Research Article

Wentao Huo and Zhong Bo Fang*

Cauchy problem for a non-Newtonian filtration equation with slowly decaying volumetric moisture content

https://doi.org/10.1515/anona-2024-0035
received May 27, 2023; accepted July 29, 2024

Abstract: This article is concerned with the qualitative properties for the Cauchy problem of a non-Newtonian filtration equation with a reaction source term and volumetric moisture content. On the basis of the slowly decaying behavior of volumetric moisture content, we establish new critical exponents that depend on the ratio of coefficients and exponent of volumetric moisture content. Meantime, under appropriate conditions, we show that the solution globally exists for small enough initial data and blows up in finite time for large enough or any nontrivial initial datum.

Keywords: non-Newtonian filtration equation, slowly decaying volumetric moisture content, global existence, blow-up

MSC 2020: 35K92, 35K15, 35B33, 35B40

1 Introduction

We consider an inhomogeneous quasilinear parabolic equation in the whole dimensional space

$$\rho(x)u_t = \Delta_p u + \rho(x)u^q, \quad (x,t) \in \mathbb{R}^N \times (0,\hat{T}), \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\Delta_p u \coloneqq \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $N \geq 3$, $2 < p < N$, $q > 1$, $\hat{T} > 0$. The coefficient $\rho(x)$ and initial data $u_0(x)$ satisfy, respectively,

(H_1) $\rho(x) \in C(\mathbb{R}^N)$, $\rho(x) > 0$, $\forall x \in \mathbb{R}^N$;

(H_2) there exist $k_1, k_2 \in (0,+\infty)$ with $k_1 \leq k_2$ and $0 \leq s < p$ such that

$$k_1\,|x|^s \leq \frac{1}{\rho(x)} \leq k_2\,|x|^s, \quad \forall x \in \mathbb{R}^N \backslash B_1(0),$$

where $B_1(0)$ is the unit ball in $\mathbb{R}^N$;

(H_3) $u_0(x) \in L^\infty(\mathbb{R}^N)$, $u_0(x) \geq 0$, $\forall x \in \mathbb{R}^N$.

* **Corresponding author: Zhong Bo Fang,** School of Mathematical Sciences, Ocean University of China, Qingdao 266100, P.R. China, e-mail: fangzb7777@ouc.edu.cn, tel: +86 0532 66787153; fax: +86 0532 66787211
Wentao Huo: School of Mathematical Sciences, Ocean University of China, Qingdao 266100, P.R. China, e-mail: huowentao@stu.ouc.edu.cn

The quasilinear parabolic equation considered in problems (1.1) and (1.2) is of the *p-Laplacian* type with the special volumetric moisture content $\rho(x)$ and a reaction term $\rho(x)u^q$. Obviously, such parabolic equation is degenerate (also called slow diffusion) due to $p > 2$. Furthermore, equation (1.1) can be written in the form

$$u_t = \frac{1}{\rho(x)}\Delta_p u + u^q, \quad (x,t) \in \mathbb{R}^N \times (0, \hat{T}),$$

and thus, the corresponding nonlinear diffusion operator is $\frac{1}{\rho(x)}\Delta_p$, and according to ($H_1$)–($H_2$), the coefficient $\frac{1}{\rho(x)}$ can positively diverge at infinity. Model (1.1) describes the flow of the compressible non-Newtonian fluids in a homogeneous isotropic rigid porous medium, where $u(x,t)$ is the density of fluid, $\rho(x)$ is the volumetric moisture content, $\rho(x)u^q$ represents a special reaction source term (see [37, Chapter 2]). Clearly, there exists the more general source term $A(x)u^q$, where $A(x)$ is not necessarily equal to $\rho(x)$.

We refer to $\rho(x)$ as a *slowly decaying volumetric moisture content* at infinity, since (H_2) implies that

$$\frac{1}{k_1|x|^s} \le \rho(x) \le \frac{1}{k_2|x|^s}, \quad \forall |x| > 1,$$

with $0 \le s < p$. Indeed, it is known from [19] that the behavior of solutions varies according to the different decaying rate of the volumetric moisture content, namely, $s < p$ and $s \ge p$. Consequently, we regard the value $s = p$ as the threshold one and focus on the case of slowly decaying.

In the past decades, many authors have been devoted to investigating the local well-posedness and qualitative properties of the solutions to p-Laplacian parabolic equations, and one can refer to monographs [7,29,37] as well as survey papers [8,12, 13,19,40] and the references therein. Among them, the global existence and blow-up phenomenon of solutions to Cauchy problem are particularly important issues. Fujita [11] first studied these issues in 1966. For example, for problems (1.1) and (1.2) with $\rho \equiv 1$, i.e.,

$$\begin{cases} u_t = \Delta_p u + u^q, & (x,t) \in \mathbb{R}^N \times (0, \hat{T}), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

and he studied the Cauchy problem for the classical semilinear parabolic equation ($p = 2$) and showed that there exists a critical Fujita exponent $q_c = 1 + \frac{2}{N}$ such that the positive solution blows up in finite time for any nontrivial initial data, whenever $1 < q < q_c$; while there are global solutions for small initial data and non-global solution for large initial data, if $q > q_c$. After that, there have been many kinds of extensions of Fujita's result. It is noted that the critical case $q = q_c$ also belongs to blow-up case (cf. [14,36]). In addition, with regard to the latest advances on the Cauchy problem of semilinear parabolic equations with sufficiently small initial value, one can see [30,31]. Meanwhile, for the extension of studying the influence of the size of initial energy on the qualitative properties and global dynamical behavior of solutions to the initial boundary value problems for semilinear pseudo-parabolic equations and finitely degenerate parabolic equations, we refer to [4,18,38,39]. From then on, the Cauchy problem to quasilinear degenerate parabolic equation ($p > 2$) is considered in literature [12,13,32,33] and monograph [29], in which they obtained the critical Fujita exponent $q_c = p - 1 + \frac{p}{N}$. We remark that monographs [29,37] have mentioned Barenblatt-type super- and sub-solution in such a form that

$$w(x,t) = C\xi(t)\left[1 - \frac{|x|^{\frac{p}{p-1}}}{a}\eta(t)\right]_+^{\frac{p-1}{p-2}}, \quad \forall (x,t) \in \mathbb{R}^N \times [0,T),$$

where $\xi = \xi(t)$, $\eta = \eta(t)$ are appropriate auxiliary functions and constants $C > 0$, $a > 0$.

Concerning problems (1.1) and (1.2) with volumetric moisture content, without reaction source term, i.e.,

$$\begin{cases} \rho(x)u_t = \Delta_p u, & (x,t) \in \mathbb{R}^N \times (0, \hat{T}), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

it has been investigated in detail. In particular, depending on the behavior of $\rho(x)$ at infinity, local and global solvability, the interface blow-up phenomenon and large time behavior of the solutions have been studied (including more general doubly degenerate operators and the case of Riemannnian manifold), see [15,16] ($p = 2$) and [5,6,9,34,35] (quasilinear operator).

For problems (1.1) and (1.2) with volumetric moisture content and a general reaction term, i.e.,

$$\begin{cases} \rho(x)u_t = \Delta_p u + A(x)u^q, & (x,t) \in \mathbb{R}^N \times (0,\hat{T}), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

when the behavior of $\rho(x)$ at infinity is given, there have been some advances on the well-posedness, blow-up, and large time behavior of the solutions for $p = 2$ (cf. [2,10,17, 26,27]), $p > 2$ (cf. [40]), and doubly degenerate operators (cf. [1,3,19–25]), which involved the case of $p > 2$. Especially, Li and Xiang [17] have recently considered the case of $p = 2$ (also refer to [26,27]), and $\rho(x)$ and $A(x)$ satisfy the following decay rates, respectively

$$k_1(1+|x|^2)^{-\frac{s_1}{2}} \le \rho(x) \le k_2(1+|x|^2)^{-\frac{s_1}{2}}, \quad 0 < s_1 < 2, \; 0 < k_1 \le k_2;$$

$$K_1(1+|x|^2)^{-\frac{s_2}{2}} \le A(x) \le K_2(1+|x|^2)^{-\frac{s_2}{2}}, \quad 0 < s_2 < 2, \; 0 < K_1 \le K_2,$$

where $s_2 \le s_1$. They obtained the critical Fujita exponent $q_c = 1 + \frac{2-s_1}{N-s_2}$. In fact, for problems (1.1) and (1.2) with $p = 2$ and $\rho(x) = A(x)$, the corresponding critical exponent is

$$q_c = 1 + \frac{b}{N-2+b},$$

where $b \coloneqq 2 - s$, $0 \le s < 2$. Zhao [40] investigated the case of $p > 2$, $\rho \equiv 1$ and $A(x) = (1+|x|)^{-s}$, $s \in \mathbb{R}$. On the basis of the energy method, he obtained the existence and uniqueness of solution and large time behavior of solutions. It was also shown that if $s < p$ and $p - 1 < q < p - 1 + \frac{b}{N}$ with $b \coloneqq p - s$, then the solution blows up in finite time. The doubly degenerate problem with $\rho(x) = A(x)$ have been considered in [19,22], whose special cases included problems (1.1) and (1.2). It was shown that (see [19, Theorems 1 and 3]) when $\rho(x) = |x|^{-s}$, $0 \le s < p$, $\forall x \in \mathbb{R}^N \backslash \{0\}$, if

$$q > p - 1 + \frac{b}{N+b-p},$$

initial value $u_0 \ge 0$ and

$$\int_{\mathbb{R}^N} \rho(u_0^{\bar{s}}(x) + u_0(x))\mathrm{d}x \le \delta,$$

where $\bar{s} > \frac{(N-p+b)(q-p+1)}{b}$ and $\delta > 0$ small enough, then there exists a global solution of problems (1.1) and (1.2) and the long-time asymptotic behavior is derived. On the other hand, when $\rho(x) = |x|^{-s}$ or $\rho(x) = (1+|x|)^{-s}$, $0 \le s < p$, if $u_0(x) \not\equiv 0$ and

$$q < p - 1 + \frac{b}{N+b-p},$$

then the solution blows up in finite time, in the sense that $\exists 0 < \theta < 1$, $0 < R < \infty$, $\hat{T} > 0$ such that

$$\int_{B_R} \rho(x)u^\theta(x,t)\mathrm{d}x \to \infty, \quad t \to \hat{T}^- < \infty.$$

Such results have also been generalized to more general initial data, decaying at infinity with a certain rate (cf. [22]).

In this article, continuing with the model considered in the literature [19,40], we consider the Cauchy problems (1.1) and (1.2). However, the key ingredient of the technique in proof is the method of directly constructing the Barenblatt-type super- and sub-solutions, and this article is a first attempt to research the asymptotic behavior of (1.1) and (1.2) involving degenerate p-Laplacian operator and slowly decaying volumetric moisture content. We mention that the methods and results used in this article are completely different from [19], see Remarks 3.2, 4.1, and 5.1. The main difficulties can be listed as follows. First, the methods used in

[11,14,17,36] cannot work in the degenerate situation ($p > 2$) as they strongly require $p = 2$. Second, since equation (1.1) has not scaling invariant structure, we cannot apply the method of analyzing self-similar solutions. To overcome these difficulties, we modify the method of Barenblatt-type super- and sub-solutions (cf. [29,37]) and establish the new asymptotic behaviors of solution. Indeed, we construct suitable super- and sub-solutions, which crucially rely on the slowly decaying behavior of volumetric moisture content $\rho(x)$ at infinity. More accurately, whenever $|x| > 1$, they are of the following form:

$$w(x,t) = C\xi(t)\left[1 - \frac{|x|^{\frac{b}{p-1}}}{a}\eta(t)\right]_+^{\frac{p-1}{p-2}}, \quad (x,t) \in [\mathbb{R}^N \backslash B_1(0)] \times [0,T),$$

where $\xi = \xi(t)$, $\eta = \eta(t)$ are appropriate auxiliary functions and constants $C > 0$, $a > 0$. In view of the term $|x|^{\frac{b}{p-1}}$ with $b \coloneqq p - s \in (0, p]$, we cannot show that such functions are super- and sub-solutions in $B_1(0) \times [0, T)$. Hence, it is essential to extend them in a suitable way in $B_1(0) \times [0, T)$ and attach some additional conditions on $\xi = \xi(t)$, $\eta = \eta(t)$, C and a. This is a technical aspect. In addition, it reflects the interaction between the behavior of the volumetric moisture content $\rho(x)$ in compact sets, say $B_1(0)$, and its behavior for large value of $|x|$. A rough sketch of our main results is as follows (the more detailed statement see Sections 3–5):

Let $b \coloneqq p - s$, since $0 \le s < p$, we have

$$0 < b \le p.$$

At the same time, assume

$$\frac{k_2}{k_1} < p - 1 + \frac{(N-p)(p-2)}{b} \tag{1.3}$$

and define

$$\bar{q} \coloneqq \frac{(N+b-p)(p-1) + \frac{b}{p-2}\left(p - 1 - \frac{k_2}{k_1}\right)}{N - p + \frac{b}{p-2}\left(p - 1 - \frac{k_2}{k_1}\right)}. \tag{1.4}$$

It is easy to check that $\bar{q}$ is monotonically increasing with respect to the ratio $\frac{k_2}{k_1}$ and

$$\bar{q} > p - 1.$$

- (See Theorem 3.1) If

$$q > \bar{q},$$

  initial data u_0 has compact support and is small enough, then problems (1.1) and (1.2) admit a global solution.

Note that for $k_1 = k_2$,

$$\bar{q} = p - 1 + \frac{b}{N + b - p},$$

this is consistent with [19, Theorem 1] (see Remark 3.1 for more details). Furthermore, if $\rho \equiv 1$, and so $b = p$, we have

$$\bar{q} = p - 1 + \frac{p}{N}.$$

Thus, our results are in accordance with those in [12,13,29,32,33]. In addition, for $p = 2$, they are in agreement with the results established in [17], and in [11,14,36] when $\rho \equiv 1$.

- (See Theorem 4.1.) For any $q > 1$, if u_0 is sufficiently large, then the solutions of problems (1.1) and (1.2) blow up in finite time.

- (See Theorem 5.1.) If $1 < q < p - 1$, then the solutions of problems (1.1) and (1.2) blows up in finite time for any $u_0 \not\equiv 0$.
- (See Theorem 5.2.) If

$$p - 1 \le q < \underline{q},$$

where

$$\underline{q} := \frac{(N + b - p)(p - 1) + \frac{b}{p-2}\left(p - 1 - \frac{k_1}{k_2}\right)}{N - p + \frac{b}{p-2}\left(p - 1 - \frac{k_1}{k_2}\right)}, \tag{1.5}$$

then the solution blows up for any nontrivial initial data under the additional hypothesis that $s \in [0, \varepsilon)$ for $\varepsilon > 0$ small enough.

In view of (1.3), it can be easily checked that

$$\underline{q} \le \bar{q}.$$

In particular, $\underline{q} = \bar{q}$, whenever $k_1 = k_2$. We mention that it remains to be considered without restriction $s \in [0, \varepsilon)$. In addition, when hypothesis (H_2) is satisfied for general $0 < k_1 < k_2$, the blow-up results for sufficiently large initial data and any nontrivial initial data with $1 < q < p - 1$ can be stated exactly as in the previous case $k_1 = k_2$ (cf. [17,19]). However, from Theorems 3.1 and 5.2, we see that the critical exponents of problems (1.1) and (1.2) depend on the ratio of coefficients k_1 and k_2, which is a new and interesting phenomenon compared with the case of $k_1 = k_2$.

The rest of our article is organized as follows. In Section 2, we introduce some preliminaries related to problems (1.1) and (1.2). In Section 3, we construct suitable super-solution to obtain the existence of global solution. The blow-up results for sufficiently large initial data are proved in Section 4. Finally, for any initial datum, the blow-up of solutions are proved in Section 5.

2 Preliminaries

In this section, we present the definitions of weak solution, weak super- and sub-solutions, comparison of principle and some propositions, which are required in the proof of the main results to problems (1.1) and (1.2).

We first give the definition of the weak solutions of problems (1.1) and (1.2) in different domains.

Definition 2.1. Let $p > 2, q > 1, \hat{T} > 0, u_0 \ge 0$, and $u_0(x) \in L^\infty(\mathbb{R}^N)$. A nonnegative function $\bar{u} \in L^\infty(\mathbb{R}^N \times (0, S))$, $|\nabla \bar{u}|^p \in L^1(\mathbb{R}^N \times (0, S))$ for any $S < \hat{T}$ is a weak super-solution of problems (1.1) and (1.2) if the following integral inequality holds

$$\int_0^{\hat{T}} \int_{\mathbb{R}^N} (-\rho(x)\bar{u}\phi_t + |\nabla \bar{u}|^{p-2}\nabla \bar{u} \cdot \nabla \phi - \rho(x)\bar{u}^q \phi)\mathrm{d}x\mathrm{d}t \ge \int_{\mathbb{R}^N} \rho(x)u_0(x)\phi(x, 0)\mathrm{d}x \tag{2.1}$$

for any nonnegative test function $\phi \in C_0^\infty(\mathbb{R}^N \times [0, \hat{T}))$.

Similarly, a nonnegative function $\underline{u} \in L^\infty(\mathbb{R}^N \times (0, S))$ is a weak sub-solution of problems (1.1) and (1.2) if it satisfies (2.1) in the reverse order. We say $u(x, t)$ is a weak solution of problems (1.1) and (1.2) if it is both a weak super-solution and a weak sub-solution of problems (1.1) and (1.2).

For any $x_0 \in \mathbb{R}^N$ and $R > 0$, we set

$$B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\}.$$

When $x_0 = 0$, we briefly write $B_R := B_R(0)$. For every $R > 0$, we consider the following Dirichlet initial boundary value problem:

$$u_t = \frac{1}{\rho(x)}\Delta_p u + u^q, \quad (x,t) \in B_R \times (0, \hat{T}_R), \tag{2.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial B_R \times (0, \hat{T}_R), \tag{2.3}$$

$$u(x,0) = u_0(x), \quad x \in B_R. \tag{2.4}$$

Definition 2.2. Let $p > 2, q > 1, \hat{T}_R > 0, u_0 \geq 0$, and $u_0(x) \in L^\infty(B_R)$. A nonnegative function $\bar{u}_R \in L^\infty(B_R \times (0,S))$, $|\nabla \bar{u}_R|^p \in L^1(B_R \times (0,S))$ for any $S < \hat{T}_R$ is a weak super-solution of problems (2.2)–(2.4) if the following integral inequality holds

$$\int_0^{\hat{T}_R}\int_{B_R} (-\rho \bar{u}_R \phi_t + |\nabla \bar{u}_R|^{p-2}\nabla \bar{u}_R \cdot \nabla\phi - \rho \bar{u}_R^q \phi)\mathrm{d}x\mathrm{d}t \geq \int_{B_R} \rho u_0(x)\phi(x,0)\mathrm{d}x \tag{2.5}$$

for any nonnegative test function $\phi \in C_0^\infty(B_R \times [0, \hat{T}_R))$.

Meanwhile, a nonnegative function $\underline{u}_R \in L^\infty(B_R \times (0,S))$ is a weak sub-solution of problems (2.2)–(2.4) if it satisfies (2.5) in the reverse order. We say $u_R(x,t)$ is a weak solution of problems (2.2)–(2.4) if it is both a weak super-solution and a weak sub-solution of problems (2.2)–(2.4).

Proposition 2.1. *Assume* (H$_1$)–(H$_3$) *hold. Then problems* (2.2)–(2.4) *has a solution u_R, and the maximal time of existence $\hat{T}_R$ satisfies*

$$\hat{T}_R \geq T_0 := \frac{1}{(q-1)||u_0||^{q-1}_{L^\infty(B_R)}}.$$

Proof. Obviously, $\underline{u}_R \equiv 0$ is a sub-solution of problems (2.2)–(2.4). Furthermore, let $\bar{u}_R(t)$ be the solution for the following initial value problem of ordinary differential equation:

$$\begin{cases} \bar{u}'(t) = \bar{u}^q(t), & t > 0, \\ \bar{u}(0) = ||u_0||_{L^\infty(B_R)}. \end{cases}$$

Standard calculations show that

$$\bar{u}_R(t) = \frac{||u_0||_{L^\infty(B_R)}}{[1-(q-1)t||u_0||^{q-1}_{L^\infty(B_R)}]^{\frac{1}{q-1}}}, \quad t \in [0, T_0).$$

It can be easily verified that $\bar{u}_R$ is a super-solution of problems (2.2)–(2.4) for every $R > 0$. In view of (H$_1$)–(H$_2$), we obtain

$$0 < \min_{\bar{B}_R} \frac{1}{\rho} \leq \frac{1}{\rho(x)} \leq \max_{\bar{B}_R} \frac{1}{\rho}, \quad \forall x \in \bar{B}_R.$$

As a result, by applying the standard results (see [7]), problems (2.2)–(2.4) have a nonnegative solution $u_R \in L^\infty(B_R \times (0,S))$ for any $S < \hat{T}_R$, where $\hat{T}_R \geq T_0$ is the maximal time of existence, namely,

$$||u_R(t)||_\infty \to \infty, \quad t \to \hat{T}_R^-.$$

The proof is completed. □

In addition, the following comparison principle for problems (2.2)–(2.4) holds (cf. [37]).

Proposition 2.2. *Assume* (H$_1$)–(H$_3$) *hold. If $\bar{u}_R$ and $\underline{u}_R$ are a super-solution and a sub-solution of problems* (2.2)–(2.4), *respectively, then we have*

$$\underline{u}_R \leq \bar{u}_R, \quad \text{a.e. } (x,t) \in B_R \times (0, \hat{T}_R).$$

Proposition 2.3. *Assume* (H_1)–(H_3) *hold. Then problems* (1.1) *and* (1.2) *have a solution u, and the maximal time of existence* $\hat{T}$ *satisfies*

$$\hat{T} \geq \hat{T}_0 := \frac{1}{(q-1)\|u_0\|_\infty^{q-1}}.$$

Moreover, u is the minimal solution, that is, the following inequality holds

$$u \leq v, \quad (x,t) \in \mathbb{R}^N \times (0, \hat{T}),$$

for any solution v of problems (1.1) *and* (1.2).

Proof. Let u_R be the unique solution of problems (2.2)–(2.4) for every $R > 0$. It is not difficult to see that when $0 < R_1 < R_2$, we have the following inequality:

$$u_{R_1} \leq u_{R2}, \quad (x,t) \in B_R \times (0, \hat{T}_0). \tag{2.6}$$

Indeed, u_{R_2} is a super-solution, whereas u_{R_1} is a solution of problems (2.2)–(2.4) with $R = R_1$. Consequently, we obtain the inequality (2.6) by virtue of Proposition 2.2. We proceed to consider the following initial value problem of ordinary differential equation:

$$\begin{cases} \bar{u}'(t) = \bar{u}^q(t), & t > 0, \\ \bar{u}(0) = \|u_0\|_\infty. \end{cases}$$

Standard calculations show that

$$\bar{u}(t) = \frac{\|u_0\|_\infty}{[1-(q-1)t\|u_0\|_\infty^{q-1}]^{\frac{1}{q-1}}}, \quad t \in [0, \hat{T}_0).$$

It can be easily verified that $\bar{u}(t)$ is a super-solution of problems (2.2)–(2.4) for every $R > 0$. Thus, we obtain

$$0 \leq u_R(x,t) \leq \bar{u}(t), \quad (x,t) \in B_R \times (0, \hat{T}_0). \tag{2.7}$$

It follows from (2.6) and (2.7) that the family $\{u_R\}_{R>0}$ is monotone increasing with respect to R and uniformly bounded. Therefore, we derive the family $\{u_R\}_{R>0}$ converges point-wise to a function $u(x,t)$, as $R \to +\infty$, namely,

$$\lim_{R \to +\infty} u_R(x,t) = u(x,t).$$

Furthermore, according to the monotone convergence theorem, taking an equal sign in (2.4) and letting $R \to +\infty$, we arrive at

$$\int_0^{\hat{T}_0} \int_{\mathbb{R}^N} (-\rho(x) u \phi_t + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi - \rho(x) u^q \phi) \mathrm{d}x \mathrm{d}t = \int_{\mathbb{R}^N} \rho(x) u_0(x) \phi(x,0) \mathrm{d}x$$

for any nonnegative test function $\phi \in C_0^\infty(\mathbb{R}^N \times [0, \hat{T}_0))$. Consequently, $u(x,t)$ is a solution of problems (1.1) and (1.2) and $u \in L^\infty(\mathbb{R}^N \times (0,S))$ for any $S < \hat{T}$, where $\hat{T} \geq \hat{T}_0$ is the maximal time of existence, that is,

$$\|u(t)\|_\infty \to \infty, \quad t \to \hat{T}^-.$$

Next, we aim to prove that u is the minimal nonnegative solution of problems (1.1) and (1.2). Suppose v be any other solution of problems (1.1) and (1.2). It is clear that v is a super-solution of problems (2.2)–(2.4) for every $R > 0$. Therefore, in view of Proposition 2.2, one can obtain

$$u_R \leq v, \quad (x,t) \in B_R \times (0, \hat{T}).$$

Then letting $R \to \infty$, we derive

$$u \leq v, \quad (x,t) \in \mathbb{R}^N \times (0, \hat{T}).$$

As a result, u is the minimal nonnegative solution. The proof is completed. □

In a word, the following comparison principles of problems (1.1) and (1.2) can be presented.

Proposition 2.4. *Assume* (H_1)–(H_3) *hold and* $\bar{u}$ *be a super-solution of problems* (1.1) *and* (1.2). *If* u *is the minimal solution to problems* (1.1) *and* (1.2) *given in Proposition 2.3, then we have*

$$u \le \bar{u}, \quad a.e.\ (x, t) \in \mathbb{R}^N \times (0, \hat{T}).$$

In particular, if $\bar{u}$ *exists up to time* $\hat{T}$, *then also* u *exists at least up to time* $\hat{T}$.

Proof. It is clear that $\bar{u}$ is a super-solution of problems (2.2)–(2.4) for any $R > 0$. Thus, making use of Proposition 2.2, we arrive at

$$u_R \le \bar{u}, \quad (x, t) \in B_R \times (0, \hat{T}). \tag{2.8}$$

Letting $R \to \infty$ in the aforementioned inequality, from the definition of maximal existence time, it is easy to see that (2.8) holds, which trivially guarantees that u does exist at least until $\hat{T}$. The proof is completed. □

Proposition 2.5. *Assume* (H_1)–(H_3) *hold. If* $u(x, t)$ *is a solution to problems* (1.1) *and* (1.2) *for some time* $\hat{T} = t_1 > 0$, $\underline{u}$ *is a sub-solution to problems* (1.1) *and* (1.2) *for some time* $\hat{T} = t_2 > 0$ *and satisfies*

$$\operatorname{supp}\ \underline{u}|_{\mathbb{R}^N \times [0,S]} \text{ is compact for every } S \in (0, t_2),$$

then

$$u \ge \underline{u}, \quad (x, t) \in \mathbb{R}^N \times (0, \min\{t_1, t_2\}). \tag{2.9}$$

Proof. Fix any $S < \min\{t_1, t_2\}$. If $R > 0$ is large enough so that

$$\operatorname{supp}\ \underline{u}|_{\mathbb{R}^N \times [0,S]} \subseteq B_R \times [0, S],$$

then it is easy to know that u and $\underline{u}$ are a super-solution and a sub-solution to (2.2)–(2.4), respectively. Consequently, we obtain

$$u \ge \underline{u}, \quad (x, t) \in B_R \times (0, S).$$

Then letting $R \to +\infty$ in the aforementioned inequality and utilizing the arbitrariness of S, we obtain the inequality (2.9). The proof is completed. □

Remark 2.1. By the similar arguments in [28], one could show that problems (1.1) and (1.2) admits at most one bounded solution, not satisfying any additional condition at infinity, when $\rho(x) \to 0$ slowly, as $r = |x| \to \infty$.

In the following, we consider the solution of the following equation:

$$u_t = \frac{1}{\rho(x)} \Delta_p u + u^q, \quad (x, t) \in \Omega \times (0, \hat{T}), \tag{2.10}$$

where $\Omega \subset \mathbb{R}^N$. Meantime, the definition of weak solution is given in the following sense.

Definition 2.3. Let $p > 2, q > 1, \hat{T} > 0$. A nonnegative function $\bar{u} \in L^\infty(\Omega \times (0, S)), |\nabla \bar{u}|^p \in L^1(\Omega \times (0, S))$ for any $S < \hat{T}$ is a weak super-solution of (2.10) if the following integral inequality holds

$$\int_0^{\hat{T}} \int_\Omega (-\rho \bar{u} \phi_t + |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \phi - \rho \bar{u}^q \phi) \mathrm{d}x \mathrm{d}t \ge \int_\Omega \rho u_0(x) \phi(x, 0) \mathrm{d}x \tag{2.11}$$

for any nonnegative test function $\phi \in C_0^\infty(\overline{\Omega} \times [0, \hat{T}))$.

Meanwhile, a nonnegative function $\underline{u} \in L^\infty(\Omega \times (0, S))$ is a weak sub-solution of (2.10) if it satisfies (2.11) in the reverse order. We say $u(x, t)$ is a weak solution of (2.10) if it is both a weak super-solution and a weak sub-solution of (2.10).

To prove the main results and reader's convenience, we review the following well-known criterion. Assume $\Omega \subset \mathbb{R}^N$ be an open set, $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$, and $\Sigma \coloneqq \partial\Omega_1 \cap \partial\Omega_2$ is of class C^1, and ν be the unit outward normal to $\partial\Omega_1$ at Σ (ν be the unit inward normal to $\partial\Omega_2$ at Σ). Let

$$u = u_i, \quad (x, t) \in \Omega_i \times [0, \hat{T}), \ (i = 1, 2), \tag{2.12}$$

where $(u_i)_t \in C(\Omega_i \times (0, \hat{T}))$, $u_i \in C^2(\Omega_i \times (0, \hat{T})) \cap C^1(\overline{\Omega}_i \times (0, \hat{T}))$, $i = 1, 2$.

Lemma 2.1. *Assume* (H$_1$)–(H$_3$) *hold.*

(*i*) *If*

$$(u_i)_t \geq \frac{1}{\rho(x)}\Delta_p u_i + u_i^q, \quad (x, t) \in \Omega_i \times (0, \hat{T}), \ (i = 1, 2), \tag{2.13}$$

$$u_1 = u_2, \quad |\nabla u_1|^{p-2}\frac{\partial u_1}{\partial \nu} \geq |\nabla u_2|^{p-2}\frac{\partial u_2}{\partial \nu}, \quad (x, t) \in \Sigma \times (0, \hat{T}), \tag{2.14}$$

then $u(x, t)$ *defined in* (2.12) *is a super-solution of equation* (2.10) *in the sense of Definition* 2.3.

(*ii*) *If*

$$(u_i)_t \leq \frac{1}{\rho(x)}\Delta_p u_i + u_i^q, \quad (x, t) \in \Omega_i \times (0, \hat{T}), \ (i = 1, 2),$$

$$u_1 = u_2, \quad |\nabla u_1|^{p-2}\frac{\partial u_1}{\partial \nu} \leq |\nabla u_2|^{p-2}\frac{\partial u_2}{\partial \nu}, \quad (x, t) \in \Sigma \times (0, \hat{T}),$$

then $u(x, t)$ *defined in* (2.12) *is a sub-solution of equation* (2.10) *in the sense of Definition* 2.3.

Proof. Taking any nonnegative test function $\phi \in C_0^\infty(\overline{\Omega} \times [0, \hat{T}))$ with $\phi|_{\partial\Omega} = 0$ for $\forall t \in [0, \hat{T})$.

(i) Multiplying (2.13) by ϕ and utilizing integrating by parts, one can see that

$$-\int_0^{\hat{T}}\int_{\Omega_1} \rho(u_1\phi_t + u_1^q\phi)\mathrm{d}x\mathrm{d}t \geq -\int_0^{\hat{T}}\int_{\Omega_1} |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla\phi\mathrm{d}x\mathrm{d}t + \int_0^{\hat{T}}\int_{\Sigma} \phi\, |\nabla u_1|^{p-2}\frac{\partial u_1}{\partial \nu}\mathrm{d}S\mathrm{d}t,$$

$$-\int_0^{\hat{T}}\int_{\Omega_2} \rho(u_2\phi_t + u_2^q\phi)\mathrm{d}x\mathrm{d}t \geq -\int_0^{\hat{T}}\int_{\Omega_2} |\nabla u_2|^{p-2}\nabla u_2 \cdot \nabla\phi\mathrm{d}x\mathrm{d}t - \int_0^{\hat{T}}\int_{\Sigma} \phi\, |\nabla u_2|^{p-2}\frac{\partial u_2}{\partial \nu}\mathrm{d}S\mathrm{d}t$$

By summing up the previous two inequalities and employing (2.14), we can derive the inequality

$$-\int_0^{\hat{T}}\int_{\Omega} \rho(u\phi_t + u^q\phi)\mathrm{d}x\mathrm{d}t \geq -\int_0^{\hat{T}}\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi\mathrm{d}x\mathrm{d}t.$$

Thus, the conclusion holds. The result of (ii) can be obtained by using the same approach. This completes the proof. □

3 Global existence for small initial data

In view of (H$_1$), there exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 \leq \rho_2$ such that

$$\rho_1 \leq \frac{1}{\rho(x)} \leq \rho_2, \quad x \in \overline{B}_1. \tag{3.1}$$

According to assumptions (H_1)–(H_2) and (3.1), we may assume that

$$\rho_1 \geq k_1, \quad \rho_2 \leq k_2. \tag{3.2}$$

Theorem 3.1. *Suppose* (H_1)–(H_3), (1.3), (3.1), *and* (3.2) *hold. If* $2 < p < N$ *and*

$$q > \bar{q},$$

where $\bar{q}$ *is given in* (1.4), *initial data* u_0 *are small enough and has compact support, then problems* (1.1)–(1.2) *exists a global solution* $u \in L^\infty(\mathbb{R}^N \times (0, \infty))$.

More precisely, if $C > 0$ is small enough, $T > 0$ is big enough, $a > 0$ and for suitable $0 < A_0 < A_1$ such that

$$A_0 \leq \frac{C^{p-2}}{a^{p-1}} \leq A_1,$$

$$\alpha \in \left(\frac{1}{q-1}, \frac{1}{p-2}\right), \quad \delta := \frac{1 - \alpha(p-2)}{q-1}, \tag{3.3}$$

$$u_0(x) \leq CT^{-\alpha}\left[1 - \frac{\tau(x)}{a}T^{-\delta}\right]_+^{\frac{p-1}{p-2}}, \quad \forall x \in \mathbb{R}^N, \tag{3.4}$$

where

$$\tau(x) := \begin{cases} |x|^{\frac{b}{p-1}}, & \text{if } |x| \geq 1, \\ \dfrac{b|x|^{\frac{b}{p-1}} + p - b}{p}, & \text{if } |x| < 1, \end{cases}$$

then problems (1.1) and (1.2) admit a global solution $u \in L^\infty(\mathbb{R}^N \times (0, \infty))$. Furthermore,

$$u(x,t) \leq C(T+t)^{-\alpha}\left[1 - \frac{\tau(x)}{a}(T+t)^{-\delta}\right]_+^{\frac{p-1}{p-2}}, \quad \forall (x,t) \in \mathbb{R}^N \times [0, +\infty). \tag{3.5}$$

The accurate selection of the parameters $C > 0$, $T > 0$, and $a > 0$ in Theorem 3.1 is discussed in Remark 3.2 below. Note that if u_0 satisfies (3.4), then

$$\|u_0\|_\infty \leq CT^{-\alpha},$$

$$\operatorname{supp} u_0 \subseteq \{x \in \mathbb{R}^N : \tau(x) \leq aT^\delta\}.$$

In view of the selection of C, T, a (see also Remark 3.2), $\|u_0\|_\infty$ is sufficiently small, but supp u_0 can be large, since we can choose $aT^\delta > r_0$ for any fixed $r_0 > 0$.

Furthermore, from (3.5), we deduce that

$$\operatorname{supp} u(x,t) \subseteq \{x \in \mathbb{R}^N : \tau(x) \leq a(T+t)^\delta\}, \quad \forall t > 0. \tag{3.6}$$

Remark 3.1. In [19, Theorem 1], when $\rho(x) = |x|^{-s}$ for any $x \in \mathbb{R}^N \setminus \{0\}$ with $s \in [0, p)$, and for appropriate initial data u_0, which is not necessary to have compact support, a similar result of global existence is proved. Obviously, such $\rho(x)$ does not fulfill our hypotheses (H_1)–(H_2). In addition, we can consider a more general behavior of $\rho(x)$ for $|x|$ large, which influences the definition of critical exponent $\bar{q}$, as well as the selection of q. The detailed conditions in Theorem 3.1 are distinct from that in [19], and it is hard to say which is stronger. Furthermore, we can derive the estimates (3.5) and (3.6), which do not have a counterpart in [19], due to u_0 with compact support. Observe that our proofs and results are totally different with [19], since they are based on the energy method and a smoothing estimate is derived.

For the remainder of this article, we set $r \equiv |x|$. In order to show the global existence of solution, we will construct a suitable super-solution of equation

$$u_t = \frac{1}{\rho(x)}\Delta_p u + u^q, \quad (x,t) \in \mathbb{R}^N \times (0, +\infty). \tag{3.7}$$

To this end, we define the function $\overline{w}(x,t)$ as follows:

$$\overline{w}(x,t) = \overline{w}(r(x),t) := \begin{cases} \overline{u}(x,t), & \text{in } (x,t) \in [\mathbb{R}^N \backslash B_1] \times [0,+\infty), \\ \overline{v}(x,t), & \text{in } (x,t) \in B_1 \times [0,+\infty), \end{cases} \tag{3.8}$$

where

$$\overline{u}(x,t) = \overline{u}(r(x),t) := C\xi(t)\left[1 - \frac{r^{\beta b}}{a}\eta(t)\right]_+^{\frac{p-1}{p-2}}, \tag{3.9}$$

$$\overline{v}(x,t) = \overline{v}(r(x),t) := C\xi(t)\left[1 - \frac{br^{\theta} + p - b}{p}\frac{\eta(t)}{a}\right]_+^{\frac{p-1}{p-2}}, \tag{3.10}$$

with $\xi, \eta \in C^1([0,+\infty);[0,+\infty))$, $C > 0$, $a > 0$, $\beta := \frac{1}{p-1}$, $\theta := \frac{p}{p-1}$.

In addition, we denote

$$\overline{\sigma}(t) := \xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_1\frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left(N + b - p + \frac{b}{p-2}\right), \tag{3.11}$$

$$\overline{\delta}(t) := \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2\frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\frac{b}{p-2}\left(\frac{b\eta}{p-2}\right)^{p-1}, \tag{3.12}$$

$$\overline{\gamma}(t) := C^{q-1}\xi^q, \tag{3.13}$$

$$\overline{\sigma}_0(t) := \xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_1N\frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}, \tag{3.14}$$

$$\overline{\delta}_0(t) := \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2\frac{C^{p-2}}{a^{p}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p}. \tag{3.15}$$

Next, we establish the proposition that plays a crucial role in the proof of Theorem 3.1.

Proposition 3.1. *Let $\xi = \xi(t)$, $\eta = \eta(t) \in C^1([0,+\infty);[0,+\infty))$ and $\overline{\sigma}(t)$, $\overline{\delta}(t)$, $\overline{\gamma}(t)$, $\overline{\sigma}_0(t)$, $\overline{\delta}_0(t)$ be defined in* (3.11)–(3.15). *Assume* (H_1)–(H_2), (1.3), *and* (3.1) *hold, and for* $\forall t \in (0,+\infty)$,

$$\eta(t) < a, \tag{3.16}$$

$$-\frac{\eta'}{\eta^p} \geq \frac{C^{p-2}}{a^{p-1}}\xi^{p-2}\left(\frac{b}{p-2}\right)^{p-1}k_2\beta b, \tag{3.17}$$

$$\xi' + \frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left[k_1\left(N + b - p + \frac{b}{p-2}\right) - \frac{b}{p-2}k_2\right] \geq C^{q-1}\xi^q, \tag{3.18}$$

$$-\frac{\eta'}{\eta^{p+1}} \geq \frac{C^{p-2}}{a^p}\xi^{p-2}\frac{b}{p-1}\left(\frac{b}{p-2}\right)^{p-1}k_2, \tag{3.19}$$

$$\xi' + \frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left[k_1N - \frac{bk_2}{p-2}\frac{\eta}{a}\right] \geq C^{q-1}\xi^q. \tag{3.20}$$

Then $\overline{w}(x,t)$ defined in (3.8) *is a super-solution of equation* (3.7).

Proof. Let

$$F(r,t) \coloneqq 1 - \frac{r^{\beta b}}{a}\eta(t),$$

and define

$$D_1 \coloneqq \{(x,t) \in [\mathbb{R}^N \backslash B_1] \times (0,+\infty) | 0 < F(r,t) < 1\}.$$

For $\forall (x,t) \in D_1$, by a straightforward calculation, one can see that

$$\begin{aligned}\bar{u}_t &= C\xi' F^{\frac{p-1}{p-2}} + C\xi \frac{p-1}{p-2} F^{\frac{1}{p-2}} \left(-\frac{r^{\beta b}}{a}\right)\eta' \\ &= C\xi' F^{\frac{p-1}{p-2}} + C\xi \frac{p-1}{p-2} F^{\frac{1}{p-2}} \left(1 - \frac{r^{\beta b}}{a}\eta\right)\frac{\eta'}{\eta} - C\xi \frac{p-1}{p-2} F^{\frac{1}{p-2}} \frac{\eta'}{\eta} \\ &= C\xi' F^{\frac{p-1}{p-2}} + C\xi \frac{p-1}{p-2} F^{\frac{p-1}{p-2}} \frac{\eta'}{\eta} - C\xi \frac{p-1}{p-2} F^{\frac{1}{p-2}} \frac{\eta'}{\eta};\end{aligned} \tag{3.21}$$

$$\bar{u}_r = C\xi \frac{p-1}{p-2} F^{\frac{1}{p-2}} \left(-\frac{\beta b}{a}\eta\right) r^{\beta b - 1}; \tag{3.22}$$

$$\begin{aligned}\bar{u}_{rr} &= C\xi \frac{p-1}{p-2} F^{\frac{1}{p-2}} \left(-\frac{\beta b}{a}\eta\right)(\beta b - 1) r^{\beta b-2} - C\xi \frac{p-1}{(p-2)^2} F^{\frac{1}{p-2}-1} \left(-\frac{r^{\beta b}}{a}\eta\right) \frac{(\beta b)^2}{a} \eta r^{\beta b - 2} \\ &= C\xi \frac{p-1}{p-2} F^{\frac{1}{p-2}} \left(-\frac{\beta b}{a}\eta\right)(\beta b - 1) r^{\beta b-2} - C\xi \frac{p-1}{(p-2)^2} F^{\frac{1}{p-2}} \frac{(\beta b)^2}{a} \eta r^{\beta b-2} \\ &\quad + C\xi \frac{p-1}{(p-2)^2} F^{\frac{1}{p-2}-1} \frac{(\beta b)^2}{a} \eta r^{\beta b-2}.\end{aligned} \tag{3.23}$$

By virtue of (3.22), (3.23) and the fact $(p-1)(\beta b - 1) = b - p + 1$, we obtain

$$\begin{aligned}\Delta_p \bar{u} &= (|\bar{u}_r|^{p-2}\bar{u}_r)_r + \frac{N-1}{r}|\bar{u}_r|^{p-2}\bar{u}_r \\ &= C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p F^{\frac{1}{p-2}} \left(\frac{\beta b}{a}\eta\right)^{p-1} \beta b r^{(\beta b-1)(p-1)-1} \\ &\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p F^{\frac{p-1}{p-2}} \left(\frac{\beta b}{a}\eta\right)^{p-1} \beta b r^{(\beta b-1)(p-1)-1} \\ &\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1} F^{\frac{p-1}{p-2}} \left(\frac{\beta b}{a}\eta\right)^{p-1} (p-1)(\beta b - 1) r^{(\beta b-1)(p-1)-1} \\ &\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1} F^{\frac{p-1}{p-2}} \left(\frac{\beta b}{a}\eta\right)^{p-1} (N-1) r^{(\beta b-1)(p-1)-1} \\ &= C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p F^{\frac{1}{p-2}} \left(\frac{\beta b}{a}\eta\right)^{p-1} \beta b r^{b-p} \\ &\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p F^{\frac{p-1}{p-2}} \left(\frac{\beta b}{a}\eta\right)^{p-1} \beta b r^{b-p} \\ &\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1} F^{\frac{p-1}{p-2}} \left(\frac{\beta b}{a}\eta\right)^{p-1} (N + b - p) r^{b-p}.\end{aligned} \tag{3.24}$$

Then it follows from (3.21) and (3.24) that

$$\begin{aligned}
&\overline{u}_t - \frac{1}{\rho(x)}\Delta_p\overline{u} - \overline{u}^q \\
&= C\xi' F^{\frac{p-1}{p-2}} + C\xi\frac{p-1}{p-2}F^{\frac{p-1}{p-2}}\frac{\eta'}{\eta} - C\xi\frac{p-1}{p-2}F^{\frac{1}{p-2}}\frac{\eta'}{\eta} \\
&\quad + \frac{r^{b-p}}{\rho}\left\{C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}F^{\frac{p-1}{p-2}}\left(\frac{\beta b}{a}\eta\right)^{p-1}\left(N+b-p+\frac{b}{p-2}\right)\right. \\
&\quad \left. - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p}F^{\frac{1}{p-2}}\left(\frac{\beta b}{a}\eta\right)^{p-1}\beta b\right\} - C^q\xi^qF^{\frac{(p-1)q}{p-2}}.
\end{aligned} \tag{3.25}$$

By hypothesis (H$_2$), we have

$$\frac{r^{b-p}}{\rho} \geq k_1, \quad -\frac{r^{b-p}}{\rho} \geq -k_2, \quad \forall x \in \mathbb{R}^N \backslash B_1. \tag{3.26}$$

From (3.25) and (3.26), we obtain

$$\begin{aligned}
&\overline{u}_t - \frac{1}{\rho(x)}\Delta_p\overline{u} - \overline{u}^q \\
&\geq CF^{\frac{1}{p-2}}\left\{F\left[\xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_1C^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\left(\frac{\beta b}{a}\eta\right)^{p-1}\left(N+b-p+\frac{b}{p-2}\right)\right]\right. \\
&\quad \left. - \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} - k_2\beta bC^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p}\left(\frac{\beta b}{a}\eta\right)^{p-1} - C^{q-1}\xi^qF^{\frac{(p-1)q-1}{p-2}}\right\}.
\end{aligned} \tag{3.27}$$

Therefore, taking advantage of (3.27) and the definitions of $\overline{\sigma}(t)$, $\overline{\delta}(t)$, $\overline{y}(t)$, one can see

$$\overline{u}_t - \frac{1}{\rho(x)}\Delta_p\overline{u} - \overline{u}^q \geq CF^{\frac{1}{p-2}}\left[\overline{\sigma}(t)F - \overline{\delta}(t) - \overline{y}(t)F^{\frac{(p-1)q-1}{p-2}}\right].$$

For each $t > 0$, set

$$\varphi(F) \coloneqq \overline{\sigma}(t)F - \overline{\delta}(t) - \overline{y}(t)F^{\frac{(p-1)q-1}{p-2}}, \quad F \in (0,1).$$

We aim to find suitable C, a, ξ, η such that, for each $t > 0$,

$$\varphi(F) \geq 0, \quad \forall F \in (0,1).$$

Indeed, computing the second derivative of φ with regard to F, one can see that

$$\varphi''(F) = -\frac{(p-1)q-1}{p-2}\frac{(p-1)(q-1)}{p-2}F^{\frac{(p-1)(q-1)}{p-2}-1} < 0.$$

As a result, $\varphi(F)$ is concave with respect to the variable F, and so it suffices to verify that

$$\varphi(0) \geq 0, \quad \varphi(1) \geq 0, \tag{3.28}$$

which ensures $\varphi(F) \geq 0$, $\forall F \in (0,1)$. Evidently, (3.28) is equivalent to

$$-\overline{\delta}(t) \geq 0, \quad \overline{\sigma}(t) - \overline{\delta}(t) - \overline{y}(t) \geq 0,$$

namely,

$$-\frac{\eta'}{\eta^p} \geq \frac{C^{p-2}}{a^{p-1}}\xi^{p-2}\left(\frac{b}{p-2}\right)^{p-1}k_2\beta b, \tag{3.29}$$

$$\xi' + \frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left[k_1\left(N+b-p+\frac{b}{p-2}\right) - \frac{b}{p-2}k_2\right] \geq C^{q-1}\xi^q, \tag{3.30}$$

which is guaranteed by (1.3), (3.17), and (3.18). Hence, we have proved that

$$\bar{u}_t - \frac{1}{\rho(x)}\Delta_p\bar{u} - \bar{u}^q \geq 0, \quad (x,t) \in D_1.$$

Set $\Omega_1 = D_1$, $\Omega_2 = \mathbb{R}^N \backslash [B_1 \cup D_1]$, $u_1 = \bar{u}$, $u_2 = 0$, $u = \bar{u}$, by virtue of Lemma 2.1-(i), we obtain that $\bar{u}$ is a super-solution of equation (3.7) in $[\mathbb{R}^N \backslash B_1] \times (0, +\infty)$, i.e.,

$$\bar{u}_t - \frac{1}{\rho(x)}\Delta_p\bar{u} - \bar{u}^q \geq 0, \quad (x,t) \in [\mathbb{R}^N \backslash B_1] \times (0,+\infty). \tag{3.31}$$

On the other hand, set

$$G(r,t) \coloneqq 1 - \frac{br^\theta + p - b}{p}\frac{\eta(t)}{a}.$$

In view of $\eta(t) < a$, it can be easily seen that

$$0 < G(r,t) < 1, \quad \forall (x,t) \in B_1 \times (0,+\infty).$$

For any $(x,t) \in B_1 \times (0,+\infty)$, a straightforward calculation yields

$$\bar{v}_t = C\xi' G^{\frac{p-1}{p-2}} + C\xi\frac{p-1}{p-2}G^{\frac{p-1}{p-2}}\frac{\eta'}{\eta} - C\xi\frac{p-1}{p-2}G^{\frac{1}{p-2}}\frac{\eta'}{\eta}; \tag{3.32}$$

$$\bar{v}_r = C\xi\frac{p-1}{p-2}G^{\frac{1}{p-2}}\left(-\frac{\theta b}{p}\frac{\eta}{a}\right)r^{\theta-1}; \tag{3.33}$$

$$\bar{v}_{rr} = C\xi\frac{p-1}{p-2}G^{\frac{1}{p-2}}\left(-\frac{\theta b}{p}\frac{\eta}{a}\right)(\theta-1)r^{\theta-2} + C\xi\frac{p-1}{(p-2)^2}G^{\frac{1}{p-2}-1}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^2 r^{2(\theta-1)}. \tag{3.34}$$

By virtue of (3.33), (3.34), and the fact $(p-1)(\theta-1) = 1$, we obtain

$$\begin{aligned}
\Delta_p\bar{v} &= (|\bar{v}_r|^{p-2}\bar{v}_r)_r + \frac{N-1}{r}|\bar{v}_r|^{p-2}\bar{v}_r \\
&= C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p G^{\frac{1}{p-2}}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^p r^{p(\theta-1)} \\
&\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1} G^{\frac{p-1}{p-2}}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^{p-1}(p-1)(\theta-1)r^{(\theta-1)(p-1)-1} \\
&\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1} G^{\frac{p-1}{p-2}}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^{p-1}(N-1)r^{(\theta-1)(p-1)-1} \\
&= C^{p-1}\xi^{p-1}\left[\left(\frac{p-1}{p-2}\right)^p G^{\frac{1}{p-2}}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^p r^\theta - \left(\frac{p-1}{p-2}\right)^{p-1} G^{\frac{p-1}{p-2}}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^{p-1} N\right].
\end{aligned} \tag{3.35}$$

From (3.32) and (3.35), we deduce

$$\begin{aligned}
\bar{v}_t - \frac{1}{\rho(x)}\Delta_p\bar{v} - \bar{v}^q &= C\xi' G^{\frac{p-1}{p-2}} + C\xi\frac{p-1}{p-2}G^{\frac{p-1}{p-2}}\frac{\eta'}{\eta} - C\xi\frac{p-1}{p-2}G^{\frac{1}{p-2}}\frac{\eta'}{\eta} \\
&\quad + \frac{N}{\rho}C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1} G^{\frac{p-1}{p-2}}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^{p-1} \\
&\quad - \frac{r^\theta}{\rho}C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p G^{\frac{1}{p-2}}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^p - C^q\xi^q G^{\frac{(p-1)q}{p-2}}.
\end{aligned} \tag{3.36}$$

By applying (3.1), (3.36), and the fact that $r \in (0,1)$, we obtain

$$\begin{aligned}
&\overline{v}_t - \frac{1}{\rho(x)}\Delta_p\overline{v} - \overline{v}^q \\
&\geq CG^{\frac{1}{p-2}}\left\{G\left[\xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_1NC^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^{p-1}\right]\right. \\
&\quad \left. - \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} - k_2C^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p}\left(\frac{\theta b}{p}\frac{\eta}{a}\right)^{p} - C^{q-1}\xi^qG^{\frac{(p-1)q-1}{p-2}}\right\}.
\end{aligned} \tag{3.37}$$

Therefore, taking advantage of (3.37) and the definitions of $\overline{\sigma}_0(t)$, $\overline{\delta}_0(t)$, $\overline{\gamma}(t)$, one can see

$$\overline{v}_t - \frac{1}{\rho(x)}\Delta_p\overline{v} - \overline{v}^q \geq CG^{\frac{1}{p-2}}\left[\overline{\sigma}_0(t)G - \overline{\delta}_0(t) - \overline{\gamma}(t)G^{\frac{(p-1)q-1}{p-2}}\right].$$

For each $t > 0$, set

$$\psi(G) := \overline{\sigma}_0(t)G - \overline{\delta}_0(t) - \overline{\gamma}(t)G^{\frac{(p-1)q-1}{p-2}}, \quad G \in (0,1).$$

We aim to find suitable C, a, ξ, η such that, for each $t > 0$,

$$\psi(G) \geq 0, \quad \forall G \in (0,1).$$

Indeed, computing the second derivative of ψ with regard to G, one can see that

$$\psi''(G) = -\frac{(p-1)q-1}{p-2}\frac{(p-1)(q-1)}{p-2}G^{\frac{(p-1)(q-1)}{p-2}-1} < 0.$$

As a result, $\psi(G)$ is concave with respect to the variable G, and so it suffices to verify that

$$\psi(0) \geq 0, \quad \psi(1) \geq 0, \tag{3.38}$$

which ensures $\psi(G) \geq 0$, $\forall G \in (0,1)$. Evidently, (3.38) is equivalent to

$$-\overline{\delta}_0(t) \geq 0, \quad \overline{\sigma}_0(t) - \overline{\delta}_0(t) - \overline{\gamma}(t) \geq 0,$$

that is,

$$-\frac{\eta'}{\eta^{p+1}} \geq \frac{C^{p-2}}{a^p}\xi^{p-2}\frac{b}{p-1}\left(\frac{b}{p-2}\right)^{p-1}k_2, \tag{3.39}$$

$$\xi' + \frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left[k_1N - \frac{bk_2}{p-2}\frac{\eta}{a}\right] \geq C^{q-1}\xi^q, \tag{3.40}$$

which is guaranteed by (3.19) and (3.20). Hence, we have proved that

$$\overline{v}_t - \frac{1}{\rho(x)}\Delta_p\overline{v} - \overline{v}^q \geq 0, \quad \forall (x,t) \in B_1 \times (0,+\infty). \tag{3.41}$$

As a consequence, $\overline{v}(x,t)$ is a super-solution of equation (3.7) in $B_1 \times (0,+\infty)$.

Now, observe that $\overline{w} \in C(\mathbb{R}^N \times [0,+\infty))$. Indeed,

$$\overline{u} = \overline{v} = C\xi(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{p-1}{p-2}}, \quad \forall (x,t) \in \partial B_1 \times (0,+\infty). \tag{3.42}$$

Furthermore, for $\forall (x,t) \in \partial B_1 \times (0,+\infty)$,

$$|\overline{u}_r|^{p-2}\overline{u}_r = |\overline{v}_r|^{p-2}\overline{v}_r = -C^{p-1}\xi^{p-1}(t)\left(\frac{b}{p-2}\frac{\eta(t)}{a}\right)^{p-1}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{p-1}{p-2}}. \tag{3.43}$$

In conclusion, set $\Omega_1 = \mathbb{R}^N \backslash B_1$, $\Omega_2 = B_1$, $u_1 = \overline{u}$, $u_2 = \overline{v}$, $u = \overline{w}$, by virtue of (3.31), (3.41)–(3.43) and Lemma 2.1-(i), we obtain that $\overline{w}(x,t)$ is a super-solution of equation (3.7). The proof is completed. □

Remark 3.2. Let

$$q > \bar{q},$$

and (1.3) and (3.1) hold. Meanwhile, denote $A := \frac{C^{p-2}}{\alpha^{p-1}}$. In Theorem 3.1, the accurate assumptions on parameters $\alpha, \delta, C > 0$, $A > 0$, $T > 0$ are the following:

$$\alpha, \delta \text{ satisfy the condition (3.3)},$$

$$\delta - A\left(\frac{b}{p-2}\right)^{p-1} \frac{bk_2}{p-1} \geq 0, \tag{3.44}$$

$$-\alpha + A\left(\frac{b}{p-2}\right)^{p-1} \left[k_1\left(N + b - p + \frac{b}{p-2}\right) - \frac{b}{p-2}k_2\right] \geq C^{q-1}, \tag{3.45}$$

$$\delta T^{\delta} \geq A\frac{1}{\alpha}\left(\frac{b}{p-2}\right)^{p-1} \frac{b}{p-1}k_2, \tag{3.46}$$

$$T^{\delta} > \frac{r_0}{\alpha}, \quad r_0 > 1, \tag{3.47}$$

$$-\alpha + A\left(\frac{b}{p-2}\right)^{p-1} \left(k_1 N - \frac{1}{p-2}\frac{b}{\alpha}k_2 T^{-\delta}\right) \geq C^{q-1}. \tag{3.48}$$

Lemma 3.1. *All the conditions in* Remark 3.2 *can be fulfilled concurrently.*

Proof. We take α satisfying (3.3) and

$$\alpha < \min\left\{\frac{k_1(N + b - p) + \frac{b}{p-2}(k_1 - k_2)}{k_1(N + b - p)(p - 2) + k_1 b}, \frac{(p-2)k_1 N - bk_2}{(p-2)^2 k_1 N}, \frac{1}{p-2}\right\}. \tag{3.49}$$

In view of

$$q > \bar{q} > p - 1 + \frac{bk_2}{Nk_1} > p - 1,$$

then (3.49) holds. Moreover, making use of (1.3), (3.49), and the fact that $\delta = \frac{1-\alpha(p-2)}{p-1}$, we can choose $A > 0$ such that (3.44) holds, the left side of (3.45) is positive, and

$$-\alpha + A\left(\frac{b}{p-2}\right)^{p-1} (k_1 N - \varepsilon) > 0,$$

for some $\varepsilon > 0$. Then, we select $C > 0$ so small that (3.45) holds and

$$-\alpha + A\left(\frac{b}{p-2}\right)^{p-1} (k_1 N - \varepsilon) > C^{q-1}. \tag{3.50}$$

Select $T > 0$ large enough so that (3.46) and (3.47) are valid and

$$k_1 N - \frac{1}{p-2}\frac{b}{\alpha}k_2 T^{-\delta} \geq \varepsilon. \tag{3.51}$$

Therefore, from (3.50) and (3.51), inequality (3.48) follows. The proof is completed. □

We are ready to give the proof of the main Theorem 3.1.

Proof of Theorem 3.1. According to Lemma 3.1, we can assume that all the conditions of Remark 3.2 are satisfied. Set

$$\xi(t) = (T+t)^{-\alpha}, \quad \eta(t) = (T+t)^{-\delta}, \ \forall t > 0.$$

Observe that condition (3.47) implies $\eta(t) < a$. In addition, combining conditions (3.17), (3.18) of Proposition 3.1 with the selection of $\xi(t)$ and $\eta(t)$, direct calculations show that

$$\delta - \frac{C^{p-2}}{a^{p-1}}\left(\frac{b}{p-2}\right)^{p-1} \beta b k_2 (T+t)^{-\alpha(p-2)-\delta(p-1)+1} \geq 0, \tag{3.52}$$

$$-\alpha(T+t)^{-\alpha-1} - C^{q-1}(T+t)^{-\alpha q} + \frac{C^{p-2}}{a^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}\left[k_1\left(N+b-p+\frac{b}{p-2}\right) - \frac{bk_2}{p-2}\right](T+t)^{-(\alpha+\delta)(p-1)} \geq 0. \tag{3.53}$$

In view of $\delta = \frac{1-\alpha(p-2)}{p-1}$, (3.52) and (3.53) become

$$\frac{C^{p-2}}{a^{p-1}}\left(\frac{b}{p-2}\right)^{p-1} \leq \frac{1-\alpha(p-2)}{bk_2}, \tag{3.54}$$

$$\left\{-\alpha + \frac{C^{p-2}}{a^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}\left[k_1\left(N+b-p+\frac{b}{p-2}\right) - \frac{b}{p-2}k_2\right]\right\}(T+t)^{-\alpha-1} \geq C^{q-1}(T+t)^{-\alpha q}. \tag{3.55}$$

By the assumption $\alpha \in \left(\frac{1}{q-1}, \frac{1}{p-2}\right)$, we obtain

$$\delta > 0, \ -\alpha - 1 > -q\alpha. \tag{3.56}$$

Then (3.54) and (3.55) are direct consequence of (3.44), (3.45), and (3.56).

We now consider conditions (3.19) and (3.20) of Proposition 3.1. Substituting $\xi(t)$, $\eta(t)$, α and δ previously chosen, we obtain (3.46) and the following inequality:

$$\left[-\alpha + \frac{C^{p-2}}{a^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}\left(k_1 N - \frac{bk_2}{p-2}\frac{1}{a}(T+t)^{-\delta}\right)\right](T+t)^{-\alpha-1} \geq C^{q-1}(T+t)^{-\alpha q}. \tag{3.57}$$

Obviously, condition (3.57) follows from (3.48) and (3.56). Hence, we can choose suitable $\alpha, \delta, C > 0, a > 0$, and T so that (3.46), (3.54), (3.55), and (3.57) hold. By means of Propositions 2.4 and 3.1, the conclusion of Theorem 3.1 holds. The proof is completed. □

4 Blow-up for large initial data

In this section, we present the blow-up phenomena of the solution to problems (1.1) and (1.2) with sufficiently large initial data.

Theorem 4.1. *Suppose* (H_1)–(H_3), (3.1) *and* (3.2) *hold. If* $2 < p < N$ *and* $q > 1$, $T > 0$, *and initial data* u_0 *is large enough, then the solution* u *of problems* (1.1) *and* (1.2) *blows up in a finite time* $S \in (0, T]$, *in the sense that*

$$\|u(t)\|_\infty \to \infty, \ t \to S^-.$$

More precisely, we have the following two cases:

(i) Let $q > p-1$. If $C > 0$ is large enough, $a > 0$, $T > 0$,

$$u_0(x) \geq CT^{-\frac{1}{q-1}}\left[1 - \frac{s(x)}{a}T^{\frac{q-(p-1)}{(p-1)(q-1)}}\right]_+^{\frac{p-1}{p-2}}, \quad \forall x \in \mathbb{R}^N, \tag{4.1}$$

where

$$s(x) := \begin{cases} |x|^{\frac{b}{p-1}}, & \text{if } |x| \ge 1, \\ |x|^{\frac{p}{p-1}}, & \text{if } |x| < 1, \end{cases}$$

then the solutions of problems (1.1) and (1.2) blow up and satisfy the bound from below

$$u(x,t) \ge C(T-t)^{-\frac{1}{q-1}}\left[1 - \frac{s(x)}{a}(T-t)^{\frac{q-(p-1)}{(p-1)(q-1)}}\right]_+^{\frac{p-1}{p-2}}, \quad \forall (x,t) \in \mathbb{R}^N \times [0,S). \tag{4.2}$$

(ii) Let $q \le p-1$. If $\frac{C^{p-2}}{a^{p-1}} > 0$, a is large enough, $T > 0$, and (4.1) holds, then the solutions of problems (1.1) and (1.2) blows up and satisfies the bound from below (4.2).

Observe that if u_0 satisfies (4.1), then

$$\operatorname{supp} u_0 \supseteq \left\{x \in \mathbb{R}^N : s(x) \le aT^{\frac{(p-1)-q}{(p-1)(q-1)}}\right\}.$$

Furthermore, in the cases (i) and (ii), from (4.2), we infer that

$$\operatorname{supp} u(\cdot,t) \supseteq \left\{x \in \mathbb{R}^N : s(x) \le a(T-t)^{\frac{(p-1)-q}{(p-1)(q-1)}}\right\}, \quad \forall t \in [0,S). \tag{4.3}$$

The accurate selection of the parameters $C > 0$, $T > 0$ and $a > 0$ in Theorem 4.1 is discussed in Remark 4.1.

Remark 4.1. Due to our results concerning any $q > 1$ and sufficiently large initial data, there is no counterpart of Theorem 3 in [19], where some blow-up results are shown for problems (1.1) and (1.2).

To show the result of blow-up, we define

$$\underline{w}(x,t) = \underline{w}(r(x),t) := \begin{cases} \underline{u}(x,t), & \text{in } (x,t) \in [\mathbb{R}^N \backslash B_1] \times [0,T), \\ \underline{v}(x,t), & \text{in } (x,t) \in B_1 \times [0,T), \end{cases} \tag{4.4}$$

where

$$\underline{u}(x,t) = \underline{u}(r(x),t) := C\xi(t)\left[1 - \frac{r^{\beta b}}{a}\eta(t)\right]_+^{\frac{p-1}{p-2}}, \tag{4.5}$$

$$\underline{v}(x,t) = \underline{v}(r(x),t) := C\xi(t)\left[1 - \frac{r^{\theta}}{a}\eta(t)\right]_+^{\frac{p-1}{p-2}}, \tag{4.6}$$

with $\xi, \eta \in C^1([0,T);[0,+\infty))$, $C > 0$, $a > 0$, $\beta := \frac{1}{p-1}$, $\theta := \frac{p}{p-1}$.

In addition, we define

$$K := \left[\frac{p-2}{(p-1)q-1}\right]^{\frac{p-2}{(p-1)(q-1)}} - \left[\frac{p-2}{(p-1)q-1}\right]^{\frac{(p-1)q}{(p-1)(q-1)}} > 0, \tag{4.7}$$

$$\underline{\sigma}(t) := \xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2\frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left(N + b - p + \frac{b}{p-2}\right), \tag{4.8}$$

$$\underline{\delta}(t) := \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_1\frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\frac{b}{p-2}\left(\frac{b\eta}{p-2}\right)^{p-1}, \tag{4.9}$$

$$\underline{\gamma}(t) := C^{q-1}\xi^q, \tag{4.10}$$

$$\underline{\sigma}_0(t) := \xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2\frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\left(\frac{p\eta}{p-2}\right)^{p-1}\left(N + \frac{p}{p-2}\right), \tag{4.11}$$

$$\underline{\delta}_0(t) := \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_1\frac{C^{p-2}}{a^{p-1}}\xi^{p-1}\frac{p}{p-2}\left(\frac{p\eta}{p-2}\right)^{p-1}. \tag{4.12}$$

Next, we establish the proposition that plays a key role in the proof of Theorem 4.1.

Proposition 4.1. *Let $T \in (0,\infty)$, $\xi = \xi(t), \eta = \eta(t) \in C^1([0,T);[0,+\infty))$, K and $\underline{\sigma}(t)$, $\underline{\delta}(t)$, $\underline{\gamma}(t)$, $\underline{\sigma}_0(t)$, $\underline{\delta}_0(t)$ be defined in* (4.7)–(4.12). *Assume* (H_1)–(H_2), (3.1) *and* (3.2) *hold, and $\forall t \in (0,T)$,*

$$K[\underline{\sigma}(t)]^{\frac{(p-1)q-1}{(p-1)(q-1)}} \le \underline{\delta}(t)[\underline{\gamma}(t)]^{\frac{p-2}{(p-1)(q-1)}}, \tag{4.13}$$

$$(p-2)\underline{\sigma}(t) \le [(p-1)q-1]\underline{\gamma}(t), \tag{4.14}$$

$$K[\underline{\sigma}_0(t)]^{\frac{(p-1)q-1}{(p-1)(q-1)}} \le \underline{\delta}_0(t)[\underline{\gamma}(t)]^{\frac{p-2}{(p-1)(q-1)}}, \tag{4.15}$$

$$(p-2)\underline{\sigma}_0(t) \le [(p-1)q-1]\underline{\gamma}(t). \tag{4.16}$$

Then $\underline{w}(x,t)$ defined in (4.4) *is a sub-solution of equation* (3.7).

Proof. For any $(x,t) \in D_1$, from (3.21) and (3.24), one can see that

$$\begin{aligned} &\underline{u}_t - \frac{1}{\rho(x)}\Delta_p\underline{u} - \underline{u}^q \\ &= C\xi' F^{\frac{p-1}{p-2}} + C\xi\frac{p-1}{p-2}F^{\frac{p-1}{p-2}}\frac{\eta'}{\eta} - C\xi\frac{p-1}{p-2}F^{\frac{1}{p-2}}\frac{\eta'}{\eta} \\ &\quad + \frac{r^{b-p}}{\rho}\left\{C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}F^{\frac{p-1}{p-2}}\left(\frac{\beta b}{a}\eta\right)^{p-1}\left(N+b-p+\frac{b}{p-2}\right)\right. \\ &\quad \left. - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p}F^{\frac{1}{p-2}}\left(\frac{\beta b}{a}\eta\right)^{p-1}\beta b\right\} - C^q\xi^q F^{\frac{(p-1)q}{p-2}}. \end{aligned} \tag{4.17}$$

According to hypothesis (H_2), we have

$$-\frac{r^{b-p}}{\rho} \le -k_1, \quad \frac{r^{b-p}}{\rho} \le k_2, \quad \forall x \in \mathbb{R}^N \backslash B_1. \tag{4.18}$$

Combining (4.17) and (4.18), we obtain

$$\begin{aligned} &\underline{u}_t - \frac{1}{\rho(x)}\Delta_p\underline{u} - \underline{u}^q \\ &\le CF^{\frac{1}{p-2}}\left\{F\left[\xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2C^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\left(\frac{\beta b}{a}\eta\right)^{p-1}\left(N+b-p+\frac{b}{p-2}\right)\right]\right. \\ &\quad \left. - \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} - k_1\beta bC^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p}\left(\frac{\beta b}{a}\eta\right)^{p-1} - C^{q-1}\xi^q F^{\frac{(p-1)q-1}{p-2}}\right\}. \end{aligned} \tag{4.19}$$

Therefore, taking advantage of (4.19) and the definitions of $\underline{\sigma}(t)$, $\underline{\delta}(t)$, $\underline{\gamma}(t)$, we obtain

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta_p\underline{u} - \underline{u}^q \le CF^{\frac{1}{p-2}}\left[\underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{(p-1)q-1}{p-2}}\right].$$

For each $t \in (0,T)$, set

$$\varphi_0(F) := \underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{(p-1)q-1}{p-2}}, \quad F \in (0,1).$$

We aim to find suitable C, a, ξ, η such that, for each $t \in (0,T)$,

$$\varphi_0(F) \le 0, \quad \forall F \in (0,1).$$

To this end, we impose that

$$\sup_{F\in(0,1)} \varphi_0(F) = \max_{F\in(0,1)} \varphi_0(F) = \varphi_0(F_0) \le 0,$$

with $F_0 \in (0,1)$. Then we have

$$\begin{aligned}\frac{\mathrm{d}\varphi_0}{\mathrm{d}F} = 0 &\Leftrightarrow \underline{\sigma}_0(t) - \frac{(p-1)q-1}{p-2}\underline{\gamma}(t)F^{\frac{(p-1)(q-1)}{p-2}} = 0\\ &\Leftrightarrow F = F_0 = \left[\frac{p-2}{(p-1)q-1}\frac{\underline{\sigma}(t)}{\underline{\gamma}(t)}\right]^{\frac{p-2}{(p-1)(q-1)}},\end{aligned}$$

from which, we obtain

$$\varphi_0(F_0) = K\frac{[\underline{\sigma}(t)]^{\frac{(p-1)q-1}{(p-1)(q-1)}}}{[\underline{\gamma}(t)]^{\frac{p-2}{(p-1)(q-1)}}} - \underline{\delta}(t),$$

where the coefficient K depending on p and q has been defined in (4.7). By hypotheses (4.13) and (4.14), for each $t \in (0,T)$,

$$\varphi_0(F_0) \le 0, \quad F_0 \le 1,$$

which implies $\varphi_0(F) \le 0$, $\forall F \in (0,1)$. So far, we have proved that

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta_p\underline{u} - \underline{u}^q \le 0, \quad (x,t) \in D_1. \tag{4.20}$$

Set $\Omega_1 = D_1$, $\Omega_2 = \mathbb{R}^N\backslash[B_1 \cup D_1]$, $u_1 = \underline{u}$, $u_2 = 0$, $u = \underline{u}$, by virtue of Lemma 2.1-(ii), we obtain that $\underline{u}$ is a sub-solution of equation (3.7) in $[\mathbb{R}^N\backslash B_1] \times (0,T)$.

On the other hand, set

$$\widetilde{G}(r,t) \coloneqq 1 - \frac{r^\theta}{a}\eta(t),$$

and define

$$D_2 \coloneqq \{(x,t) \in B_1 \times (0,T) | 0 < \widetilde{G}(r,t) < 1\}.$$

For any $(x,t) \in D_2$, by a straightforward calculation, one can see that

$$\underline{v}_t = C\xi'\widetilde{G}^{\frac{p-1}{p-2}} + C\xi\frac{p-1}{p-2}\widetilde{G}^{\frac{p-1}{p-2}}\frac{\eta'}{\eta} - C\xi\frac{p-1}{p-2}\widetilde{G}^{\frac{1}{p-2}}\frac{\eta'}{\eta}; \tag{4.21}$$

$$\underline{v}_r = C\xi\frac{p-1}{p-2}\widetilde{G}^{\frac{1}{p-2}}\left(-\frac{\theta}{a}\eta\right)r^{\theta-1}; \tag{4.22}$$

$$\underline{v}_{rr} = C\xi\frac{p-1}{p-2}\widetilde{G}^{\frac{1}{p-2}}\left(-\frac{\theta}{a}\eta\right)(\theta-1)r^{\theta-2} - C\xi\frac{p-1}{(p-2)^2}\widetilde{G}^{\frac{1}{p-2}}\frac{\theta^2}{a}\eta r^{\theta-2} + C\xi\frac{p-1}{(p-2)^2}\widetilde{G}^{\frac{1}{p-2}-1}\frac{\theta^2}{a}\eta r^{\theta-2}. \tag{4.23}$$

By virtue of (4.22), (4.23) and the fact $(p-1)(\theta-1) = 1$, we obtain

$$\begin{aligned}\Delta_p\underline{v} &= (|\underline{v}_r|^{p-2}\underline{v}_r)_r + \frac{N-1}{r}|\underline{v}_r|^{p-2}\underline{v}_r\\ &= C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p\widetilde{G}^{\frac{1}{p-2}}\left(\frac{\theta}{a}\eta\right)^{p-1}\theta - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^p\widetilde{G}^{\frac{p-1}{p-2}}\left(\frac{\theta}{a}\eta\right)^{p-1}\theta\\ &\quad - C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\widetilde{G}^{\frac{p-1}{p-2}}\left(\frac{\theta}{a}\eta\right)^{p-1}N.\end{aligned} \tag{4.24}$$

In view of (4.21) and (4.24), we deduce

$$\begin{aligned} &\underline{v}_t - \frac{1}{\rho(x)}\Delta_p \underline{v} - \underline{v}^q \\ &= C\xi'\widetilde{G}^{\frac{p-1}{p-2}} + C\xi\frac{p-1}{p-2}\widetilde{G}^{\frac{p-1}{p-2}}\frac{\eta'}{\eta} - C\xi\frac{p-1}{p-2}\widetilde{G}^{\frac{1}{p-2}}\frac{\eta'}{\eta} \\ &\quad + \frac{1}{\rho}C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\widetilde{G}^{\frac{p-1}{p-2}}\left(\frac{\theta}{a}\eta\right)^{p-1}\left(N + \frac{p}{p-2}\right) \\ &\quad - \frac{1}{\rho}C^{p-1}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p}\widetilde{G}^{\frac{1}{p-2}}\left(\frac{\theta}{a}\eta\right)^{p-1}\theta - C^q\xi^q\widetilde{G}^{\frac{(p-1)q}{p-2}}. \end{aligned} \tag{4.25}$$

By (3.1) and (4.25), we have

$$\begin{aligned} &\underline{v}_t - \frac{1}{\rho(x)}\Delta_p \underline{v} - \underline{v}^q \\ &\le C\widetilde{G}^{\frac{1}{p-2}}\left\{\widetilde{G}\left[\xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2C^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\left(\frac{\theta}{a}\eta\right)^{p-1}\left(N + \frac{p}{p-2}\right)\right]\right. \\ &\quad \left. - \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} - k_1\theta C^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p}\left(\frac{\theta}{a}\eta\right)^{p-1} - C^{q-1}\xi^q\widetilde{G}^{\frac{(p-1)q-1}{p-2}}\right\}. \end{aligned} \tag{4.26}$$

Then, according to (4.26) and the definitions of $\underline{\sigma}_0(t)$, $\underline{\delta}_0(t)$, $\underline{\gamma}(t)$, we obtain

$$\underline{v}_t - \frac{1}{\rho(x)}\Delta_p \underline{v} - \underline{v}^q \le C\widetilde{G}^{\frac{1}{p-2}}\left[\underline{\sigma}_0(t)\widetilde{G} - \underline{\delta}_0(t) - \underline{\gamma}(t)\widetilde{G}^{\frac{(p-1)q-1}{p-2}}\right].$$

For each $t \in (0, T)$, set

$$\psi_0(\widetilde{G}) \coloneqq \underline{\sigma}_0(t)\widetilde{G} - \underline{\delta}_0(t) - \underline{\gamma}(t)\widetilde{G}^{\frac{(p-1)q-1}{p-2}}, \quad \widetilde{G} \in (0,1).$$

We aim to find suitable C, a, ξ, η such that, for each $t \in (0, T)$,

$$\psi_0(\widetilde{G}) \le 0, \quad \forall \widetilde{G} \in (0,1).$$

To this end, we impose that

$$\sup_{\widetilde{G}\in(0,1)} \psi_0(\widetilde{G}) = \max_{\widetilde{G}\in(0,1)} \psi_0(\widetilde{G}) = \psi_0(\widetilde{G}_0) \le 0,$$

with $\widetilde{G}_0 \in (0,1)$. Then we have

$$\begin{aligned} \frac{\mathrm{d}\psi_0}{d\widetilde{G}} = 0 &\Leftrightarrow \underline{\sigma}_0(t) - \frac{(p-1)q-1}{p-2}\underline{\gamma}(t)\widetilde{G}^{\frac{(p-1)(q-1)}{p-2}} = 0 \\ &\Leftrightarrow \widetilde{G} = \widetilde{G}_0 = \left[\frac{p-2}{(p-1)q-1}\frac{\underline{\sigma}_0(t)}{\underline{\gamma}(t)}\right]^{\frac{p-2}{(p-1)(q-1)}}, \end{aligned}$$

from which, we obtain

$$\psi_0(\widetilde{G}_0) = K\frac{[\underline{\sigma}_0(t)]^{\frac{(p-1)q-1}{(p-1)(q-1)}}}{[\underline{\gamma}(t)]^{\frac{p-2}{(p-1)(q-1)}}} - \underline{\delta}_0(t),$$

where the coefficient K depending on p and q has been defined in (4.7). By hypotheses (4.13) and (4.14), for each $t \in (0, T)$,

$$\psi_0(\widetilde{G}_0) \le 0, \quad \widetilde{G}_0 \le 1,$$

which implies $\psi_0(\widetilde{G}) \le 0$, $\forall \widetilde{G} \in (0,1)$. Hence, we have proved that

$$\underline{v}_t - \frac{1}{\rho(x)}\Delta_p \underline{v} - \underline{v}^q \le 0, \quad \forall (x,t) \in D_2.$$

Set $\Omega_1 = D_2$, $\Omega_2 = B_1 \backslash D_2$, $u_1 = \underline{v}$, $u_2 = 0$, $u = \underline{v}$, by virtue of Lemma 2.1-(ii), we obtain that $\underline{v}$ is a sub-solution of equation (3.7) in $B_1 \times (0,T)$.

Now, observe that $\underline{w} \in C(\mathbb{R}^N \times [0,T))$. Indeed,

$$\underline{u} = \underline{v} = C\xi(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{p-1}{p-2}}, \quad \forall (x,t) \in \partial B_1 \times (0,T).$$

Moreover, in view of $b \in (0,p]$, for $\forall (x,t) \in \partial B_1 \times (0,T)$,

$$|\underline{u}_r|^{p-2}\underline{u}_r \ge |\underline{v}_r|^{p-2}\underline{v}_r = -C^{p-1}\xi^{p-1}(t)\left(\frac{p}{p-2}\frac{\eta(t)}{a}\right)^{p-1}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{p-1}{p-2}}.$$

In conclusion, set $\Omega_1 = B_1$, $\Omega_2 = \mathbb{R}^N \backslash B_1$, $u_1 = \underline{v}$, $u_2 = \underline{u}$, $u = \underline{w}$, by virtue of Lemma 2.1-(ii), we obtain that $\underline{w}(x,t)$ is a sub-solution of equation (3.7). The proof is completed. □

Remark 4.2. Denote $A \coloneqq \frac{C^{p-2}}{a^{p-1}}$. In Theorem 4.1, the accurate assumptions on parameters $C > 0$, $A > 0$, $T > 0$ are the following:

(i) Let $q > p - 1$. We require that

$$K\left[\frac{1}{p-2} + A\left(\frac{b}{p-2}\right)^{p-1} k_2\left(N + b - p + \frac{b}{p-2}\right)\right]^{\frac{(p-1)q-1}{(p-1)(q-1)}} \le \left[\frac{q-(p-1)}{(p-2)(q-1)} + A\left(\frac{b}{p-2}\right)^p k_1\right] C^{\frac{p-2}{p-1}}, \tag{4.27}$$

$$1 + A\left(\frac{b}{p-2}\right)^{p-1} k_2\left(N + b - p + \frac{b}{p-2}\right)(p-2) \le [(p-1)q - 1]C^{q-1}, \tag{4.28}$$

$$K\left[\frac{1}{p-2} + A\left(\frac{p}{p-2}\right)^{p-1} k_2\left(N + \frac{p}{p-2}\right)\right]^{\frac{(p-1)q-1}{(p-1)(q-1)}} \le \left[\frac{q-(p-1)}{(p-2)(q-1)} + A\left(\frac{p}{p-2}\right)^p k_1\right] C^{\frac{p-2}{p-1}}, \tag{4.29}$$

$$1 + A\left(\frac{p}{p-2}\right)^{p-1} k_2\left(N + \frac{p}{p-2}\right)(p-2) \le [(p-1)q - 1]C^{q-1}. \tag{4.30}$$

(ii) Let $q \le p - 1$. We require that

$$A > \frac{(p-1)-q}{k_1(p-2)(q-1)}\left(\frac{p-2}{b}\right)^p, \tag{4.31}$$

$$a \ge \max\left\{\frac{K\left[\frac{1}{p-2} + A\left(\frac{b}{p-2}\right)^{p-1} k_2\left(N + b - p + \frac{b}{p-2}\right)\right]^{\frac{(p-1)q-1}{(p-1)(q-1)}}}{A^{\frac{1}{p-1}}\left[A\left(\frac{b}{p-2}\right)^p k_1 - \frac{(p-1)-q}{(p-2)(q-1)}\right]}, \right.$$
$$\left. \times \frac{K\left[\frac{1}{p-2} + A\left(\frac{p}{p-2}\right)^{p-1} k_2\left(N + \frac{p}{p-2}\right)\right]^{\frac{(p-1)q-1}{(p-1)(q-1)}}}{A^{\frac{1}{p-1}}\left[A\left(\frac{p}{p-2}\right)^p k_1 - \frac{(p-1)-q}{(p-2)(q-1)}\right]}\right\}, \tag{4.32}$$

$$[(p-1)q-1](a^{p-1}A)^{\frac{q-1}{p-2}} \\ \geq \max\left\{1 + A\left(\frac{b}{p-2}\right)^{p-1} k_2\left(N + b - p + \frac{b}{p-2}\right)(p-2), 1 + A\left(\frac{p}{p-2}\right)^{p-1} k_2\left(N + \frac{p}{p-2}\right)(p-2)\right\}. \tag{4.33}$$

Lemma 4.1. *All the conditions in* Remark 4.2 *can be fulfilled concurrently.*

Proof.
(i) For any $A > 0$, we select $C > 0$ large enough (therefore, $a > 0$ is also fixed, owing to the definition of A) so that (4.27)–(4.30) hold.
(ii) We select $A > 0$ so that (4.31) holds and $a > 0$ sufficiently large to guarantee (4.32) and (4.33) (hence, $C > 0$ is also fixed).

The proof is completed. □

We are ready to give a proof of the main Theorem 4.1.

Proof of Theorem 4.1. According to Lemma 4.1, we can assume that all the conditions of Remark 4.2 are satisfied. Set

$$\xi(t) = (T-t)^{-\alpha_0}, \quad \eta(t) = (T-t)^{\delta_0},$$

where

$$\alpha_0 \coloneqq \frac{1}{q-1}, \quad \delta_0 \coloneqq \frac{(p-1)-q}{(p-1)(q-1)}.$$

Then

$$\begin{aligned}
\underline{\sigma}(t) &= \left[\frac{1}{p-2} + \frac{C^{p-2}}{a^{p-1}}\left(\frac{b}{p-2}\right)^{p-1} k_2\left(N + b - p + \frac{b}{p-2}\right)\right](T-t)^{\frac{-q}{q-1}},\\
\underline{\delta}(t) &= \left[\frac{q-(p-1)}{(p-2)(q-1)} + \frac{C^{p-2}}{a^{p-1}}\left(\frac{b}{p-2}\right)^{p} k_1\right](T-t)^{\frac{-q}{q-1}},\\
\underline{\gamma}(t) &= C^{q-1}(T-t)^{\frac{-q}{q-1}},\\
\underline{\sigma}_0(t) &= \left[\frac{1}{p-2} + \frac{C^{p-2}}{a^{p-1}}\left(\frac{p}{p-2}\right)^{p-1} k_2\left(N + \frac{p}{p-2}\right)\right](T-t)^{\frac{-q}{q-1}},\\
\underline{\delta}_0(t) &= \left[\frac{q-(p-1)}{(p-2)(q-1)} + \frac{C^{p-2}}{a^{p-1}}\left(\frac{p}{p-2}\right)^{p} k_1\right](T-t)^{\frac{-q}{q-1}}.
\end{aligned}$$

Case 1: Let $q > p-1$. Conditions (4.27) and (4.28) imply (4.13) and (4.14), whereas (4.29) and (4.30) imply (4.15) and (4.16). Hence, by Propositions 3.1 and 4.1, the conclusion follows in this case.
Case 2: Let $q \leq p-1$. Conditions (4.32) and (4.33) imply (4.13)–(4.16). Therefore, the conclusion holds in this case by means of Propositions 3.1 and 4.1.

The proof is completed. □

5 Blow-up for any nontrivial initial data

In this section, we deal with a further result with regard to the blow-up of solution to problems (1.1) and (1.2) for any nonnegative and nontrivial initial data u_0. We distinguish between two cases:

(i) $1 < q < p - 1$,
(ii) $p - 1 \le q < \underline{q}$.

In case **(ii)**, we need an additional assumption. In fact, we suppose that (H_1)–(H_2) hold with

$$s \in (0, \varepsilon), \tag{5.1}$$

for some $\varepsilon > 0$ to be fixed small enough later. Then, by the definition of b, we obtain

$$p - \varepsilon < b < p. \tag{5.2}$$

Theorem 5.1. *Suppose* (H_1)–(H_2) *hold. If* $2 < p < N$, $1 < q < p - 1$, *the nonnegative initial data* $u_0 \in C(\mathbb{R}^N)$ *and* $u_0(x) \not\equiv 0$, *then for any sufficiently large* $T > 0$, *the solution* u *of problems* (1.1) *and* (1.2) *blows up in a finite time* $S \in (0, T]$, *in the sense that*

$$||u(t)||_\infty \to \infty, \quad t \to S^-.$$

More precisely, the bound from (4.2) holds, with C, a, ξ, η as in Theorem 4.1-(ii).

Proof. Since $u_0(x) \not\equiv 0$ and $u_0 \in C(\mathbb{R}^N)$, there exist $\sigma > 0$, $r_1 > 0$ and $x_0 \in \mathbb{R}^N$ such that

$$u_0(x) \ge \sigma, \quad \forall x \in B_{r_1}(x_0).$$

Without loss of generality, we can suppose that $x_0 = 0$. Let $\underline{w}$ be the sub-solution of problems (1.1) and (1.2) considered in Theorem 4.1. We can choose $T > 0$ sufficiently large in the way that

$$CT^{-\frac{1}{q-1}} \le \sigma, \quad aT^{-\frac{(p-1)-q}{(p-1)(q-1)}} \le \min\left\{r_1^{\frac{b}{p-1}}, r_1^{\frac{p}{p-1}}\right\}. \tag{5.3}$$

From inequalities in (5.3), we deduce that

$$\underline{w}(x, 0) \le u_0(x), \quad \forall x \in \mathbb{R}^N.$$

Hence, by Theorem 4.1 and the comparison principle, the result follows. The proof is completed. □

Theorem 5.2. *Let assumptions* (H_1)–(H_2) *and* (5.1) *be satisfied for* $\varepsilon > 0$ *small enough. For any* $2 < p < N$, *the nonnegative initial data* $u_0 \in C(\mathbb{R}^N)$ *and* $u_0(x) \not\equiv 0$. *If*

$$p - 1 \le q < \underline{q}, \tag{5.4}$$

where $\underline{q}$ *is defined in* (1.5), *then there exist* $t_1 > 0$ *and sufficiently large* $T > 0$, *the solution* u *of problems* (1.1) *and* (1.2) *blows up in a finite time* $S \in (0, T + t_1]$, *in the sense that*

$$||u(t)||_\infty \to \infty, \quad t \to S^-.$$

More precisely, when $S > t_1$, we have the bound from below

$$u(x, t) \ge C(T + t_1 - t)^{-\frac{1}{q-1}}\left[1 - \frac{s(x)}{a}(T + t_1 - t)^{\frac{q-(p-1)}{(p-1)(q-1)}}\right]_+^{\frac{p-1}{p-2}}, \quad \forall (x, t) \in \mathbb{R}^N \times (t_1, S),$$

with C, a and $s(x)$ as in Theorem 4.1-(i).

Remark 5.1. Observe that an integral blow-up result of problems (1.1) and (1.2) is shown when $\rho(x) = |x|^{-s}$ or $\rho(x) = (1 + |x|)^{-s}$ with $s \in [0, p)$ in [19, Theorem 3], namely, for some $R > 0$, $\theta \in (0, 1)$, $\hat{T} > 0$, $\int_{B_R} \rho(x)u^\theta(x, t)\mathrm{d}x \to +\infty$ as $t \to \hat{T}^-$. Indeed, under the assumption $\rho(x)$ integrable in B_R this implies L^∞ blow-up. Therefore, our Theorem 5.2 is similar to that under the extra assumption (5.1). Nevertheless, our approach is completely different with the methods of proofs in [19] as they are mainly based on the selection of a special test function and integration by parts.

In what follows, let us explain the strategy of the proof of Theorem 5.2. Let $u(x,t)$ be a solution of problems (1.1) and (1.2) and $\underline{w}(x,t)$ be given in Theorem 4.1. However, as Theorem 5.2 considers the arbitrary initial data and $q \geq p-1$, it cannot be determined that $\underline{w}(x,t)$ is the blow-up sub-solution of problems (1.1) and (1.2). The main reason is that the size relation between the initial value $u_0(x)$ and $\underline{w}(x,0)$ is unknown. Consequently, our goal is find some time $t_1 > 0$ such that $u(x,t_1) \geq \underline{w}(x,0)$, so that we can directly obtain that the solution blows up in a finite time by virtue of Theorem 4.1. To this end, we find a sub-solution $z(x,t)$ of the equation

$$y_t = \frac{1}{\rho(x)}\Delta_p y, \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{5.5}$$

such that

$$z(x,0) \leq u_0(x), \quad \forall x \in \mathbb{R}^N \tag{5.6}$$

and

$$z(x,t_1) \geq \underline{w}(x,0), \quad \forall x \in \mathbb{R}^N, \tag{5.7}$$

for $t_1 > 0$ and sufficiently large $T > 0$. Let $\hat{T}$ be the maximal existence time of $u(x,t)$. If $\hat{T} \leq t_1 < \infty$, then nothing has to be proved, and $u(x,t)$ blows up in a finite time. If $\hat{T} > t_1$, since $z(x,t)$ is also a sub-solution to (1.1), by (5.6) and the comparison principle, we deduce

$$z(x,t) \leq u(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times (0,\hat{T}). \tag{5.8}$$

Combining (5.7) and (5.8), one can see that

$$u(x,t_1) \geq z(x,t_1) \geq \underline{w}(x,0), \quad \forall x \in \mathbb{R}^N,$$

Hence, $u(x,t+t_1)$ and $\underline{w}(x,t)$ are the super-solution and sub-solution of the following problem, respectively,

$$h_t = \frac{1}{\rho(x)}\Delta_p h + h^q, \quad (x,t) \in \mathbb{R}^N \times (0,+\infty),$$

$$h(x,0) = \underline{w}(x,0), \quad x \in \mathbb{R}^N.$$

As a result, it follows from the finite time blow-up property of $\underline{w}(x,t)$ that $u(x,t)$ also blows up in a finite time $S \in (t_1, T+t_1]$.

Next, we construct a sub-solution of equation (5.5). We define

$$z(x,t) = z(r(x),t) := \begin{cases} \mu(x,t), & \text{in } (x,t) \in [\mathbb{R}^N \backslash B_1] \times [0,+\infty), \\ \upsilon(x,t), & \text{in } (x,t) \in B_1 \times [0,+\infty), \end{cases} \tag{5.9}$$

where

$$\mu(x,t) = \mu(r(x),t) := C_1\xi(t)\left[1 - \frac{r^{\beta b}}{a_1}\eta(t)\right]_+^{\frac{p-1}{p-2}}, \tag{5.10}$$

$$\upsilon(x,t) = \upsilon(r(x),t) := C_1\xi(t)\left[1 - \frac{br^{\theta} + p - b}{p}\frac{\eta(t)}{a_1}\right]_+^{\frac{p-1}{p-2}}, \tag{5.11}$$

with $\xi, \eta \in C^1([0,+\infty);[0,+\infty))$, $C_1 > 0$, $a_1 > 0$, $\beta := \frac{1}{p-1}$, $\theta := \frac{p}{p-1}$.

Let

$$F_1(r,t) := 1 - \frac{r^{\beta b}}{a_1}\eta(t), \quad G_1(r,t) := 1 - \frac{br^{\theta} + p - b}{p}\frac{\eta(t)}{a_1},$$

and we define

$$D_3 := \{(x,t) \in [\mathbb{R}^N \backslash B_1] \times (0,+\infty) | 0 < F_1(r,t) < 1\},$$
$$D_4 := \{(x,t) \in B_1 \times (0,+\infty) | 0 < G_1(r,t) < 1\}.$$

In addition, for $\varepsilon_0 > 0$ small enough, let

$$\delta_0 := \frac{\frac{k_1}{k_2} b}{(p-1)[(p-2)(N-p)+(p-1)b]}, \tag{5.12}$$

$$\alpha_0 := \frac{1-\delta_0(p-1)}{p-2} = \frac{N-p+\frac{b}{p-2}\left(p-1-\frac{k_1}{k_2}\right)}{(p-2)(N-p)+(p-1)b}, \tag{5.13}$$

$$\widetilde{\delta}_0 := \frac{\frac{k_1}{k_2} p - \varepsilon_0}{(p-1)[(p-2)N+p]}, \tag{5.14}$$

$$\widetilde{\alpha}_0 := \frac{1-\widetilde{\delta}_0(p-1)}{p-2} = \frac{N(p-2)+p\left(1-\frac{k_1}{k_2}\right)+\varepsilon_0}{(p-1)[(p-2)N+p]}. \tag{5.15}$$

Observe that

$$0 < \delta_0 < \frac{1}{p-1}, \quad 0 < \widetilde{\delta}_0 < \frac{1}{p-1}, \tag{5.16}$$

and if ε_0 is small enough, then

$$0 < \delta_0 < \widetilde{\delta}_0. \tag{5.17}$$

We establish the proposition that plays a crucial role in the proof of Theorem 5.2.

Proposition 5.1. *Assume* (H_1)–(H_2), (5.1) *holds for* $\varepsilon > 0$ *small enough and*

$$\overline{\delta} \in (0, \delta_0), \tag{5.18}$$

$$\overline{\alpha} := \frac{1-\overline{\delta}(p-1)}{p-2}. \tag{5.19}$$

Suppose that

$$1 < q < p-1+(p-1)\frac{\overline{\delta}}{\overline{\alpha}}. \tag{5.20}$$

Let $T_1 \in (0,+\infty)$, *we choose*

$$\xi(t) = (T_1+t)^{-\overline{\alpha}}, \quad \eta(t) = (T_1+t)^{-\overline{\delta}}.$$

Then there exist $\widetilde{A} := \frac{C_1^{p-2}}{a_1^{p-1}} > 0$, $t_1 > 0$ *and* $T > 0$ *such that* z *defined in* (5.9) *is a sub-solution of equation* (5.5) *and satisfies* (5.6) *and* (5.7).

Proof. For any $(x,t) \in D_3$, by the similar calculations to (4.17), we deduce

$$\begin{aligned}
&\mu_t - \frac{1}{\rho(x)}\Delta_p \mu \\
&\leq C_1 F_1^{\frac{1}{p-2}} \left\{ F_1 \left[\xi' + \xi \frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2 C_1^{p-2} \xi^{p-1} \left(\frac{p-1}{p-2}\right)^{p-1} \left(\frac{\beta b}{a_1}\eta\right)^{p-1} \left(N+b-p+\frac{b}{p-2}\right)\right] \right. \\
&\qquad \left. - \xi \frac{p-1}{p-2}\frac{\eta'}{\eta} - k_1 \beta b C_1^{p-2} \xi^{p-1} \left(\frac{p-1}{p-2}\right)^{p} \left(\frac{\beta b}{a_1}\eta\right)^{p-1} \right\}.
\end{aligned} \tag{5.21}$$

Now, we define

$$\sigma(t) := \xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_2\frac{C_1^{p-2}}{a_1^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left(N+b-p+\frac{b}{p-2}\right), \tag{5.22}$$

$$\delta(t) := \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + k_1\frac{C_1^{p-2}}{a_1^{p-1}}\xi^{p-1}\frac{b}{p-2}\left(\frac{b\eta}{p-2}\right)^{p-1}. \tag{5.23}$$

Hence, (5.21) becomes

$$\mu_t - \frac{1}{\rho(x)}\Delta_p\mu \le C_1F_1^{\frac{1}{p-2}}\varphi_1(F_1),$$

where

$$\varphi_1(F_1) := \sigma(t)F_1 - \delta(t), \quad F_1 \in (0,1).$$

Now we aim to seek suitable C_1, a_1 such that, for each $t > 0$,

$$\varphi_1(F_1) \le 0, \quad \forall F_1 \in (0,1).$$

We observe that it is sufficient that for each $t > 0$, satisfies

$$\sigma(t) > 0, \tag{5.24}$$

$$\delta(t) > 0, \tag{5.25}$$

$$\sigma(t) - \sigma(t) \le 0, \tag{5.26}$$

which guarantees $\varphi_1(F_1) \ge 0$, $\forall F_1 \in (0,1)$.

By virtue of the definitions of $\xi(t)$ and $\eta(t)$, we obtain

$$\sigma(t) = \left(-\bar{\alpha} - \frac{p-1}{p-2}\bar{\delta}\right)(T_1+t)^{-\bar{\alpha}-1} + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}k_2\left(N+b-p+\frac{b}{p-2}\right)(T_1+t)^{-(\bar{\alpha}+\bar{\delta})(p-1)},$$

$$\delta(t) = -\frac{p-1}{p-2}\bar{\delta}(T_1+t)^{-\bar{\alpha}-1} + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}\frac{k_1b}{p-2}(T_1+t)^{-(\bar{\alpha}+\bar{\delta})(p-1)}.$$

By (5.16), (5.18), and (5.19), we have

$$0 < \bar{\delta} < \frac{1}{p-1}, \quad \bar{\alpha} > 0.$$

By applying (5.19), (5.24)–(5.26) can be rewritten as follows:

$$-1 + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}(p-2)k_2\left(N+b-p+\frac{b}{p-2}\right) > 0,$$

$$-\bar{\delta} + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}k_1\beta b > 0,$$

$$\bar{\delta}(p-1) - 1 + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}(p-2)\left[k_2\left(N+b-p+\frac{b}{p-2}\right) - \frac{k_1b}{p-2}\right] \le 0,$$

which reduces to

$$\frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}(p-2) > \max\left\{\frac{1}{k_2\left(N+b-p+\frac{b}{p-2}\right)}, \frac{\bar{\delta}(p-2)}{k_1\beta b}\right\}, \tag{5.27}$$

$$\frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}(p-2) \le \frac{1-\bar{\delta}(p-1)}{k_2\left(N+b-p+\frac{b}{p-2}\right)-\frac{bk_1}{p-2}}. \tag{5.28}$$

If (5.27) and (5.28) are verified, then $\mu(x,t)$ is a sub-solution to equation (5.5) in D_1. Thus, we show that it is possible to find $\widetilde{A} \coloneqq \frac{C_1^{p-2}}{a_1^{p-1}}$ such that (5.27) and (5.28) hold. If

$$\frac{1}{k_2\left(N+b-p+\frac{b}{p-2}\right)} < \frac{1-\bar{\delta}(p-1)}{k_2\left(N+b-p+\frac{b}{p-2}\right)-\frac{bk_1}{p-2}} \tag{5.29}$$

and

$$\frac{\bar{\delta}(p-2)}{k_1\beta b} < \frac{1-\bar{\delta}(p-1)}{k_2\left(N+b-p+\frac{b}{p-2}\right)-\frac{bk_1}{p-2}}, \tag{5.30}$$

then $\widetilde{A}$ can be selected to hold up for (5.27) and (5.28). If

$$\bar{\delta} < \delta_0,$$

then conditions (5.29) and (5.30) are satisfied. As a consequence, by virtue of Lemma 2.1-(ii), we obtain that $\mu(x,t)$ is a sub-solution of equation (5.5) in $[\mathbb{R}^N \backslash B_1] \times (0,+\infty)$.

On the other hand, for any $(x,t) \in D_4$, by the similar arguments to (3.35), we deduce

$$\begin{aligned}
&v_t - \frac{1}{\rho(x)}\Delta_p v \\
&\le C_1 G_1^{\frac{1}{p-2}}\left\{G_1\left[\xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + C_1^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\left(\frac{\theta b}{p}\frac{\eta}{a_1}\right)^{p-1}k_2\left(N+\frac{p}{p-2}\right)\right]\right. \\
&\quad \left. - \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} - C_1^{p-2}\xi^{p-1}\left(\frac{p-1}{p-2}\right)^{p-1}\left(\frac{\theta b}{p}\frac{\eta}{a_1}\right)^{p-1}\left[\frac{pk_1}{p-2} - \frac{\eta}{a_1}\frac{p-b}{p-2}k_2\right]\right\}.
\end{aligned} \tag{5.31}$$

Now, we define

$$\sigma_0(t) \coloneqq \xi' + \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + \frac{C_1^{p-2}}{a_1^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}k_2\left(N+\frac{p}{p-2}\right), \tag{5.32}$$

$$\delta_0(t) \coloneqq \xi\frac{p-1}{p-2}\frac{\eta'}{\eta} + \frac{C_1^{p-2}}{a_1^{p-1}}\xi^{p-1}\left(\frac{b\eta}{p-2}\right)^{p-1}\left[\frac{pk_1}{p-2} - \frac{\eta}{a_1}\frac{p-b}{p-2}k_2\right]. \tag{5.33}$$

Hence, (5.33) becomes

$$v_t - \frac{1}{\rho(x)}\Delta_p v \le C_1 G_1^{\frac{1}{p-2}}\psi_1(G_1),$$

where

$$\psi_1(G_1) \coloneqq \sigma_0(t)G_1 - \delta_0(t), \quad G_1 \in (0,1).$$

Now we aim to seek suitable C_1, a_1 such that, for each $t > 0$,

$$\psi_1(G_1) \le 0, \quad \forall G_1 \in (0,1)$$

we observe that it is sufficient that for each $t > 0$, satisfies

$$\sigma_0(t) > 0, \tag{5.34}$$

$$\delta_0(t) > 0, \tag{5.35}$$

$$\sigma_0(t) - \sigma_0(t) \le 0, \tag{5.36}$$

which guarantees $\psi_1(G_1) \le 0$, $\forall G_1 \in (0,1)$.

By virtue of the definitions of $\xi(t)$ and $\eta(t)$, we obtain

$$\sigma_0(t) = \left(-\bar{\alpha} - \frac{p-1}{p-2}\bar{\delta}\right)(T_1+t)^{-\bar{\alpha}-1} + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1} k_2\left(N + \frac{p}{p-2}\right)(T_1+t)^{-(\bar{\alpha}+\bar{\delta})(p-1)},$$

$$\delta_0(t) = -\frac{p-1}{p-2}\bar{\delta}(T_1+t)^{-\bar{\alpha}-1} + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}\left[\frac{k_1 p}{p-2} - \frac{k_2}{a_1}\frac{p-b}{p-2}(T_1+t)^{-\bar{\delta}}\right](T_1+t)^{-(\bar{\alpha}+\bar{\delta})(p-1)}.$$

Hence, (5.34)–(5.36) become

$$-1 + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}(p-2)k_2\left(N + \frac{p}{p-2}\right) > 0,$$

$$-\bar{\delta} + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}\left(\frac{pk_1}{p-1} - \frac{k_2}{a_1}\frac{p-b}{p-1}(T_1+t)^{-\bar{\delta}}\right) > 0,$$

$$\bar{\delta}(p-1) - 1 + \frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1}(p-2)k_2\left[N + \frac{p}{p-2}\left(1 - \frac{k_1}{k_2}\right) + \frac{(p-b)(T_1+t)^{-\bar{\delta}}}{a_1(p-2)}\right] \le 0,$$

which reduces to

$$\frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1} > \max\left\{\frac{1}{k_2(p-2)\left(N + \frac{p}{p-2}\right)}, \frac{\bar{\delta}(p-1)}{k_2\left[p\frac{k_1}{k_2} - \frac{p-b}{a_1}(T_1+t)^{-\bar{\delta}}\right]}\right\}, \tag{5.37}$$

$$\frac{C_1^{p-2}}{a_1^{p-1}}\left(\frac{b}{p-2}\right)^{p-1} \le \frac{1 - \bar{\delta}(p-1)}{k_2(p-2)\left[N + \frac{p}{p-2}\left(1 - \frac{k_1}{k_2}\right) + \frac{p-b}{a_1(p-2)}(T_1+t)^{-\bar{\delta}}\right]}. \tag{5.38}$$

If (5.37) and (5.38) are verified, then $v(x,t)$ is a sub-solution to equation (5.5) in D_4. To find $\widetilde{A} := \frac{C_1^{p-2}}{a_1^{p-1}}$ such that (5.37) and (5.38) hold, we require

$$\frac{1}{k_2(p-2)\left(N + \frac{p}{p-2}\right)} < \frac{1 - \bar{\delta}(p-1)}{k_2(p-2)\left[N + \frac{p}{p-2}\left(1 - \frac{k_1}{k_2}\right) + \frac{p-b}{a_1(p-2)}(T_1+t)^{-\bar{\delta}}\right]}, \tag{5.39}$$

and

$$\frac{\bar{\delta}(p-1)}{k_2\left[p\frac{k_1}{k_2} - \frac{p-b}{a_1}(T_1+t)^{-\bar{\delta}}\right]} < \frac{1 - \bar{\delta}(p-1)}{k_2(p-2)\left[N + \frac{p}{p-2}\left(1 - \frac{k_1}{k_2}\right) + \frac{p-b}{a_1(p-2)}(T_1+t)^{-\bar{\delta}}\right]}. \tag{5.40}$$

Now we select in (5.1) $\varepsilon = \varepsilon(a_1, T_1) > 0$ so that

$$\frac{\varepsilon}{a_1}T_1^{-\bar{\delta}} \le \varepsilon_0, \tag{5.41}$$

with ε_0 used in (5.14) and (5.15) to be appropriately fixed. By (5.1), (5.2), and (5.41), we obtain

$$\frac{p-b}{a_1}(T_1+t)^{-\bar{\delta}} < \frac{\varepsilon}{a_1}T_1^{-\bar{\delta}} \le \varepsilon_0.$$

Hence, if

$$\frac{1}{k_2(p-2)\left(N+\frac{p}{p-2}\right)} < \frac{1-\bar{\delta}(p-1)}{k_2(p-2)\left[N+\frac{p}{p-2}\left(1-\frac{k_1}{k_2}\right)+\frac{\varepsilon_0}{p-2}\right]} \tag{5.42}$$

and

$$\frac{\bar{\delta}(p-1)}{k_2\left[p\frac{k_1}{k_2}-\varepsilon_0\right]} < \frac{1-\bar{\delta}(p-1)}{k_2(p-2)\left[N+\frac{p}{p-2}\left(1-\frac{k_1}{k_2}\right)+\frac{\varepsilon_0}{p-2}\right]}, \tag{5.43}$$

then conditions (5.37) and (5.38) are satisfied. Finally, for ε_0 is small enough, if

$$\bar{\delta} < \tilde{\delta}_0, \tag{5.44}$$

then conditions (5.42) and (5.43) are satisfied. Observe that (5.44) is ensured owing to assumptions (5.17) and (5.18). As a consequence, by virtue of Lemma 2.1(ii), we obtain that $v(x,t)$ is a sub-solution of equation (5.5) in $B_1 \times (0,+\infty)$.

Now, observe that $z \in C(\mathbb{R}^N \times [0,\infty))$. Indeed,

$$\mu = v = C_1\xi(t)\left[1-\frac{\eta(t)}{a_1}\right]_+^{\frac{p-1}{p-2}}, \quad \forall (x,t) \in \partial B_1 \times (0,\infty). \tag{5.45}$$

Moreover, for $\forall (x,t) \in \partial B_1 \times (0,+\infty)$,

$$|\mu_r|^{p-2}\mu_r = |v_r|^{p-2}v_r = -C_1^{p-1}\xi^{p-1}(t)\left(\frac{b}{p-2}\frac{\eta(t)}{a_1}\right)^{p-1}\left[1-\frac{\eta(t)}{a_1}\right]_+^{\frac{p-1}{p-2}}. \tag{5.46}$$

In conclusion, set $\Omega_1 = B_1, \Omega_2 = \mathbb{R}^N\backslash B_1, u_1 = \mu, u_2 = v, u = z$, by virtue of (5.45), (5.46) and Lemma 2.1-(ii), we obtain that $z(x,t)$ is a sub-solution of equation (5.5).

As before, since $u_0(x) \not\equiv 0$ and $u_0 \in C(\mathbb{R}^N)$, there exist $\sigma > 0$ and $r_1 > 0$ such that

$$u_0(x) \geq \sigma, \quad \forall x \in B_{r_1}.$$

Thus, if

$$\operatorname{supp} z(\cdot,0) \subset B_{r_1} \tag{5.47}$$

and

$$z(\cdot,0) \leq \sigma, \quad \forall x \in B_{r_1}, \tag{5.48}$$

then (5.6) holds. Furthermore, if

$$\operatorname{supp} \underline{w}(\cdot,0) \subset \operatorname{supp} z(\cdot,t_1) \tag{5.49}$$

and

$$\underline{w}(\cdot,0) \leq z(x,t_1), \quad \forall x \in \mathbb{R}^N, \tag{5.50}$$

then (5.7) follows.

We first show that $z(x,t)$ satisfies condition (5.47) and (5.48). If

$$a_1 T_1^{\bar{\delta}} \leq \frac{r_1^2}{2}, \tag{5.51}$$

then

$$\operatorname{supp} z(\cdot,0) \cap B_1 \subset B_{r_1}$$

and

$$\operatorname{supp} z(\cdot,0) \cap [\mathbb{R}^N \backslash B_1] \subset B_{r_1},$$

hence (5.47) follows. Furthermore, if

$$(a_1^{p-1}\widetilde{A})^{\frac{1}{p-2}} \leq \sigma T_1^{\overline{\alpha}}, \tag{5.52}$$

then (5.48) holds. Clearly, for any $T_1 > 0$, we can select $a_1 = a_1(T_1)$ such that (5.51) and (5.52) are valid. On the other hand, if

$$a_1(T_1 + t_1)^{\overline{\delta}} \geq aT^{-\frac{(p-1)-1}{(p-1)(q-1)}}, \tag{5.53}$$

then

$$\operatorname{supp} \underline{w}(\cdot,0) \cap B_1 \subset \operatorname{supp} z(\cdot,t_1) \cap B_1$$

and

$$\operatorname{supp} \underline{w}(\cdot,0) \cap [\mathbb{R}^N \backslash B_1] \subset \operatorname{supp} z(\cdot,t_1) \cap [\mathbb{R}^N \backslash B_1],$$

thus (5.48) holds. If

$$C_1(T_1 + t_1)^{-\overline{\alpha}} \geq CT^{-\frac{1}{q-1}}, \tag{5.54}$$

then (5.49) holds. If we select

$$T_1 + t_1 = \left(\frac{C}{C_1}\right)^{-\overline{\alpha}} T^{\frac{1}{(q-1)\overline{\alpha}}},$$

then (5.53) becomes

$$a_1\left(\frac{C}{C_1}\right)^{-\frac{\overline{\delta}}{\overline{\alpha}}} T^{\frac{\overline{\delta}}{(q-1)\overline{\alpha}}} \geq aT^{\frac{q-(p-1)}{(p-1)(q-1)}}. \tag{5.55}$$

Therefore, if

$$T^{\frac{1}{q-1}\left(\frac{q-(p-1)}{p-1}-\frac{\overline{\delta}}{\overline{\alpha}}\right)} \leq \left(\frac{C}{C_1}\right)^{-\frac{\overline{\delta}}{\overline{\alpha}}} \frac{a_1}{a}, \tag{5.56}$$

then (5.55) holds. Condition (5.56) is satisfied according to (5.4), for $T > 0$ sufficiently large. The proof is completed. □

We are in a position to give a proof of the main Theorem 5.2.

Proof of Theorem 5.2. Let $\hat{T}$ be the maximal existence time of $u(x, t)$. Based on the aforementioned explanation of the strategy of the proof of Theorem 5.2, we can divide it into the following two cases.

If $\hat{T} \leq t_1 < \infty$, then nothing has to be proved, and $u(x, t)$ blows up in a finite time.

If $\hat{T} > t_1$, we consider the sub-solution $z(x, t)$ of equation (5.5) as defined in (5.9). In view of $q < \underline{q}$, we can find $\overline{\delta}$ and $\overline{\alpha}$ such that (5.18), (5.19), and (5.20) hold. $z(x, t)$ satisfies (5.6) and (5.7) by virtue of Proposition 5.1. According to condition (5.6) and the comparison principle, we obtain (5.8). It follows from (5.7) and (5.8) that

$$u(x, t_1) \geq z(x, t_1) \geq \underline{w}(x, 0), \quad \forall x \in \mathbb{R}^N.$$

Hence, $u(x, t + t_1)$ and $\underline{w}(x, t)$ are the super-solution and sub-solution of the following problem, respectively,

$$h_t = \frac{1}{\rho(x)}\Delta_p h + h^q, \quad (x, t) \in \mathbb{R}^N \times (0, +\infty),$$

$$h(x, 0) = \underline{w}(x, 0), \quad x \in \mathbb{R}^N.$$

As a result, it follows from the finite time blow-up property of $\underline{w}(x, t)$ that $u(x, t)$ also blows up in a finite time $S \in (t_1, T + t_1]$. The proof is completed. □

Acknowledgments: The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

Funding information: This work was supported by the Natural Science Foundation of Shandong Province of China (No. ZR2019MA072).

Author contributions: All authors contributed equally to the manuscript and read and approved the final manuscript.

Conflict of interest: The authors declare that they have no competing interests.

Ethical approval: Not applicable.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

[1] D. Andreucci and A. F. Tedeev, *Universal bounds at the blow-up time for nonlinear parabolic equations*, Adv. Differential Equ. **10** (2005), no. 1, 89–120.
[2] P. Baras and R. Kersner, *Local and global solvability of a class of semilinear parabolic equations*, J. Differ. Equations **68** (1987), 238–252.
[3] P. Cianci, A. V. Martynenko, and A. F. Tedeev, *The blow-up phenomenon for degenerate parabolic equations with variable coefficients and nonlinear source*, Nonlinear Anal. **73** (2010), no. 7, 2310–2323.
[4] Y. X. Chen, *Global dynamical behavior of solutions for finite degenerate fourth-order parabolic equations with mean curvature nonlinearity*, Commun. Anal. Mech. **15** (2023), no. 4, 658–694.
[5] D. Andreucci and A. F. Tedeev, *Asymptotic properties of solutions to the Cauchy problem for degenerate parabolic equations with inhomogeneous density on manifolds*, Milan J. Math. **89** (2021), 295–327.
[6] D. Andreucci and A. F. Tedeev, *Existence of solutions of degenerate parabolic equations with inhomogeneous density and growing data on manifolds*, Nonlinear Anal. **219** (2022), 112818, DOI: https://doi.org/10.1016/J.NA.2022.112818.
[7] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.
[8] E. DiBenedetto and M. A. Herrero, *On the Cauchy problem and initial traces for a degenerate parabolic equation*, Trans. Amer. Math. Soc. **314** (1989), 187–224.
[9] L. F. Dzagoeva and A. F. Tedeev, *Asymptotic behavior of the solution of doubly degenerate parabolic equations with inhomogeneous density*, Vladikavkaz. Mat. Zh. **24** (2022), no. 3, 78–86.
[10] S. Eidelman, S. Kamin, and F. Porper, *Uniqueness of solutions of the Cauchy problem for parabolic equations degenerating at infinity*, Asymptotic Anal. **22** (2000), 349–358.
[11] H. Fujita, *On the blowing up of solutions to the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), no. 2, 109–124.
[12] V. A. Galaktionov, *Blow-up for quasilinear heat equations with critical Fujita's exponents*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 517–525.
[13] V. A. Galaktionov, *Conditions for nonexistence as a whole and localization of the solutions of Cauchyas problem for a class of nonlinear parabolic equations*, Zh. Vychisl. Mat. Mat. Fiz. **23** (1985), 1341–1354.
[14] K. Hayakawa, *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Jpn. Acad. **49** (1973), no. 7, 503–505.
[15] S. Kamin and R. Kersner, *Disappearance of interfaces in finite time*, Meccanica **28** (1993), 117–120.
[16] S. Kamin, A. Pozio, and A. Tessi, *Admissible conditions for parabolic equations degenerating at infinity*, St. Petersburg Math. J. **19** (2008), 239–251.
[17] X. Li and Z. Y. Xiang, *Existence and nonexistence of local/global solutions for a nonhomogeneous heat equation*, Commun. Pur. Appl. Anal. **13** (2014), 1465–1480.
[18] W. Lian, J. Wang, and R. Z. Xu, *Global existence and blow up of solutions for pseudo-parabolic equation with singular potential*, J. Differ. Equations **269** (2020), 4914–4959.

[19] A. V. Martynenko and A. F. Tedeev, *On the behavior of solutions to the Cauchy problem for a degenerate parabolic equation with inhomogeneous density and a source*, Comput. Math. Math. Phys. **48** (2008), no. 7, 1145–1160.
[20] A. V. Martynenko and A. F. Tedeev, *Cauchy problem for a quasilinear parabolic equation with a source term and an inhomogeneous density*, Comput. Math. Math. Phys. **47** (2007), 238–248.
[21] A. V. Martynenko and A. F. Tedeev, *Regularity of solutions of degenerate parabolic equation with inhomogenious density*, Ukr. Mat. Visn. **5** (2008), 116–145.
[22] A. V. Martynenko, A. F. Tedeev, and V. N. Shramenko, *The Cauchy problem for a degenerate parabolic equation with inhomogeneous density and a source in the class of slowly vanishing initial functions*, Izv. Math. **76** (2012), no. 3, 563–580.
[23] A. V. Martynenko, A. F. Tedeev, and V. N. Shramenko, *On the behavior of solutions of the Cauchy problem for a degenerate parabolic equation with source in the case where the initial function slowly vanishes*, Ukr. Math. J. **64** (2013), no. 11, 1698–1715.
[24] A. V. Martynenko, *Global solvability for quasilinear parabolic equation with inhomogeneous density and a source*, Appl. Anal. **92** (2013), no. 9, 1863–1888.
[25] P. Mastrolia, D. D. Monticelli, and F. Punzo, *Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds*, Math. Ann. **367** (2017), 929–963.
[26] A. D. Pablo, G. Reyes, and A. Sanchez, *The Cauchy problem for a nonhomogeneous heat equation with reaction*, Discrete Contin. Dyn. Syst., Ser. A **33** (2013), 643–662.
[27] R. G. Pinsky, *Existence and nonexistence of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbb{R}^d$*, J. Differ. Equations **133** (1997), 152–177.
[28] F. Punzo, *On the Cauchy problem for nonlinear parabolic equations with variable density*, J. Evol. Equ. **9** (2009), 429–447.
[29] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin, 1995.
[30] P. Souplet, *Morrey spaces and classification of global solutions for a supercritical semilinear heat equation in $\mathbb{R}^N$*, J. Funct. Anal. **272** (2017), 2005–2037.
[31] P. Quittner, *The decay of global solutions of a semilinear heat equation*, Discrete Contin. Dyn. Syst. **21** (2008), 307–318.
[32] Y. W. Qi, *Critical exponents of degenerate parabolic equations*, Sci. China Ser. A **38** (1995), 1153–1162.
[33] Y. W. Qi, *The global existence and nonuniqueness of a nonlinear degenerate equation*, Nonlinear Anal. **31** (1998), 117–136.
[34] A. F. Tedeev, *Conditions for the time global existence and nonexistence of a compact support of solutions to the Cauchy problem for quasilinear degenerate parabolic equations*, Siberian Math. J. **45** (2004), no. 1, 155–164.
[35] A. F. Tedeev, *The interface blow-up phenomenon and local estimates for doubly degenerate parabolic equations*, Appl. Anal. **86** (2007), no. 6, 755–782.
[36] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Isr. J. Math. **38** (1981), no. 1–2, 29–40.
[37] Z. Q. Wu, J. N. Zhao, J. X. Yin, and H. L. Li, *Nonlinear Diffusion Equations*, World Scientific, Singapore, 2001.
[38] R. Z. Xu and J. Su, *Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations*, J. Funct. Anal. **264** (2013), no. 12, 2732–2763.
[39] H. Y. Xu, *Existence and blow-up of solutions for finitely degenerate semilinear parabolic equations with singular potentials*, Commun. Anal. Mech. **15** (2023), no. 2, 132–161.
[40] J. N. Zhao, *On the Cauchy problem and initial traces for the evolution p-Laplacian equations with strongly nonlinear sources*, J. Differ. Equations **121** (1995), no. 2, 329–383.