 : h_2(t,0) = \Gamma_{2p}'(t) + s_2 h_1(t,0), \tag{7.9}$$

$$h_3(t,0) = \Gamma_{3p}'(t) + s_3 h_1(t,0), \tag{7.10}$$

and

$$x = L : h_1^{(T)}(t,L) = \Gamma_{1p}'(t) + s_1 h_2^{(T)}(t,L) + s_1^* h_3^{(T)}(t,L), \tag{7.11}$$

$$x = 0 : h_2^{(T)}(t,0) = \Gamma_{2p}'(t) + s_2 h_1^{(T)}(t,0), \tag{7.12}$$

$$h_3^{(T)}(t,0) = \Gamma_{3p}'(t) + s_3 h_1^{(T)}(t,0). \tag{7.13}$$

Subtracting (7.11) from (7.8), and noticing (7.4), it shows

$$\begin{aligned}&\sup_{t\in[NT_0,\tau]}|h_1(t,L)-h_1^{(T)}(t,L)|\\&\le |s_1||h_2(t,L)-h_2^{(T)}(t,L)|+|s_1^*||h_3(t,L)-h_3^{(T)}(t,L)|\\&\le\ sJ_S^*\beta^N\varepsilon.\end{aligned}\tag{7.14}$$

Similarly,

$$\sup_{t\in[NT_0,\tau]}|h_2(t,0)-h_2^{(T)}(t,0)|\le sJ_S^*\beta^N\varepsilon,\tag{7.15}$$

$$\sup_{t\in[NT_0,\tau]}|h_2(t,0)-h_2^{(T)}(t,0)|\le sJ_S^*\beta^N\varepsilon,\tag{7.16}$$

For $(t,x)\in\mathbb{R}_+\times(0,L)$, we take the temporal derivative on (4.4) and (5.14) to have

$$\begin{aligned}&\partial_x h_1+\mu_1(\Gamma+\tilde\Gamma)\partial_t h_1\\&\quad=-\sum_{j=1}^3\frac{\partial\mu_1(\Gamma+\tilde\Gamma)}{\partial\Gamma_j}h_jh_1-\frac{\gamma-1}{8}a(x)\Psi_1(x)\mu_1(\Gamma+\tilde\Gamma)\cdot h_1\\&\qquad-\frac{\gamma-1}{8}a(x)\Psi_1(x)\cdot\sum_{j=1}^3\frac{\partial\mu_1(\Gamma+\tilde\Gamma)}{\partial\Gamma_j}h_j\Gamma_1+\frac{\gamma-1}{8}a(x)\Psi_2(x)\mu_1(\Gamma+\tilde\Gamma)h_3\\&\qquad+\frac{\gamma-1}{8}a(x)\Psi_2(x)\sum_{j=1}^3\frac{\partial\mu_1(\Gamma+\tilde\Gamma)}{\partial\Gamma_j}h_j\Gamma_3-\frac{1}{4\gamma}(h_3-h_1)(\partial_x\Gamma_2+\mu_1(\Gamma+\tilde\Gamma)\partial_t\Gamma_2)\\&\qquad-\frac{1}{4\gamma}(\Gamma_3-\Gamma_1+\Psi_5(x))\cdot\sum_{j=1}^3\frac{\partial\mu_1(\Gamma+\tilde\Gamma)}{\partial\Gamma_j}h_jh_2-\frac{1}{4\gamma}(\Gamma_3-\Gamma_1+\Psi_5(x))(\partial_x h_2+\mu_1(\Gamma+\tilde\Gamma)\partial_t h_2)\\&\qquad+\frac{\gamma-1}{8}a(x)\sum_{j=1}^3\frac{\partial\mu_1(\Gamma+\tilde\Gamma)}{\partial\Gamma_j}h_j((\Gamma_3)^2-(\Gamma_1)^2)+\frac{\gamma-1}{4}a(x)\mu_1(\Gamma+\tilde\Gamma)(\Gamma_3h_3-\Gamma_1h_1),\end{aligned}\tag{7.17}$$

and

$$\begin{aligned}&\partial_x h_1^{(T)}+\mu_1(\Gamma^{(T)}+\tilde\Gamma)\partial_t h_1^{(T)}\\&\quad=-\sum_{j=1}^3\frac{\partial\mu_1(\Gamma^{(T)}+\tilde\Gamma)}{\partial\Gamma_j^{(T)}}h_j^{(T)}h_1^{(T)}-\frac{\gamma-1}{8}a(x)\Psi_1(x)\mu_1(\Gamma^{(T)}+\tilde\Gamma)\cdot h_1^{(T)}\\&\qquad-\frac{\gamma-1}{8}a(x)\Psi_1(x)\cdot\sum_{j=1}^3\frac{\partial\mu_1(\Gamma^{(T)}+\tilde\Gamma)}{\partial\Gamma_j^{(T)}}h_j^{(T)}\Gamma_1^{(T)}+\frac{\gamma-1}{8}a(x)\Psi_2(x)\mu_1(\Gamma^{(T)}+\tilde\Gamma)h_3^{(T)}\\&\qquad+\frac{\gamma-1}{8}a(x)\Psi_2(x)\sum_{j=1}^3\frac{\partial\mu_1(\Gamma^{(T)}+\tilde\Gamma)}{\partial\Gamma_j^{(T)}}h_j^{(T)}\Gamma_3^{(T)}-\frac{1}{4\gamma}(h_3^{(T)}-h_1^{(T)})(\partial_x\Gamma_2^{(T)}+\mu_1(\Gamma^{(T)}+\tilde\Gamma)\partial_t\Gamma_2^{(T)})\\&\qquad-\frac{1}{4\gamma}(\Gamma_3^{(T)}-\Gamma_1^{(T)}+\Psi_5(x))\cdot\sum_{j=1}^3\frac{\partial\mu_1(\Gamma^{(T)}+\tilde\Gamma)}{\partial\Gamma_j^{(T)}}h_j^{(T)}h_2^{(T)}\\&\qquad-\frac{1}{4\gamma}(\Gamma_3^{(T)}-\Gamma_1^{(T)}+\Psi_5(x))(\partial_x h_2^{(T)}+\mu_1(\Gamma^{(T)}+\tilde\Gamma)\partial_t h_2^{(T)})\\&\qquad+\frac{\gamma-1}{8}a(x)\sum_{j=1}^3\frac{\partial\mu_1(\Gamma^{(T)}+\tilde\Gamma)}{\partial\Gamma_j^{(T)}}h_j^{(T)}((\Gamma_3^{(T)})^2-(\Gamma_1^{(T)})^2)\\&\qquad+\frac{\gamma-1}{4}a(x)\mu_1(\Gamma^{(T)}+\tilde\Gamma)(\Gamma_3^{(T)}h_3^{(T)}-\Gamma_1^{(T)}h_1^{(T)}).\end{aligned}\tag{7.18}$$

Subtracting (7.18) from (7.17) and using similar method in the proof of (5.17), we have

$$\begin{aligned}
&\|h_1(\hat{t},\hat{x}) - h_1^{(T)}(\hat{t},\hat{x})\|_{C^0} \\
&\le\ sJ_S^*\beta^N \hat{+} \sup_{\Gamma}|\nabla\mu_1|(1+B_0^2)H_RJ_S\beta^N \hat{+} C\varepsilon J_S\beta^N\varepsilon + C\varepsilon J_S^*\beta^N\varepsilon + C\varepsilon_0 J_S^*\beta^N\varepsilon \\
&\le\ \hbar_4 J_S^*\beta^{N+1}\varepsilon, \quad \forall(\hat{t},\hat{x}) \in [(N+1)T_0,\tau]\times[0,L]
\end{aligned} \tag{7.19}$$

where we assume the constant $\hbar_4 \in (0,1)$ and take

$$J_S^* > 100 \max_{i=1,2,3} \sup_{\Gamma}|\nabla\mu_i|(1+B_0^2)H_RJ_S + J_S.$$

By similar method, it follows:

$$\|h_2(\hat{t},\hat{x}) - h_2^{(T)}(\hat{t},\hat{x})\|_{C^0} \le \hbar_4 J_S^*\beta^{N+1}\varepsilon, \quad \forall(\hat{t},\hat{x}) \in [(N+1)T_0,\tau]\times[0,L], \tag{7.20}$$

$$\|h_3(\hat{t},\hat{x}) - h_3^{(T)}(\hat{t},\hat{x})\|_{C^0} \le \hbar_4 J_S^*\beta^{N+1}\varepsilon, \quad \forall(\hat{t},\hat{x}) \in [(N+1)T_0,\tau]\times[0,L]. \tag{7.21}$$

According to (5.15),

$$\begin{aligned}
g_1 - g_1^{(T)} = &-\mu_1(\Gamma+\tilde{\Gamma})(h_1 - h_1^{(T)}) - (\mu_1(\Gamma+\tilde{\Gamma}) - \mu_1(\Gamma^{(T)}+\tilde{\Gamma}))h_1^{(T)} \\
&- \frac{\gamma-1}{8}a(x)\Psi_1(x)\mu_1(\Gamma+\tilde{\Gamma})(\Gamma_1 - \Gamma_1^{(T)}) \\
&- \frac{\gamma-1}{8}a(x)\Psi_1(x)(\mu_1(\Gamma+\tilde{\Gamma}) - \mu_1(\Gamma^{(T)}+\tilde{\Gamma}))\Gamma_1^{(T)} \\
&+ \frac{\gamma-1}{8}a(x)\Psi_2(x)\mu_1(\Gamma+\tilde{\Gamma})(\Gamma_3 - \Gamma_3^{(T)}) \\
&+ \frac{\gamma-1}{8}a(x)\Psi_2(x)(\mu_1(\Gamma+\tilde{\Gamma}) - \mu_1(\Gamma^{(T)}+\tilde{\Gamma}))\Gamma_3^{(T)} \\
&- \frac{1}{4\gamma}(\Gamma_3 - \Gamma_1 + \Psi_5(x))((g_2 - g_2^{(T)}) + \mu_1(\Gamma+\tilde{\Gamma})(h_2 - h_2^{(T)})) \\
&- \frac{1}{4\gamma}(\Gamma_3 - \Gamma_1 + \Psi_5(x))(\mu_1(\Gamma+\tilde{\Gamma}) - \mu_1(\Gamma^{(T)}+\tilde{\Gamma}))h_2^{(T)} \\
&- \frac{1}{4\gamma}(\Gamma_3 - \Gamma_1 - \Gamma_3^{(T)} + \Gamma_1^{(T)})(g_2^{(T)} + \mu_1(\Gamma^{(T)}+\tilde{\Gamma})h_2^{(T)}) \\
&+ \frac{\gamma-1}{8}a(x)\mu_1(\Gamma+\tilde{\Gamma})((\Gamma_3^2 - \Gamma_1^2) - ((\Gamma_3^{(T)})^2 - (\Gamma_1^{(T)})^2)) \\
&+ \frac{\gamma-1}{8}a(x)(\mu_1(\Gamma+\tilde{\Gamma}) - \mu_1(\Gamma^{(T)}+\tilde{\Gamma}))((\Gamma_3^{(T)})^2 - (\Gamma_1^{(T)})^2).
\end{aligned} \tag{7.22}$$

With the help of (1.2), (4.3), (4.7), (4.16), (7.1)–(7.3), (7.5), and (7.19)–(7.20), it yields

$$\begin{aligned}
\|g_1(t,\cdot) - g_1^{(T)}(t,\cdot)\|_{C^0} &\le B_0\hbar_4 J_S^*\beta^{N+1}\varepsilon + C(\varepsilon+\varepsilon_0)J_S^*\beta^N\varepsilon + C(\varepsilon+\varepsilon_0)J_S\beta^{N+1}\varepsilon + C\varepsilon^2 \\
&\le B_0 J_S^*\beta^{N+1}\varepsilon, \quad \forall t \in [(N+1)T_0,\tau].
\end{aligned} \tag{7.23}$$

Similarly to have

$$\|g_2(t,\cdot) - g_2^{(T)}(t,\cdot)\|_{C^0} \le B_0 J_S^*\beta^{N+1}\varepsilon, \quad \forall t \in [(N+1)T_0,\tau], \tag{7.24}$$

$$\|g_3(t,\cdot) - g_3^{(T)}(t,\cdot)\|_{C^0} \le B_0 J_S^*\beta^{N+1}\varepsilon, \quad \forall t \in [(N+1)T_0,\tau]. \tag{7.25}$$

Hence, we prove Theorem 3.4 completely.

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