

Research Article

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Peaked solitary waves and shock waves of the Degasperis-Procesi-Kadomtsev-Petviashvili equation

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Abstract: In this study, we establish the existence and nonexistence of smooth and peaked solitary wave solutions (or periodic) to the Degasperis-Procesi-Kadomtsev-Petviashvili (DP-KP) equation with a weak transverse effect. We have also shown that DP-KP equation possesses periodic shock waves similar to that of the Degasperis-Procesi equation.

Keywords: solitary waves, shock waves, Degasperis-Procesi-Kadomtsev-Petviashvili equation, Degasperis-Procesi equation

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1 Introduction

In this work, we show the existence of peaked solitary waves and shock waves of the Degasperis-Procesi-Kadomtsev-Petviashvili (DP-KP) equation

$$(u_t - u_{xxt} + \kappa u_x + 4uu_x - (3u_x u_{xx} + uu_{xxx}))_x + u_{yy} = 0, \quad (1.1)$$

where κ is a real parameter and the real function $u(t, x, y)$ depends on the spatial variables $x, y \in \mathbb{R}$ and the temporal variable $t > 0$. Equation (1.1) may be viewed as the Degasperis-Procesi (DP) counterpart of the two-dimensional generalization of the Korteweg-de Vries (KdV) equation [19] in much the same way as Camassa-Holm-Kadomtsev-Petviashvili (CH-KP) equation [15], which was formally derived from the incompressible and irrotational three-dimensional shallow water under the CH regime. Local well-posedness, formation of singularities, existence of traveling wave solutions, and the Liouville-type property for CH-KP equation were presented in [15]. The recent related results for other models with respect to transverse effects can be found in [3,13,16].

In the case of no y -dependence, equations (1.1) and CH-KP equation take the form of the integrable DP equation [9]

$$u_t - u_{xxt} + \kappa u_x + 4uu_x - (3u_x u_{xx} + uu_{xxx}) = 0 \quad (1.2)$$

and the Camassa-Holm (CH) shallow-water equation [2,12]

$$u_t - u_{xxt} + \kappa u_x + 3uu_x - (2u_x u_{xx} + uu_{xxx}) = 0, \quad (1.3)$$

respectively.

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During the last decade, the DP and CH equations have attracted much attention due to their integrable structure. Both equations arise in the modeling of the propagation of shallow water waves over a flat bed. The DP and CH equations capture stronger nonlinear effects than the classical nonlinear dispersive Benjamin-Bona-Mahony (BBM) [1] and KdV equations [19]. They accommodate wave breaking phenomena (i.e., wave profile remains bounded while its slope becomes unbounded in finite time [30]). From a mathematical physics viewpoint, it is interesting to study these equations sharing the peaked solitary wave solution on the line and circle in the limiting case of vanishing linear dispersion ($\kappa = 0$)

$$u(t, x) = ce^{-|x-ct|}, \quad c > 0, \quad (1.4)$$

$$u(t, x) = \frac{c}{\cosh\left(\frac{1}{2}\right)} \cosh\left((x - ct) - \left[x - ct\right] - \frac{1}{2}\right), \quad x \in \mathbb{R}, \quad t \geq 0, \quad c > 0. \quad (1.5)$$

Here the notation $[x]$ denotes the greatest integer part of the real number $x \in \mathbb{R}$. The orbital stability of peaked solitary wave solutions (1.4) and (1.5) of the CH and DP equations has been established [7,21,23]. It was shown [27] that peakons and a train of peakons for CH equation are asymptotically stable (see also [18] for DP equation). Note that the peaked solitary wave solutions (1.4) and (1.5) are not classical solutions due to the fact that they have a peak at their crest (a feature that explains why they are called “peakons”). It is also worth pointing out that one of the most relevant motivations to look for peaked solitary waves (or periodic) is the fact that the governing equations for irrotational water waves do admit peaked solitary waves (or periodic), namely, the celebrated Stokes waves of greatest height [5,6,29]. There has recently been a growing interest in finding various CH-type equations with higher-order nonlinearity admitting peaked solitary wave solutions (e.g. [4,17,25,28,31,32] and references therein). It is worth mentioning that the recent research works [14,25,28] where peaked waves were shown to be unstable in the dynamical evolution of the CH-related equations are important for the complete picture about (in)stability of peaked waves in the CH-related equations.

Although the DP equation has a similarity to the CH equation, these two equations have truly different features. An important difference between the DP and the CH equation for $\kappa = 0$ is that the DP equation also enjoys shock peaked waves [24]

$$u(t, x) = -\frac{1}{t+c} \operatorname{sgn}(x) e^{-|x|}, \quad c > 0 \quad (1.6)$$

and periodic shock waves [11]

$$u(t, x) = \begin{cases} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{\sinh\left(x - [x] - \frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}, & x \in \mathbb{R} \setminus \mathbb{Z}, \quad c > 0, \\ 0, & x \in \mathbb{Z}. \end{cases} \quad (1.7)$$

It is unsure that whether such a shock waves (or periodic) (1.6) and (1.7) is stable or not in proper setting. On the other hand, DP equation has the third-order equation in the Lax pair for the isospectral problem [8]

$$\psi_x - \psi_{xxx} - \lambda m \psi = 0, \quad m = u - u_{xx},$$

while CH equation has the second-order equation for the isospectral problem [2]

$$\psi_{xx} - \frac{1}{4} \psi - \lambda m \psi = 0, \quad m = u - u_{xx}.$$

Moreover, the DP and CH equations have entirely different forms of conservation laws due to the fact that there is no simple transformation of the DP equation into the CH equation [2,8]. Furthermore, the CH equation has several nice geometric formulations [26], while only a non-metric geometric derivation of the DP equation is available [10]. Their similarities and differences can be found in related recent results [22].

In the case of non-vanishing linear dispersion ($\kappa \neq 0$), the DP and CH equations have localized smooth solitary waves [20,22]. Existence and stability result of smooth solitary waves for the DP and CH equations are established in [7,22].

The aim of this work is to prove whether the DP-KP equation (1.1) has properties for existence of solitary waves and existence of shock waves similar to that of the CH-KP equation and DP equation (1.2), respectively. Recently, it is known from [15] that the CH-KP equation admit peaked solitary wave solution of the form

$$u(t, x, y) = ce^{-|x+\beta y-ct|} \quad \text{if and only if } \kappa + \beta^2 = 0.$$

Since the CH-KP equation and DP equation (1.2) have peaked solitary waves (or periodic) and shock waves, respectively, one can expect that equation (1.1) possesses peaked solitary waves (or periodic) and shock waves. It is a reasonable expectation due to the fact that DP-KP equation (1.1) was able to transform to DP equation (1.2) (Remark 2.1). Until now, it is unclear whether the DP-KP equation (1.1) is integrable and an inverse scattering method is available for it or not. This will be the objective for future work.

Our work is motivated by the works of Gui et al. [15] and Escher et al. [11]. In [15], the existence of peaked solitary wave solutions for the CH-KP equation has been proven for the first time. On the other hand, in [11], the authors present a periodic shock waves of the DP equation (1.2) with $\kappa = 0$. Combining the ideas from both papers we show existence of periodic shock waves of the DP-KP (1.1).

This work is organized as follows. Section 2 is devoted to the existence and non-existence of peaked solitary wave solutions for the DP-KP equation (1.1) for certain cases. Existence of periodic shock waves are presented in Section 3.

2 Peaked solitary waves

In this section, we prove the existence of peaked solitary waves of the DP-KP equation (1.1) in the following form:

$$u(t, x, y) = \alpha_1 e^{-|x+\beta_1 y-ct|}, \quad c \in \mathbb{R} \quad (2.1)$$

and

$$u(t, x, y) = \alpha_2 \cosh\left(\frac{1}{2} - (x + \beta_2 y - ct) + [x + \beta_2 y - ct]\right), \quad c \in \mathbb{R}, \quad (2.2)$$

where $[x]$ stands for the greatest integer of $x \in \mathbb{R}$. Note that the inverse operator $(1 - \partial_x^2)^{-1}$ can be obtained by convolution with the corresponding Green's function, so that

$$(1 - \partial_x^2)^{-1}f = G*f \quad \text{for all } f \in L^2(X), \quad (2.3)$$

where

$$G(x) = \frac{1}{2}e^{-|x|} \quad \text{for the non-periodic case } X = \mathbb{R}, \quad (2.4)$$

while

$$G(x) = \frac{\cosh\left(\frac{1}{2} - x + [x]\right)}{2 \sinh\left(\frac{1}{2}\right)} \quad \text{for the periodic case } X = \mathbb{S}, \quad (2.5)$$

and the convolution product is defined by

$$f*g(x) = \int_X f(y)g(x-y)dy.$$

We can rewrite (1.1) with the initial data u_0 as the following weak form:

$$\begin{aligned} u_t + uu_x + (1 - \partial_x^2)^{-1} \partial_x \left(\frac{3}{2} u^2 + \kappa u \right) + (1 - \partial_x^2)^{-1} v_y &= 0, \quad t > 0, \quad (x, y) \in X = \mathbb{R}^2 \text{ or } \mathbb{S}^2 \\ u_y &= v_x. \\ u|_{t=0} &= u_0(x, y), \quad (x, y) \in X = \mathbb{R}^2 \text{ or } \mathbb{S}^2. \end{aligned} \quad (2.6)$$

The formulation (2.6) allows us to define the notion of a weak solution as follows.

Definition 2.1. Given initial data $u_0 \in H^1(X)$, a function $u \in C([0, T]; H_{\text{loc}}^1(X))$ is said to be a weak solution of the initial value problem (2.6) if it satisfies the following identity:

$$\int_0^T \int_X \left[-\partial_t \partial_x \phi u + \partial_x \phi \left(u \partial_x u + \partial_x G^* \left(\frac{3}{2} u^2 + \kappa u \right) \right) - \partial_y^2 \phi G^* u \right] dx dy dt + \int_X u_0(x, y) \partial_x \phi(0, x, y) dx dy = 0, \quad (2.7)$$

for any smooth test function $\phi(t, x, y) \in C_c^\infty([0, T] \times X)$. If u is a weak solution on $[0, T)$ for every $T > 0$, then it is called a global weak solution.

The following theorem deals with the existence of peaked solitary waves for the DP-KP equation (1.1).

Theorem 2.1. The DP-KP equation (1.1) possesses the peaked solitary wave solutions of the form $u(t, x, y) = ce^{-|x+\beta_1 y-ct|}$ for constant $\beta_1 \in \mathbb{R}$ if and only if it satisfies $\kappa + \beta_1^2 = 0$. These peakons are global weak solution to (2.6) in the sense of Definition 2.1.

Proof. Suppose that

$$u_c(t, x, y) = ce^{-|x+\beta_1 y-ct|}. \quad (2.8)$$

Then, we have

$$\partial_t u_c(t, x, y) = c \cdot \text{sign}(x + \beta_1 y - ct) u_c(t, x, y), \quad \partial_x u_c(t, x, y) = -\text{sign}(x + \beta_1 y - ct) u_c(t, x, y). \quad (2.9)$$

Combining (2.9) with the fact $\partial_x G(x) = -\frac{1}{2} \text{sign}(x) e^{-|x|}$ for $x \in \mathbb{R}$, for any test function $\phi(t, x, y) \in C_c^\infty([0, T] \times \mathbb{R}^2)$, we have

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} \left[-\partial_t \partial_x \phi u_c + \partial_x \phi \left(u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 + \kappa u_c \right) \right) - \partial_y^2 \phi G^* u_c \right] dx dy dt + \int_{\mathbb{R}^2} u_{c,0}(x, y) \partial_x \phi(0, x, y) dx dy \\ &= \int_0^\infty \int_{\mathbb{R}^2} \partial_x \phi \left[\partial_t u_c + u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 - (\kappa + \beta_1^2) u_c \right) \right] dx dy dt \\ &= \int_0^\infty \int_{\mathbb{R}^2} \partial_x \phi \left[c^2 \cdot \text{sign}(x + \beta_1 y - ct) e^{-|x+\beta_1 y-ct|} - c^2 \cdot \text{sign}(x + \beta_1 y - ct) e^{-2|x+\beta_1 y-ct|} \right] dx dy dt \\ &\quad + \int_0^\infty \int_{\mathbb{R}^2} \partial_x \phi \left[\partial_x G^* \left(\frac{3}{2} c^2 e^{-2|x+\beta_1 y-ct|} - (\kappa + \beta_1^2) c e^{-|x+\beta_1 y-ct|} \right) \right] dx dy dt, \end{aligned} \quad (2.10)$$

where we use the fact $G^* \partial_y u_c = \beta_1 \partial_x G^* u_c$.

Using the explicit form (2.4) and the convolution product, we deduce

$$\begin{aligned} \partial_x G^* \left(\frac{3}{2} c^2 e^{-2|x+\beta_1 y-ct|} - (\kappa + \beta_1^2) c e^{-|x+\beta_1 y-ct|} \right) &= -\frac{1}{2} \int_{-\infty}^\infty \text{sign}(x + \beta_1 y - z) e^{-|x+\beta_1 y-z|} \left(\frac{3}{2} c^2 e^{-2|z-ct|} \right. \\ &\quad \left. - (\kappa + \beta_1^2) c e^{-|z-ct|} \right) dz. \end{aligned} \quad (2.11)$$

For $x + \beta_1 y > ct$, we can split the right-hand side of (2.11) into the following three parts:

$$\begin{aligned}
& \partial_x G^* \left(\frac{3}{2} c^2 e^{-2|x+\beta_1 y-ct|} - (\kappa + \beta_1^2) c e^{-|x+\beta_1 y-ct|} \right) \\
&= -\frac{1}{2} \left[\int_{-\infty}^{ct} + \int_{ct}^{x+\beta_1 y} + \int_{x+\beta_1 y}^{\infty} \right] \text{sign}(x + \beta_1 y - z) e^{-|x+\beta_1 y-z|} \left(\frac{3}{2} c^2 e^{-2|z-ct|} - (\kappa + \beta_1^2) c e^{-|z-ct|} \right) dz. \\
&= J_1 + J_2 + J_3.
\end{aligned} \tag{2.12}$$

A direct calculation yields

$$J_1 = \int_{-\infty}^{ct} e^{-(x+\beta_1 y-z)} \left(-\frac{3}{4} c^2 e^{2(z-ct)} + \frac{1}{2} (\kappa + \beta_1^2) c e^{(z-ct)} \right) dz = \left\{ -\frac{1}{4} c^2 + \frac{1}{4} (\kappa + \beta_1^2) c \right\} e^{-(x+\beta_1 y-ct)}, \tag{2.13}$$

$$\begin{aligned}
J_2 &= \int_{ct}^{x+\beta_1 y} e^{-(x+\beta_1 y-z)} \left(-\frac{3}{4} c^2 e^{-2(z-ct)} + \frac{1}{2} (\kappa + \beta_1^2) c e^{-(z-ct)} \right) dz \\
&= \frac{3}{4} c^2 e^{-2(x+\beta_1 y-ct)} + \left\{ -\frac{3}{4} c^2 + \frac{1}{2} (\kappa + \beta_1^2) c (x + \beta_1 y - ct) \right\} e^{-(x+\beta_1 y-ct)},
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
J_3 &= \int_{x+\beta_1 y}^{\infty} e^{(x+\beta_1 y-z)} \left(\frac{3}{4} c^2 e^{-2(z-ct)} - \frac{1}{2} (\kappa + \beta_1^2) c e^{-(z-ct)} \right) dz \\
&= \frac{1}{4} c^2 e^{-2(x+\beta_1 y-ct)} - \frac{1}{4} (\kappa + \beta_1^2) c e^{-(x+\beta_1 y-ct)}.
\end{aligned} \tag{2.15}$$

Plugging the above equalities $J_1 - J_3$ into (2.12), we find that for $x + \beta_1 y > ct$,

$$\begin{aligned}
& \partial_x G^* \left(\frac{3}{2} c^2 e^{-2|x+\beta_1 y-ct|} - (\kappa + \beta_1^2) c e^{-|x+\beta_1 y-ct|} \right) \\
&= c^2 e^{-2(x+\beta_1 y-ct)} + \left\{ -c^2 + \frac{1}{2} (\kappa + \beta_1^2) c (x + \beta_1 y - ct) \right\} e^{-(x+\beta_1 y-ct)}.
\end{aligned} \tag{2.16}$$

While for $x + \beta_1 \leq ct$, we can also split the right-hand side of (2.11) into three parts:

$$\begin{aligned}
& \partial_x G^* \left(\frac{3}{2} c^2 e^{-2|x+\beta_1 y-ct|} - (\kappa + \beta_1^2) c e^{-|x+\beta_1 y-ct|} \right) \\
&= -\frac{1}{2} \left[\int_{-\infty}^{x+\beta_1 y} + \int_{x+\beta_1 y}^{ct} + \int_{ct}^{\infty} \right] \text{sign}(x + \beta_1 y - z) e^{-|x+\beta_1 y-z|} \left(\frac{3}{2} c^2 e^{-2|z-ct|} - (\kappa + \beta_1^2) c e^{-|z-ct|} \right) dz. \\
&= \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3.
\end{aligned} \tag{2.17}$$

Applying a similar computation as J_1, J_2, J_3 , we find that

$$\begin{aligned}
\tilde{J}_1 &= \int_{-\infty}^{x+\beta_1 y} e^{-(x+\beta_1 y-z)} \left(-\frac{3}{4} c^2 e^{2(z-ct)} + \frac{1}{2} (\kappa + \beta_1^2) c e^{(z-ct)} \right) dz \\
&= -\frac{1}{4} c^2 e^{2(x+\beta_1 y-ct)} + \frac{1}{4} (\kappa + \beta_1^2) c e^{(x+\beta_1 y-ct)},
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
\tilde{J}_2 &= \int_{x+\beta_1 y}^{ct} e^{(x+\beta_1 y-z)} \left(\frac{3}{4} c^2 e^{2(z-ct)} - \frac{1}{2} (\kappa + \beta_1^2) c e^{(z-ct)} \right) dz \\
&= -\frac{3}{4} c^2 e^{2(x+\beta_1 y-ct)} + \left\{ \frac{3}{4} c^2 + \frac{1}{2} (\kappa + \beta_1^2) c (x + \beta_1 y - ct) \right\} e^{(x+\beta_1 y-ct)}
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\tilde{J}_3 &= \int_{ct}^{\infty} e^{(x+\beta_1 y-z)} \left(\frac{3}{4} c^2 e^{-2(z-ct)} - \frac{1}{2} (\kappa + \beta_1^2) c e^{-(z-ct)} \right) dz \\ &= \frac{1}{4} c^2 e^{(x+\beta_1 y-ct)} - \frac{1}{4} (\kappa + \beta_1^2) c e^{(x+\beta_1 y-ct)}.\end{aligned}\quad (2.20)$$

Substituting (2.18)–(2.20) in (2.12) yields for $x + \beta_1 y \leq ct$,

$$\partial_x G^* \left(\frac{3}{2} c^2 e^{-2|x+\beta_1 y-ct|} - (\kappa + \beta_1^2) c e^{-|x+\beta_1 y-ct|} \right) = -c^2 e^{2(x+\beta_1 y-ct)} + \left\{ c^2 + \frac{1}{2} (\kappa + \beta_1^2) c (x + \beta_1 y - ct) \right\} e^{(x+\beta_1 y-ct)}, \quad (2.21)$$

which together with (2.16) leads to

$$\begin{aligned}\partial_x G^* \left(\frac{3}{2} u_c^2 - (\kappa + \beta_1^2) u_c \right) (t, x, y) \\ = \begin{cases} c^2 e^{-2(x+\beta_1 y-ct)} + \left\{ -c^2 + \frac{1}{2} (\kappa + \beta_1^2) c (x + \beta_1 y - ct) \right\} e^{-(x+\beta_1 y-ct)}, & x + \beta_1 y > ct, \\ -c^2 e^{2(x+\beta_1 y-ct)} + \left\{ c^2 + \frac{1}{2} (\kappa + \beta_1^2) c (x + \beta_1 y - ct) \right\} e^{(x+\beta_1 y-ct)}, & x + \beta_1 y \leq ct. \end{cases}\end{aligned}\quad (2.22)$$

On the other hand, one can deduce

$$\begin{aligned}\partial_t u_c + u_c \partial_x u_c &= c^2 \cdot \text{sign}(x + \beta_1 y - ct) e^{-|x+\beta_1 y-ct|} - c^2 \cdot \text{sign}(x + \beta_1 y - ct) e^{-2|x+\beta_1 y-ct|} \\ &= \begin{cases} -c^2 e^{-2(x+\beta_1 y-ct)} + c^2 e^{-(x+\beta_1 y-ct)}, & x + \beta_1 y > ct, \\ c^2 e^{2(x+\beta_1 y-ct)} - c^2 e^{-(x+\beta_1 y-ct)}, & x + \beta_1 y \leq ct. \end{cases}\end{aligned}\quad (2.23)$$

Thanks to (2.22), (2.23), and (2.10), we conclude that for any $\phi(t, x, y) \in C_c^\infty([0, \infty) \times \mathbb{R}^2)$,

$$\begin{aligned}& \int_0^\infty \int_{\mathbb{R}^2} \left[-\partial_t \partial_x \phi u_c + \partial_x \phi \left(u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 - (\kappa + \beta_1^2) u_c \right) \right) - \partial_y^2 \phi G^* u_c \right] dx dy dt + \int_{\mathbb{R}^2} u_{c,0}(x, y) \partial_x \phi(0, x, y) dx dy \\ &= \int_0^\infty \int_{\mathbb{R}^2} \partial_x \phi \left[\partial_t u_c + u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 - (\kappa + \beta_1^2) u_c \right) \right] dx dy dt \\ &= 0\end{aligned}\quad (2.24)$$

if and only if $\kappa + \beta_1^2 = 0$. This completes the proof of Theorem 2.1. $\square$

Next we give a concrete example of weak solutions of the DP-KP equation (1.1), which may be considered as periodic peaked solitary waves.

Theorem 2.2. *Let*

$$u(t, x, y) = \frac{c}{\cosh\left(\frac{1}{2}\right)} \cosh\left[\frac{1}{2} - (x + \beta_2 y - ct) + [x + \beta_2 y - ct]\right], \quad x \in \mathbb{R}, \quad \beta_2 \in \mathbb{R}, \quad t \geq 0. \quad (2.25)$$

The DP-KP equation (1.1) possesses the periodic peaked solitary waves of the form $u(t, x, y)$ if and only if it satisfies $\kappa + \beta_2^2 = 0$. These peakons are global weak solution to (2.6) in the sense of Definition 2.1.

Proof. Assume that

$$u_c(t, x, y) = \frac{c}{\cosh\left(\frac{1}{2}\right)} \cosh\left[\frac{1}{2} - (x + \beta_2 y - ct) + [x + \beta_2 y - ct]\right]. \quad (2.26)$$

Then, we have

$$\partial_x u_c(t, x, y) = -\frac{c \sinh \chi}{\cosh\left(\frac{1}{2}\right)}, \quad \partial_t u_c(t, x, y) = \frac{c^2 \sinh \chi}{\cosh\left(\frac{1}{2}\right)}, \quad (2.27)$$

where $\chi = \frac{1}{2} - (x + \beta_2 y - ct) + [x + \beta_2 y - ct]$. Using the fact $\partial_x G(x) = -\frac{\sinh\left(\frac{1}{2} - x + [x]\right)}{2 \sinh\left(\frac{1}{2}\right)}$, $x \in \mathbb{S}$, for any test

function $\phi(t, x, y) \in C_c^\infty([0, T) \times \mathbb{S}^2)$, we have

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{S}^2} \left[-\partial_t \partial_x \phi u_c + \partial_x \phi \left(u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 + \kappa u_c \right) \right) - \partial_y^2 \phi G^* u_c \right] dx dy dt + \int_{\mathbb{R}^2} u_{c,0}(x, y) \partial_x \phi(0, x, y) dx dy \\ &= \int_0^\infty \int_{\mathbb{S}^2} \partial_x \phi \left[\partial_t u_c + u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 - (\kappa + \beta_1^2) u_c \right) \right] dx dy dt \\ &= \int_0^\infty \int_{\mathbb{S}^2} \partial_x \phi \left[\frac{c^2}{\cosh\left(\frac{1}{2}\right)} \sinh \chi - \frac{c^2}{\cosh^2\left(\frac{1}{2}\right)} \cosh \chi \sinh \chi \right] dx dy dt \\ &\quad + \int_0^\infty \int_{\mathbb{S}^2} \partial_x \phi \left[\partial_x G^* \left(\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2 \chi - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh \chi \right) \right] dx dy dt. \end{aligned} \quad (2.28)$$

Here we use the relation $G^* \partial_y u_c = \beta_2 \partial_x G^* u_c$. We compute

$$\begin{aligned} & \partial_x G(x)^* \left[\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2 \chi - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh \chi \right] (t, x, y) \\ &= -\frac{1}{2 \sinh\left(\frac{1}{2}\right)} \int_{\mathbb{S}} \sinh\left(\frac{1}{2} - (x + \beta_2 y - q) + [x + \beta_2 y - q]\right) \left[\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2\left(\frac{1}{2} - (q - ct) + [q - ct]\right) \right. \\ &\quad \left. - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} - (q - ct) - [q + ct]\right) \right] dq. \end{aligned} \quad (2.29)$$

For $x + \beta_2 y > ct$, we divide (2.29) as follows:

$$\begin{aligned} & \partial_x G(x)^* \left[\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2 \chi - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh \chi \right] (t, x, y) \\ &= -\frac{1}{2 \sinh\left(\frac{1}{2}\right)} \left(\int_0^{ct} + \int_{ct}^{x+\beta_2 y} + \int_{x+\beta_2 y}^1 \right) \sinh\left(\frac{1}{2} - (x + \beta_2 y - q) + [x + \beta_2 y - q]\right) \end{aligned}$$

$$\times \left[\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2\left(\frac{1}{2} - (q - ct) + [q - ct]\right) - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} - (q - ct) - [q + ct]\right) \right] dq$$

$$=: K_1 + K_2 + K_3.$$

Using the identities $\cosh^2 X = \frac{1}{2} \cosh 2X + \frac{1}{2}$ and $\sinh X \cosh Y = \frac{1}{2} \sinh(X + Y) + \frac{1}{2} \sinh(X - Y)$, a direct calculation yields

$$\begin{aligned} K_1 &= -\frac{1}{2 \sinh\left(\frac{1}{2}\right)} \int_0^{ct} \sinh\left(\frac{1}{2} - (x + \beta_2 y - q)\right) \left[\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2\left(\frac{1}{2} + (q - ct)\right) \right. \\ &\quad \left. - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} + (q - ct)\right) \right] dq \\ &= -\frac{3c^2}{4 \sinh\left(\frac{1}{2}\right) \cosh^2\left(\frac{1}{2}\right)} \int_0^{ct} \left[\frac{1}{4} \sinh\left(\frac{3}{2} - (x + \beta_2 y + 2ct) + 3q\right) \right. \\ &\quad \left. + \frac{1}{4} \sinh\left(-\frac{1}{2} - (x + \beta_2 y - 2ct) - q\right) + \frac{1}{2} \sinh\left(\frac{1}{2} - (x + \beta_2 y) + q\right) \right] dq \\ &\quad + \frac{(\kappa + \beta_2^2)c}{2 \sinh\left(\frac{1}{2}\right) \cosh\left(\frac{1}{2}\right)} \int_0^{ct} \left[\frac{1}{2} \sinh(1 - (x + \beta_2 y + ct) + 2q) + \frac{1}{2} \sinh(-(x + \beta_2 y - ct)) \right] dq \\ &= -\frac{3c^2}{4 \sinh\left(\frac{1}{2}\right) \cosh^2\left(\frac{1}{2}\right)} \left[\frac{1}{12} \cosh\left(\frac{3}{2} - (x + \beta_2 y - ct)\right) - \frac{1}{12} \cosh\left(\frac{3}{2} - (x + \beta_2 y + 2ct)\right) \right. \\ &\quad \left. - \frac{1}{4} \cosh\left(-\frac{1}{2} - (x + \beta_2 y - ct)\right) + \frac{1}{4} \cosh\left(-\frac{1}{2} - (x + \beta_2 y - 2ct)\right) + \frac{1}{2} \cosh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \right. \\ &\quad \left. - \frac{1}{2} \cosh\left(\frac{1}{2} - (x + \beta_2 y)\right) \right] + \frac{(\kappa + \beta_2^2)c}{2 \cosh\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2}\right)} \left[\frac{1}{4} \cosh(1 - (x - ct + \beta_2 y)) \right. \\ &\quad \left. - \frac{1}{4} \cosh(1 - (x + ct + \beta_2 y)) + \frac{ct}{2} \sinh(-(x - ct + \beta_2 y)) \right], \end{aligned} \tag{2.30}$$

$$\begin{aligned} K_2 &= -\frac{1}{2 \sinh\left(\frac{1}{2}\right)} \int_{ct}^{x+\beta_2 y} \sinh\left(\frac{1}{2} - (x + \beta_2 y - q)\right) \\ &\quad \times \left[\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2\left(\frac{1}{2} - (q - ct)\right) - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} - (q - ct)\right) \right] dq \\ &= -\frac{3c^2}{4 \sinh\left(\frac{1}{2}\right) \cosh^2\left(\frac{1}{2}\right)} \int_{ct}^{x+\beta_2 y} \left[\frac{1}{4} \sinh\left(\frac{3}{2} - (x + \beta_2 y - 2ct) - q\right) \right. \\ &\quad \left. + \frac{1}{4} \sinh\left(-\frac{1}{2} - (x + \beta_2 y + 2ct) + 3q\right) + \frac{1}{2} \sinh\left(\frac{1}{2} - (x + \beta_2 y) + q\right) \right] dq \end{aligned} \tag{2.31}$$

$$\begin{aligned}
& + \frac{(\kappa + \beta_2^2)c}{2 \cosh\left(\frac{1}{2}\right) \sinh\left(\frac{1}{2}\right)} \int_{ct}^{x+\beta_2 y} \left(\frac{1}{2} \sinh(1 - (x - ct + \beta_2 y)) + \frac{1}{2} \sinh(-(x + ct + \beta_2 y) + 2q) \right) dq \\
& = - \frac{3c^2}{4 \sinh\left(\frac{1}{2}\right) \cosh^2\left(\frac{1}{2}\right)} \left[-\frac{1}{4} \cosh\left(\frac{3}{2} - 2(x + \beta_2 y - ct)\right) + \frac{1}{4} \cosh\left(\frac{3}{2} - (x + \beta_2 y - ct)\right) \right. \\
& \quad + \frac{1}{12} \cosh\left(-\frac{1}{2} + 2(x + \beta_2 y - ct)\right) - \frac{1}{12} \cosh\left(-\frac{1}{2} - (x - ct + \beta_2 y)\right) + \frac{1}{2} \cosh\left(\frac{1}{2}\right) \\
& \quad \left. - \frac{1}{2} \cosh\left(\frac{1}{2} - (x - ct + \beta_2 y)\right) \right] + \frac{(\kappa + \beta_2^2)c}{2 \sinh\left(\frac{1}{2}\right) \cosh\left(\frac{1}{2}\right)} \left[\frac{1}{2} (x + \beta_2 y - ct) \sinh(1 - (x - ct + \beta_2 y)) \right]
\end{aligned}$$

and

$$\begin{aligned}
K_3 & = - \frac{1}{2 \sinh\left(\frac{1}{2}\right)} \int_{x+\beta_2 y}^1 \sinh\left(-\frac{1}{2} - (x + \beta_2 y - q)\right) \\
& \quad \times \left(\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2\left(\frac{1}{2} - (q - ct)\right) - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} - (q - ct)\right) \right) dq \\
& = - \frac{3c^2}{4 \sinh\left(\frac{1}{2}\right) \cosh^2\left(\frac{1}{2}\right)} \int_{x+\beta_2 y}^1 \left(\frac{1}{4} \sinh\left(\frac{1}{2} - (x + \beta_2 y - 2ct) - q\right) \right. \\
& \quad \left. + \frac{1}{4} \sinh\left(-\frac{3}{2} - (x + \beta_2 y + 2ct) + 3z\right) + \frac{1}{2} \sinh\left(-\frac{1}{2} - (x + \beta_2 y) + q\right) \right) dq \\
& \quad + \frac{(\kappa + \beta_2^2)c}{2 \sinh\left(\frac{1}{2}\right) \cosh\left(\frac{1}{2}\right)} \int_{x+\beta_2 y}^1 \left(\frac{1}{2} \sinh(-(x - ct + \beta_2 y)) + \frac{1}{2} \sinh(-1 - (x + ct + \beta_2 y) + 2q) \right) dq \\
& = - \frac{3c^2}{4 \sinh\left(\frac{1}{2}\right) \cosh^2\left(\frac{1}{2}\right)} \left[-\frac{1}{4} \cosh\left(-\frac{1}{2} - (x + \beta_2 y - 2ct)\right) + \frac{1}{4} \cosh\left(\frac{1}{2} - 2(x - ct + \beta_2 y)\right) \right. \\
& \quad + \frac{1}{12} \cosh\left(\frac{3}{2} - (x + 2ct + \beta_2 y)\right) - \frac{1}{12} \cosh\left(-\frac{3}{2} + 2(x + \beta_2 y - ct)\right) + \frac{1}{2} \cosh\left(\frac{1}{2} - (x + \beta_2 y)\right) \\
& \quad \left. - \frac{1}{2} \cosh\left(-\frac{1}{2}\right) \right] + \frac{(\kappa + \beta_2^2)c}{2 \sinh\left(\frac{1}{2}\right) \cosh\left(\frac{1}{2}\right)} \left[\frac{1}{4} \cosh(1 - (x + ct + \beta_2 y)) - \frac{1}{4} \cosh((x - ct + \beta_2 y) - 1) \right. \\
& \quad \left. + \frac{1}{2} (1 - (x + \beta_2 y)) \sinh(-(x - ct + \beta_2 y)) \right]. \tag{2.32}
\end{aligned}$$

Plugging the above equalities $K_1 - K_3$ into (2.29) gives us for $x + \beta_2 y > ct$,

$$\begin{aligned}
& \partial_x G(x)^* \left(\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2 \chi - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh \chi \right) (t, x, y) \\
& = - \frac{3c^2}{4 \sinh\left(\frac{1}{2}\right) \cosh^2\left(\frac{1}{2}\right)} \left[\frac{2}{3} \sinh(1) \sinh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) - \frac{2}{3} \sinh\left(\frac{1}{2}\right) \sinh(1 - 2(x + \beta_2 y - ct)) \right]
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
& + \frac{(\kappa + \beta_2^2)c}{2 \sinh\left(\frac{1}{2}\right) \cosh\left(\frac{1}{2}\right)} \left[\frac{1}{2} \sinh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \cosh\left(\frac{1}{2}\right) - \cosh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \sinh\left(\frac{1}{2}\right) \right. \\
& \quad \left. \times \left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \right] \\
& = - \frac{c^2}{\cosh\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) + \frac{c^2}{\cosh^2\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \cosh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \\
& \quad + \frac{(\kappa + \beta_2^2)c}{4 \sinh\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) - \frac{(\kappa + \beta_2^2)c}{2 \cosh\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \left(\frac{1}{2} - (x + \beta_2 y - ct)\right).
\end{aligned}$$

While for $x + \beta_2 y \leq ct$, we also divide (2.29) into three parts,

$$\begin{aligned}
& \partial_x G(x)^* \left(\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2 \chi - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh \chi \right) (t, x, y) \\
& = - \frac{1}{2 \sinh\left(\frac{1}{2}\right)} \left(\int_0^{x+\beta_2 y} + \int_{x+\beta_2 y}^{ct} + \int_{ct}^1 \right) \sinh\left(\frac{1}{2} - (x + \beta_2 y - q) + [x + \beta_2 y - q]\right) \\
& \quad \times \left(\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2\left(\frac{1}{2} - (q - ct) + [q - ct]\right) - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} - (q - ct) - [q + ct]\right) \right) dq \\
& =: \widetilde{K}_1 + \widetilde{K}_2 + \widetilde{K}_3.
\end{aligned} \tag{2.34}$$

Applying a similar computation as $K_1 - K_3$ to the terms $\widetilde{K}_1 - \widetilde{K}_3$ on the right-hand side of (2.34), we find that for $x + \beta_2 y \leq ct$,

$$\begin{aligned}
& \partial_x G(x)^* \left(\frac{3c^2}{2 \cosh^2\left(\frac{1}{2}\right)} \cosh^2 \chi - \frac{(\kappa + \beta_2^2)c}{\cosh\left(\frac{1}{2}\right)} \cosh \chi \right) (t, x, y) \\
& = \frac{c^2}{\cosh\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} + (x + \beta_2 y - ct)\right) - \frac{c^2}{\cosh^2\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} + (x + \beta_2 y - ct)\right) \cosh\left(\frac{1}{2} + (x + \beta_2 y - ct)\right) \\
& \quad - \frac{(\kappa + \beta_2^2)c}{4 \sinh\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} + (x + \beta_2 y - ct)\right) + \frac{(\kappa + \beta_2^2)c}{2 \cosh\left(\frac{1}{2}\right)} \cosh\left((x + \beta_2 y - ct) + \frac{1}{2}\right) \left((x + \beta_2 y - ct) + \frac{1}{2}\right).
\end{aligned} \tag{2.35}$$

For the terms $\partial_t u_c + u_c \partial_x u_c$, we have

$$\begin{aligned}
& \partial_t u_c + u_c \partial_x u_c \\
& = \frac{c^2}{\cosh\left(\frac{1}{2}\right)} \sinh \chi - \frac{c^2}{\cosh^2\left(\frac{1}{2}\right)} \cosh \chi \sinh \chi
\end{aligned} \tag{2.36}$$

$$= \begin{cases} \frac{c^2}{\cosh\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \\ - \frac{c^2}{\cosh^2\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right) \sinh\left(\frac{1}{2} - (x + \beta_2 y - ct)\right), & x + \beta_2 y > ct, \\ - \frac{c^2}{\cosh\left(\frac{1}{2}\right)} \sinh\left(\frac{1}{2} + (x + \beta_2 y - ct)\right) \\ + \frac{c^2}{\cosh^2\left(\frac{1}{2}\right)} \cosh\left(\frac{1}{2} + (x + \beta_2 y - ct)\right) \sinh\left(\frac{1}{2} + (x + \beta_2 y - ct)\right), & x + \beta_2 y \leq ct. \end{cases}$$

Using the property of linear independence of functions $\cosh\chi$ and $\sinh\chi$ and (2.33), (2.35), (2.36), it follows from (2.27) that for every test function $\phi(t, x, y) \in C_c^\infty([0, T] \times \mathbb{S}^2)$,

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{S}^2} \left[-\partial_t \partial_x \phi u_c + \partial_x \phi \left(u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 + \kappa u_c \right) \right) - \partial_y^2 \phi G^* u_c \right] dx dy dt + \int_{\mathbb{R}^2} u_{c,0}(x, y) \partial_x \phi(0, x, y) dx dy \\ &= \int_0^\infty \int_{\mathbb{S}^2} \partial_x \phi \left[\frac{(\kappa + \beta_2^2)c}{4 \sinh\left(\frac{1}{2}\right)} \sinh\chi - \frac{(\kappa + \beta_2^2)c}{2 \cosh\left(\frac{1}{2}\right)} \chi \cosh\chi \right] dx dy dt \\ &= 0 \end{aligned} \quad (2.37)$$

if and only if $\kappa + \beta_2^2 = 0$. This completes the proof of Theorem 2.2. $\square$

Remark 2.1. For the initial value problem equation (1.1) with the initial data $u_0(x, y) = F_0(x + \gamma y)$, function $u(x, y, t) = F(x + \gamma y, t)$ is a solution to equation (1.1) by the uniqueness. Here the function $F(t, \eta)$ is a solution of the following initial value problem for the DP equation

$$F_t - F_{t\eta\eta} + (\kappa + \gamma^2)F_\eta + 4FF_\eta - (3F_\eta F_{\eta\eta} + FF_{\eta\eta\eta}) = 0 \quad (2.38)$$

with initial data $F(0, \eta) = F_0(\eta)$.

In view of the proof of existence result of smooth solitons of the DP equation in [22], we may also have the following result and detailed proof is omitted.

Theorem 2.3. *If $\kappa + \gamma^2 > 0$, then equation (2.38) possesses a localized smooth solitary wave solution.*

3 Periodic shock waves

It is our purpose in this section to show that the DP-KP equation (1.1) admits the periodic shock waves of the form

$$u(t, x, y) = \begin{cases} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{\sinh\left(x + \gamma y - [x + \gamma y] - \frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}, & (x, y) \in \mathbb{R}^2 \setminus \mathbb{Z}^2, \quad c > 0, \\ 0, & (x, y) \in \mathbb{Z}^2. \end{cases} \quad (3.1)$$

Such a type of solution is a weak solution in the sense of Definition 2.1. The following result on the existence of the periodic shock waves can be obtained by verifying the definition of weak solution.

Theorem 3.1. *The DP-KP equation (1.1) admits a global weak solution in the periodic shock wave form of (3.1) for some constant $\beta_3 \in \mathbb{R}$ if and only if it satisfies that $\kappa + \beta_3^2 = 0$.*

Proof. Assume that

$$u_c(t, x, y) = \begin{cases} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{\sinh\left(x + \beta_3 y - [x + \beta_3 y] - \frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}, & (x, y) \in \mathbb{R}^2 \setminus \mathbb{Z}^2, \quad c > 0, \\ 0, & (x, y) \in \mathbb{Z}^2. \end{cases} \quad (3.2)$$

Then, we have

$$\partial_x u_c(t, x, y) = \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{\cosh \zeta}{\sinh\left(\frac{1}{2}\right)} \quad (3.3)$$

and

$$\partial_t u_c(t, x, y) = - \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{\cosh\left(\frac{1}{2}\right) \sinh \zeta}{\sinh^2\left(\frac{1}{2}\right)}, \quad (3.4)$$

where $\zeta = (x + \beta_3 y) - [x + \beta_3 y] - \frac{1}{2}$. Using (3.2), (3.3), (3.4), and integration by parts, we deduce from the Definition 2.1 that for every test function $\phi(t, x, y) \in C_c^\infty([0, T] \times \mathbb{S}^2)$,

$$\begin{aligned} & \iint_{0 \leq t \leq T} \left[-\partial_t \partial_x \phi u_c + \partial_x \phi \left(u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 + \kappa u_c \right) \right) - \partial_y^2 \phi G^* u_c \right] dx dy dt + \int_{\mathbb{R}^2} u_{c,0}(x, y) \partial_x \phi(0, x, y) dx dy \\ &= \iint_{0 \leq t \leq T} \partial_x \phi \left[\partial_t u_c + u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 - (\kappa + \beta_3^2) u_c \right) \right] dx dy dt \\ &= \iint_{0 \leq t \leq T} \partial_x \phi \left[- \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{\cosh\left(\frac{1}{2}\right) \sinh \zeta}{\sinh^2\left(\frac{1}{2}\right)} + \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{\cosh \zeta \sinh \zeta}{\sinh^2\left(\frac{1}{2}\right)} \right] dx dy dt \\ &\quad + \iint_{0 \leq t \leq T} \partial_x \phi \left[\partial_x G^* \left[\left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{3 \sinh^2 \zeta}{2 \sinh^2\left(\frac{1}{2}\right)} - \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{(\kappa + \beta_3^2) \sinh \zeta}{\sinh\left(\frac{1}{2}\right)} \right] \right] dx dy dt, \end{aligned} \quad (3.5)$$

where we use the relation $G^* \partial_y u_c = \beta_3 \partial_x G^* u_c$. Note that

$$\partial_x G(x) = - \frac{\sinh\left(\frac{1}{2} - x + [x]\right)}{2 \sinh\left(\frac{1}{2}\right)}, \quad x \in \mathbb{R} \setminus \mathbb{Z}.$$

It then follows that

$$\begin{aligned}
 & \partial_x G^* \left[\left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{3 \sinh^2 \zeta}{2 \sinh^2\left(\frac{1}{2}\right)} - \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{(\kappa + \beta_3^2) \sinh \zeta}{\sinh\left(\frac{1}{2}\right)} \right] (t, x, y) \\
 &= - \frac{1}{2 \sinh\left(\frac{1}{2}\right)} \int_{\mathbb{S}} \sinh\left(\frac{1}{2} - (x + \beta_3 y - z) + [x + \beta_3 y - z]\right) \\
 &\quad \times \left[\left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{3 \sinh^2\left(z - [z] - \frac{1}{2}\right)}{2 \sinh^2\left(\frac{1}{2}\right)} - \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{(\kappa + \beta_3^2) \sinh\left(z - [z] - \frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} \right] dz \\
 &= \int_0^{x+\beta_3 y} \sinh\left(\frac{1}{2} - (x + \beta_3 y - z)\right) \\
 &\quad \times \left[- \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{3 \sinh^2\left(z - \frac{1}{2}\right)}{4 \sinh^3\left(\frac{1}{2}\right)} + \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{(\kappa + \beta_3^2) \sinh\left(z - \frac{1}{2}\right)}{2 \sinh^2\left(\frac{1}{2}\right)} \right] dz \\
 &\quad + \int_{x+\beta_3 y}^1 \sinh\left(-\frac{1}{2} - (x + \beta_3 y - z)\right) \\
 &\quad \times \left[- \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \frac{3 \sinh^2\left(z - \frac{1}{2}\right)}{4 \sinh^3\left(\frac{1}{2}\right)} + \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{(\kappa + \beta_3^2) \sinh\left(z - \frac{1}{2}\right)}{2 \sinh^2\left(\frac{1}{2}\right)} \right] dz.
 \end{aligned} \tag{3.6}$$

A simple computation shows that

$$\begin{aligned}
 & - \frac{3}{4 \sinh^3\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \int_0^{x+\beta_3 y} \sinh\left(\frac{1}{2} - (x + \beta_3 y - z)\right) \sinh^2\left(z - \frac{1}{2}\right) dz \\
 & + \frac{\kappa + \beta_3^2}{2 \sinh^2\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \int_0^{x+\beta_3 y} \sinh\left(\frac{1}{2} - (x + \beta_3 y - z)\right) \sinh\left(z - \frac{1}{2}\right) dz \\
 &= - \frac{3}{4 \sinh^3\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \left[\cosh\left(\frac{1}{2}\right) \sinh^2\left(x + \beta_3 y - \frac{1}{2}\right) \right. \\
 &\quad - \cosh\left(\frac{1}{2} - (x + \beta_3 y)\right) \sinh^2\left(-\frac{1}{2}\right) + \frac{1}{3} \sinh\left(\frac{1}{2}\right) \sinh(2(x + \beta_3 y) - 1) \\
 &\quad \left. - \frac{1}{3} \sinh\left(\frac{1}{2} - (x + \beta_3 y)\right) \sinh(-1) - \frac{2}{3} \cosh\left(\frac{1}{2}\right) \cosh(2(x + \beta_3 y) - 1) + \frac{2}{3} \cosh\left(\frac{1}{2} - (x + \beta_3 y)\right) \cosh(-1) \right] \\
 &\quad + \frac{\kappa + \beta_3^2}{2 \sinh^2\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \left[\frac{1}{2} \sinh(x + \beta_3 y) - \frac{1}{2} \cosh(1 - (x + \beta_3 y))(x + \beta_3 y) \right]
 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
 & -\frac{3}{4 \sinh^3\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \int_{x+\beta_3 y}^1 \sinh\left(\frac{1}{2} - (x + \beta_3 y - z)\right) \sinh^2\left(z - \frac{1}{2}\right) dz \\
 & + \frac{\kappa + \beta_3^2}{2 \sinh^2\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \int_{x+\beta_3 y}^1 \sinh\left(\frac{1}{2} - (x + \beta_3 y - z)\right) \sinh\left(z - \frac{1}{2}\right) dz \\
 & = -\frac{3}{4 \sinh^3\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \left[\cosh\left(\frac{1}{2} - (x + \beta_3 y)\right) \sinh^2\left(\frac{1}{2}\right) \right. \\
 & \quad - \cosh\left(-\frac{1}{2}\right) \sinh^2\left(x + \beta_3 y - \frac{1}{2}\right) + \frac{1}{3} \sinh\left(\frac{1}{2} - (x + \beta_3 y)\right) \sinh(1) \\
 & \quad - \frac{1}{3} \sinh\left(-\frac{1}{2}\right) \sinh(2(x + \beta_3 y) - 1) - \frac{2}{3} \cosh\left(\frac{1}{2} - (x + \beta_3 y)\right) \cosh(1) \\
 & \quad \left. + \frac{2}{3} \cosh\left(-\frac{1}{2}\right) \cosh(2(x + \beta_3 y) - 1) \right] \\
 & + \frac{\kappa + \beta_3^2}{2 \sinh^2\left(\frac{1}{2}\right)} \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \left[\frac{1}{2} \sinh(1 - (x + \beta_3 y)) + \frac{1}{2} \cosh(-(x + \beta_3 y))(x + \beta_3 y - 1) \right].
 \end{aligned} \tag{3.8}$$

By (3.6) and (3.7), we have

$$\begin{aligned}
 & \partial_x G(x)^* \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \left[\frac{3 \sinh^2 \zeta}{2 \sinh^2\left(\frac{1}{2}\right)} - \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \frac{(\kappa + \beta_3^2) \sinh \zeta}{\sinh\left(\frac{1}{2}\right)} \right] (t, x, y) \\
 & = \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-2} \left[-\frac{\sinh\left(x + \beta_3 y - \frac{1}{2}\right) \cosh\left(x + \beta_3 y - \frac{1}{2}\right)}{\sinh^2\left(\frac{1}{2}\right)} + \frac{\cosh\left(\frac{1}{2}\right) \sinh\left(x + \beta_3 y - \frac{1}{2}\right)}{\sinh^2\left(\frac{1}{2}\right)} \right] \\
 & + \left(\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right)^{-1} \left[\frac{(\kappa + \beta_3^2) \left(2 \sinh\left(\frac{1}{2}\right) - \cosh\left(\frac{1}{2}\right) \right) \cosh\left(x + \beta_3 y - \frac{1}{2}\right)}{4 \sinh^2\left(\frac{1}{2}\right)} \right. \\
 & \quad \left. + \frac{(\kappa + \beta_3^2) \left(x + \beta_3 y - \frac{1}{2} \right) \sinh\left(x + \beta_3 y - \frac{1}{2}\right)}{2 \sinh\left(\frac{1}{2}\right)} \right].
 \end{aligned} \tag{3.9}$$

It is found from (3.4), (3.5), and (3.8) that for every test function $\phi(t, x, y) \in C_c^\infty([0, T] \times \mathbb{S}^2)$,

$$\begin{aligned}
& \iint_{\mathbb{S}^2} \left[-\partial_t \partial_x \phi u_c + \partial_x \phi \left(u_c \partial_x u_c + \partial_x G^* \left(\frac{3}{2} u_c^2 + \kappa u_c \right) \right) - \partial_y^2 \phi G^* u_c \right] dx dy dt + \int_{\mathbb{R}^2} u_{c,0}(x, y) \partial_x \phi(0, x, y) dx dy \\
&= \iint_{\mathbb{S}^2} \partial_x \phi \left[\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} t + c \right]^{-1} \left[\frac{(\kappa + \beta_3^2) \left(2 \sinh\left(\frac{1}{2}\right) - \cosh\left(\frac{1}{2}\right) \right) \cosh \zeta}{4 \sinh^2\left(\frac{1}{2}\right)} + \frac{(\kappa + \beta_3^2) \zeta \sinh \zeta}{2 \sinh\left(\frac{1}{2}\right)} \right] dx dy dt \\
&= 0
\end{aligned}$$

if and only if $\kappa + \beta_3^2 = 0$. This completes the proof of Theorem 3.1. $\square$

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