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Research Article

Huyuan Chen* and Li Cheng

Bounds for the sum of the first *k*-eigenvalues of Dirichlet problem with logarithmic order of Klein-Gordon operators

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Abstract: We provide bounds for the sequence of eigenvalues $\{\lambda_i(\Omega)\}_i$ of the Dirichlet problem

$$(I-\Delta)^{\ln} u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega,$$

where $(I-\Delta)^{\ln}$ is the Klein-Gordon operator with Fourier transform symbol $\ln(1+|\xi|^2)$. The purpose of this study is to obtain the upper and lower bounds for the sum of the first k-eigenvalues by extending the Li-Yau's method and Kröger's method, respectively.

Keywords: Dirichlet eigenvalues, logarithmic Klein-Gordon operator

MSC 2020: 35P15, 35R09

1 Introduction and main results

The purpose of this study, is to consider bounds for the spectrum of Klein-Gordon operator with the logarithmic order, subject to the Dirichlet boundary condition, i.e.,

$$\begin{cases} (I-\Delta)^{\ln} u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded regular domain $\mathbb{R}^N$ with $N \ge 2$ and the logarithmic Kein-Gordon operator $(I-\Delta)^{\ln}$ is an integro-differential operator with the Fourier symbol $\ln(1+|\cdot|^2)$.

In recent years, problems involving nonlocal differential order operators have been studied extensively and deeply, see [4,6,16,18,19,26,27] and references therein. The prototype model is the fractional power of the Laplacian. Recall that for $s \in (0,1)$ the fractional Laplacian of a function $u \in C_c^\infty(\mathbb{R}^N)$ is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) \coloneqq (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\mathrm{i}x\cdot\xi}((-\Delta)^s u)(x)\mathrm{d}x = |\xi|^{2s}\hat{u}(\xi) \quad \text{for all } \ \xi \in \mathbb{R}^N,$$

where and in the sequel both $\mathcal{F}$ and $\hat{\cdot}$ denote the Fourier transform. Equivalently, $(-\Delta)^s$ can be written as a singular integral operator under the following form:

$$(-\Delta)^s u(x) = c_{N,s} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N \backslash B_\varepsilon(0)} \frac{u(x)-u(x+y)}{|y|^{N+2s}}\mathrm{d}y,$$

where $c_{N,s} = 2^{2s}\pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$ and Γ is the Gamma function.

*** Corresponding author: Huyuan Chen,** School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, PR China, e-mail: chenhuyuan@yeah.net

Li Cheng: School of Mathematics and Statistics, Jiangxi Normal University, Nanchang, Jiangxi 330022, PR China, e-mail: Chengli152793@163.com

Another important class of nonlocal differential operators is the fractional power of the relativistic Shrodinger operator. It is known that the pseudo-differential operator $(m^2-\Delta)^{\frac{1}{2}}$ appears in the prescriptions of quantum mechanics to the Klein-Gordon Hamiltonian $\sqrt{|P|^2+m^2}$, which is a first-order pseudo-differential operator used to model relativistic particles in quantum mechanics. Here m stands for the mass of particles. Setting $m=1$, the normalized operator $(I-\Delta)^{\frac{1}{2}}$ is also called as the Kein-Gordon operator with order $\frac{1}{2}$, corresponding to the Fourier symbol $(1+|\cdot|^2)^{\frac{1}{2}}$ and $(I-\Delta)^s$ is called as the s-order Klein-Gordon operator.

Moreover, $(I-\Delta)^s$ could be represented via hypersingular integral ([28, page 548] and [12]) that for a regular function $u:\mathbb{R}^N\to\mathbb{R}$,

$$(I-\Delta)^s u(x) = u(x) + d_{N,s}\,\text{p.v.}\int_{\mathbb{R}^N}\frac{u(x)-u(x+y)}{|y|^{N+2s}}\omega_s(|y|)\mathrm{d}y, \tag{1.2}$$

where $d_{N,s}=\frac{\pi^{-\frac{N}{2}}4^s}{-\Gamma(-s)}$ is a normalization constant and the function ω_s is given by

$$\omega_s(|y|) = 2^{1-\frac{N+2s}{2}}|y|^{\frac{N+2s}{2}}K_{\frac{N+2s}{2}}(|y|) = \int_0^\infty t^{-1+\frac{N+2s}{2}}e^{-t-\frac{|y|^2}{4t}}\mathrm{d}t.$$

Here the function K_ν is the modified Bessel function of the second kind with index $\nu>0$ and it is given by the expression

$$K_\nu(r) = \frac{(\pi/2)^{\frac{1}{2}}r^\nu e^{-r}}{\Gamma(\frac{2\nu+1}{2})}\int_0^\infty e^{-rt}t^{\nu-\frac{1}{2}}(1+t/2)^{\nu-\frac{1}{2}}\mathrm{d}t.$$

The normalization constant $d_{N,s}$ in (1.2) is chosen such that the operator $(I-\Delta)^s$ is equivalently defined via its Fourier representation given by

$$\mathcal{F}((I-\Delta)^s u)(\xi) = (1+|\xi|^2)^s\mathcal{F}(u)(\xi)\quad\text{for a.e } \xi\in\mathbb{R}^N.$$

The fractional Laplacian has the following limiting properties when s approaches the values 0 and 1:

$$\lim_{s\to1^-}(-\Delta)^s u(x) = -\Delta u(x)\quad\text{and}\quad \lim_{s\to0^+}(-\Delta)^s u(x) = u(x)\quad\text{for}\ u\in C_c^2(\mathbb{R}^N),$$

e.g., [28]. It is proved in [8] that a remarkable expansion at $s=0$ is valid for $u\in C_c^2(\mathbb{R}^N)$ and $x\in\mathbb{R}^N$,

$$(-\Delta)^s u(x) = u(x) + s(-\Delta)^{\ln}u(x) + o(s)\quad\text{as}\ s\to0^+,$$

where, formally, the operator

$$(-\Delta)^{\ln} \coloneqq \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(-\Delta)^s$$

is given as a *logarithmic Laplacian*. For compactly supported Dini continuous functions $\phi:\mathbb{R}^N\to\mathbb{R}$, the logarithmic Laplacian $(-\Delta)^{\ln}$ has the integro-differential formula defined by

$$(-\Delta)^{\ln}\phi(x) = c_N\lim_{\varepsilon\to0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{\phi(x)1_{B_1(x)}(y)-\phi(y)}{|x-y|^N}\mathrm{d}y + \rho_N\phi(x)$$

with the constants $c_N\coloneqq\frac{\Gamma(N/2)}{\pi^{N/2}}$ and $\rho_N\coloneqq 2\ln 2+\psi(\frac{N}{2})-\gamma$.

It is easy to observe that

$$\lim_{s\to0^+}(I-\Delta)^s u = u\quad\text{for}\ u\in C^2(\mathbb{R}^N).$$

Inspired by the logarithmic Laplacian $(-\Delta)^{\ln}$, the logarithmic Schrödinger operator $(I-\Delta)^{\ln}$ has been introduced in [13] in a Taylor expansion with respect to the parameter s of the operator $(I-\Delta)^s$ near zero, i.e., for $u\in C^2(\mathbb{R}^N)$ and $x\in\mathbb{R}^N$

$$(I-\Delta)^s u(x) = u(x) + s(I-\Delta)^{\ln}u(x) + o(s)\quad\text{as}\ s\to0^+,$$

where the logarithmic Schrödinger operator $(I-\Delta)^{\ln}$ appears as the first-order term in the above expansion and it has the following properties: For $u \in C^{\alpha}(\mathbb{R}^N)$ for some $\alpha > 0$ and any $x \in \mathbb{R}^N$,

$$
\begin{aligned}
(I-\Delta)^{\ln}u(x) &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0^+}[(I-\Delta)^s u](x) \\
&= d_N \int_{\mathbb{R}^N} \frac{u(x)-u(x+y)}{|y|^N}\omega(|y|)\mathrm{d}y \\
&= \int_{\mathbb{R}^N} (u(x)-u(x+y))J(y)\mathrm{d}y,
\end{aligned}
\tag{1.3}
$$

where $d_N := \pi^{-\frac{N}{2}} = -\lim_{s\to 0^+}\frac{d_{N,s}}{s}$, $J(y) = d_N\frac{\omega(|y|)}{|y|^N}$ and

$$
\omega(|y|) := 2^{1-\frac{N}{2}}|y|^{\frac{N}{2}}K_{\frac{N}{2}}(|y|) = \int_0^{\infty} t^{-1+\frac{N}{2}}e^{-t-\frac{|y|^2}{4t}}\mathrm{d}t. \tag{1.4}
$$

Moreover, the author obtained a sequence of Dirichlet eigenvalues for the logarithmic order Klein-Gordon operator. We denote by $\mathbb{H}_0^{\ln}(\Omega)$, the completion of $C_c^{\infty}(\Omega)$ with respect to the norm $\|u\|_{\ln} := \sqrt{\mathcal{E}_{\omega}(u,u)}$, where $\mathcal{E}_{\omega}$ is the bilinear form

$$
\mathcal{E}_{\omega}(u,v) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(v(x)-v(y))J(x-y)\mathrm{d}x\mathrm{d}y.
$$

It is known that the logarithmic Schrödinger operator appears in the probability literature as the characteristic exponent of the symmetric variance gamma process in $\mathbb{R}^N$ and can be seen as a sub-class of increasing Lévy process [2]. As a particular case of geometric stable processes $\ln(1+|\cdot|^{2s})$ for $s \in (0,1)$, it plays an important role in the study of Markov process [1,30]. Very recently, [5] obtained the regularity result of Poisson problem with the logarithmic Schrödinger operator.

We call a function $u \in \mathbb{H}_0^{\ln}(\Omega)$ an eigenfunction of (1.1) corresponding to the eigenvalue λ if

$$
\mathcal{E}_{\omega}(u,\phi) = \lambda\int_{\Omega} u\phi\mathrm{d}x \quad \text{for all } \phi \in C_c^{\infty}(\Omega). \tag{1.5}
$$

Feulefack [13, Theorem 1.3] provided the following characterization of the eigenvalues and eigenfunctions for the operator $(I-\Delta)^{\ln}$ in an open bounded set Ω of $\mathbb{R}^N$:

(i) Problem (1.1) admits an eigenvalue $\lambda_1(\Omega) > 0$, which is characterized by

$$
\lambda_1(\Omega) = \inf_{u\in\mathcal{P}_1(\Omega)} \mathcal{E}_{\omega}(u,u), \tag{1.6}
$$

with $\mathcal{P}_1(\Omega) := \{u \in \mathbb{H}_0^{\ln}(\Omega) : \|u\|_{L^2(\Omega)} = 1\}$ and there exists a positive function $\phi_1 \in \mathbb{H}_0^{\ln}(\Omega)$, which is an eigenfunction corresponding to $\lambda_1(\Omega)$ and that attains the minimum in (1.6), i.e., $\|\phi_1\|_{L^2(\Omega)} = 1$ and $\lambda_1(\Omega) = \mathcal{E}_{\omega}(\phi_1,\phi_1)$.

(ii) The first eigenvalue $\lambda_1(\Omega)$ is simple, i.e., if $u \in \mathbb{H}_0^{\ln}(\Omega)$ satisfies (1.5) with $\lambda = \lambda_1(\Omega)$, then $u = \alpha\phi_1$ for some $\alpha \in \mathbb{R}$.

(iii) Problem (1.1) admits a sequence of eigenvalues $\{\lambda_k(\Omega)\}_{k\in\mathbb{N}}$ with

$$
0 < \lambda_1(\Omega) < \lambda_2(\Omega) \le \cdots \le \lambda_k(\Omega) \le \lambda_{k+1}(\Omega), \le \cdots
$$

with the corresponding eigenfunctions ϕ_k, $k \in \mathbb{N}$ and $\lim_{k\to\infty}\lambda_k(\Omega) = +\infty$. Moreover, for any $k \in \mathbb{N}$, the eigenvalue $\lambda_k(\Omega)$ can be characterized as

$$
\lambda_k(\Omega) = \inf_{u\in\mathcal{P}_k(\Omega)} \mathcal{E}_{\omega}(u,u), \tag{1.7}
$$

where $\mathcal{P}_k(\Omega)$ is given by

$$\mathcal{P}_k(\Omega) := \left\{u \in \mathbb{H}_0^{\ln}(\Omega) : \int_\Omega u\phi_j dx = 0 \quad \text{for } j = 1, 2, \cdots k-1 \quad \text{and} \quad \|\phi_k\|_{L^2(\Omega)} = 1\right\}.$$

(iv) The sequence $\{\phi_k\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to eigenvalues $\lambda_k(\Omega)$ forms a complete orthonormal basis of $L^2(\Omega)$ and an orthogonal system of $\mathbb{H}_0^{\ln}(\Omega)$.

The asymptotic of the Dirichlet eigenvalues is one of the hottest topics in the area of partial differential equations. It is well-known that the Hilbert-Pólya conjecture is to associate the zero of the Riemann zeta function with the eigenvalue of a Hermitian operator. This quest initiated the mathematical interest for estimating the sum of Dirichlet eigenvalues of the Laplacian while in physics, the question is related to count the number of bound states of a one body Schrödinger operator and to bound their asymptotic distribution.

The eigenvalues of the Laplacian subject to the zero boundary condition has Weyl's limit [31] and Pólya lower bound [25] for "plane-covering domain." The well-known Pólya's conjecture has partially been solved in [23] by considering the sum of the first k-eigenvalues and in a ball [14] very recently. More bounds for eigenvalues could be seen in [15,20,21,24]. The eigenvalues of fractional Laplacians

$$(-\Delta)^s u = \mu u \quad \text{in } \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \tag{1.8}$$

have been studied in [3,9,11,32] and references therein. The related lower bound is

$$\mu_{s,k}(\Omega) \ge \frac{N}{N+2s} C_N \left(\frac{k}{|\Omega|}\right)^{\frac{2s}{N}}. \tag{1.9}$$

For the $\frac{1}{2}$-order Klein-Gordon operator $(I-\Delta)^{\frac{1}{2}}$, Weyl's limit of related Dirichlet eigenvalues has been studied in [17] in a trace sense.

Very recently, Chen and Weth in [8] showed the existence of a sequence of Dirichlet eigenvalues $\{\lambda_i^L(\Omega)\}_{i\in\mathbb{N}}$ of logarithmic Laplacian, i.e.,

$$\begin{cases} (-\Delta)^{\ln} u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.10}$$

Later on, Laptev and Weth obtained in [23, Corollary 6.2] Weyl's limit

$$\lim_{k\to+\infty} \frac{\lambda_k^L(\Omega)}{\ln k} = \frac{2}{N}.$$

Furthermore, lower bounds for the first eigenvalue are considered there by a particular scaling property. Recently, Chen and Véron [7, Theorems 1.2 and 1.3] show that the bounds of related eigenvalues are as follows:

(i) for any $k \in \mathbb{N}^*$, setting $d_N = \frac{2\omega_N}{N^2(2\pi)^N}$, $\sum_{i=1}^k \lambda_i^L(\Omega) \ge -d_N|\Omega|$;

(ii) if $k > \frac{eNd_N}{2}|\Omega|$, $\sum_{i=1}^k \lambda_i^L(\Omega) > 0$;

(iii) if $k \ge \frac{e^{e+1}Nd_N}{2}|\Omega|$;

$$\sum_{i=1}^k \lambda_i^L(\Omega) \ge \frac{2k}{N}\left(\ln k - \ln\ln\left(\frac{2k}{eNd_N|\Omega|} + \ln\left(\frac{2}{eNd_N|\Omega|}\right)\right)\right);$$

(iv) upper bound, setting $p_N = \frac{2(2\pi)^N N}{\omega_N}$, for $k > \frac{eNd_N}{2}|\Omega|$,

$$\sum_{i=1}^k \lambda_i^L(\Omega) \le \frac{2k}{N}\left(\ln(k+1) + \ln\left(\frac{p_N}{|\Omega|}\right) + \frac{\omega_N}{\sqrt{|\Omega|}} \ln\ln\left(\frac{p_N(k+1)}{|\Omega|}\right)\right).$$

Due to the lack of homogeneity of the logarithmic order operators, there is no scaling property. As a consequence, homothety of the domain is inoperative for studying the variations in principle eigenvalue as it is the case for the Laplacian or the fractional Laplacian. Different from the logarithmic Laplacian, the logarithmic Klein-Gordon operator combines the homogeneity and the logarithmic order differentiating, which makes it more complicated to study the asymptotics of the Dirichlet eigenvalues.

Our purpose in this article is to provide the lower and upper bounds for the sum of the first k-eigenvalues of (1.1) by developing the Berezin-Li-Yau methods and Kröger's result for the Laplacian.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain, $\{\lambda_i(\Omega)\}_{i\in N}$ be the sequence of eigenvalues of problem* (1.1).

Then, there exist positive constants C_1, C_2, which only depend on N such that for integer $k > \frac{e^{e^2}\omega_N \mid \Omega \mid}{2(2\pi)^N}$,

$$\sum_{i=1}^{k} \lambda_i(\Omega) > \frac{2k}{N}\left(\ln\frac{k}{|\Omega|} - \ln\ln\left(1 + \frac{2(2\pi)^N k}{\omega_N|\Omega|}\right)\right) + C_1 k \tag{1.11}$$

and for positive integer $k \leq \frac{e^{e^2}\omega_N \mid \Omega \mid}{2(2\pi)^N}\left(if \ \frac{e^{e^2}\omega_N \mid \Omega \mid}{2(2\pi)^N} \geq 1\right)$,

$$\sum_{i=1}^{k} \lambda_i(\Omega) > C_2 \ |\Omega|^{-\frac{2}{N}} \ k^{1+\frac{2}{N}}.$$

In particular, it holds that

$$\lambda_1(\Omega) > \begin{cases} \frac{2}{N}\left(\ln\frac{1}{|\Omega|} - \ln\ln\left(1 + \frac{2(2\pi)^N}{\omega_N|\Omega|}\right)\right) + C_1, & if \quad \frac{e^{e^2}\omega_N|\Omega|}{2(2\pi)^N} < 1, \\ C_2 \ |\Omega|^{-\frac{2}{N}}, & if \quad \frac{e^{e^2}\omega_N|\Omega|}{2(2\pi)^N} \geq 1. \end{cases} \tag{1.12}$$

Our second interest is to give an upper bound for the sum of the first k-eigenvalues. Motivated by Kröger's result for the Laplacian [20], we shall construct an upper bound by computing the related Rayleigh quotient via a particular complex valued function. Together with the lower bound (3.2), we can derive the limit of the sum of the first k-eigenvalues as $k \to +\infty$. The results state as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\{\lambda_i(\Omega)\}_{i\in N}$ be the sequence of eigenvalues of problem* (1.1). *Then, for $k \geq 2$,*

$$\sum_{j=1}^{k} \lambda_j(\Omega) \leq \frac{2}{N} k\ln k + C(\ln\ln k + 1)k, \tag{1.13}$$

where $C = C(|\Omega|, |\partial\Omega|, N) > 0$ is independent of k.

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\{\lambda_i(\Omega)\}_{i\in\mathbb{N}}$ be the sequence of eigenvalues of problem* (1.1). *Then,*

$$\lim_{k\to+\infty} (k\ln k)^{-1} \sum_{i=1}^{k} \lambda_i(\Omega) = \frac{2}{N} \tag{1.14}$$

and the Weyl's limit

$$\lim_{k\to+\infty} \frac{\lambda_k(\Omega)}{\ln k} = \frac{2}{N}. \tag{1.15}$$

Remark 1.4. Note that $\frac{\lambda_k(\Omega)}{\ln k}$ and $(k\ln k)^{-1}\sum_{i=1}^{k}\lambda_i(\Omega)$ have the same limit $\frac{2}{N}$ as $k \to +\infty$, which is independent of Ω.

Due to the inhomogeneity of the expression of the Fourier symbol of $\ln(1 + |\cdot|^2)$, our bounds are inspired by [7], where the lower one is based on a developed Berezin-Li-Yau method for the non-homogeneous operator and the upper one is based on the analysis of functions $\sum_{i=1}^{k} e^{-\mathrm{i}x\cdot z}\chi_\Omega \phi_i(x)\phi_i(y)$.

The rest of this work is organized as follows. In Section 2, we prove the following Li-Yau-type lower bound for the sums of the first k-eigenvalues. Sections 3 and 4 are devoted to the lower and upper bounds for the sum of the first k-eigenvalues of Theorems 1.1 and 1.2, respectively. Finally, we obtain the limits of the eigenvalues in Corollary 1.3.

2 Preliminary

The section is devoted to show Li-Yau-type lower bound for the sums of the first k-eigenvalues to the Logarithmic Schrödinger operator. To be convenient for the analysis, we denote

$$c_1 = \frac{N-1}{N+2}\left(\frac{2}{N+2}\right)^{\frac{N+2}{2}} \tag{2.1}$$

and

$$c_2 = \left(e^{\frac{2}{N}+1} - 1\right)^{\frac{N}{2}}. \tag{2.2}$$

2.1 Li-Yau-type lemma

Proposition 2.1. *Let f be a real measurable function defined in $\mathbb{R}^N$ with $0 \le f \le M_1$ a.e. in $\mathbb{R}^N$ and*

$$\int_{\mathbb{R}^N} \ln(1+|z|^2) f(z)\mathrm{d}z = M_2.$$

Then,

$$\int_{\mathbb{R}^N} f(z)\mathrm{d}z < \max\left\{\frac{M_2 N}{2}\left(\ln\frac{M_2 N}{M_1\omega_N} - \ln\left(\ln\left(1+\frac{M_2 N}{M_1\omega_N}\right)\right) + \ln\frac{N}{2} - 1\right)^{-1}, \frac{A_1}{A_2}\frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2 N}{M_1\omega_N}\right)^{\frac{N}{N+2}}\right\},$$

where $A_1 = \frac{M_1\omega_N}{2}\frac{c_2^2}{c_1}\left(\ln c_2 - \ln(\ln(1+c_2)) + \ln\frac{N}{2} - 1\right)^{-1}$, $A_2 = \frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}c_1\right)^{\frac{N}{N+2}}$.

To prove Proposition 2.1, we need some auxiliary lemmas.

Lemma 2.2. *Let*

$$g_0(r) = r\log\left(1 + r^{\frac{2}{N}}\right) - \frac{2}{N+2}r^{\frac{N+2}{N}}, \quad \forall r \in [0, t_0],$$

with $t_0 = \left(\sqrt{1+(\frac{N}{4})^2} + \frac{N}{4} - 1\right)^{\frac{N}{2}}$, *the function*

$$g_1(r) = \begin{cases} g_0(r), & r \in [0, t_0], \\ g_0(t_0), & r > t_0 \end{cases} \tag{2.3}$$

and c_1 be given by (2.1).

Then, for any $c \in (0, c_1]$, there exists a unique real number r_c such that

$$g_1(r_c) = c.$$

Moreover, it holds that

$$r_c \le \left(\frac{N+2}{N-1}c\right)^{\frac{N}{N+2}}. \tag{2.4}$$

Proof. By direct computation, we have that

$$g_0'(r) = \log\left(1 + r^{\frac{2}{N}}\right) + \frac{2}{N}\frac{r^{\frac{2}{N}}}{1 + r^{\frac{2}{N}}} - \frac{2}{N}r^{\frac{2}{N}},$$

$$g_0''(r) = \frac{2}{N}\frac{r^{\frac{2}{N}-1}}{\left(1 + r^{\frac{2}{N}}\right)^2}\left(1 + r^{\frac{2}{N}} - \frac{4}{N}r^{\frac{2}{N}} - \frac{2}{N}r^{\frac{4}{N}}\right)$$

and observe that $g_0''(r) > 0$ when $r \in (0, t_0)$. Then, g_0' is increasing in $[0, t_0]$, combining with the fact that $g_0'(0) = 0$, we obtain that g_0 is increasing in $(0, t_0]$, so is g_1.

Denote $t_1 = \left(\frac{2}{N+2}\right)^{\frac{N}{2}}$. Note that $t_0 > t_1$ and $g_1(t_0) > g_1(t_1) \geq c_1$, then for any $c \in (0, c_1]$, there is a unique real number $r_c \in (0, t_1)$ such that

$$g_1(r_c) = c.$$

Now, we do the bound of r_c. Since $\log(1 + r) \geq r - \frac{1}{2}r^2$ for any $r > 0$, we have that

$$\begin{aligned} c = g_1(r_c) &= r_c \log(1 + r_c^{\frac{2}{N}}) - \frac{2}{N+2}r_c^{\frac{N+2}{N}} \\ &\geq r_c\left(r_c^{\frac{2}{N}} - \frac{1}{2}r_c^{\frac{4}{N}}\right) - \frac{2}{N+2}r_c^{\frac{N+2}{N}} \\ &= r_c^{\frac{N+2}{N}}\left(\frac{N}{N+2} - \frac{1}{2}r_c^{\frac{2}{N}}\right) \\ &\geq \frac{N-1}{N+2}r_c^{\frac{N+2}{N}}, \end{aligned}$$

where the last inequality holds by the fact that $r_c < t_1 = \left(\frac{2}{N+2}\right)^{\frac{N}{2}}$. Thus, (2.4) holds. The proof is completed. □

Lemma 2.3. *Assume that*

$$g_2(r) = r\ln\left(1 + r^{\frac{2}{N}}\right) - \frac{2}{N}r, \quad \forall r \geq 0. \tag{2.5}$$

Then, for any $c > 0$, there exists a unique real number $\tilde{r}_c$ such that

$$g_2(\tilde{r}_c) = c.$$

Furthermore, let c_1 and c_2 be given by (2.1) *and* (2.2), *then*

(*i*) *for $c \geq c_2$, it holds that*

$$\tilde{r}_c < \frac{N}{2}\frac{c}{\ln c - \ln(\ln(1 + c)) + \ln\frac{N}{2} - 1}; \tag{2.6}$$

(*ii*) *for $c \in (c_1, c_2)$, it holds that*

$$\tilde{r}_c < \frac{N}{2}\frac{c_2}{c_1}\frac{c}{\ln c_2 - \ln(\ln(1 + c_2)) + \ln\frac{N}{2} - 1}. \tag{2.7}$$

Proof. By direct computation, we have that

$$g_2'(r) = \ln\left(1 + r^{\frac{2}{N}}\right) + \frac{2}{N}\frac{r^{\frac{2}{N}}}{1 + r^{\frac{2}{N}}} - \frac{2}{N}$$

and

$$g_2''(r) = \frac{\left(\frac{2}{N} + \frac{4^2}{N^2}\right) r^{\frac{2}{N}-1} + \frac{2}{N} r^{\frac{4}{N}-1}}{\left(1 + r^{\frac{2}{N}}\right)^2} > 0.$$

Then, the function g_2' is increasing in $(0, +\infty)$ and then g_2 is increasing in $[\tilde{t}_0, +\infty)$ with $\tilde{t}_0 = (e^{\frac{2}{N}} - 1)^{\frac{N}{2}}$. Observe that $g_2(\tilde{t}_0) = g_2(0) = 0$ and $\lim_{r\to+\infty} g_2(r) = +\infty$. Then, for any $c > 0$, there exists a unique real number $\tilde{r}_c > \tilde{t}_0$ such that

$$g_2(\tilde{r}_c) = c,$$

i.e.,

$$\tilde{r}_c = \frac{c}{\ln\left(1 + \tilde{r}_c^{\frac{2}{N}}\right) - \frac{2}{N}}. \tag{2.8}$$

Moreover, $\tilde{r}_c$ is increasing with respect to c.

Now, we do bounds for $\tilde{r}_c$.

(i) In the case of $c \ge c_2$. Note that $g_2(c_2) = c_2$ and $c_2 > \tilde{t}_0 > 0$. Letting

$$\psi(c) = g_2(c) - c = c \ln\left(1 + c^{\frac{2}{N}}\right) - \frac{2}{N} c - c, \quad \forall c \ge c_2.$$

By direct computation, it yields that

$$\psi'(c) = g_2'(c) - 1 = \ln\left(1 + c^{\frac{2}{N}}\right) + \frac{2}{N} \frac{c^{\frac{2}{N}}}{1 + c^{\frac{2}{N}}} - \frac{2}{N} - 1$$

and

$$\psi''(c) = g_2''(c) = \frac{\left(\frac{2}{N} + \frac{4^2}{N^2}\right) c^{\frac{2}{N}-1} + \frac{2}{N} c^{\frac{4}{N}-1}}{\left(1 + c^{\frac{2}{N}}\right)^2} > 0.$$

Then, ψ' is increasing and by the fact that $\psi(0) = \psi(c_2) = 0$ and $\psi'(c_2) = \frac{2}{N} \frac{e^{\frac{2}{N}+1} - 1}{e^{\frac{2}{N}+1}} > 0$, we have that ψ is increasing in $[c_2, +\infty)$. Thus, $\psi \ge 0$ in $[c_2, +\infty)$. Then, for any $c \ge c_2$, we obtain that $g_2(c) \ge c = g_2(\tilde{r}_c)$, by the monotonicity of g_2, which implies that $c \ge \tilde{r}_c$. Combining with (2.8), it holds that

$$\tilde{r}_c = \frac{c}{\ln\left(1 + \tilde{r}_c^{\frac{2}{N}}\right) - \frac{2}{N}} \ge \frac{c}{\ln\left(1 + c^{\frac{2}{N}}\right) - \frac{2}{N}}. \tag{2.9}$$

For $N \ge 1$, we have that for any $a, b > 0$,

$$a^{\frac{2}{N}} + b^{\frac{2}{N}} \le 2^{\frac{2}{N}} (a + b)^{\frac{2}{N}}.$$

Then,

$$1 + c^{\frac{2}{N}} \le 2^{\frac{2}{N}} (1 + c)^{\frac{2}{N}}$$

and

$$\frac{2}{N} \ln(1 + c) + \frac{2}{N} \ln 2 - \frac{2}{N} \le \frac{2}{N} \ln(1 + c).$$

Combining with (2.9), it yields that

$$\tilde{r}_c > \frac{N}{2} \frac{c}{\ln(1 + c)}. \tag{2.10}$$

Together with (2.8) and (2.10), for $c \geq c_2$, it holds that

$$\tilde{r}_c < \frac{c}{\ln\left(1+\left(\frac{N}{2}\frac{c}{\ln(1+c)}\right)^{\frac{2}{N}}\right)-\frac{2}{N}} < \frac{c}{\frac{2}{N}\ln\left(\frac{N}{2}\frac{c}{\ln(1+c)}\right)-\frac{2}{N}} = \frac{N}{2}\frac{c}{\ln c-\ln(\ln(1+c))+\ln\frac{N}{2}-1}. \tag{2.11}$$

(ii) In the case of $c \in (c_1, c_2)$, observe that $\tilde{r}_{c_1} < \tilde{r}_c < \tilde{r}_{c_2}$, by (2.11), we have that

$$\tilde{r}_c < \frac{N}{2}\frac{c_2}{\ln c_2-\ln(\ln(1+c_2))+\ln\frac{N}{2}-1} < \frac{N}{2}\frac{\frac{c_2}{c_1}c}{\ln c_2-\ln(\ln(1+c_2))+\ln\frac{N}{2}-1}.$$

The proof is completed. □

Now, we are ready to give the proof of Proposition 2.1.

Proof of Proposition 2.1. Let $R > 0$, we observe that

$$(\ln(1+|z|^2)-\ln(1+R^2))(f(z)-M_1\mathbf{1}_{B_R}) \geq 0.$$

By integration over $\mathbb{R}^N$, we obtain that

$$M_2+\frac{2M_1\omega_N}{N}\int_0^R\frac{r^{N+1}}{1+r^2}\mathrm{d}r \geq \ln(1+R^2)\int_{\mathbb{R}^N}f(z)\mathrm{d}z.$$

Since R is arbitrary, $\int_{\mathbb{R}^N}f(z)\mathrm{d}z$ satisfies the following upper bound:

$$\int_{\mathbb{R}^N}f(z)\mathrm{d}z \leq \sup\left\{A>0 \quad \text{s.t. } M_2+\frac{2M_1\omega_N}{N}\int_0^R\frac{r^{N+1}}{1+r^2}\mathrm{d}r-A\ln(1+R^2)>0\right\}. \tag{2.12}$$

Set

$$\mathcal{I}_A(R)=M_2+\frac{2M_1\omega_N}{N}\int_0^R\frac{r^{N+1}}{1+r^2}\mathrm{d}r-A\ln(1+R^2),$$

then $\mathcal{I}_A$ achieves the minimum if $\frac{M_1\omega_N R^N}{N}=A$, that is

$$R=\left(\frac{NA}{\omega_N M_1}\right)^{\frac{1}{N}}.$$

Let

$$t_A=\frac{NA}{\omega_N M_1},$$

then,

$$\mathcal{I}_A(t_A^{\frac{1}{N}})=M_2+\frac{2M_1\omega_N}{N}\int_0^{t_A^{\frac{1}{N}}}\frac{r^{N+1}}{1+r^2}\mathrm{d}r-A\ln\left(1+t_A^{\frac{2}{N}}\right)\geq 0,$$

which implies that

$$A\ln\left(1+t_A^{\frac{2}{N}}\right)-\frac{2M_1\omega_N}{N}\int_0^{t_A^{\frac{1}{N}}}\frac{r^{N+1}}{1+r^2}\mathrm{d}r \leq M_2. \tag{2.13}$$

On the one hand, we denote

$$h(t) = t\ln\left(1 + t^{\frac{2}{N}}\right) - 2\int_0^{t^{\frac{1}{N}}} \frac{r^{N+1}}{1+r^2}\mathrm{d}r,$$

by direct computation, it holds that

$$h'(t) = \ln\left(1 + t^{\frac{2s}{N}}\right) > 0,$$

then h is increasing in $(0, +\infty)$. Moreover, we note that

$$h(t) \geq t\ln\left(1 + t^{\frac{2}{N}}\right) - 2\int_0^{t^{\frac{1}{N}}} r^{N+1}\mathrm{d}r = t\ln(1 + t^{\frac{2}{N}}) - \frac{2}{N+2}t^{\frac{N+2}{N}},$$

combining with the monotonicity of h, it yields that $h \geq g_1$ in $(0, +\infty)$, where g_1 is given by (2.3). Together with (2.13), we have that

$$\frac{M_2 N}{M_1\omega_N} \geq h(t_A) \geq g_1(t_A).$$

(i) When $\frac{M_2}{M_1} \leq \frac{c_1\omega_N}{N}$, i.e., $c \leq c_1$, there exists a unique real number r_c such that $g_1(r_c) = c$. Then, $g_1(r_c) = \frac{M_2 N}{M_1\omega_N} \geq g_1(t_A)$ and by the monotonicity of g_1, it holds that $t_A \leq r_c$, by (2.4), we obtain that

$$t_A \leq \left(\frac{N+2}{N-1}c\right)^{\frac{N}{N+2}},$$

and then

$$A \leq \frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2 N}{M_1\omega_N}\right)^{\frac{N}{N+2}},$$

thus,

$$\int_{\mathbb{R}^N} f(z)\mathrm{d}z < \frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2 N}{M_1\omega_N}\right)^{\frac{N}{N+2}}.$$

On the other hand, by the fact that $\frac{r^{N+1}}{1+r^2} < r^{N-1}$, Inequality (2.13) could be replaced by

$$A\ln\left(1 + t_A^{\frac{2}{N}}\right) - \frac{2M_1\omega_N}{N^2}t_A \leq M_2$$

and

$$t_A\ln(1 + t_A^{\frac{2}{N}}) - \frac{2}{N}t_A \leq \frac{M_2 N}{M_1\omega_N}.$$

(ii) When $\frac{c_1\omega_N}{N} < \frac{M_2}{M_1} < \frac{c_2\omega_N}{N}$, i.e., $c_1 < c < c_2$, applying Lemma 2.3 (ii), we have that

$$t_A < \frac{N}{2}\frac{c_2}{c_1}\frac{c}{\ln c_2 - \ln(\ln(1+c_2)) + \log\frac{N}{2} - 1},$$

then

$$A < \frac{M_1\omega_N}{2}\frac{c_2}{c_1}\frac{c_2}{\ln c_2 - \ln(\ln(1+c_2)) + \ln\frac{N}{2} - 1}.$$

Thus, we obtain that

$$\int_{\mathbb{R}^N} f(z)\mathrm{d}z < \frac{M_1\omega_N}{2}\frac{c_2^2}{c_1}\left(\ln c_2 - \ln(\ln(1+c_2)) + \ln\frac{N}{2} - 1\right)^{-1} = A_1$$

and

$$\frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2N}{M_1\omega_N}\right)^{\frac{N}{N+2}} > \frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}c_1\right)^{\frac{N}{N+2}} = A_2.$$

Then,

$$\frac{M_1\omega_N}{2}\frac{c_2^2}{c_1}\left(\ln c_2 - \ln(\ln(1+c_2)) + \ln\frac{N}{2} - 1\right)^{-1} < \frac{A_1}{A_2}\frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2N}{M_1\omega_N}\right)^{\frac{N}{N+2}}.$$

Therefore, we have that

$$\int_{\mathbb{R}^N} f(z)\mathrm{d}z < \frac{A_1}{A_2}\frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2N}{M_1\omega_N}\right)^{\frac{N}{N+2}}.$$

(iii) When $\frac{M_2}{M_1} \geq \frac{c_2\omega_N}{N}$, i.e., $c = \frac{M_2N}{M_1\omega_N} \geq c_2$, by Lemma 2.3 (i), we obtain that

$$t_A < \frac{N}{2}\frac{c}{\ln c - \ln(\ln(1+c)) + \ln\frac{N}{2} - 1},$$

and then,

$$A < \frac{M_1\omega_N}{2}\frac{c}{\ln c - \ln(\log(1+c)) + \ln\frac{N}{2} - 1},$$

which implies that

$$\int_{\mathbb{R}^N} f(z)\mathrm{d}z < \frac{M_2N}{2}\left(\ln\frac{M_2N}{M_1\omega_N} - \ln\left(\ln\left(1+\frac{M_2N}{M_1\omega_N}\right)\right) + \ln\frac{N}{2} - 1\right)^{-1}.$$

Thus, we have that

$$\int_{\mathbb{R}^N} f(z)\mathrm{d}z < \max\left\{\frac{M_2N}{2}\left(\ln\frac{M_2N}{M_1\omega_N} - \ln\left(\ln\left(1+\frac{M_2N}{M_1\omega_N}\right)\right) + \ln\frac{N}{2} - 1\right)^{-1}, \frac{A_1}{A_2}\frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2N}{M_1\omega_N}\right)^{\frac{N}{N+2}}\right\}.$$

The proof is complete. □

2.2 Dual bounds

Lemma 2.4. *Let*

$$\tilde{g}(r) = \frac{r}{\ln r - \ln\ln(1+r) + a_0} \quad \text{for } r > e^{2e},$$

where $a_0 = \min\{0, \ln\frac{N}{2} - 1\}$. *Then, for* $t > e^{e^2}$, *there exists a unique point* $r_t > e^{2e}$ *such that* $\tilde{g}(r_t) = t$.

Furthermore,

$$r_t > t(\ln t - \ln\ln(1+t) + a_0). \tag{2.14}$$

Proof. Since

$$\begin{aligned}\tilde g'(r) &= \frac{1}{\ln r - \ln\ln(1+r) + a_0} - \frac{1 - \frac{r}{\ln(1+r)(1+r)}}{(\ln r - \ln\ln(1+r) + a_0)^2}\\ &> \frac{1}{\ln r - \ln\ln(1+r) + a_0}\Big(1 - \frac{1}{\ln r - \ln\ln(1+r) + a_0}\Big).\end{aligned}$$

Since $r > e^{2e}$, $\ln r - \ln\ln(1+r) + a_0 > 1$, so

$$\tilde g'(r) > 0,$$

the function $\tilde g$ is increasing from $(e^{2e}, +\infty)$. Setting $r_t^* = t(\ln t - \ln\ln(1+t) + a_0)$, then

$$\begin{aligned}\tilde g(r_t^*) &= \frac{t(\ln t - \ln\ln(1+t) + a_0)}{\ln t + \ln(\ln t - \ln\ln(1+t) + a_0) - \ln\ln(1 + t(\ln t - \ln\ln(1+t) + a_0)) + a_0}\\ &\le \frac{t(\ln t - \ln\ln(1+t) + a_0)}{\ln t + \ln(\ln t - \ln\ln(1+t) + a_0) - \ln\ln(t\ln t) + a_0}\\ &= \frac{\ln t - \ln\ln(1+t) + a_0}{\ln t - \ln\frac{\ln(t\ln t)}{\ln t - \ln\ln(1+t) + a_0} + a_0}t < t,\end{aligned}$$

where the last inequality holds if

$$\frac{\ln(t\ln t)}{\ln t - \ln\ln(1+t) + a_0} < \ln(1+t),$$

and this inequality is equivalent to

$$\frac{\ln(t\ln t)}{\ln t - \ln\ln(1+t) + a_0} < \ln t,$$

which is equivalent to

$$\tilde h(\tau) \coloneqq \tau^2 - \ln\ln(e^\tau + 1)\tau - \tau + a_0\tau - \ln\tau > 0, \quad \tau = \ln t.$$

Thanks to $a_0 > -2$, we have that

$$\tilde h'(\tau) > 2\tau - \ln\ln(e^\tau + 1) - \frac{e^\tau}{\ln(e^\tau + 1)(e^\tau + 1)} - 3 - \frac{1}{\tau} > 0 \quad \text{if } \tau > e$$

and

$$\tilde h(e^2) > e^4 - \ln\ln(e^{e^2} + 1)e^2 - 3e^2 - 2 > 0.$$

Thus, for $t > e^{e^2}$, it yields that

$$r_t > t(\ln t - \ln\ln(1+t) + a_0).$$

The proof is complete. □

3 Proof of lower bound

This section gives the proof of lower bound in Theorem 1.1.

Proof of Theorem 1.1. Denote

$$\Phi_k(x,y) = \sum_{j=1}^{k}\phi_j(x)\phi_j(y), \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}^N$$

and

$$\widehat{\Phi}_k(y,z) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \Phi_k(x,y) e^{-ix\cdot z} \mathrm{d}x,$$

where $\widehat{\Phi}_k$ is the Fourier transform with respect to x. Since $\phi_{j_{i\in N}}$ is orthogonal in $L^2(\Omega)$, we have the identity

$$\int_{R^N}\int_{\Omega} |\widehat{\Phi}_k(y,z)|^2 \mathrm{d}z\mathrm{d}y = \int_{\Omega}\int_{\Omega} |\Phi_k(x,y)|^2 \mathrm{d}x\mathrm{d}y = k.$$

Furthermore, we note that

$$\begin{aligned}\int_{\Omega} |\widehat{\Phi}_k(y,z)|^2 \mathrm{d}y &= \int_{\Omega} \left(\sum_{j=1}^{k} \widehat{\phi}_j(z)\phi_j(y)\right)\left(\sum_{j=1}^{k} \overline{\widehat{\phi}_j(z)\phi_j(y)}\right)\mathrm{d}y \\ &= \int_{\Omega}\left(\sum_{j,\ell=1}^{k} \widehat{\phi}_j(z)\overline{\widehat{\phi}_\ell(z)}\phi_j(y)\phi_\ell(y)\right)\mathrm{d}y = \sum_{j=1}^{k} |\widehat{\phi}_j(z)|^2.\end{aligned}$$

Using again the orthonormality of the $\{\phi_j\}_{i\in N}$ in $L^2(\Omega)$, we infer by the k-dim Pythagorean theorem,

$$\begin{aligned}\int_{\Omega} |\widehat{\Phi}_k(y,z)|^2 \mathrm{d}y &= (2\pi)^{-N} \int_{\Omega} \left|\sum_{j=1}^{k}\left(\int_{\Omega} e^{-ix.z}\phi_j(x)\mathrm{d}x\right)\phi_j(y)\right|^2 \mathrm{d}y \\ &= (2\pi)^{-N} \left|\sum_{j=1}^{k}\left(\int_{\Omega} e^{-ix.z}\phi_j(x)\mathrm{d}x\right)\right|^2 \\ &\le (2\pi)^{-N} \left|\sum_{j=1}^{\infty}\left(\int_{\Omega} e^{-ix.z}\phi_j(x)\mathrm{d}x\right)\right|^2 \\ &= (2\pi)^{-N}\int_{\Omega} |e^{-ix.z}|^2 \mathrm{d}x = (2\pi)^{-N}|\Omega|.\end{aligned}$$

The Fourier expression of $(I-\Delta)^{\ln}$ applied in the variable x holds that

$$\begin{aligned}\sum_{j=1}^{k} \lambda_j(\Omega) &= \int_{\Omega}\int_{\Omega} \Phi_k(x,y)\Phi_k(x,y)\mathrm{d}y\mathrm{d}x \\ &= \sum_{j=1}^{k}\int_{\mathbb{R}^N} |\widehat{\phi}_j(z)|^2 \ln(1+|z|^2)\mathrm{d}z = \int_{\mathbb{R}^N}\left(\int_{\Omega} |\widehat{\Phi}_k(y,z)|^2\mathrm{d}y\right)\ln(1+|z|^2)\mathrm{d}z.\end{aligned}$$

Then, we obtain that

$$\begin{aligned}k &= \int_{\mathbb{R}^N} f(z)\mathrm{d}z \\ &< \max\left\{\frac{M_2N}{2}\left(\ln\frac{M_2N}{M_1\omega_N} - \ln\left(\ln\left(1+\frac{M_2N}{M_1\omega_N}\right)\right) + \ln\frac{N}{2} - 1\right)^{-1}, \frac{A_1}{A_2}\frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2N}{M_1\omega_N}\right)^{\frac{N}{N+2}}\right\}.\end{aligned}$$

Note that

$$\frac{M_2N}{2}\left(\ln\frac{M_2N}{M_1\omega_N} - \ln\left(\ln\left(1+\frac{M_2N}{M_1\omega_N}\right)\right) + \ln\frac{N}{2} - 1\right)^{-1} > k,$$

setting

$$r = \frac{NM_2}{M_1\omega_N}, \quad t = \frac{2k}{M_1\omega_N} = \frac{2(2\pi)^N k}{\omega_N|\Omega|}$$

and

$$\frac{r}{\ln r - \ln\ln(1+r) + \ln\frac{N}{2} - 1} = \frac{\frac{NM_2}{2} \times \frac{2}{M_1\omega_N}}{\ln r - \ln\ln(1+r) + \ln\frac{N}{2} - 1}.$$

Then, we have that

$$\frac{r}{\ln r - \ln\ln(1+r) + \ln\frac{N}{2} - 1} > \frac{2k}{M_1\omega_N} = \frac{2(2\pi)^N k}{\omega_N|\Omega|} = t.$$

By Lemma 2.4, if

$$\frac{2(2\pi)^N k}{\omega_N|\Omega|} > e^{e^2},$$

i.e.,

$$k > \frac{e^{e^2}\omega_N|\Omega|}{2(2\pi)^N},$$

then

$$r > r_t > t\left(\ln t - \ln\ln(1+t) + \ln\frac{N}{2} - 1\right),$$

which implies that

$$\frac{NM_2}{M_1\omega_N} > \frac{2(2\pi)^N k}{\omega_N|\Omega|}\left(\ln\left(\frac{2(2\pi)^N k}{\omega_N|\Omega|}\right) - \ln\ln\left(1 + \frac{2(2\pi)^N k}{\omega_N|\Omega|}\right) + \ln\frac{N}{2} - 1\right).$$

Then,

$$\sum_{j=1}^{k}\lambda_j(\Omega) > \frac{2k}{N}\left(\ln\frac{k}{|\Omega|} - \ln\ln\left(1 + \frac{2(2\pi)^N k}{\omega_N|\Omega|}\right) + C_1\right), \tag{3.1}$$

where $C_1 = C(N)$.

Letting $k = 1$ and $|\Omega| < \frac{2(2\pi)^N}{e^{e^2}\omega_N}$, we have that

$$\lambda_1(\Omega) > \frac{2}{N}\left(\ln\frac{1}{|\Omega|} - \ln\ln\left(1 + \frac{2(2\pi)^N}{\omega_N|\Omega|}\right) + C_1\right)$$

and

$$\frac{A_1}{A_2}\frac{M_1\omega_N}{N}\left(\frac{N+2}{N-1}\frac{M_2 N}{M_1\omega_N}\right)^{\frac{N}{N+2}} > k,$$

which leads to

$$\sum_{i=1}^{k}\lambda_i(\Omega) > C_2\,|\Omega|^{-\frac{2}{N}}k^{1+\frac{2}{N}}, \tag{3.2}$$

where $C_2 = C(N)$.

Particularly, take $k = 1$, we have that

$$\lambda_1(\Omega) > C_2\,|\Omega|^{-\frac{2}{N}}.$$

When $k > \frac{e^{e^2}\omega_N\,|\,\Omega\,|}{2(2\pi)^N}$, we compare (3.1) and (3.2) and obtain

$$\frac{2k}{N}\left(\ln\frac{k}{|\Omega|}-\ln\ln\left(1+\frac{2(2\pi)^N k}{\omega_N|\Omega|}\right)+C_1\right)<C_2\,|\Omega|^{-\frac{2}{N}}k^{1+\frac{2}{N}}.$$

Therefore, (i) if $k>\frac{e^{e^2}\omega_N\,|\,\Omega\,|}{2(2\pi)^N}$,

$$\sum_{j=1}^{k}\lambda_j(\Omega)>\frac{2k}{N}\left(\ln\frac{k}{|\Omega|}-\ln\ln\left(1+\frac{2(2\pi)^N k}{\omega_N|\Omega|}\right)+C_1\right).$$

(ii) If $k\le\frac{e^{e^2}\omega_N\,|\,\Omega\,|}{2(2\pi)^N}$,

$$\sum_{i=1}^{k}\lambda_i(\Omega)>C_2\,|\Omega|^{-\frac{2}{N}}k^{1+\frac{2}{N}}.$$

The proof is complete. □

4 Upper bound

The definition of $(I-\triangle)^{ln}$, i.e.,

$$(I-\triangle)^{ln}u(x)=d_N\int_{R^N}\frac{u(x)-u(x+y)}{|y|^N}\omega(|y|)\mathrm{d}y.$$

For any bounded Hölder continuous complex valued functions u, v admissible at infinity, there holds

$$(I-\triangle)^{ln}(uv)(x)=u(x)(I-\triangle)^{ln}v(x)+d_N\int_{\mathbb{R}^N}\frac{u(x)-u(\eta)}{|x-\eta|^N}v(\eta)\omega(|x-\eta|)\mathrm{d}\eta. \tag{4.1}$$

Lemma 4.1. *For $z\in\mathbb{R}^N\backslash\{0\}$, we denote*

$$\mu_z(x)=e^{-\mathrm{i}x\cdot z},\quad \forall x\in\mathbb{R}^N,$$

then

$$(I-\triangle)^{ln}\mu_z(x)=(\ln(1+|z|^2))\mu_z(x),\quad \forall x\in\mathbb{R}^N. \tag{4.2}$$

Proof. Since $s\in(0,1)$,

$$(I-\triangle)^s\mu_z(x)=(1+|z|^2)^s\mu_z(x),\quad \forall x\in\mathbb{R}^N. \tag{4.3}$$

μ_z is bounded, we can consider the distribution T_{μ_z} in $\mathbb{R}^N$ defined by

$$\langle T_{\mu_z}\zeta\rangle\coloneqq\int_{\mathbb{R}^N}\mu_z(x)\zeta(x)\mathrm{d}x\quad\text{for all }\ \zeta\in C_c^\infty(\mathbb{R}^N).$$

Then, $\langle(I-\triangle)^sT_{\mu_z}\zeta\rangle\coloneqq\langle T_{\mu_z},(I-\triangle)^s\zeta\rangle$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}(I-\triangle)^s\zeta\bigg|_{s=0}=(I-\triangle)^{ln}\zeta,$$

there holds in the sense of distributions

$$\frac{d}{\mathrm{d}s}(I-\triangle)^sT_{\mu_z}\bigg|_{s=0}=(I-\triangle)^{ln}T_{\mu_z}$$

in the sense that

$$\langle (I-\triangle)^{ln} T_{\mu_z}, \zeta\rangle = \langle T_{\mu_z}, (I-\triangle)^{ln}\zeta\rangle = \int_{\mathbb{R}^N} \mu_z (I-\triangle)^{ln}\zeta \,\mathrm{d}x.$$

By definition of the Fourier transform of distributions $\mathcal{F}$ in the class $\mathcal{S}'(\mathbb{R}^N)$ [29], there holds

$$\begin{aligned}\langle (I-\triangle)^{ln} T_{\mu_z}, \zeta\rangle &= (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\mathrm{i}z\cdot x}(I-\triangle)^{ln}\zeta(x)\mathrm{d}x \\ &= \mathcal{F}((I-\triangle)^{ln}\zeta)(z) = \ln(1+|z|^2)\mathcal{F}(\zeta)(z) = \ln(1+|z|^2)\int_{\mathbb{R}^N}\mu_z(x)\zeta(x)\mathrm{d}x.\end{aligned}$$

Since $x \mapsto (I-\triangle)^{ln}\mu_z(x)$ is locally integrable, $(I-\triangle)^{ln}T_{\mu_z} = T_{(I-\triangle)^{ln}T_{\mu_z}}$, hence

$$\langle T_{(I-\triangle)^{ln}T_{\mu_z}}, \zeta\rangle = \int_{\mathbb{R}^N} (I-\triangle)^{ln}T_{\mu_z}\zeta \mathrm{d}x = \ln(1+|z|^2)\int_{\mathbb{R}^N}\mu_z\zeta\mathrm{d}x.$$

Because ζ is arbitrary, this implies $T_{(I-\triangle)^{ln}T_{\mu_z}} = \ln(1+|z|^2)\mu_z$, a.e. in $\mathbb{R}^N$ and finally everywhere by continuity, which is the claim. □

For a given $0 < \sigma \ll 1$, we take $\mathcal{O} \subset \Omega$ be a C^2 domain such that

$$|\Omega| - |\mathcal{O}| < \sigma \quad \text{and} \quad |\partial\mathcal{O}| \le |\partial\Omega| + 1.$$

Let $\rho(x) = \mathrm{dist}(x, \mathbb{R}^N\setminus\mathcal{O})$, then it is C^2 near the boundary $\partial\mathcal{O}$ and zero in $\mathbb{R}^N\setminus\mathcal{O}$. Now we denote

$$w_\sigma(x) = \eta_0(\sigma^{-1}\rho(x)), \quad \forall x \in \mathcal{O}, \tag{4.4}$$

where $\eta_0 : [0,+\infty) \mapsto [0,1]$ is a nondecreasing C^2 function such that

$$\eta_0(t) = 1 \quad \text{if } t \ge 1, \quad \eta_0(t) = 0 \quad \text{if } t \le 0.$$

Here we can assume more that $\|\eta_0'\|_{L^\infty} \le 2$.

Observe that $w_\sigma \in C^2(\mathbb{R}^N)$ for $\sigma > 0$ small enough and

$$w_\sigma \to 1 \quad \text{in} \quad \mathcal{O} \quad \text{as } \sigma \to 0^+,$$

then there exists a positive constant $\sigma_1 \ll 1$ such that for $\sigma \in (0, \sigma_1]$,

$$\int_\Omega w_\sigma^2 \mathrm{d}x > |\{x \in \mathcal{O} : \rho(x) > \sigma\}| \ge |\mathcal{O}| - |\partial\mathcal{O}|\sigma \ge |\Omega| - (|\partial\Omega|+1)\sigma$$

and

$$\int_\Omega w_\sigma \mathrm{d}x < |\mathcal{O}| \le |\Omega| - \sigma,$$

thus

$$|\Omega| - (|\partial\Omega|+1)\sigma < \int_\Omega w_\sigma^2\mathrm{d}x \le \int_\Omega w_\sigma \mathrm{d}x < |\Omega| - \sigma.$$

Lemma 4.2. *Let*

$$\mathcal{I}_z w_\sigma(x) = \int_{\mathbb{R}^N} \frac{w_\sigma(x) - w_\sigma(\zeta)}{|x-\zeta|^N} e^{-i\zeta\cdot z}\omega(|x-\zeta|)\mathrm{d}\zeta,$$

then there holds

$$|\mathcal{I}_z w_\sigma(x)| \le \frac{2}{\sigma}\int_{\mathbb{R}^N}\frac{Ce^{-|x-\zeta|}}{|x-\zeta|^{N-1}}\mathrm{d}\zeta \le \frac{2}{\sigma}C \quad \textit{for } x \in \Omega$$

with $\omega(|x-\zeta|) = \int_0^\infty t^{-1+\frac{N}{2}}e^{-t-\frac{|x-\zeta|^2}{4t}}\mathrm{d}t$.

Proof. Actually, if $x \in \Omega$,

$$|w_\sigma(x) - w_\sigma(\zeta)| \le \|Dw_\sigma\|_{L^\infty}|x-\zeta| \le \sigma^{-1}\|\eta_0'\|_{L^\infty}|x-\zeta|$$

and

$$\omega(|x-\zeta|) = \int_0^\infty t^{-1+\frac{N}{2}} e^{-t-\frac{|x-\zeta|^2}{4t}}\mathrm{d}t \le Ce^{-|x-\zeta|},$$

then there exists $C = C(|\partial\Omega|) > 0$ such that

$$\left|\int_{\mathbb{R}^N} \frac{w_\sigma(x) - w_\sigma(\zeta)}{|x-\zeta|^N} e^{-\mathrm{i}\zeta\cdot z}\omega(|x-\zeta|)\mathrm{d}\zeta\right| \le \frac{\|\eta_0'\|_{L^\infty}}{\sigma}\int_{\mathbb{R}^N}\frac{Ce^{-|x-\zeta|}}{|x-\zeta|^{N-1}}\mathrm{d}\zeta \le \frac{2}{\sigma}C,$$

since $\|\eta_0'\|_{L^\infty} \le 2$, where $C > 0$ is independent of σ. □

Proof of Theorem 1.2. We recall that $\Phi_k(x,y)$ and $\widehat{\Phi}_k(y,z)$ have been defined in the proof of Theorem 1.1. If we denote

$$\tilde{G}_{\sigma,z}(x) \coloneqq G_\sigma(x,z) = w_\sigma(x)e^{-\mathrm{i}x\cdot z},$$

the projection of G_σ onto the subspace of $L^2(\Omega)$ spanned by the ϕ_j for $1 \le j \le k$ can be written in terms of the Fourier transform of $w_\sigma\Phi_k$ with respect to the x-variable:

$$\int_\Omega G_\sigma(x,z)\Phi_k(x,y)\mathrm{d}x = (2\pi)^{N/2}\mathcal{F}_x(w_\sigma\Phi_k)(y,z).$$

Put

$$G_{\sigma,k}(y,z) = G_\sigma(y,z) - (2\pi)^{N/2}\mathcal{F}_x(w_\sigma\Phi_k)(y,z)$$

and the Rayleigh-Ritz formula shows that for any $z \in \mathbb{R}^N$ and $\sigma > 0$,

$$\lambda_{k+1}(\Omega)\int_\Omega |G_{\sigma,k}(y,z)|^2\mathrm{d}y \le \int_\Omega \overline{G_{\sigma,k}(y,z)}(I-\triangle)^{\log}_{\Delta,y}G_{\sigma,k}(y,z)\mathrm{d}y,$$

where the right-hand side is a real value

$$\begin{aligned}\int_\Omega \overline{G_{\sigma,k}(y,z)}(I-\triangle)^{\log}_{\Delta,y}G_{\sigma,k}(y,z)\mathrm{d}y &= \int_{\mathbb{R}^N} \overline{G_{\sigma,k}(y,z)}(I-\triangle)^{\log}_{\Delta,y}G_{\sigma,k}(y,z)\mathrm{d}y\\ &= \int_{\mathbb{R}^N} \ln(1+|\xi|^2)|\mathcal{F}(G_{\sigma,k})(z,\xi)|^2\mathrm{d}\xi,\end{aligned}$$

although $G_{\sigma,k}$ is a complex valued function. Then, integrating this last inequality with respect to z in B_r, for $r > 1$, we obtain

$$\lambda_{k+1}(\Omega) \le \inf_{\sigma>0}\frac{\int_{B_r}\int_\Omega \overline{G_{\sigma,k}(y,z)}(I-\triangle)^{\log}_{\Delta,y}G_{\sigma,k}(y,z)\mathrm{d}y\mathrm{d}z}{\int_{B_r}\int_\Omega |G_{\sigma,k}(y,z)|^2\mathrm{d}y\mathrm{d}z}.$$

By Pythagorean theorem, we have that

$$\int_\Omega |G_{\sigma,k}(y,z)|^2\mathrm{d}y = \int_\Omega |G_\sigma(y,z)|^2\mathrm{d}y - (2\pi)^N\int_\Omega\sum_{j=1}^k|\mathcal{F}_x(w_\sigma\phi_i)(z)|^2\phi_i(y)^2\mathrm{d}y.$$

Integrating over B_r we obtain that

$$\int_{B_r}\int_\Omega |G_{\sigma,k}(y,z)|^2\mathrm{d}y\mathrm{d}z \ge \frac{\omega_N r^N}{N}\int_\Omega w_\sigma^2(y)\mathrm{d}y - (2\pi)^N\sum_{j=1}^k\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_i)(z)|^2\mathrm{d}z.$$

On the other hand, we see that

$$\begin{aligned}&\int_{B_r}\int_{\Omega}\overline{G_{\sigma,k}(y,z)}(I-\triangle)^{\log}_{\Delta,y}G_{\sigma,k}(y,z)\mathrm{d}y\mathrm{d}z\\&=\int_{B_r}\int_{\Omega}\overline{G_{\sigma}(y,z)}(I-\triangle)^{\log}_{\Delta,y}G_{\sigma}(y,z)\mathrm{d}y\mathrm{d}z-(2\pi)^N\int_{B_r}\int_{\Omega}\overline{\mathcal{F}_x(w_\sigma\Phi_k)(y,z)}(I-\triangle)^{\log}_{\Delta,y}\mathcal{F}_x(w_\sigma\Phi_k)(y,z)\mathrm{d}y\mathrm{d}z,\end{aligned}$$

where

$$\int_{B_r}\int_{\Omega}\overline{\mathcal{F}_x(w_\sigma\Phi_k)(y,z)}(I-\triangle)^{\log}_{\Delta,y}\mathcal{F}_x(w_\sigma\Phi_k)(y,z)\mathrm{d}y\mathrm{d}z=\sum_{j=1}^{k}\lambda_j(\Omega)\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2\mathrm{d}z$$

and

$$\begin{aligned}&\int_{B_r}\int_{\Omega}\overline{G_{\sigma}(y,z)}(I-\triangle)^{\log}_{\Delta,y}G_{\sigma}(y,z)\mathrm{d}y\mathrm{d}z\\&\le\int_{B_r}\int_{\Omega}w_\sigma^2(y)|(I-\triangle)^{\log}_{\Delta,y}e^{-\mathrm{i}y\cdot z}|\mathrm{d}y\mathrm{d}z+\int_{B_r}\int_{\Omega}w_\sigma(y)|\mathcal{I}_z w_\sigma(y)|\mathrm{d}y\mathrm{d}z\\&\le\int_{B_r}\int_{\Omega}w_\sigma^2(y)\ln(1+|z|^2)\mathrm{d}y\mathrm{d}z+\frac{2C}{\sigma}\int_{B_r}\int_{\Omega}w_\sigma(y)\mathrm{d}y\mathrm{d}z\\&=\frac{\omega_N}{N}\int_{\Omega}w_\sigma^2(y)\mathrm{d}y\left(r^N\ln(1+r^2)-2\int_0^r\frac{\varepsilon^{N+1}}{1+\varepsilon^2}\mathrm{d}\varepsilon\right)+\frac{2\omega_N C}{N\sigma}\int_{\Omega}w_\sigma(y)\mathrm{d}y r^N\\&<\frac{\omega_N}{N}(r^N\ln(1+r^2))\int_{\Omega}w_\sigma^2(y)\mathrm{d}y+C_0\frac{\omega_N}{N\sigma}r^N|\Omega|.\end{aligned}$$

Because of Parseval's identity, there holds

$$\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_i)(z)|^2\mathrm{d}z\le\int_{\mathbb{R}^N}|\mathcal{F}_x(w_\sigma\phi_i)(z)|^2\mathrm{d}z=\int_{\Omega}(w_\sigma\phi_i)^2\mathrm{d}x\le 1.$$

Denote

$$A_1=\frac{\omega_N}{N}(r^N\ln(1+r^2))\int_{\Omega}w_\sigma^2(y)\mathrm{d}y+C_0\frac{\omega_N}{N\sigma}|\Omega|r^N\quad A_2=\frac{\omega_N r^N}{N}\int_{\Omega}w_\sigma^2\mathrm{d}x,$$

then

$$\begin{aligned}0&\le\frac{A_1-(2\pi)^N\sum_{j=1}^{k}\lambda_j(\Omega)\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2\mathrm{d}z}{A_2-(2\pi)^N\sum_{j=1}^{k}\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2\mathrm{d}z}-\lambda_{k+1}(\Omega)\\&=\frac{(A_1-A_2\lambda_{k+1}(\Omega))+(2\pi)^N\sum_{j=1}^{k}(\lambda_{k+1}(\Omega)-\lambda_j(\Omega))\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2\mathrm{d}z}{A_2-(2\pi)^N\sum_{j=1}^{k}\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2\mathrm{d}z}\\&\le\frac{(A_1-A_2\lambda_{k+1}(\Omega))+(2\pi)^N\sum_{j=1}^{k}(\lambda_{k+1}(\Omega)-\lambda_j(\Omega))}{A_2-(2\pi)^N k},\end{aligned}$$

since $\lambda_{k+1}(\Omega)\ge\lambda_j(\Omega)$ for $j<k+1$ and $\int_{B_r}|\mathcal{F}_x(w_\sigma\phi_j)(z)|^2\mathrm{d}z\in(0,1)$. As a consequence, we obtain that

$$\lambda_{k+1}(\Omega)\le\frac{A_1-(2\pi)^N\sum_{j=1}^{k}\lambda_j(\Omega)}{A_2-(2\pi)^N k}.$$

Note that

$$\sigma \coloneqq \frac{1}{2\,|\Omega|^{-1}(|\partial\Omega|+1)}\frac{1}{\ln\ln(1+k)} < \frac{1}{|\Omega|^{-1}(|\partial\Omega|+1)}\frac{1}{\ln\ln(1+k)}$$

and let

$$r^N \coloneqq \frac{(2\pi)^N N}{\omega_N\int_\Omega w_\sigma^2 \mathrm{d}x}(k+1) < \frac{(2\pi)^N N}{\omega_N}(k+1)\frac{1}{|\Omega|-(|\partial\Omega|+1)\sigma} < \frac{2(2\pi)^N N}{\omega_N|\Omega|}(k+1),$$

where

$$\int_\Omega w_\sigma^2 \mathrm{d}x > |\Omega| - (|\partial\Omega|+1)\sigma.$$

Then,

$$\begin{aligned}
(2\pi)^N\sum_{j=1}^{k+1}\lambda_j(\Omega) \le A_1 &= \frac{\omega_N}{N}(r^N\ln(1+r^2))\int_\Omega w_\sigma^2(y)\mathrm{d}y + \frac{\omega_N C_0}{N\sigma}|\Omega|r^N\\
&= \frac{\omega_N\int_\Omega w_\sigma^2(y)\mathrm{d}y}{N}\frac{(2\pi)^N N}{\omega_N\int_\Omega w_\sigma^2\mathrm{d}y}(k+1)\ln\left(1+\left(\frac{(2\pi)^N N}{\omega_N\int_\Omega w_\sigma^2\mathrm{d}y}(k+1)\right)^{\frac{2}{N}}\right) + \frac{\omega_N C_0}{N\sigma}|\Omega|r^N\\
&\le (2\pi)^N\frac{2}{N}(k+1)\left(\ln(k+1)+\ln\left(\frac{C_0(2\pi)^N N}{\omega_N|\Omega|}\right)\right) + C_1(k+1)\ln\ln(1+k),
\end{aligned}$$

where $C_1 = (2\pi)^N C_0\,|\Omega|^{-1}(|\partial\Omega|+1)$.

As a consequence, we have that

$$\sum_{j=1}^{k+1}\lambda_j(\Omega) \le \frac{2}{N}(k+1)\ln(k+1) + C_1(k+1)\ln\ln(1+k) + C_2(k+1),$$

where $C_2 = \frac{2}{N}\ln\left(\frac{C_0(2\pi)^N N}{\omega_N\,|\,\Omega\,|}\right) > 0$. □

5 Limits

Proof of Corollary 1.3. The Limit (1.14) follows by (1.13) and (3.2) directly. Now, we prove (1.15). On the one hand, it follows by the monotonicity of $k \mapsto \lambda_k(\Omega)$ and (3.2) that

$$\lambda_k(\Omega) \ge \frac{1}{k}\sum_{i=1}^{k}\lambda_i(\Omega) > \frac{2}{N}\left(\ln k - \ln\ln\left(1+\frac{k}{T_N|\Omega|}\right) - \ln(T_N|\Omega|) + \ln\frac{N}{2} - 1\right). \tag{5.1}$$

On the other hand, letting $m = k$, we obtain that

$$\begin{aligned}
\lambda_{k+1}(\Omega) &\le \frac{1}{m}\left(\sum_{j=1}^{k+m}\lambda_j(\Omega) - \sum_{j=1}^{k}\lambda_j(\Omega)\right)\\
&\le \frac{2}{Nm}(k+m)\left(\ln(k+m)+\ln\left(\frac{2(2\pi)^N N}{|\Omega|\omega_N}\right) + c(\ln\ln(k+m))\right)\\
&\quad - \frac{2}{Nm}k\left(\ln k - \ln\ln\left(1+\frac{k}{T_N|\Omega|}\right) - \ln(T_N|\Omega|) + \ln\frac{N}{2} - 1\right)\\
&< \frac{2}{N}(\ln(k+1) + c + c\ln\ln(k+1)),
\end{aligned}$$

where $c > 0$, independent of k. Thus, we have that

$$\lim_{k\to+\infty}\frac{\lambda_k(\Omega)}{\ln k}=\frac{2}{N}.$$

The proof is completed □

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