

Research Article

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Time decay estimates of solutions to a two-phase flow model in the whole space

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Abstract: In this article, we aim to establish the optimal time decay rates of strong solutions to a two-phase flow model derived from a type of coupled fluid-kinetic equation. It is proved that the strong solutions converge to the given constant states with algebraic time decay rates under some additional assumptions on the initial data.

Keywords: two-phase flow model, Navier-Stokes equations, negative Sobolev space, decay rate

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1 Introduction

The two-phase flow model [2,24,40] has been drawing more attention due to their wide applications in the study of rain formation, chemical engineering, diesel engines, waste water treatment, etc. A type of coupled fluid-kinetic equation to the two-phase flow model, the so-called Navier-Stokes-Vlasov-Fokker-Planck (NS-VFP) system, is given as

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x P(\rho) = \operatorname{div}_x(2\tilde{\mu} \mathbb{D}(u) + \tilde{\nu} \operatorname{div} u \mathbb{I}_3) + \int_{\mathbb{R}^3} (v - u) f dv, \\ \partial_t f + v \cdot \nabla_x f = \operatorname{div}_v((v - u) f + \nabla_v f), \end{cases} \quad (1.1)$$

where $\rho = \rho(x, t)$ and $u = u(x, t)$ denote the density and velocity of the fluid, respectively; $f = f(x, v, t)$ represents the distribution function of the particles for $(x, v, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$. The pressure $P(\rho)$ is taken the form

$$P(\rho) = \rho^\gamma, \quad \text{with } \gamma > 1, \quad (1.2)$$

and the stress tensor $\mathbb{D}(u)$ is defined by

$$\mathbb{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top). \quad (1.3)$$

The constant viscosity coefficients $\tilde{\mu}$ and $\tilde{\nu}$ satisfy

$$\tilde{\mu} > 0 \quad \text{and} \quad \frac{2}{3}\tilde{\mu} + \tilde{\nu} \geq 0. \quad (1.4)$$

$\mathbb{I}_3$ is the 3×3 identity matrix.

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In this article, we consider the viscous two-phase flow model, which can be formally derived from system (1.1) by applying the Chapman-Enskog expansion near the local Maxwellian [26] and ignoring the microscopic effect. The model can be written as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}(2\tilde{\mu}D(u) + \tilde{\nu}\operatorname{div}u\mathbb{I}_3) - n(u - w), \\ \partial_t n + \operatorname{div}(nw) = 0, \\ \partial_t(nw) + \operatorname{div}(nw \otimes w) + \nabla n = \operatorname{div}(nD(w)) + n(u - w), \end{cases} \quad (1.5)$$

where $n = n(x, t)$ and $w = w(x, t)$ denote the density and velocity for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$, respectively. The pressure $P(\rho)$ and the stress tensor $D(u)$ are given in (1.2) and (1.3), respectively.

We impose (1.5) with the following initial data:

$$(\rho, u, n, w)(x, 0) = (\rho_0, u_0, n_0, w_0)(x), \quad \inf_{x \in \mathbb{R}^3} \rho_0 > 0, \quad \text{and} \quad \inf_{x \in \mathbb{R}^3} n_0 > 0, \quad (1.6)$$

and the far field state

$$\lim_{|x| \rightarrow +\infty} (\rho_0, u_0, n_0, w_0)(x) = (\rho_*, 0, n_*, 0), \quad (1.7)$$

where $\rho_* > 0$ and $n_* > 0$ are the positive constants.

There is much important progress made recently on the global existence and large time behavior of solutions to the compressible flows (see [1,3–10,12–23,25,27–34,36,38,39,41,43,44] and references therein). For the one-phase flow, Matsumura and Nishida [33,34] considered the Cauchy problem for the non-isentropic compressible Navier-Stokes equations with small initial perturbations and obtained the global existence and L^2 time decay rates of classical solutions. Ponce [36] extended Matsumura-Nishida's results to L^p ($p \geq 2$) time decay rates. Recently, Chen et al. [4] established the energy balance criterion of weak solutions to the isentropic compressible Navier-Stokes equations in a bounded domain without any additional regularity assumption of the velocity near the boundary. Ma et al. [31] established the global existence and stability of the smooth solution near the steady supersonic flow to the compressible non-isentropic Euler system with a source term in a bounded domain. It is also important to study the pointwise behavior of the solution so as to understand the spread of diffusion waves for the compressible fluid. Considering the one-dimensional case, Zeng [43] investigated Green's function of the isentropic compressible Navier-Stokes equations and has shown that the solutions are approximated by diffusion waves in L^1 space. Moreover, Liu and Zeng [30] obtained the pointwise space-time behavior for the Cauchy problem of general hyperbolic systems. In the case of three-dimensions, Hoff and Zumbrun [12,13] studied Green's function of the isentropic compressible Navier-Stokes equations with artificial viscosities and gained the pointwise estimates of solutions to the Cauchy problem. Furthermore, Liu and Wang [29] investigated Green's function of the isentropic compressible Navier-Stokes equations and made use of complex analysis to derive the pointwise behavior of solutions to the Cauchy problem. They observed the weaker Huygens' principle in odd dimensional space, due to the stronger dispersion effects.

For the two-phase flow, Li and Zhao [27] investigated the existence and nonlinear stability of steady states to the inflow problem of the viscous two-phase flow for supersonic, sonic, and subsonic cases. Later, they continued previous work on the outflow problem of the viscous two-phase flow and established exponential time decay rates to the supersonic case and algebraic time decay rates to both the supersonic and sonic cases in [28]. Choi [5] proved the global existence and uniqueness for the strong solution to Cauchy problem in $\mathbb{R}^3$ and gained the exponential decay rates of the solution converging to the constant state in $\mathbb{T}^3$ for the compressible Euler-Navier-Stokes (E-NS) system. Then, Tang and Zhang [38] and Wu et al. [41] derived the optimal algebraic convergence rates of the solution to Cauchy problem for E-NS in $\mathbb{R}^3$. Jung [14] established the global existence, uniqueness, and large time behavior of strong solutions to the initial boundary value problem for the E-NS system in a bounded domain. In addition, Bresch et al. [3] obtained the global existence of weak solutions to the initial value problem of a generic two-phase flow model in $\mathbb{T}^3$.

We define a new unknown function $m = \log n$, and then, problems (1.5)–(1.7) can be rewritten as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \tilde{L}_1 u - e^m(u - w), \\ \partial_t m + \operatorname{div} w + w \cdot \nabla m = 0, \\ \partial_t w + w \cdot \nabla w + \nabla m = \nabla m \cdot \mathbb{D}(w) + L_2 w + (u - w), \end{cases} \quad (1.8)$$

with the initial data

$$(\rho, u, m, w)(x, 0) = (\rho_0, u_0, m_0, w_0)(x), \quad (1.9)$$

and the far field state

$$\lim_{|x| \rightarrow +\infty} (\rho_0, u_0, m_0, w_0)(x) = (\rho_*, 0, m_*, 0), \quad (1.10)$$

where $m_0 = \log n_0$ and $m_* = \log n_*$. The operators $\tilde{L}_1$ and L_2 are given by

$$\tilde{L}_1 = \tilde{\mu} \Delta + (\tilde{\mu} + \tilde{\nu}) \nabla \operatorname{div} \quad \text{and} \quad L_2 = \frac{1}{2} \Delta + \frac{1}{2} \nabla \operatorname{div}.$$

With the help of clear spectrum analysis to the linear operator and energy estimate to the nonlinear system, we obtained that if $(\rho_0 - \rho_*, u_0 - m_*, w_0)$ is sufficiently small in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, then for $k \leq 3$, the strong solution (ρ, u, m, w) satisfies

$$\|\nabla^k(\rho - \rho_*, u, m - m_*, w)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (1.11)$$

where $C > 0$ is a positive constant independent of time, cf. [42]. It should be mentioned that all the results cited earlier need the smallness of L^1 norm for the initial data and complicated spectrum analysis for linear operator.

Motivated by Guo and Wang's work [11], when the initial data are a small perturbation around the equilibrium state in $H^3(\mathbb{R}^3)$, we make use of energy method to establish the global existence and uniqueness of the strong solution, instead of using the spectrum analysis method. As the initial data further belong to $\dot{H}^{-s}(\mathbb{R}^3)$ with $s \in [0, \frac{3}{2})$, by pure energy method without any linear time decay analysis, we show the optimal decay rate of solutions, which is consistent with the results obtained in [42].

This article is organized as follows. Some notations and significant lemmas are given in Section 2. The main result is presented in Section 3. The local existence and uniqueness of the strong solution to system (1.8)–(1.10) are established in Section 4. Finally, in Section 5, we prove the global existence and obtain the nonlinear decay estimates of the solution.

2 Preliminaries

Throughout this article, we denote by C a generic positive constant, which may vary in different estimates. We employ $A \lesssim B$ to mean that $A \leq CB$ for a universal constant $C > 0$. With an integer $k \geq 0$, ∇^k represents the usual spatial derivatives of order k and H^k denotes the k -th order L^2 Sobolev space on $\mathbb{R}^3$. Particularly, $H^0 = L^2$.

The pseudodifferential operator $\mathfrak{P}^s$ with $s \in \mathbb{R}$ is defined by

$$\mathfrak{P}^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad (2.1)$$

where $\hat{f}$ represents the Fourier transform of f , i.e., $\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx$, for $\xi \in \mathbb{R}^3$. Meanwhile, we define the homogenous Sobolev space $\dot{H}^s$ of f such that

$$\|f\|_{\dot{H}^s} \triangleq \|\mathfrak{P}^s f\|_{L^2} = \| |\xi|^s \hat{f} \|_{L^2} < \infty. \quad (2.2)$$

We introduce some inequalities as follows.

Lemma 2.1. Let $0 \leq \alpha, \beta \leq \gamma$, and it holds

$$\|\nabla^\beta f\|_{L^p} \lesssim \|\nabla^\alpha f\|_{L^q}^{1-\theta} \|\nabla^\gamma f\|_{L^r}^\theta, \quad (2.3)$$

where $0 \leq \theta \leq 1$ and θ satisfies

$$\frac{\beta}{3} - \frac{1}{p} = \left(\frac{\alpha}{3} - \frac{1}{q} \right) (1 - \theta) + \left(\frac{\gamma}{3} - \frac{1}{r} \right) \theta. \quad (2.4)$$

Here, if $p = \infty$, we require that $0 < \theta < 1$.

Proof. The proof can be found in [35]. □

Lemma 2.2. For $s \geq 0$ and integer $k \geq 0$, it holds for $\theta = \frac{1}{1+k+s}$ that

$$\|\nabla^k f\|_{L^2} \lesssim \|\nabla^{k+1} f\|_{L^2}^{1-\theta} \|\mathfrak{P}^{-s} f\|_{L^2}^\theta. \quad (2.5)$$

Proof. According to Parseval's theorem, the definition of $\|\mathfrak{P}^{-s} f\|_{L^2}$, and Hölder's inequality, we obtain

$$\|\nabla^k f\|_{L^2} = \| |\xi|^k \hat{f} \|_{L^2} \lesssim \| |\xi|^{k+1} \hat{f} \|_{L^2}^{1-\theta} \| |\xi|^{-s} \hat{f} \|_{L^2}^\theta = \|\nabla^{k+1} f\|_{L^2}^{1-\theta} \|\mathfrak{P}^{-s} f\|_{L^2}^\theta. \quad (2.6)$$

Thus, we complete the proof of this lemma. □

It should be pointed out that if $s \in (0, 3)$, $\mathfrak{P}^{-s} f$ defined by (2.1) is the Riesz potential. According to the Hardy-Littlewood-Sobolev theorem of fractional integration, we obtain the following L^p -type inequality for the Riesz potential.

Lemma 2.3. Let $0 < s < 3$, $1 < p < q < \infty$, and $\frac{1}{q} + \frac{s}{3} = \frac{1}{p}$, then

$$\|\mathfrak{P}^{-s} f\|_{L^q} \lesssim \|f\|_{L^p}. \quad (2.7)$$

Proof. We refer to [37] for details. □

3 Main results

Theorem 3.1. Assume the initial data (ρ_0, u_0, m_0, w_0) satisfy $(\rho_0 - \rho_*, u_0, m_0 - m_*, w_0) \in H^3(\mathbb{R}^3)$, and there exists a small positive constant $\delta_0 > 0$ such that

$$\|(\rho_0 - \rho_*, u_0, m_0 - m_*, w_0)\|_{H^3(\mathbb{R}^3)} \leq \delta_0. \quad (3.1)$$

Then, the Cauchy problem (1.8)–(1.10) has a unique global strong solution (ρ, u, m, w) satisfying

$$\begin{aligned} \rho - \rho_* &\in C(\mathbb{R}_+, H^3(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^2(\mathbb{R}^3)), \\ u &\in C(\mathbb{R}_+, H^3(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^1(\mathbb{R}^3)), \\ m - m_* &\in C(\mathbb{R}_+, H^3(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^2(\mathbb{R}^3)), \\ w &\in C(\mathbb{R}_+, H^3(\mathbb{R}^3)) \cap C^1(\mathbb{R}_+, H^1(\mathbb{R}^3)), \end{aligned} \quad (3.2)$$

and for any $t \geq 0$ that

$$\|(\rho - \rho_*, u, m - m_*, w)(t)\|_{H^3}^2 + \int_0^t (\|\nabla(\rho, m)(\tau)\|_{H^2}^2 + \|\nabla(u, w)(\tau)\|_{H^3}^2) d\tau \leq \delta_0^2. \quad (3.3)$$

Moreover, if $(\rho_0 - \rho_*, u_0, m_0 - m_*, w_0) \in \dot{H}^{-s}$ for some $s \in [0, \frac{3}{2})$, then for all $t \geq 0$,

$$\|(\rho - \rho_*, u, m - m_*, w)(t)\|_{\dot{H}^{-s}}^2 \leq C. \quad (3.4)$$

Furthermore, for $k = 0, 1, 2$, it holds

$$\|\nabla^k(\rho - \rho_*, u, m - m_*, w)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k+s}{2}}, \quad (3.5)$$

with $C > 0$ a positive constant independent of time.

We note that the Hardy-Littlewood-Sobolev theorem implies that for $p \in (1, 2]$, L^p is a subset of H^{-s} with $s = 3\left(\frac{1}{p} - \frac{1}{2}\right) \in \left[0, \frac{3}{2}\right]$. Then applying Theorem 3.1 yields the following result of the usual L^p – L^2 -type optimal decay estimates of the strong solution to a given constant state.

Remark 3.2. If we replace the $\dot{H}^{-s}$ assumption on the initial data by L^p for some $p \in (1, 2]$ in Theorem 3.1, then for $k = 0, 1, 2$, the following time decay estimates hold:

$$\|\nabla^k(\rho - \rho_*, u, m - m_*, w)(t)\|_{H^{3-k}} \leq C(1+t)^{-\sigma_{p,k}}, \quad (3.6)$$

where $\sigma_{p,k} = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{k}{2}$.

4 Local existence

To obtain the global existence, we first state the local existence of system (1.8)–(1.10) as follows.

Theorem 4.1. *If $(\rho_0 - \rho_*, u_0, m_0 - m_*, w_0) \in H^3(\mathbb{R}^3)$, then the Cauchy problem (1.8)–(1.10) has a unique strong solution (ρ, u, m, w) satisfying*

$$\begin{aligned} \rho - \rho_* &\in C([0, T_*], H^3(\mathbb{R}^3)) \cap C^1([0, T_*], H^2(\mathbb{R}^3)), \\ u &\in C([0, T_*], H^3(\mathbb{R}^3)) \cap C^1([0, T_*], H^1(\mathbb{R}^3)), \\ m - m_* &\in C([0, T_*], H^3(\mathbb{R}^3)) \cap C^1([0, T_*], H^2(\mathbb{R}^3)), \\ w &\in C([0, T_*], H^3(\mathbb{R}^3)) \cap C^1([0, T_*], H^1(\mathbb{R}^3)), \end{aligned} \quad (4.1)$$

where $T_* > 0$ is a short time.

Proof. The construction of the local-in-time solution is based on an iteration scheme. The details can be found in [33]. $\square$

5 Global existence and nonlinear decay estimates

In this article, we study the small perturbations of solution (ρ, u, m, w) to the Cauchy problem (1.8)–(1.10) near the constant state $(\rho_*, 0, m_*, 0)$. Denote

$$a = \frac{\rho - \rho_*}{\rho_*}, \quad q = \frac{u}{c}, \quad b = \frac{k(m - m_*)}{c}, \quad \text{and} \quad \sigma = \frac{kw}{c}, \quad (5.1)$$

with

$$k = \sqrt{\frac{n_*}{\rho_*}} \quad \text{and} \quad c = \sqrt{P'(\rho_*)}.$$

Then, the system (1.8)–(1.10) is reformulated as

$$\begin{cases} \partial_t a + c \operatorname{div} q = f_1, \\ \partial_t q + c \nabla a - k \sigma + k^2 q = L_1 q + f_2, \\ \partial_t b + \operatorname{div} \sigma = f_3, \\ \partial_t \sigma + \nabla b + \sigma - k q = L_2 \sigma + f_4, \end{cases} \quad (5.2)$$

with the following initial data:

$$(a, q, b, \sigma)(x, 0) = (a_0, q_0, b_0, \sigma_0)(x) \rightarrow (0, 0, 0, 0), \quad \text{as } |x| \rightarrow +\infty. \quad (5.3)$$

The operators L_1 and L_2 are defined as

$$L_1 = \mu \Delta + (\mu + \nu) \nabla \operatorname{div} \quad \text{and} \quad L_2 = \frac{1}{2} \Delta + \frac{1}{2} \nabla \operatorname{div}, \quad (5.4)$$

with $\mu = \frac{\bar{\mu}}{\rho_*}$ and $\nu = \frac{\bar{\nu}}{\rho_*}$. The nonlinear term f_i ($i = 1, 2, 3, 4$) is denoted by

$$\begin{aligned} f_1 &= -c \operatorname{div}(aq), \\ f_2 &= -cq \cdot \nabla q - \frac{\nabla P((1+a)\rho_*)}{c\rho_*(1+a)} + c\nabla a - \frac{aL_1 q}{1+a} + \frac{(e^{\frac{\varepsilon}{k}b} - 1 - a)}{1+a}(k\sigma - k^2q), \\ f_3 &= -\frac{c}{k} \sigma \cdot \nabla b, \\ f_4 &= -\frac{c}{k} \sigma \cdot \nabla \sigma + \frac{c}{k} \nabla b \cdot \mathbb{D}(\sigma). \end{aligned} \quad (5.5)$$

5.1 A priori estimates

In order to extend the local solution to be global, we shall establish the uniform estimates. Therefore, it is natural to provide the *a priori* assumption, namely, there exists a suitable small constant $\delta > 0$ such that

$$\|(a, q, b, \sigma)\|_{H^3} \leq \delta. \quad (5.6)$$

We first derive the following estimates for system (5.2) under the *a priori* assumption.

Lemma 5.1. *Suppose that (a, q, b, σ) is the smooth solution of system (5.2) with the initial data (5.3), then for $k = 0, 1, 2$, we have*

$$\frac{d}{dt} \|\nabla^k(a, q, b, \sigma)\|_{H^{3-k}}^2 + \tilde{C}(\|\nabla^{k+1}(a, b)\|_{H^{2-k}}^2 + \|\nabla^{k+1}(q, \sigma)\|_{H^{3-k}}^2) \leq 0, \quad (5.7)$$

where $\tilde{C} > 0$ is a positive constant independent of time.

Proof. For simplicity, we denote the following symbols for $k = 0, 1, 2$:

$$\mathcal{E}_k(t) = \|\nabla^k(a, q, b, \sigma)\|_{H^{3-k}}^2 \quad \text{and} \quad \mathcal{D}_k(t) = \|\nabla^{k+1}(a, b)\|_{H^{2-k}}^2 + \|\nabla^{k+1}(q, \sigma)\|_{H^{3-k}}^2. \quad (5.8)$$

The case of $k = 0$ is taken into consideration at first, which means we need to prove that

$$\frac{d}{dt} \|(a, q, b, \sigma)\|_{H^3}^2 + \tilde{C}(\|\nabla(a, b)\|_{H^2}^2 + \|\nabla(q, \sigma)\|_{H^3}^2) \leq 0. \quad (5.9)$$

Let integer α satisfy $0 \leq \alpha \leq 3$. Applying ∇^α to (5.2) and multiplying by $\nabla^\alpha a$, $\nabla^\alpha q$, $\nabla^\alpha b$, and $\nabla^\alpha \sigma$, respectively, and then integrating the resulting equation with respect to x over $\mathbb{R}^3$ yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla^\alpha(a, q, b, \sigma)\|_{L^2}^2 + \|\nabla^\alpha(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla^{\alpha+1} q\|_{L^2}^2 + \frac{1}{2} \|\nabla^{\alpha+1} \sigma\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} \nabla^\alpha a \cdot \nabla^\alpha f_1 \, dx + \int_{\mathbb{R}^3} \nabla^\alpha q \cdot \nabla^\alpha f_2 \, dx + \int_{\mathbb{R}^3} \nabla^\alpha b \cdot \nabla^\alpha f_3 \, dx + \int_{\mathbb{R}^3} \nabla^\alpha \sigma \cdot \nabla^\alpha f_4 \, dx \\ &\triangleq I_1^\alpha + I_2^\alpha + I_3^\alpha + I_4^\alpha, \end{aligned} \quad (5.10)$$

where we used a simple fact that by constraints (5.4), there exists $C_1 > 0$ a positive constant such that

$$\mu \int_{\mathbb{R}^3} |\nabla \nabla^\alpha q|^2 \, dx + (\mu + \nu) \int_{\mathbb{R}^3} |\operatorname{div} \nabla^\alpha q|^2 \, dx \geq C_1 \|\nabla^{\alpha+1} q\|_{L^2}^2. \quad (5.11)$$

If $\alpha = 0$, with the help of (5.5) and Lemma 2.1, the nonhomogeneous term I_1^0 is estimated as follows:

$$\begin{aligned} |I_1^0| &\leq C\|a\|_{L^6}(\|a\|_{L^3}\|\nabla q\|_{L^2} + \|q\|_{L^3}\|\nabla a\|_{L^2}) \\ &\leq C\|\nabla a\|_{L^2}(\|a\|_{L^2}^{\frac{1}{2}}\|\nabla a\|_{L^2}^{\frac{1}{2}}\|\nabla q\|_{L^2} + \|q\|_{L^2}^{\frac{1}{2}}\|\nabla q\|_{L^2}^{\frac{1}{2}}\|\nabla a\|_{L^2}) \\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \end{aligned} \quad (5.12)$$

The definitions of $\mathcal{E}_0(t)$ and $\mathcal{D}_0(t)$ are given by (5.8). According to Hölder's and Young's inequalities, the second term is estimated as follows:

$$\begin{aligned} |I_2^0| &\leq C\|q\|_{L^6}(\|q\|_{L^3}\|\nabla q\|_{L^2} + \|\nabla a\|_{L^2}\|a\|_{L^3} + \|a\|_{L^3}\|\nabla^2 q\|_{L^2} + (\|a\|_{L^3} + \|b\|_{L^3})\|\sigma - kq\|_{L^2}) \\ &\leq C\|\nabla q\|_{L^2} \left(\|q\|_{L^2}^{\frac{1}{2}}\|\nabla q\|_{L^2}^{\frac{3}{2}} + \|a\|_{L^2}^{\frac{1}{2}}\|\nabla a\|_{L^2}^{\frac{3}{2}} + \|a\|_{L^2}^{\frac{1}{2}}\|\nabla a\|_{L^2}^{\frac{1}{2}}\|\nabla^2 q\|_{L^2} \right. \\ &\quad \left. + (\|a\|_{L^2}^{\frac{1}{2}}\|\nabla a\|_{L^2}^{\frac{1}{2}} + \|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}})\|\sigma - kq\|_{L^2} \right) \\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2 + \frac{1}{2}\|\sigma - kq\|_{L^2}^2. \end{aligned} \quad (5.13)$$

For the third term, we also have

$$|I_3^0| \leq C\|b\|_{L^6}\|\sigma\|_{L^3}\|\nabla b\|_{L^2} \leq C\|\nabla b\|_{L^2}\|\sigma\|_{L^2}^{\frac{1}{2}}\|\nabla\sigma\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2} \leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \quad (5.14)$$

Then, we consider the last term as

$$\begin{aligned} |I_4^0| &\leq C\|\sigma\|_{L^6}(\|\sigma\|_{L^3}\|\nabla\sigma\|_{L^2} + \|\nabla\sigma\|_{L^3}\|\nabla b\|_{L^2}) \\ &\leq C\|\nabla\sigma\|_{L^2}(\|\sigma\|_{L^2}^{\frac{1}{2}}\|\nabla\sigma\|_{L^2}^{\frac{3}{2}} + \|\nabla^2\sigma\|_{L^2}\|\nabla b\|_{L^2}) \\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \end{aligned} \quad (5.15)$$

Combining these estimates of $I_1^0 - I_4^0$ together, we conclude

$$\frac{d}{dt}\|(a, q, b, \sigma)\|_{L^2}^2 + \|\sigma - kq\|_{L^2}^2 + C_1\|\nabla(q, \sigma)\|_{L^2}^2 \leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \quad (5.16)$$

Then, we follow a similar argument for the case $\alpha = 1$. After a direct calculation, we obtain

$$\begin{aligned} |I_1^1| &\leq C\|\nabla a\|_{L^2}(\|\nabla(\operatorname{div} q)\|_{L^2} + \|\nabla(q \cdot \nabla a)\|_{L^2}) \\ &\leq C\|\nabla a\|_{L^2}(\|\nabla q\|_{L^2}\|\nabla a\|_{L^\infty} + \|a\|_{L^\infty}\|\nabla^2 q\|_{L^2} + \|\nabla a\|_{L^2}\|\nabla q\|_{L^\infty} + \|q\|_{L^\infty}\|\nabla^2 a\|_{L^2}) \\ &\leq C\|\nabla a\|_{L^2}(\|\nabla q\|_{L^2}\|\nabla^2 a\|_{H^1} + \|\nabla a\|_{H^1}\|\nabla^2 q\|_{L^2} + \|\nabla a\|_{L^2}\|\nabla^2 q\|_{H^1} + \|\nabla q\|_{H^1}\|\nabla^2 a\|_{L^2}) \\ &\leq C\|\nabla a\|_{L^2}(\|\nabla a\|_{H^2}\|\nabla q\|_{H^1} + \|\nabla q\|_{H^2}\|\nabla a\|_{H^1}) \\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \end{aligned} \quad (5.17)$$

The estimate of the second term is given by

$$\begin{aligned} |I_2^1| &\leq C\|\nabla q\|_{L^2}(\|\nabla(q \cdot \nabla q)\|_{L^2} + \|\nabla(\nabla a \cdot a)\|_{L^2} + \|\nabla(a \nabla^2 q)\|_{L^2} + \|\nabla((a - b)(\sigma - kq))\|_{L^2}) \\ &\leq C\|\nabla q\|_{L^2}(\|\nabla q\|_{H^2}\|\nabla q\|_{H^1} + \|\nabla a\|_{H^2}\|\nabla a\|_{H^1} + \|\nabla a\|_{H^2}\|\nabla^2 q\|_{H^1} + \|\nabla(a - b)\|_{H^1}\|\nabla(\sigma - kq)\|_{H^1}) \\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \end{aligned} \quad (5.18)$$

For the third term, we obtain

$$\begin{aligned} |I_3^1| &\leq C\|\nabla b\|_{L^2}\|\nabla(\sigma \cdot \nabla b)\|_{L^2} \\ &\leq C\|\nabla b\|_{L^2}(\|\nabla^2\sigma\|_{H^1}\|\nabla b\|_{L^2} + \|\nabla\sigma\|_{H^1}\|\nabla^2 b\|_{L^2}) \\ &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \end{aligned} \quad (5.19)$$

The last term is bounded by

$$\begin{aligned}
 |I_4^1| &\leq C\|\nabla\sigma\|_{L^2}(\|\nabla(\sigma\cdot\nabla\sigma)\|_{L^2} + \|\nabla(\nabla b\cdot\mathbb{D}(\sigma))\|_{L^2}) \\
 &\leq C\|\nabla\sigma\|_{L^2}(\|\nabla\sigma\|_{L^2}\|\nabla\sigma\|_{L^\infty} + \|\sigma\|_{L^\infty}\|\nabla^2\sigma\|_{L^2} + \|\nabla^2b\|_{L^2}\|\nabla\sigma\|_{L^\infty} + \|\nabla b\|_{L^\infty}\|\nabla^2\sigma\|_{L^2}) \\
 &\leq C\|\nabla\sigma\|_{L^2}(\|\nabla\sigma\|_{L^2}\|\nabla^2\sigma\|_{H^1} + \|\nabla\sigma\|_{H^1}\|\nabla^2\sigma\|_{L^2} + \|\nabla^2b\|_{L^2}\|\nabla^2\sigma\|_{H^1} + \|\nabla^2b\|_{H^1}\|\nabla^2\sigma\|_{L^2}) \\
 &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2.
 \end{aligned} \tag{5.20}$$

Thus, we combine the aforementioned estimates together to show that

$$\frac{d}{dt}\|\nabla(a, q, b, \sigma)\|_{L^2}^2 + \|\nabla(\sigma - kq)\|_{L^2}^2 + C_1\|\nabla^2(q, \sigma)\|_{L^2}^2 \leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \tag{5.21}$$

Similarly, for the case $\alpha = 2$, we also obtain

$$\begin{aligned}
 |I_1^2| &\leq C\|\nabla^2a\|_{L^2}(\|\nabla^2(a\operatorname{div}q)\|_{L^2} + \|\nabla^2(q\cdot\nabla a)\|_{L^2}) \\
 &\leq C\|\nabla^2a\|_{L^2}(\|\nabla^2a\|_{L^2}\|\operatorname{div}q\|_{L^\infty} + \|\nabla a\|_{L^\infty}\|\nabla\operatorname{div}q\|_{L^2} + \|\nabla^2\operatorname{div}q\|_{L^2}\|a\|_{L^\infty} + \|q\|_{L^\infty}\|\nabla^3a\|_{L^2} \\
 &\quad + \|\nabla^2a\|_{L^2}\|\nabla q\|_{L^\infty} + \|\nabla^2q\|_{L^\infty}\|\nabla a\|_{L^2}) \\
 &\leq C\|\nabla^2a\|_{L^2}(\|\nabla^2a\|_{L^2}\|\nabla^2q\|_{H^1} + \|\nabla^2a\|_{H^1}\|\nabla^2q\|_{L^2} + \|\nabla^3q\|_{L^2}\|\nabla a\|_{H^1} + \|\nabla q\|_{H^1}\|\nabla^3a\|_{L^2} + \|\nabla^2a\|_{L^2}\|\nabla^2q\|_{H^1} \\
 &\quad + \|\nabla^3q\|_{H^1}\|\nabla a\|_{L^2}) \\
 &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2.
 \end{aligned} \tag{5.22}$$

The estimate of the second term is given by

$$\begin{aligned}
 |I_2^2| &\leq C\|\nabla^2q\|_{L^2}(\|\nabla^2(\nabla q\cdot q)\|_{L^2} + \|\nabla^2(\nabla a\cdot a)\|_{L^2} + \|\nabla^2(a\nabla^2q)\|_{L^2} + \|\nabla^2((a-b)(\sigma-kq))\|_{L^2}) \\
 &\leq C\|\nabla^2q\|_{L^2}(\|\nabla q\|_{H^2}\|\nabla^2q\|_{H^1} + \|\nabla a\|_{H^2}\|\nabla^2a\|_{H^1} + \|\nabla a\|_{H^2}\|\nabla^3q\|_{H^1} + \|\nabla(a-b)\|_{H^2}\|\nabla(\sigma-kq)\|_{H^1}) \\
 &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2.
 \end{aligned} \tag{5.23}$$

For the third term, we have

$$\begin{aligned}
 |I_3^2| &\leq C\|\nabla^2b\|_{L^2}\|\nabla^2(\sigma\cdot\nabla b)\|_{L^2} \\
 &\leq C\|\nabla^2b\|_{L^2}(\|\nabla^2\sigma\|_{L^2}\|\nabla b\|_{L^\infty} + \|\nabla\sigma\|_{L^\infty}\|\nabla^2b\|_{L^2} + \|\sigma\|_{L^\infty}\|\nabla^3b\|_{L^2}) \\
 &\leq C\|\nabla^2b\|_{L^2}(\|\nabla^2\sigma\|_{L^2}\|\nabla^2b\|_{H^1} + \|\nabla^2\sigma\|_{H^1}\|\nabla^2b\|_{L^2} + \|\nabla\sigma\|_{H^1}\|\nabla^3b\|_{L^2}) \\
 &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2.
 \end{aligned} \tag{5.24}$$

The last term is estimated as

$$\begin{aligned}
 |I_4^2| &\leq C\|\nabla^2\sigma\|_{L^2}\|\nabla^2(\sigma\cdot\nabla\sigma + \nabla b\cdot\mathbb{D}(\sigma))\|_{L^2} \\
 &\leq C\|\nabla^2\sigma\|_{L^2}(\|\nabla^2\sigma\|_{L^2}\|\nabla\sigma\|_{L^\infty} + \|\sigma\|_{L^\infty}\|\nabla^3\sigma\|_{L^2} + \|\nabla^3\sigma\|_{L^2}\|\nabla b\|_{L^\infty} + \|\nabla^2\sigma\|_{L^\infty}\|\nabla^2b\|_{L^2} + \|\nabla^3b\|_{L^2}\|\nabla\sigma\|_{L^\infty}) \\
 &\leq C\|\nabla^2\sigma\|_{L^2}(\|\nabla\sigma\|_{H^1}\|\nabla^2\sigma\|_{H^1} + \|\nabla^2\sigma\|_{H^2}\|\nabla^2b\|_{H^1}) \\
 &\leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2.
 \end{aligned} \tag{5.25}$$

Collecting the estimates of I_1^2 – I_4^2 together, we deduce

$$\frac{d}{dt}\|\nabla^2(a, q, b, \sigma)\|_{L^2}^2 + \|\nabla^2(\sigma - kq)\|_{L^2}^2 + C_1\|\nabla^3(q, \sigma)\|_{L^2}^2 \leq C\mathcal{E}_0(t)\mathcal{D}_0(t)^2. \tag{5.26}$$

Then, the case $\alpha = 3$ is taken into account as follows:

$$\begin{aligned}
 |I_1^3| &= \left| \int_{\mathbb{R}^3} \nabla^3 a \cdot \nabla^3 (-c \operatorname{div}(aq)) dx \right| \\
 &\leq C \left| \int_{\mathbb{R}^3} \nabla^3 a \cdot \nabla^3 (a \operatorname{div} q + q \cdot \nabla a) dx \right| \\
 &\leq C \|\nabla^3 a\|_{L^2} (\|\nabla^3 a\|_{L^2} \|\nabla q\|_{L^\infty} + \|\nabla^2 q\|_{L^\infty} \|\nabla^2 a\|_{L^2} + \|\nabla^3 q\|_{L^2} \|\nabla a\|_{L^\infty} + \|a\|_{L^\infty} \|\nabla^4 q\|_{L^2} + \|\nabla^3 q\|_{L^2} \|\nabla a\|_{L^\infty} \\
 &\quad + \|\nabla^2 q\|_{L^\infty} \|\nabla^2 a\|_{L^2} + \|\nabla q\|_{L^\infty} \|\nabla^3 a\|_{L^2}) + C \left| \int_{\mathbb{R}^3} \nabla^3 a \cdot (q \cdot \nabla^4 a) dx \right| \\
 &\leq C \|\nabla^3 a\|_{L^2} \|\nabla a\|_{H^2} \|\nabla^2 q\|_{H^2} + C \|\nabla^3 a\|_{L^2} \|\nabla^2 q\|_{H^2} \|\nabla^2 a\|_{H^1} + C \left| \int_{\mathbb{R}^3} \nabla^3 a \cdot (q \cdot \nabla^4 a) dx \right| \\
 &\leq C \|\nabla^3 a\|_{L^2} \|\nabla a\|_{H^2} \|\nabla^2 q\|_{H^2} + C \|\nabla^3 a\|_{L^2} \|\nabla^2 q\|_{H^2} \|\nabla^2 a\|_{H^1} + C \|\operatorname{div} q\|_{L^\infty} \|\nabla^3 a\|_{L^2}^2 \\
 &\leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2,
 \end{aligned} \tag{5.27}$$

where we make use of the following inequality:

$$\left| \int_{\mathbb{R}^3} \nabla^3 a \cdot (q \cdot \nabla^4 a) dx \right| \leq C \|\operatorname{div} q\|_{L^\infty} \|\nabla^3 a\|_{L^2}^2 \leq C \|\nabla^2 q\|_{H^1} \|\nabla^3 a\|_{L^2}^2 \leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2. \tag{5.28}$$

The estimate of the second term is given by

$$\begin{aligned}
 |I_2^3| &\leq C \|\nabla^4 q\|_{L^2} (\|\nabla^2(\nabla q \cdot q)\|_{L^2} + \|\nabla^2(\nabla a \cdot a)\|_{L^2} + \|\nabla^2(a \nabla^2 q)\|_{L^2} + \|\nabla^2((a-b)(\sigma - kq))\|_{L^2}) \\
 &\leq C \|\nabla^4 q\|_{L^2} (\|\nabla q\|_{H^2} \|\nabla^2 q\|_{H^1} + \|\nabla a\|_{H^2} \|\nabla^2 a\|_{H^1} + \|\nabla a\|_{H^2} \|\nabla^3 q\|_{H^1} + \|\nabla(a-b)\|_{H^2} \|\nabla(\sigma - kq)\|_{H^1}) \\
 &\leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2.
 \end{aligned} \tag{5.29}$$

For the third term, we have

$$\begin{aligned}
 |I_3^3| &= \left| \int_{\mathbb{R}^3} \nabla^3 b \cdot \nabla^3 \left(-\frac{c}{k} \sigma \cdot \nabla b \right) dx \right| \\
 &\leq C \int_{\mathbb{R}^3} \nabla^3 b \cdot (\nabla^3 \sigma \cdot \nabla b + \nabla^2 \sigma \cdot \nabla^2 b + \nabla \sigma \cdot \nabla^3 b + \sigma \cdot \nabla^4 b) dx \\
 &\leq C \|\nabla^3 b\|_{L^2} (\|\nabla^3 \sigma\|_{L^2} \|\nabla b\|_{L^\infty} + \|\nabla^2 \sigma\|_{L^3} \|\nabla^2 b\|_{L^6} + \|\nabla \sigma\|_{L^\infty} \|\nabla^3 b\|_{L^2}) + C \mathcal{E}_0(t) \mathcal{D}_0(t)^2 \\
 &\leq C \|\nabla^3 b\|_{L^2} (\|\nabla^2 \sigma\|_{H^1} \|\nabla^2 b\|_{H^1} + \|\nabla^2 \sigma\|_{L^2}^{\frac{1}{2}} \|\nabla^3 \sigma\|_{L^2}^{\frac{1}{2}} \|\nabla^3 b\|_{L^2}) + C \mathcal{E}_0(t) \mathcal{D}_0(t)^2 \\
 &\leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2,
 \end{aligned} \tag{5.30}$$

where we have used the fact that

$$\left| \int_{\mathbb{R}^3} \nabla^3 b \cdot (\sigma \cdot \nabla^4 b) dx \right| \leq C \|\operatorname{div} \sigma\|_{L^\infty} \|\nabla^3 b\|_{L^2}^2 \leq C \|\nabla^2 \sigma\|_{H^1} \|\nabla^3 b\|_{L^2}^2 \leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2. \tag{5.31}$$

The last term is bounded by

$$\begin{aligned}
 |I_4^3| &= \left| \int_{\mathbb{R}^3} \nabla^3 \sigma \cdot \nabla^3 \left(-\frac{c}{k} \sigma \cdot \nabla \sigma + \frac{c}{k} \nabla b \cdot \mathbb{D}(\sigma) \right) dx \right| \\
 &\leq C \|\nabla^3 \sigma\|_{L^2} (\|\nabla^2 \sigma\|_{H^2} \|\nabla^2 \sigma\|_{H^2} + \|\nabla^2 \sigma\|_{L^\infty} \|\nabla^3 b\|_{L^2} + \|\nabla^2 b\|_{L^3} \|\nabla^3 \sigma\|_{L^6} + \|\nabla b\|_{L^\infty} \|\nabla^4 \sigma\|_{L^2}) \\
 &\quad + C \left| \int_{\mathbb{R}^3} \nabla^3 \sigma \cdot (\nabla^4 b \cdot \nabla \sigma) dx \right| \\
 &\leq C \|\nabla^3 \sigma\|_{L^2} (\|\nabla^2 \sigma\|_{H^2} \|\nabla^2 \sigma\|_{H^2} + \|\nabla^2 b\|_{H^1} \|\nabla^3 \sigma\|_{H^1}) + C \left| \int_{\mathbb{R}^3} \nabla^3 \sigma \cdot (\nabla^4 b \cdot \nabla \sigma) dx \right| \\
 &\leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2,
 \end{aligned} \tag{5.32}$$

where we make use of the following estimate:

$$\begin{aligned}
 \left| \int_{\mathbb{R}^3} \nabla^3 \sigma \cdot (\nabla^4 b \cdot \nabla \sigma) dx \right| &= \left| \int_{\mathbb{R}^3} \nabla^3 \sigma \cdot (\nabla(\nabla^3 b \cdot \nabla \sigma) - \nabla^3 b \cdot \nabla^2 \sigma) dx \right| \\
 &\leq C (\|\nabla^4 \sigma\|_{L^2} \|\nabla^3 b\|_{L^2} \|\nabla^2 \sigma\|_{H^1} + \|\nabla^3 \sigma\|_{L^2} \|\nabla^3 \sigma\|_{H^1} \|\nabla^3 b\|_{L^2}) \\
 &\leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2.
 \end{aligned} \tag{5.33}$$

Thus, we collect the aforementioned estimates together to show that

$$\frac{d}{dt} \|\nabla^3(a, q, b, \sigma)\|_{L^2}^2 + \|\nabla^3(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla^4(q, \sigma)\|_{L^2}^2 \leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2. \tag{5.34}$$

Moreover, summing α from 0 to 3, we conclude

$$\frac{d}{dt} \|(a, q, b, \sigma)\|_{H^3}^2 + \|\sigma - kq\|_{H^3}^2 + C_1 \|\nabla(q, \sigma)\|_{H^3}^2 \leq C \mathcal{E}_0(t) \mathcal{D}_0(t)^2. \tag{5.35}$$

Let s be an integer satisfying $0 \leq s \leq 2$. Applying ∇^s to (5.2)₂ and (5.2)₄ and multiplying by $\nabla^{s+1}a$ and $\nabla^{s+1}b$, then integrating the resulting equations by parts in $\mathbb{R}^3$,

$$\begin{aligned}
 &\frac{d}{dt} \int_{\mathbb{R}^3} (\nabla^s q \cdot \nabla^{s+1} a + \nabla^s \sigma \cdot \nabla^{s+1} b) dx + c \|\nabla^{s+1} a\|_{L^2}^2 + \|\nabla^{s+1} b\|_{L^2}^2 \\
 &\leq C (\|\nabla^{s+1} \sigma\|_{L^2}^2 + \|\nabla^{s+1} q\|_{L^2}^2 + \|\nabla^{s+1} q\|_{L^2} \|\nabla^s f_1\|_{L^2} + \|\nabla^{s+1} \sigma\|_{L^2} \|\nabla^s f_3\|_{L^2}) \\
 &\quad + C (\|\nabla^{s+2} q\|_{L^2} \|\nabla^{s+1} a\|_{L^2} + \|\nabla^{s+2} \sigma\|_{L^2} \|\nabla^{s+1} b\|_{L^2} + \|\nabla^{s+1} a\|_{L^2} \|\nabla^s f_2\|_{L^2} \\
 &\quad + \|\nabla^{s+1} b\|_{L^2} \|\nabla^s f_4\|_{L^2}) + C (\|\nabla^{s+1} b\|_{L^2} + k \|\nabla^{s+1} a\|_{L^2}) \|\nabla^s(\sigma - kq)\|_{L^2}.
 \end{aligned} \tag{5.36}$$

By the Cauchy-Swartz inequality, we find that

$$\begin{aligned}
 &\frac{d}{dt} \int_{\mathbb{R}^3} (\nabla^s q \cdot \nabla^{s+1} a + \nabla^s \sigma \cdot \nabla^{s+1} b) dx + \frac{c}{2} \|\nabla^{s+1} a\|_{L^2}^2 + \frac{1}{2} \|\nabla^{s+1} b\|_{L^2}^2 \\
 &\leq C \mathcal{D}_0(t)^2 + C \mathcal{E}_0(t) \mathcal{D}_0(t)^2 + C \|\nabla^s(\sigma - kq)\|_{L^2}^2.
 \end{aligned} \tag{5.37}$$

Summing all the derivatives together, we have

$$\begin{aligned}
 &\frac{d}{dt} \left\{ \sum_{s=0}^2 \int_{\mathbb{R}^3} (\nabla^s q \cdot \nabla^{s+1} a + \nabla^s \sigma \cdot \nabla^{s+1} b) dx \right\} + \frac{c}{2} \|\nabla a\|_{H^2}^2 + \frac{1}{2} \|\nabla b\|_{H^2}^2 \\
 &\leq C \mathcal{D}_0(t)^2 + C \mathcal{E}_0(t) \mathcal{D}_0(t)^2 + C \|\sigma - kq\|_{H^2}^2.
 \end{aligned} \tag{5.38}$$

Taking the summation (5.35) + $\theta \times$ (5.38) with $\theta > 0$, it holds

$$\begin{aligned} & \frac{d}{dt} (\|(a, q, b, \sigma)\|_{H^3}^2 + \theta \sum_{s=0}^2 \int_{\mathbb{R}^3} (\nabla^s q \cdot \nabla^{s+1} a + \nabla^s \sigma \cdot \nabla^{s+1} b) dx) \\ & + \|(\sigma - kq)\|_{H^3}^2 + C_1 \|\nabla(q, \sigma)\|_{H^3}^2 + \theta \left(\frac{1}{2} \|\nabla a\|_{H^2}^2 + \frac{c}{2} \|\nabla b\|_{H^2}^2 \right) \\ & \leq C\theta \mathcal{D}_0(t)^2 + C\theta \mathcal{E}_0(t) \mathcal{D}_0(t)^2 + C\mathcal{E}_0(t) \mathcal{D}_0(t)^2 + C\theta \|\sigma - kq\|_{H^2}^2. \end{aligned} \quad (5.39)$$

Choosing θ suitably small satisfying $0 < \theta \leq 1$, we obtain

$$\begin{aligned} & \|(a, q, b, \sigma)\|_{H^3}^2 + \theta \sum_{s=0}^2 \int_{\mathbb{R}^3} (\nabla^s q \cdot \nabla^{s+1} a + \nabla^s \sigma \cdot \nabla^{s+1} b) dx \\ & \leq \|(a, q, b, \sigma)\|_{H^3}^2 + \theta (\|q\|_{H^2} \|\nabla a\|_{H^2} + \|\sigma\|_{H^2} \|\nabla b\|_{H^2}) \\ & \leq \frac{3}{2} \|(a, q, b, \sigma)\|_{H^3}^2 \end{aligned} \quad (5.40)$$

and

$$\begin{aligned} & \|(a, q, b, \sigma)\|_{H^3}^2 + \theta \sum_{s=0}^2 \int_{\mathbb{R}^3} (\nabla^s q \cdot \nabla^{s+1} a + \nabla^s \sigma \cdot \nabla^{s+1} b) dx \\ & \geq \|(a, q, b, \sigma)\|_{H^3}^2 - \theta (\|q\|_{H^2} \|\nabla a\|_{H^2} + \|\sigma\|_{H^2} \|\nabla b\|_{H^2}) \\ & \geq \frac{1}{2} \|(a, q, b, \sigma)\|_{H^3}^2, \end{aligned} \quad (5.41)$$

which implies that there exists a positive constant $C_2 > 0$ such that

$$\|(a, q, b, \sigma)\|_{H^3}^2 + \theta \sum_{s=0}^2 \int_{\mathbb{R}^3} (\nabla^s q \cdot \nabla^{s+1} a + \nabla^s \sigma \cdot \nabla^{s+1} b) dx = C_2 \|(a, q, b, \sigma)\|_{H^3}^2. \quad (5.42)$$

By choosing $C\theta \leq \frac{1}{2}$, on the basis of (5.42), it gives rise to

$$\begin{aligned} & C_2 \frac{d}{dt} \|(a, q, b, \sigma)\|_{H^3}^2 + \frac{1}{2} \|\sigma - kq\|_{H^3}^2 + C_1 \|\nabla(q, \sigma)\|_{H^3}^2 + \theta \left(\frac{1}{2} \|\nabla a\|_{H^2}^2 + \frac{c}{2} \|\nabla b\|_{H^2}^2 \right) \\ & \leq C\theta \mathcal{D}_0(t)^2 + C\theta \mathcal{E}_0(t) \mathcal{D}_0(t)^2 + C\mathcal{E}_0(t) \mathcal{D}_0(t)^2. \end{aligned} \quad (5.43)$$

After a direct calculation, it is easy to verify that there exists a positive constant $C_3 > 0$ such that

$$\frac{1}{4} \|\sigma - kq\|_{H^3}^2 + \frac{1}{2} C_1 \|\nabla(q, \sigma)\|_{H^3}^2 \geq C_3 \|\nabla(q, \sigma)\|_{H^3}^2, \quad (5.44)$$

which, together with (5.43), also leads to

$$\begin{aligned} & \frac{d}{dt} \|(a, q, b, \sigma)\|_{H^3}^2 + C_4 (\|\sigma - kq\|_{H^3}^2 + \|\nabla(a, b)\|_{H^2}^2 + \|\nabla(q, \sigma)\|_{H^3}^2) \\ & \leq C\theta \mathcal{D}_0(t)^2 + C\theta \mathcal{E}_0(t) \mathcal{D}_0(t)^2 + C\mathcal{E}_0(t) \mathcal{D}_0(t)^2, \end{aligned} \quad (5.45)$$

where

$$C_4 = \min \left\{ \frac{1}{4C_2}, \frac{\theta}{2C_2}, \frac{c\theta}{2C_2}, \frac{C_1}{2C_2}, \frac{C_3}{C_2} \right\}.$$

By the definition of $\mathcal{E}_0(t)$, $\mathcal{D}_0(t)$, and the *a priori* assumption (5.6), we find that

$$\frac{d}{dt} \mathcal{E}_0(t)^2 + C_4 \mathcal{D}_0(t)^2 \leq C\theta \mathcal{D}_0(t)^2 + C\theta \delta \mathcal{D}_0(t)^2 + C\delta \mathcal{D}_0(t)^2. \quad (5.46)$$

Due to the smallness assumptions on δ and θ , we immediately obtain

$$\frac{d}{dt} \mathcal{E}_0(t)^2 + \tilde{C} \mathcal{D}_0(t)^2 \leq 0, \quad (5.47)$$

with $\tilde{C} = \frac{C_4}{2}$ a positive constant. Hence, we have

$$\frac{d}{dt} \|(a, q, b, \sigma)\|_{H^3}^2 + \tilde{C}(\|\nabla(a, b)\|_{H^2}^2 + \|\nabla(q, \sigma)\|_{H^3}^2) \leq 0, \quad (5.48)$$

and we complete the proof of (5.9). It should be mentioned that for $k = 1, 2$, we deal them in a similar argument. This completes the proof of Lemma 5.1. $\square$

5.2 Nonlinear decay rates

Lemma 5.2. *Let (a, q, b, σ) be the smooth solution to system (5.2)–(5.3), and then, for $s \in (0, \frac{1}{2}]$, it holds*

$$\begin{aligned} & \frac{d}{dt} \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2}^2 + \|\mathfrak{P}^{-s}(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla \mathfrak{P}^{-s}q\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathfrak{P}^{-s}\sigma\|_{L^2}^2 \\ & \leq C \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2} (\|\nabla(a, q, b, \sigma)\|_{H^2}^2 + \|\sigma - kq\|_{L^2}^2). \end{aligned} \quad (5.49)$$

And for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, it holds

$$\begin{aligned} & \frac{d}{dt} \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2}^2 + \|\mathfrak{P}^{-s}(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla \mathfrak{P}^{-s}q\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathfrak{P}^{-s}\sigma\|_{L^2}^2 \\ & \leq C \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2} (\|(a, q, b)\|_{L^2}^{s-\frac{1}{2}} + \|\sigma\|_{H^1}^{s-\frac{1}{2}}) (\|\nabla(a, q, b)\|_{L^2}^{\frac{3}{2}-s} + \|\nabla \sigma\|_{H^1}^{\frac{3}{2}-s}) \\ & \quad \times (\|\nabla(a, q, b, \sigma)\|_{L^2} + \|\nabla^2 q\|_{L^2} + \|\sigma - kq\|_{L^2}). \end{aligned} \quad (5.50)$$

Proof. Applying $\mathfrak{P}^{-s}$ to (5.2), and multiplying the resulting by $\mathfrak{P}^{-s}a$, $\mathfrak{P}^{-s}q$, $\mathfrak{P}^{-s}b$, and $\mathfrak{P}^{-s}\sigma$, respectively, summing up, and then integrating by parts over $\mathbb{R}^3$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2}^2 + \|\mathfrak{P}^{-s}(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla \mathfrak{P}^{-s}q\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathfrak{P}^{-s}\sigma\|_{L^2}^2 \\ & = \int_{\mathbb{R}^3} \mathfrak{P}^{-s}a \cdot \mathfrak{P}^{-s}f_1 dx + \int_{\mathbb{R}^3} \mathfrak{P}^{-s}q \cdot \mathfrak{P}^{-s}f_2 dx + \int_{\mathbb{R}^3} \mathfrak{P}^{-s}b \cdot \mathfrak{P}^{-s}f_3 dx + \int_{\mathbb{R}^3} \mathfrak{P}^{-s}\sigma \cdot \mathfrak{P}^{-s}f_4 dx \\ & \triangleq J_1 + J_2 + J_3 + J_4. \end{aligned} \quad (5.51)$$

Next, we will estimate the terms on the right-hand side of (5.51). It is easy to check that if $s \in (0, \frac{1}{2}]$, then $\frac{1}{2} + \frac{s}{3} < 1$ and $\frac{3}{s} \geq 6$. Applying Lemmas 2.1 and 2.3 together with Young's and Hölder's inequalities, we have

$$\begin{aligned} |J_1| & \leq C \|\mathfrak{P}^{-s}(a \operatorname{div} q + q \cdot \nabla a)\|_{L^2} \|\mathfrak{P}^{-s}a\|_{L^2} \\ & \leq C \|a \operatorname{div} q + q \cdot \nabla a\|_{L^{\frac{6}{3+2s}}} \|\mathfrak{P}^{-s}a\|_{L^2} \\ & \leq C (\|a\|_{L^{\frac{3}{s}}} \|\operatorname{div} q\|_{L^2} + \|\nabla a\|_{L^2} \|q\|_{L^{\frac{3}{s}}}) \|\mathfrak{P}^{-s}a\|_{L^2} \\ & \leq C (\|\nabla a\|_{L^2}^{\frac{1}{2}+s} \|\nabla^2 a\|_{L^2}^{\frac{1}{2}-s} \|\nabla q\|_{L^2} + \|\nabla a\|_{L^2} \|\nabla q\|_{L^2}^{\frac{1}{2}+s} \|\nabla^2 q\|_{L^2}^{\frac{1}{2}-s}) \|\mathfrak{P}^{-s}a\|_{L^2} \\ & \leq C (\|\nabla a\|_{L^2}^2 + \|\nabla^2 a\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 + \|\nabla^2 q\|_{L^2}^2) \|\mathfrak{P}^{-s}a\|_{L^2} \\ & \leq C (\|\nabla a\|_{H^1}^2 + \|\nabla q\|_{H^1}^2) \|\mathfrak{P}^{-s}a\|_{L^2}. \end{aligned} \quad (5.52)$$

We also have the following estimate of J_2 :

$$\begin{aligned} |J_2| & \leq C \|\mathfrak{P}^{-s}(q \nabla q + a \nabla a + a \nabla^2 q + (a - b)(\sigma - kq))\|_{L^2} \|\mathfrak{P}^{-s}q\|_{L^2} \\ & \leq C \|q \nabla q + a \nabla a + a \nabla^2 q + (a - b)(\sigma - kq)\|_{L^{\frac{6}{3+2s}}} \|\mathfrak{P}^{-s}q\|_{L^2} \\ & \leq C \left(\|q\|_{L^{\frac{3}{s}}} \|\nabla q\|_{L^2} + \|\nabla a\|_{L^2} \|a\|_{L^{\frac{3}{s}}} + \|a\|_{L^{\frac{3}{s}}} \|\nabla^2 q\|_{L^2} + (\|a\|_{L^{\frac{3}{s}}} + \|b\|_{L^{\frac{3}{s}}}) \|\sigma - kq\|_{L^2} \right) \|\mathfrak{P}^{-s}q\|_{L^2} \\ & \leq C (\|\nabla a\|_{H^1}^2 + \|\nabla b\|_{H^1}^2 + \|\nabla q\|_{H^1}^2 + \|\sigma - kq\|_{L^2}^2) \|\mathfrak{P}^{-s}q\|_{L^2}. \end{aligned} \quad (5.53)$$

The estimate of J_3 holds

$$\begin{aligned}
 |J_3| &\leq C \|\mathfrak{P}^{-s}(\sigma \cdot \nabla b)\|_{L^2} \|\mathfrak{P}^{-s}b\|_{L^2} \\
 &\leq C \|\sigma \cdot \nabla b\|_{L^{\frac{6}{3+2s}}} \|\mathfrak{P}^{-s}b\|_{L^2} \\
 &\leq C \|\sigma\|_{L^{\frac{3}{s}}} \|\nabla b\|_{L^2} \|\mathfrak{P}^{-s}b\|_{L^2} \\
 &\leq C \|\nabla \sigma\|_{L^2}^{\frac{1}{2}+s} \|\nabla^2 \sigma\|_{L^2}^{\frac{1}{2}-s} \|\nabla b\|_{L^2} \|\mathfrak{P}^{-s}b\|_{L^2} \\
 &\leq C(\|\nabla \sigma\|_{L^2}^2 + \|\nabla^2 \sigma\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \|\mathfrak{P}^{-s}b\|_{L^2} \\
 &\leq C(\|\nabla \sigma\|_{H^1}^2 + \|\nabla b\|_{L^2}^2) \|\mathfrak{P}^{-s}b\|_{L^2}.
 \end{aligned} \tag{5.54}$$

The last term can be estimated as

$$\begin{aligned}
 |J_4| &\leq C \|\mathfrak{P}^{-s}(\sigma \cdot \nabla \sigma + \nabla b \cdot \mathbb{D}(\sigma))\|_{L^2} \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\
 &\leq C(\|\sigma \cdot \nabla \sigma\|_{L^{\frac{6}{3+2s}}} + \|\nabla b \cdot \nabla \sigma\|_{L^{\frac{6}{3+2s}}}) \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\
 &\leq C(\|\sigma\|_{L^{\frac{3}{s}}} \|\nabla \sigma\|_{L^2} + \|\nabla \sigma\|_{L^{\frac{3}{s}}} \|\nabla b\|_{L^2}) \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\
 &\leq C(\|\nabla \sigma\|_{L^2}^{\frac{1}{2}+s} \|\nabla^2 \sigma\|_{L^2}^{\frac{1}{2}-s} \|\nabla \sigma\|_{L^2} + \|\nabla^2 \sigma\|_{L^2}^{\frac{1}{2}+s} \|\nabla^3 \sigma\|_{L^2}^{\frac{1}{2}-s} \|\nabla b\|_{L^2}) \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\
 &\leq C(\|\nabla \sigma\|_{L^2}^2 + \|\nabla^2 \sigma\|_{L^2}^2 + \|\nabla^3 \sigma\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\
 &\leq C(\|\nabla \sigma\|_{H^2}^2 + \|\nabla b\|_{L^2}^2) \|\mathfrak{P}^{-s}\sigma\|_{L^2}.
 \end{aligned} \tag{5.55}$$

For $s \in \left(0, \frac{1}{2}\right]$, combining all the aforementioned estimates yields that

$$\begin{aligned}
 &\frac{d}{dt} \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2}^2 + \|\mathfrak{P}^{-s}(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla \mathfrak{P}^{-s}q\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathfrak{P}^{-s}\sigma\|_{L^2}^2 \\
 &\leq C(\|\nabla a\|_{H^1}^2 + \|\nabla q\|_{H^1}^2 + \|\nabla b\|_{H^1}^2 + \|\nabla \sigma\|_{H^2}^2 + \|\sigma - kq\|_{L^2}^2) \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2}.
 \end{aligned} \tag{5.56}$$

Therefore, we complete the proof of (5.49).

As for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, we obtain that $\frac{1}{2} + \frac{s}{3} < 1$ and $2 < \frac{3}{s} < 6$. Applying the Sobolev interpolation inequality, we obtain

$$\begin{aligned}
 |J_1| &\leq C \|\mathfrak{P}^{-s}(a \operatorname{div} q + q \cdot \nabla a)\|_{L^2} \|\mathfrak{P}^{-s}a\|_{L^2} \\
 &\leq C \|a \operatorname{div} q + q \cdot \nabla a\|_{L^{\frac{6}{3+2s}}} \|\mathfrak{P}^{-s}a\|_{L^2} \\
 &\leq C(\|a\|_{L^{\frac{3}{s}}} \|\operatorname{div} q\|_{L^2} + \|q\|_{L^{\frac{3}{s}}} \|\nabla a\|_{L^2}) \|\mathfrak{P}^{-s}a\|_{L^2} \\
 &\leq C(\|a\|_{L^2}^{s-\frac{1}{2}} \|\nabla a\|_{L^2}^{\frac{3}{2}-s} \|\nabla q\|_{L^2} + \|q\|_{L^2}^{s-\frac{1}{2}} \|\nabla q\|_{L^2}^{\frac{3}{2}-s} \|\nabla a\|_{L^2}) \|\mathfrak{P}^{-s}a\|_{L^2}.
 \end{aligned} \tag{5.57}$$

In a similar fashion, we have the following estimate for J_2 :

$$\begin{aligned}
 |J_2| &\leq C \|\mathfrak{P}^{-s}(q \nabla q + a \nabla a + a \nabla^2 q + (a - b)(\sigma - kq))\|_{L^2} \|\mathfrak{P}^{-s}q\|_{L^2} \\
 &\leq C \|q \nabla q + a \nabla a + a \nabla^2 q + (a - b)(\sigma - kq)\|_{L^{\frac{6}{3+2s}}} \|\mathfrak{P}^{-s}q\|_{L^2} \\
 &\leq C \left(\|q\|_{L^{\frac{3}{s}}} \|\nabla q\|_{L^2} + \|\nabla a\|_{L^2} \|a\|_{L^{\frac{3}{s}}} + \|a\|_{L^{\frac{3}{s}}} \|\nabla^2 q\|_{L^2} + \left(\|a\|_{L^{\frac{3}{s}}} + \|b\|_{L^{\frac{3}{s}}} \right) \|\sigma - kq\|_{L^2} \right) \|\mathfrak{P}^{-s}q\|_{L^2} \\
 &\leq C \left(\|q\|_{L^2}^{s-\frac{1}{2}} \|\nabla q\|_{L^2}^{\frac{5}{2}-s} + \|a\|_{L^2}^{s-\frac{1}{2}} \|\nabla a\|_{L^2}^{\frac{5}{2}-s} + \|a\|_{L^2}^{s-\frac{1}{2}} \|\nabla a\|_{L^2}^{\frac{3}{2}-s} \|\nabla^2 q\|_{L^2} \right. \\
 &\quad \left. + \left(\|a\|_{L^2}^{s-\frac{1}{2}} \|\nabla a\|_{L^2}^{\frac{3}{2}-s} + \|b\|_{L^2}^{s-\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{3}{2}-s} \right) \|\sigma - kq\|_{L^2} \right) \|\mathfrak{P}^{-s}q\|_{L^2}.
 \end{aligned} \tag{5.58}$$

As for J_3 , it is written as

$$\begin{aligned} |J_3| &\leq C \|\mathfrak{P}^{-s}(\sigma \cdot \nabla b)\|_{L^2} \|\mathfrak{P}^{-s}b\|_{L^2} \\ &\leq C \|\sigma \cdot \nabla b\|_{L^{\frac{6}{3+2s}}} \|\mathfrak{P}^{-s}b\|_{L^2} \\ &\leq C \|\sigma\|_{L^{\frac{3}{s}}} \|\nabla b\|_{L^2} \|\mathfrak{P}^{-s}b\|_{L^2} \\ &\leq C \|\sigma\|_{L^2}^{s-\frac{1}{2}} \|\nabla \sigma\|_{L^2}^{\frac{3}{2}-s} \|\nabla b\|_{L^2} \|\mathfrak{P}^{-s}b\|_{L^2}. \end{aligned} \quad (5.59)$$

The last term can be estimated as

$$\begin{aligned} |J_4| &\leq C \|\mathfrak{P}^{-s}(\sigma \cdot \nabla \sigma + \nabla b \cdot \mathbb{D}(\sigma))\|_{L^2} \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\ &\leq C \left(\|\sigma \cdot \nabla \sigma\|_{L^{\frac{6}{3+2s}}} + \|\nabla b \cdot \nabla \sigma\|_{L^{\frac{6}{3+2s}}} \right) \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\ &\leq C \left(\|\sigma\|_{L^{\frac{3}{s}}} \|\nabla \sigma\|_{L^2} + \|\nabla \sigma\|_{L^{\frac{3}{s}}} \|\nabla b\|_{L^2} \right) \|\mathfrak{P}^{-s}\sigma\|_{L^2} \\ &\leq C \left(\|\sigma\|_{L^2}^{s-\frac{1}{2}} \|\nabla \sigma\|_{L^2}^{\frac{5}{2}-s} + \|\nabla \sigma\|_{L^2}^{s-\frac{1}{2}} \|\nabla^2 \sigma\|_{L^2}^{\frac{3}{2}-s} \|\nabla b\|_{L^2} \right) \|\mathfrak{P}^{-s}\sigma\|_{L^2}. \end{aligned} \quad (5.60)$$

Combining J_1 – J_4 together, it gives rise to

$$\begin{aligned} &\frac{d}{dt} \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2}^2 + \|\mathfrak{P}^{-s}(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla \mathfrak{P}^{-s}q\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathfrak{P}^{-s}\sigma\|_{L^2}^2 \\ &\leq C \|\mathfrak{P}^{-s}(a, q, b, \sigma)\|_{L^2} \left(\|(a, q, b)\|_{L^2}^{s-\frac{1}{2}} + \|\sigma\|_{H^1}^{s-\frac{1}{2}} \left(\|\nabla(a, q, b)\|_{L^2}^{\frac{3}{2}-s} + \|\nabla \sigma\|_{H^1}^{\frac{3}{2}-s} \right) \right) \\ &\quad \times (\|\nabla(a, q, b, \sigma)\|_{L^2} + \|\nabla^2 q\|_{L^2} + \|\sigma - kq\|_{L^2}). \end{aligned} \quad (5.61)$$

This completes the proof of (5.50), and hence, we finish the proof. $\square$

Then, we are able to provide the decay estimates of strong solution for the nonlinear system (5.2)–(5.3).

Proposition 5.3. *Let $s \in \left[0, \frac{3}{2}\right]$. If the initial data $(a_0, q_0, b_0, \sigma_0) \in \dot{H}^{-s}$, then for $k = 0, 1, 2$, the solution (a, q, b, σ) satisfies*

$$\|\nabla^k(a, q, b, \sigma)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k+s}{2}}. \quad (5.62)$$

Proof. We first consider the case $s \in \left[0, \frac{1}{2}\right]$. For simplicity, we define

$$M(t) = \|\mathfrak{P}^{-s}(a, q, b, \sigma)(t)\|_{L^2}^2. \quad (5.63)$$

When $s = 0$,

$$M(t) = \|(a, q, b, \sigma)\|_{L^2}^2 \leq C\delta_0^2. \quad (5.64)$$

When $s \in \left[0, \frac{1}{2}\right]$, according to Lemma 5.2, we obtain

$$\begin{aligned} &\frac{d}{dt} M(t) + \|\mathfrak{P}^{-s}(\sigma - kq)\|_{L^2}^2 + C_1 \|\nabla \mathfrak{P}^{-s}q\|_{L^2}^2 + \frac{1}{2} \|\nabla \mathfrak{P}^{-s}\sigma\|_{L^2}^2 \\ &\leq C(\|\nabla(a, q, b, \sigma)\|_{H^2}^2 + \|\sigma - kq\|_{L^2}^2) \sqrt{M(t)}. \end{aligned} \quad (5.65)$$

Integrating (5.47) with respect to time from 0 to t , it yields

$$\mathcal{E}_0(t)^2 + \tilde{C} \int_0^t \mathcal{D}_0(\tau)^2 d\tau \leq C\delta_0^2. \quad (5.66)$$

Then, by the *a priori* assumption (5.6) and (5.45), it holds

$$\|(a, q, b, \sigma)\|_{H^3}^2 + C_4 \int_0^t (\|\sigma - kq\|_{H^3}^2 + \|\nabla(a, b)\|_{H^2}^2 + \|\nabla(q, \sigma)\|_{H^3}^2) d\tau \leq C\delta_0^2. \quad (5.67)$$

Making use of (5.67) and Lemma 5.2, we easily obtain that

$$\begin{aligned} M(t) &\leq M(0) + C \int_0^t (\|\nabla(a, q, b, \sigma)\|_{H^1}^2 + \|\sigma - kq\|_{L^2}^2) \sqrt{M(\tau)} d\tau \\ &\leq M(0) + C\delta_0^2 \sup_{0 \leq \tau \leq t} \sqrt{M(\tau)}. \end{aligned} \quad (5.68)$$

As a result of the Cauchy-Schwarz inequality, we directly obtain $M(t) \leq C$, which implies for $s \in [0, \frac{1}{2}]$ that

$$M(t) = \|\mathfrak{P}^{-s}(a, q, b, \sigma)(t)\|_{L^2}^2 \leq C. \quad (5.69)$$

If $k = 0, 1, 2$, we may apply Lemma 2.2 to have

$$\|\nabla^{k+1}f\|_{L^2} \geq C \|\mathfrak{P}^{-s}f\|_{L^2}^{-\frac{1}{k+s}} \|\nabla^k f\|_{L^2}^{1+\frac{1}{k+s}}. \quad (5.70)$$

According to this fact, we obtain

$$\|\nabla^{k+1}a\|_{L^2}^2 + \|\nabla^{k+1}q\|_{L^2}^2 + \|\nabla^{k+1}b\|_{L^2}^2 + \|\nabla^{k+1}\sigma\|_{L^2}^2 \geq C(\|\nabla^k a\|_{L^2}^2 + \|\nabla^k q\|_{L^2}^2 + \|\nabla^k b\|_{L^2}^2 + \|\nabla^k \sigma\|_{L^2}^2)^{1+\frac{1}{k+s}} \quad (5.71)$$

and

$$\begin{aligned} &\|\nabla^{k+1}a\|_{H^{3-k}}^2 + \|\nabla^{k+1}q\|_{H^{3-k}}^2 + \|\nabla^{k+1}b\|_{H^{3-k}}^2 + \|\nabla^{k+1}\sigma\|_{H^{3-k}}^2 \\ &\geq C(\|\nabla^k a\|_{H^{3-k}}^2 + \|\nabla^k q\|_{H^{3-k}}^2 + \|\nabla^k b\|_{H^{3-k}}^2 + \|\nabla^k \sigma\|_{H^{3-k}}^2)^{1+\frac{1}{k+s}}. \end{aligned} \quad (5.72)$$

With the help of Lemma 5.1, we can verify that

$$\frac{d}{dt} \mathcal{E}_k(t)^2 + \tilde{C} \mathcal{D}_k(t)^2 \leq 0, \quad \|\mathcal{D}_k(t)\|_{L^2}^2 \geq C(\mathcal{E}_k(t)^2)^{1+\frac{1}{k+s}}. \quad (5.73)$$

Furthermore, for $k = 0, 1, 2$, it is derived from (5.73) that

$$\frac{d}{dt} \mathcal{E}_k(t)^2 + C(\mathcal{E}_k(t)^2)^{1+\frac{1}{k+s}} \leq 0. \quad (5.74)$$

By solving this inequality directly, we finally obtain

$$\mathcal{E}_k(t) \leq C(1+t)^{-\frac{k+s}{2}}. \quad (5.75)$$

As a result, we find that for $s \in [0, \frac{1}{2}]$ and $k = 0, 1, 2$,

$$\|\nabla^k(a, q, b, \sigma)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k+s}{2}}. \quad (5.76)$$

For the case $s \in [\frac{1}{2}, \frac{3}{4}]$, observing that $(a_0, q_0, b_0, \sigma_0) \in \dot{H}^{-\frac{1}{2}}$ since $\dot{H}^{-s} \cap L^2$ is a subset of $\dot{H}^{-s'}$ for any $s' \in [0, s]$, from what we have proved for (3.4)–(3.5) with $s = \frac{1}{2}$. Then, we deduce that

$$\|\nabla^k(a, q, b, \sigma)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k}{2}-\frac{1}{4}}. \quad (5.77)$$

Applying (5.67), (5.77), and Lemma 5.2, it holds

$$\begin{aligned}
 M(t) &\leq M(0) + C \int_0^t (\| (a, q, b) \|_{L^2}^{s-\frac{1}{2}} + \|\sigma\|_{H^1}^{s-\frac{1}{2}}) (\|\nabla(a, q, b)\|_{L^2}^{\frac{3}{2}-s} + \|\nabla\sigma\|_{H^1}^{\frac{3}{2}-s}) \|\nabla(a, q, b, \sigma)\|_{L^2} \sqrt{M(\tau)} d\tau \\
 &\quad + C \int_0^t (\| (a, q, b) \|_{L^2}^{s-\frac{1}{2}} + \|\sigma\|_{H^1}^{s-\frac{1}{2}}) \left(\|\nabla(a, q, b)\|_{L^2}^{\frac{3}{2}-s} + \|\nabla\sigma\|_{H^1}^{\frac{3}{2}-s} \right) (\|\nabla^2 q\|_{L^2} + \|\sigma - kq\|_{L^2}) \sqrt{M(\tau)} d\tau \\
 &\leq M(0) + C \sup_{0 \leq \tau \leq t} \sqrt{M(\tau)} \left(\delta_0^2 \int_0^t (1+\tau)^{-\left(\frac{7}{4}-\frac{s}{2}\right)} d\tau + \delta_0 \int_0^t (1+\tau)^{-(2-s)} d\tau \right)^{\frac{1}{2}} \left(\int_0^t (\|\nabla^2 q\|_{L^2} + \|\sigma - kq\|_{L^2}^2) d\tau \right)^{\frac{1}{2}} \\
 &\leq M(0) + C \delta_0^2 \sup_{0 \leq \tau \leq t} \sqrt{M(\tau)}.
 \end{aligned}$$

By the Cauchy-Schwarz inequality, we directly obtain $M(t) \leq C$, which implies that

$$\|\mathfrak{P}^{-s}(a, q, b, \sigma)(t)\|_{L^2}^2 \leq C. \quad (5.78)$$

Then, we repeat the same process as (5.70)–(5.75). As a result, we have

$$\|\nabla^k(a, q, b, \sigma)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k+s}{2}}. \quad (5.79)$$

As for the case $s \in \left[\frac{3}{4}, 1\right]$, observing that $(a_0, q_0, b_0, \sigma_0) \in \dot{H}^{-\frac{3}{4}}$ since $\dot{H}^{-s} \cap L^2$ is a subset of $\dot{H}^{-s'}$ for any $s' \in [0, s]$, from what we have proved for (3.4)–(3.5) with $s = \frac{3}{4}$, we then deduce that

$$\|\nabla^k(a, q, b, \sigma)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k}{2}-\frac{3}{8}}. \quad (5.80)$$

It follows from Lemma 5.2, (5.67), and (5.80) that

$$\begin{aligned}
 M(t) &\leq M(0) + C \int_0^t (\| (a, q, b) \|_{L^2}^{s-\frac{1}{2}} + \|\sigma\|_{H^1}^{s-\frac{1}{2}}) \left(\|\nabla(a, q, b)\|_{L^2}^{\frac{3}{2}-s} + \|\nabla\sigma\|_{H^1}^{\frac{3}{2}-s} \right) \|\nabla(a, q, b, \sigma)\|_{L^2} \sqrt{M(\tau)} d\tau \\
 &\quad + C \int_0^t (\| (a, q, b) \|_{L^2}^{s-\frac{1}{2}} + \|\sigma\|_{H^1}^{s-\frac{1}{2}}) \left(\|\nabla(a, q, b)\|_{L^2}^{\frac{3}{2}-s} + \|\nabla\sigma\|_{H^1}^{\frac{3}{2}-s} \right) (\|\nabla^2 q\|_{L^2} + \|\sigma - kq\|_{L^2}) \sqrt{M(\tau)} d\tau \\
 &\leq M(0) + C \sup_{0 \leq \tau \leq t} \sqrt{M(\tau)} \left(\delta_0^2 \int_0^t (1+\tau)^{-(2-\frac{s}{2})} d\tau + \delta_0 \int_0^t (1+\tau)^{-\left(\frac{9}{4}-s\right)} d\tau \right)^{\frac{1}{2}} \left(\int_0^t (\|\nabla^2 q\|_{L^2} + \|\sigma - kq\|_{L^2}^2) d\tau \right)^{\frac{1}{2}} \\
 &\leq M(0) + C \delta_0^2 \sup_{0 \leq \tau \leq t} \sqrt{M(\tau)}.
 \end{aligned}$$

By the Cauchy-Schwarz inequality, we directly obtain $M(t) \leq C$, which implies

$$\|\mathfrak{P}^{-s}(a, q, b, \sigma)(t)\|_{L^2}^2 \leq C. \quad (5.81)$$

Then, we repeat the same process as (5.70)–(5.75). As a result, we have

$$\|\nabla^k(a, q, b, \sigma)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k+s}{2}}. \quad (5.82)$$

For the case $s \in \left[1, \frac{3}{2}\right]$, observing that $(a_0, w_0, b_0, q_0) \in \dot{H}^{-1}$ since $\dot{H}^{-s} \cap L^2$ is a subset of $\dot{H}^{-s'}$ for $s' \in [0, s]$, we then deduce from what we have proved for (3.4)–(3.5) with $s = 1$ that

$$\|\nabla^k(a, q, b, \sigma)(t)\|_{H^{3-k}} \leq C(1+t)^{-\frac{k}{2}-\frac{1}{2}}. \quad (5.83)$$

Applying Lemma 5.2, (5.67), and (5.83), we obtain

$$\begin{aligned}
 M(t) &\leq M(0) + C \int_0^t (\| (a, q, b) \|_{L^2}^{s-\frac{1}{2}} + \| \sigma \|_{H^1}^{s-\frac{1}{2}}) (\| \nabla(a, q, b) \|_{L^2}^{\frac{3}{2}-s} + \| \nabla \sigma \|_{H^1}^{\frac{3}{2}-s}) \| \nabla(a, q, b, \sigma) \|_{L^2} \sqrt{M(\tau)} d\tau \\
 &\quad + C \int_0^t (\| (a, q, b) \|_{L^2}^{s-\frac{1}{2}} + \| \sigma \|_{H^1}^{s-\frac{1}{2}}) (\| \nabla(a, q, b) \|_{L^2}^{\frac{3}{2}-s} + \| \nabla \sigma \|_{H^1}^{\frac{3}{2}-s}) (\| \nabla^2 q \|_{L^2} + \| \sigma - kq \|_{L^2}) \sqrt{M(\tau)} d\tau \\
 &\leq M(0) + C \sup_{0 \leq \tau \leq t} \sqrt{M(\tau)} \left(\delta_0^2 \int_0^t (1 + \tau)^{-\left(\frac{9}{4}-\frac{s}{2}\right)} d\tau + \delta_0 \int_0^t (1 + \tau)^{-\left(\frac{5}{2}-s\right)} d\tau \right)^{\frac{1}{2}} \left(\int_0^t (\| \nabla^2 q \|_{L^2} + \| \sigma - kq \|_{L^2}^2) d\tau \right)^{\frac{1}{2}} \\
 &\leq M(0) + C \delta_0^2 \sup_{0 \leq \tau \leq t} \sqrt{M(\tau)}.
 \end{aligned}$$

Again, by $M(t) \leq C$, it yields

$$\| \mathfrak{P}^{-s}(a, q, b, \sigma)(t) \|_{L^2}^2 \leq C. \quad (5.84)$$

Then, we repeat the same process as (5.70)–(5.75), which gives rise to

$$\| \nabla^k(a, q, b, \sigma)(t) \|_{H^{3-k}} \leq C(1 + t)^{-\frac{k+s}{2}}. \quad (5.85)$$

Combining (5.76), (5.79), (5.82), and (5.85) together, we conclude (5.62) and the proof is complete. $\square$

5.3 Proof of Theorem 3.1

Proof. Integrating (5.7) with respect to time leads to

$$\| (a, q, b, \sigma)(t) \|_{H^3}^2 + \tilde{C} \int_0^t (\| (\nabla a, \nabla b)(\tau) \|_{H^2}^2 + \| (\nabla q, \nabla \sigma)(\tau) \|_{H^3}^2) d\tau \leq C \delta_0^2. \quad (5.86)$$

By choosing $C \delta_0^2 \leq \frac{1}{4} \delta^2$, we enclose the *a priori* assumption (5.6), and hence, we obtain the uniform bound of energy. Applying the standard continuation argument, we obtain the global existence of the strong solution (see [33] for details). Combining Proposition 5.3 and (5.1) together, we have for $s \in \left[0, \frac{3}{2}\right]$,

$$\| (\rho - \rho_*, u, m - m_*, w)(t) \|_{H^{-s}}^2 \leq C. \quad (5.87)$$

Meanwhile, for $k = 0, 1, 2$, the time decay estimates (3.5) hold. Hence, we complete the proof. $\square$

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