Research Article

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On optimal control in a nonlinear interface problem described by hemivariational inequalities

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Abstract: This article is devoted to the existence of optimal controls in various control problems associated with a novel nonlinear interface problem on an unbounded domain with non-monotone set-valued transmission conditions. This interface problem involves a nonlinear monotone partial differential equation in the interior domain and the Laplacian in the exterior domain. Such a scalar interface problem models non-monotone frictional contact of elastic infinite media. The variational formulation of the interface problem leads to a hemivariational inequality (HVI), which, however, lives on the unbounded domain, and thus cannot be analyzed in a reflexive Banach space setting. Boundary integral methods lead to another HVI that is amenable to functional analytic methods using standard Sobolev spaces on the interior domain and Sobolev spaces of fractional order on the coupling boundary. Broadening the scope of this article, we consider extended real-valued HVIs augmented by convex extended real-valued functions. Under a smallness hypothesis, we provide existence and uniqueness results; moreover, we establish a stability result with respect to the extended real-valued function as a parameter. Based on the latter stability result, we prove the existence of optimal controls for four kinds of optimal control problems: distributed control on the bounded domain, boundary control, simultaneous boundary-distributed control governed by the interface problem, and control of the obstacle driven by a related bilateral obstacle interface problem.

Keywords: monotone operator, non-monotone transmission conditions, unbounded domain, extended real-valued hemivariational inequality, boundary-distributed control, obstacle control

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1 Introduction

Optimal control of partial differential equations (PDEs) is a vast field of applied mathematics. Here, we focus on the control of elliptic PDEs, governed by a hemivariational inequality (HVI) in a weak formulation.

The theory of HVIs was introduced and has been studied since the 1980s by Panagiotopoulos [44], as a generalization of variational inequalities with the aim to model many problems coming from mechanics when the energy functionals are nonconvex, but locally Lipschitz, so Clarke's generalized differentiation calculus [12] can be used [18,19,40]. For more recent monographs on HVIs with application to contact problems, we refer to [38,55].

While optimal control in variational inequalities has already been treated for a longer time (see the monograph [5] and, e.g., the articles [1,13,15,30,37,47]), optimal control in HVIs has been more recently studied (see, e.g., [28,35,48,52–54]). In particular, let us mention the very recent work on optimal control in HVIs and on

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related inverse problems in HVIs presented in the articles [34] on optimal control for elliptic bilateral obstacle problems; [60] on well-posedness, optimal control, and sensitivity analysis for a class of differential variational-hemivariational inequalities, [59] on optimal control in nonlinear quasi-hemivariational inequalities; and [10] on inverse problems for generalized quasi-variational inequalities with application to elliptic mixed boundary value systems. Contrary to the work cited above, the underlying state problem of this article is not a boundary value problem on a bounded domain, but an interface problem involving a PDE on an unbounded domain. For the simplicity of presentation, we consider a scalar interface problem with a monotone PDE on the interior domain and the Laplacian on the exterior domain, connected by non-monotone set-valued transmission conditions as a novelty. This scalar problem models nonlinear contact problems with non-monotone friction in infinite elastic media that arise in various fields of science and technology; let us mention geophysics (see, e.g., [51]), soil mechanics, in particular soil-structure interaction problems (see, e.g., [16]), and civil engineering of underground structures (see, e.g., [57]).

It should be underlined that such interface problems involving a PDE on an unbounded domain are more difficult than standard boundary value problems on bounded domains, since a direct variational formulation of the former problems leads to a HVI, which lives on the unbounded domain, and thus cannot be analyzed in a reflexive Banach space setting. Thanks to boundary integral methods (see the monograph [29]), we provide another HVI that is amenable to functional analytic methods using standard Sobolev spaces on the interior domain and Sobolev spaces of fractional order on the coupling boundary. Let us note in passing that these integral methods lay the basis for the numerical treatment of such interface problems by the well-known coupling of boundary elements and finite elements (see [27, Chapter 12]).

A main novel ingredient of our analysis is a stability theorem that considerably improves a related result in the recent article [52] and extends it to more general extended real-valued HVIs augmented by convex extended real-valued functions. This stability theorem provides the key to a unified approach to the existence of optimal controls in various optimal control problems (OCPs): distributed control on the bounded domain, boundary control, simultaneous distributed-boundary control governed by the interface problem, as well as the control of the obstacle in a related bilateral obstacle interface problem.

The plan of this study is as follows. Section 2 provides preliminaries and consists of three parts: a collection of some basic tools of Clarke's generalized differential calculus for the analysis of the non-monotone transmission conditions, a description of the interface problem in strong form and in weak HVI formulation, and existence and uniqueness results for a class of abstract HVIs using an equilibrium approach. Section 3 establishes well-posedness results, in particular a stability theorem for a more general class of extended real-valued HVIs. Based on this stability theorem, Section 4 presents a unified approach to the existence of optimal controls in four OCPs: distributed, boundary, boundary-distributed, and obstacle control. Section 5 shortly summarizes our findings, gives some concluding remarks, and sketches some directions of further research.

2 Some preliminaries – Clarke's generalized differential calculus, the interface problem, and an equilibrium approach to HVIs

2.1 Some preliminaries from Clarke's generalized differential calculus

From Clarke's generalized differential calculus [12], we need the concept of the *generalized directional derivative* of a locally Lipschitz function $\phi : X \to \mathbb{R}$ on a real Banach space X at $x \in X$ in the direction $z \in X$ defined by:

$$\phi^0(x; z) \coloneqq \limsup_{y \to x;\ t \downarrow 0} \frac{\phi(y + tz) - \phi(y)}{t}.$$

Note that the function $z \in X \mapsto \phi^0(x; z)$ is finite, sublinear, hence convex, and continuous; furthermore, the function $(x, z) \mapsto \phi^0(x; z)$ is upper semicontinuous. The *generalized gradient* of the function ϕ at x, denoted by (simply) $\partial\phi(x)$, is the unique nonempty weak* compact convex subset of the dual space X^*, whose support function is $\phi^0(x; .)$. Thus,

$$\xi \in \partial\phi(x) \Leftrightarrow \phi^0(x; z) \geq \langle \xi, z\rangle, \ \ \forall z \in X,$$
$$\phi^0(x; z) = \max\{\langle \xi, z\rangle : \xi \in \partial\phi(x)\}, \ \ \forall z \in X.$$

When X is finite dimensional, according to Rademacher's theorem, ϕ is differentiable almost everywhere, and the generalized gradient of ϕ at a point $x \in \mathbb{R}^n$ can be characterized by:

$$\partial\phi(x) = \mathrm{co}\left\{\xi \in \mathbb{R}^n : \xi = \lim_{k\to\infty} \nabla\phi(x_k), \ \ x_k \to x, \ \phi \ \text{ is differentiable at } \ x_k\right\},$$

where "co" denotes the convex hull.

2.2 Interface problem

Let $\Omega \subset \mathbb{R}^d (d \geq 2)$ be a bounded domain with Lipschitz boundary $\Gamma = \mathrm{cl}\Gamma_s \cup \mathrm{cl}\Gamma_t$ with non-empty open disjoint boundary parts Γ_s and Γ_t. Let n denote the unit normal on Γ defined almost everywhere pointing from Ω into $\Omega^c := \mathbb{R}^d \backslash \overline{\Omega}$. Let the data $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $q \in L^2(\Gamma)$ be given.

In the interior part Ω, consider the nonlinear PDE

$$\mathrm{div}(p(|\nabla u|)\cdot\nabla u) + f = 0, \quad \text{in } \Omega, \tag{2.1}$$

where $p : [0, \infty) \to [0, \infty)$ is a continuous function with $t \cdot p(t)$ being monotonously increasing with t.

In the exterior part Ω^c, consider the Laplace equation

$$\Delta u = 0, \quad \text{in } \Omega^c, \tag{2.2}$$

with the radiation condition at infinity for $|x| \to \infty$,

$$u(x) = \begin{cases} a + o(1), & \text{if } d = 2, \\ O(|x|^{2-d}), & \text{if } d > 2, \end{cases} \tag{2.3}$$

where a is a real constant for any u, but may vary with u.

With $u_1 := u|_\Omega$ and $u_2 := u|_{\Omega^c}$, the tractions on the coupling boundary Γ are given by the traces of $p(|\nabla u_1|)\frac{\partial u_1}{\partial n}$ and $-\frac{\partial u_2}{\partial n}$, respectively. Prescribe classical transmission conditions on Γ_t:

$$u_1|_{\Gamma_t} = u_2|_{\Gamma_t} + u_0|_{\Gamma_t} \quad \text{and} \quad p(|\nabla u_1|)\frac{\partial u_1}{\partial n}\bigg|_{\Gamma_t} = \frac{\partial u_2}{\partial n}\bigg|_{\Gamma_t} + q|_{\Gamma_t} \tag{2.4}$$

and on Γ_s analogously for the tractions:

$$p(|\nabla u_1|)\frac{\partial u_1}{\partial n}\bigg|_{\Gamma_s} = \frac{\partial u_2}{\partial n}\bigg|_{\Gamma_s} + q|_{\Gamma_s} \tag{2.5}$$

and the generally non-monotone, set-valued transmission condition:

$$p(|\nabla u_1|)\frac{\partial u_1}{\partial n}\bigg|_{\Gamma_s} \in \partial j(\cdot, u_0 + (u_2 - u_1)|_{\Gamma_s}). \tag{2.6}$$

Here, the function $j : \Gamma_s \times \mathbb{R} \to \mathbb{R}$ is such that $j(\cdot,\xi) : \Gamma_s \to \mathbb{R}$ is measurable on Γ_s for all $\xi \in \mathbb{R}$ and $j(s, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz for almost all (a.a.) $s \in \Gamma_s$ with $\partial j(s, \xi) := \partial j(s, \cdot)(\xi)$, the generalized gradient of $j(s, \cdot)$ at ξ.

Furthermore, require the following growth condition on the so-called superpotential j: there exist positive constants $c_{j,1}$ and $c_{j,2}$ such that for a.a. $s \in \Gamma_s$, all $\xi \in \mathbb{R}$ and for all $\eta \in \partial j(s, \xi)$, the following inequalities hold

$$(i) \quad |\eta| \le c_{j,1}(1 + |\xi|), \quad \text{and} \quad (ii) \quad \eta\xi \ge -c_{j,2}|\xi|. \tag{2.7}$$

Note that it follows from (2.7)(i) and (2.7)(ii), respectively, that for a.a. $s \in \Gamma_s$,

$$|j^0(s, \xi; \varsigma)| \le c_{j,1}(1 + |\xi|)|\varsigma|, \quad \forall \xi, \varsigma \in \mathbb{R}, \tag{2.8}$$

and

$$j^0(s, \xi; -\xi) \le c_{j,2}|\xi|, \quad \forall \xi \in \mathbb{R}. \tag{2.9}$$

Altogether, the interface problem consists in finding $u_1 \in H^1(\Omega)$ and $u_2 \in H^1_{\mathrm{loc}}(\Omega^c)$ that satisfy (2.1)–(2.6) in a weak form.

To exhibit the relation of the above scalar interface problem to an elastic transmission problem with frictional contact, we insert the following remark.

Remark 1. In elasticity – to simplify focus to the case $d = 2$ – instead of the unknown scalar field u, there is the displacement field u that decomposes in its normal component $u^n = u \cdot n$ and its tangential component $u^t = (u - u^n n) \cdot t$, where $t = (-n_2, n_1)^T$ for $n = (n_1, n_2)^T$. Similarly, as dual variable, the flux $q_\nu = \frac{\partial u}{\partial n}$ at the boundary is to be replaced by the boundary stress vector T with its normal component T^n and its tangential component T^t. This leads in local coordinates to $u_i = (u_i^t, u_i^n)$; $T_i = (T_i^n, T_i^t)$ with $i = (1, 2)$ for the elastic body in the bounded domain ($i = 1$) and the exterior elastic medium ($i = 2$). Then, the set-valued transmission condition (2.6) includes a transmission condition of Tresca's type analogous to Tresca's friction boundary condition (given friction model) (see [14,31]). Indeed, choose $j(\cdot,\xi) = g(\cdot)|\xi|$ with given nonnegative friction force $g \in L^\infty(\Gamma_s)$, then

$$\partial j(\cdot,\xi) = \begin{cases} -g, & \text{if } \xi < 0, \\ [-g, g], & \text{if } \xi = 0, \\ g, & \text{if } \xi > 0, \end{cases}$$

is monotone set-valued and with

$$\delta u := u_0 + (u_2^n - u_1^n)|_{\Gamma_s}$$

(2.6) becomes

$$\begin{cases} |p(|T_1|)T_1^t| \le g, & \text{if } \delta u = 0, \\ p(|T_1|)T_1^t = g\dfrac{\delta u}{|\delta u|}, & \text{if } \delta u \ne 0. \end{cases}$$

In this sense, (2.6) gives a simplified (scalar) model of an elastic transmission problem with frictional contact.

To arrive at a first variational formulation of the interface problem in form of a HVI, introduce some function spaces. For the bounded Lipschitz domain Ω, use the standard Sobolev space $H^s(\Omega)$ and the Sobolev spaces on the bounded Lipschitz boundary Γ (see [50, Sect 2.4.1]),

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+1/2}(\mathbb{R}^d)\}, & (0 < s \le 1), \\ L^2(\Gamma), & (s = 0), \\ (H^{-s}(\Gamma))^* \text{(dual space)}, & (-1 \le s < 0). \end{cases}$$

Furthermore, for the unbounded domain $\Omega^c = \mathbb{R}^d \backslash \overline{\Omega}$, introduce the Frechet space (see, e.g., [29, Section 4.1, (4.1.43)]):

$$H^s_{\mathrm{loc}}(\Omega^c) = \{u \in \mathcal{D}^*(\Omega^c) : \chi u \in H^s(\mathbb{R}^d) \ \ \forall \chi \in C_0^\infty(\Omega^c)\}.$$

By the trace theorem, $u|_\Gamma \in H^{1/2}(\Gamma)$ for $u \in H^1_{\mathrm{loc}}(\Omega^c)$. Next, define $\Phi : H^1(\Omega) \times H^1_{\mathrm{loc}}(\Omega^c) \to \mathbb{R} \cup \{\infty\}$ by:

$$\Phi(u_1, u_2) := \int_\Omega g(|\nabla u_1|)\mathrm{d}x + \frac{1}{2}\int_{\Omega^c} |\nabla u_2|^2 \mathrm{d}x - L(u_1, u_2|_\Gamma). \tag{2.10}$$

Here the data $f \in L^2(\Omega)$, $q \in L^2(\Gamma)$ enter the linear functional:

$$L(u, v) := \int_\Omega f \cdot u\mathrm{d}x + \int_\Gamma q \cdot v\mathrm{d}s. \tag{2.11}$$

Furthermore, in (2.10), the function g is given by p (see (2.1)) through

$$g : [0, \infty) \to [0, \infty), t \mapsto g(t) = \int_0^t s \cdot p(s)\mathrm{d}s,$$

assume that p is C^1, $0 \le p(t) \le p_0 < \infty$, and $t \mapsto t \cdot p(t)$ is strictly monotonic increasing. Then, $0 \le g(t) \le \frac{1}{2}p_0 \cdot t^2$ and the real-valued functional

$$G(u) := \int_\Omega g(|\nabla u|)\mathrm{d}x, \quad u \in H^1(\Omega),$$

is strictly convex. The Frechet derivative of G,

$$DG(u;\, v) = \int_\Omega p(|\nabla u|)(\nabla u)^T \cdot \nabla v \mathrm{d}x, \quad u, v \in H^1(\Omega), \tag{2.12}$$

is Lipschitz continuous and strongly monotone in $H^1(\Omega)$ with respect to the semi-norm:

$$|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)},$$

i.e., there exists a constant $c_G > 0$ such that

$$c_G\, |u - v|^2_{H^1(\Omega)} \le DG(u;\, u - v) - DG(v;\, u - v) \quad \forall u, v \in H^1(\Omega). \tag{2.13}$$

Analogously to [8,36], we first define

$$\mathcal{L}_0 := \{v \in H^1_{\mathrm{loc}}(\Omega^c) : \Delta v = 0 \text{ in } H^{-1}(\Omega^c) \text{ (and for } d = 2\ \exists a \in \mathbb{R} \text{ such that } v \text{ satisfies (2.3))}\},$$

and then, the affine, hence convex set of admissible functions:

$$C := \{(u_1, u_2) \in H^1(\Omega) \times H^1_{\mathrm{loc}}(\Omega^c) : u_1|_{\Gamma_t} = u_2|_{\Gamma_t} + u_0|_{\Gamma_t} \quad \text{and} \quad u_2 \in \mathcal{L}_0\}.$$

According to [8, Remark 4], C is closed in $H^1(\Omega) \times H^1_{\mathrm{loc}}(\Omega^c)$. Furthermore, it holds

$$D\Phi((\hat{u}_1, \hat{u}_2);(u_1, u_2)) = DG(\hat{u}_1;\, u_1) + \int_{\Omega^c} \nabla \hat{u}_2 \cdot \nabla u_2 \mathrm{d}x - \int_\Omega f \cdot u_1 \mathrm{d}x - \int_{\Gamma_t} q \cdot u_2|_{\Gamma_t} \mathrm{d}s.$$

Then, it can be proved [26, Theorem 1] that the interface problem (2.1)–(2.6) is equivalent in the sense of distributions to the HVI problem (P_Φ): find $(\hat{u}_1, \hat{u}_2) \in C$ such that for all $(u_1, u_2) \in C$, there holds for $\delta u_1 := u_1 - \hat{u}_1$, $\delta u_2 := u_2 - \hat{u}_2$,

$$D\Phi((\hat{u}_1, \hat{u}_2);(\delta u_1, \delta u_2)) + J^0(\gamma(\hat{u}_2 - \hat{u}_1 + u_0);\, \gamma(\delta u_2 - \delta u_1)) \ge 0. \tag{2.14}$$

However, since this HVI lives on the unbounded domain $\Omega \times \Omega^c$ (as the original problem), this HVI cannot be treated in a reflexive Banach space setting and therefore provides only an intermediate step in the analysis. Therefore, employ boundary integral operator theory [27,29] to reformulate the interface problem (2.1)–(2.6) in the weak sense as a boundary-domain variational inequality on $\Gamma \times \Omega$. From now on, concentrate on the analysis to the case of dimension $d = 3$, since as already the distinction in the radiation condition (2.3) indicates, in the case $d = 2$, some peculiarities of boundary integral methods for exterior problems come up that need extra attention (see, e.g., [8], [27, Sec. 12.2]). As a result, arrive at an equivalent hemivariational formulation of the original interface problem (2.1)–(2.6) that lives on $\Omega \times \Gamma$ and consists of a weak formulation

of the nonlinear differential operator in the bounded domain Ω, the Poincaré–Steklov operator on the bounded boundary Γ, and a nonsmooth functional on the boundary part Γ_s.

To this end, recall the Poincaré-Steklov operator for the exterior problem: $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a selfadjoint operator with the defining property:

$$S(u_2|_\Gamma) = -\partial_n u_2|_\Gamma, \tag{2.15}$$

for solutions $u_2 \in \mathcal{L}_0$ of the Laplace equation on Ω^c. The operator S enjoys the important property that it can be expressed as:

$$S = \frac{1}{2}[W + (I - K')V^{-1}(I - K)],$$

where I, V, K, K', and W denote the identity, the single-layer boundary integral operator, the double-layer boundary integral operator, its formal adjoint, and the hypersingular integral operator, respectively (see [27, Section 12.2] for details).

Furthermore, S gives rise to the positive definite bilinear form $\langle S\cdot,\cdot\rangle$, i.e., there exists a constant $c_S > 0$ such that

$$\langle Sv, v\rangle \geq c_S \|v\|^2_{H^{1/2}(\Gamma)}, \quad \forall v \in H^{1/2}(\Gamma), \tag{2.16}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$ extends the L^2 duality on Γ.

Let $E := H^1(\Omega) \times \widetilde{H}^{1/2}(\Gamma_s)$ with $\widetilde{H}^{1/2}(\Gamma_s) := \{w \in H^{1/2}(\Gamma) | \text{supp } w \subseteq \overline{\Gamma}_s\}$. Next, define the linear functional $\lambda \in E^*$ by:

$$\lambda(u, v) := \int_\Omega f \cdot u \mathrm{d}x + \langle q + Su_0, u|_\Gamma + v\rangle, \quad (u, v) \in E.$$

Using the representation formula of potential theory (see [17,36] for similar nonlinear interface problems), it can be proved [26, Theorem 2] that the intermediate HVI (P_Φ) is equivalent to the following HVI problem $(P_\mathcal{A})$: find $(\hat{u}, \hat{v}) \in E$ such that for all $(u, v) \in E$,

$$\mathcal{A}(\hat{u}, \hat{v};\, u - \hat{u}, v - \hat{v}) + J^0(\gamma\hat{v};\, \gamma(v - \hat{v})) \geq \lambda(u - \hat{u}, v - \hat{v}), \tag{2.17}$$

where $\mathcal{A} : E \to E^*$ is defined for all $(u, v), (u', v') \in E$ by:

$$\mathcal{A}(u, v)\,(u', v') = \mathcal{A}(u, v;\, u', v') := DG(u, u') + \langle S(u|_\Gamma + v), u'|_\Gamma + v'\rangle.$$

2.3 An equilibrium approach to a class of HVIs – existence and uniqueness results

Next, describe the functional analytic setting for the interface problem and provide the existence and uniqueness results using an equilibrium approach. To this end, let $X := L^2(\Gamma_s)$ and introduce the real-valued locally Lipschitz functional:

$$J(y) := \int_{\Gamma_s} j(s, y(s))\mathrm{d}s, \quad y \in X. \tag{2.18}$$

Then, by Lebesgue's theorem of majorized convergence,

$$J^0(y;\, z) = \int_{\Gamma_s} j^0(s, y(s);\, z(s))\mathrm{d}s, \quad (y, z) \in X \times X, \tag{2.19}$$

where $j^0(s, \cdot;\cdot)$ denotes the generalized directional derivative of $j(s, \cdot)$.

As seen in the previous subsection, the weak formulation of problem (2.1)–(2.6) leads, in an abstract setting, to a HVI with a nonlinear operator $\mathcal{A}$ and the nonsmooth functional J, namely: find $\hat{v} \in C$ such that

$$\mathcal{A}(\hat{v})(v - \hat{v}) + J^0(\gamma\hat{v}; \gamma v - \gamma\hat{v}) \geq \lambda(v - \hat{v}), \quad \forall v \in C. \tag{2.20}$$

Here, $C \neq \emptyset$ is a closed convex subset of a real reflexive Banach space E, $\gamma := \gamma_{E\to X}$ is a linear continuous operator, the linear form λ belongs to the dual E^*, and the nonlinear monotone operator $\mathcal{A} : E \to E^*$ is Lipschitz continuous and strongly monotone with some monotonicity constant $c_{\mathcal{A}} > 0$, which results from the strong monotonicity of the nonlinear operator DG in $H^1(\Omega)$ with respect to the semi-norm $|\cdot|_{H^1(\Omega)} = \|\nabla\cdot\|_{L^2(\Omega)}$ and the positive definiteness of the Poincaré-Steklov operator S (see [8, Lemma 4.1]).

On the other hand, by (2.8) and the compactness of the operator γ, the real-valued upper semicontinuous bivariate function, shortly bifunction:

$$\psi(v, w) := J^0(\gamma v; \gamma w - \gamma v), \quad \forall (v, w) \in E \times E,$$

becomes pseudomonotone (see [42, Lemma 1], [25, Lemma 4.1]). The latter result also shows that (2.9) implies a linear growth of $\psi(\cdot,0)$. This and the strong monotonicity of $\mathcal{A}$ imply coercivity. Therefore, by the theory of pseudomonotone VIs [20, Theorem 3], [58] (see [43] for the application to HVIs), we have solvability of (2.20).

Furthermore, suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition: there exists $c_J > 0$ such that

$$J^0(y_1; y_2 - y_1) + J^0(y_2; y_1 - y_2) \leq c_J\|y_1 - y_2\|_X^2, \quad \forall y_1, y_2 \in X. \tag{2.21}$$

Then, the smallness condition

$$c_J\|\gamma\|_{E\to X}^2 < c_{\mathcal{A}} \tag{2.22}$$

implies unique solvability of (2.20) (see, e.g., [41, Theorem 5.1] and [55, Theorem 83]).

It is noteworthy that under the smallness condition (2.22) together with (2.21), fixed point arguments [7] or the theory of set-valued pseudomonotone operators [55] are not needed, but simpler monotonicity arguments are sufficient to conclude unique solvability. Moreover, the compactness of the linear operator γ is not needed either. In fact, (2.20) can be framed as a *monotone equilibrium problem* in the sense of Blum-Oettli [6]:

Proposition 1. *Suppose* (2.21) *and* (2.22). *Then, the bifunction* $\varphi : C \times C \to \mathbb{R}$ *defined by*:

$$\varphi(v, w) := \mathcal{A}(v)(w - v) + J^0(\gamma v; \gamma w - \gamma v) - \lambda(w - v) \tag{2.23}$$

has the following properties:

- $\varphi(v, v) = 0$ *for all* $v \in C$;
- $\varphi(v, \cdot)$ *is convex and lower semicontinuous for all* $v \in C$;
- *there exists some* $\mu > 0$ *such that* $\varphi(v, w) + \varphi(w, v) \leq -\mu\|v - w\|_E^2$ *for all* $v, w \in C$ (*strong monotonicity*);
- *the function* $t \in [0, 1] \mapsto \varphi(tw + (1 - t)v, w)$ *is upper semicontinuous at* $t = 0$ *for all* $v, w \in C$ (*hemicontinuity*).

Proof. Obviously, φ vanishes on the diagonal and is convex and lower semicontinuous with respect to the second variable. To show strong monotonicity, estimate

$$\begin{aligned}\varphi(v, w) + \varphi(w, v) &= (\mathcal{A}(v) - \mathcal{A}(w))(w - v) + J^0(\gamma v; \gamma w - \gamma v) + J^0(\gamma w; \gamma v - \gamma w)\\ &\leq -c_{\mathcal{A}}\ \|v - w\|_E^2 + c_J\|\gamma v - \gamma w\|_X^2\\ &\leq -(c_{\mathcal{A}} - c_J\|\gamma\|_{E\to X}^2)\|v - w\|_E^2.\end{aligned}$$

To show hemicontinuity, it is enough to consider the bifunction $(y, z) \in X \times X \mapsto J^0(y; z - y)$. Then, for $(y, z) \in X \times X$ fixed, $t \in [0, 1]$, one has

$$J^0(y + t(z - y); z - (y + t(z - y))) = (1 - t)J^0(y + t(z - y); z - y),$$

and thus, hemicontinuity follows from upper semicontinuity of J^0:

$$\limsup_{t\downarrow 0} J^0(y + t(z - y); z - y) \leq J^0(y; z - y).\square$$

Since strong monotonicity implies coercivity and uniqueness, the fundamental existence result [6, Theorem 1] applies to the HVI (2.20) to conclude the following.

Theorem 1. *Suppose* (2.21) *and* (2.22). *Then, the HVI* (2.20) *is uniquely solvable.*

Thus, under the smallness condition, unique solvability holds for $(P_{\mathcal{A}})$.

3 Extended real-valued HVIs – existence, uniqueness, and stability

In view of the subsequent study of OCPs in Section 4 governed by the interface problem which we have described in the previous section, we broaden the scope of analysis and consider the extended real-valued HVIs: find $\hat{v} \in \mathrm{dom} F$ such that

$$\mathcal{A}(\hat{v})(v - \hat{v}) + J^0(\gamma\hat{v}; \gamma v - \gamma\hat{v}) + F(v) - F(\hat{v}) \geq 0, \quad \forall v \in V, \tag{3.1}$$

where V is a real reflexive Banach space, the nonlinear operator $\mathcal{A} : V \to V^*$ is a monotone operator, $\gamma := \gamma_{V\to X}$ with X a real Hilbert space (in the interface problem we have $X = L^2(\Gamma_s)$) denotes a linear continuous operator, J^0 stands for the generalized directional derivative of a real-valued locally Lipschitz functional J, and now in addition, $F : V \to \mathbb{R} \cup \{+\infty\}$ is a convex lower semicontinuous function that is supposed to be proper (i.e., $F \not\equiv \infty$ on V). This means that the effective domain of F in the sense of convex analysis [49],

$$\mathrm{dom}\ F := \{v \in V : F(v) < +\infty\}$$

is nonempty, closed, and convex. To resume the HVI (2.20) of Section 2.2, let $F(v) := \lambda(v) + \chi_C(v)$, where $\lambda \in V^*$ and

$$\chi_C(v) := \begin{cases} 0, & \text{if } v \in C, \\ +\infty, & \text{elsewhere}, \end{cases}$$

is the indicator function on C in the sense of convex analysis [49].

Next, similar to (2.23) in Section 2.2, define

$$\varphi(v, w) := \mathcal{A}(v)(w - v) + J^0(\gamma v; \gamma w - \gamma v) \tag{3.2}$$

and apply Proposition 1. Thus, under Assumptions (2.21) and (2.22), the aforementioned HVI (3.1) falls into the framework of an *extended real-valued equilibrium problem of monotone type* in the sense of [23]. Clearly, strong monotonicity implies uniqueness. Note by the separation theorem it can be shown that any convex proper lower semicontinuous function $\phi : V \to \mathbb{R} \cup \{+\infty\}$ is conically minorized, i.e., it enjoys the estimate

$$\phi(v) \geq -c_\phi(1 + \|v\|), \quad v \in V,$$

with some $c_\phi > 0$. Hence, strong monotonicity implies the asymptotic coercivity condition in [23], too. Thus the existence result [23, Theorem 5.9] applies to the HVI (3.1) to conclude the following.

Theorem 2. *Suppose* (2.21) *and* (2.22). *Then, the HVI* (3.1) *is uniquely solvable.*

By this solvability result, we can introduce the solution map $\mathcal{S}$ by $\mathcal{S}(F) := \hat{v}$, the solution of (3.1). Next, we investigate the stability of the solution map $\mathcal{S}$ with respect to the extended real-valued function F. Here, we follow the concept of epi-convergence in the sense of Mosco [3,39] ("Mosco convergence"). Let $F_n (n \in \mathbb{N}), F : V \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous proper functions. Then, F_n are called to converge to F in the Mosco sense, written $F_n \xrightarrow{\mathrm{M}} F$, if and only if the subsequent two hypotheses hold:
(M1) If $v_n \in V (n \in \mathbb{N})$ weakly converge to v for $n \to \infty$, then

$$F(v) \leq \liminf_{n\to\infty} F_n(v_n).$$

(M2) For any $v \in V$, there exist $v_n \in V (n \in \mathbb{N})$ strongly converging to v for $n \to \infty$ such that

$$F(v) = \lim_{n\to\infty} F_n(v_n).$$

In view of our later applications, it is not hard to require that the functions F_n are uniformly conically minorized, i.e., there holds the estimate

$$F_n(v) \geq -d_0(1 + \|v\|), \quad \forall n \in \mathbb{N},\ v \in V, \tag{3.3}$$

with some $d_0 \geq 0$. Moreover, similar to [52], in addition to the one-sided Lipschitz continuity (2.21), we assume that the locally Lipschitz function J satisfies the following growth condition:

$$\|\zeta\|_{X^*} \leq d_J(1 + \|z\|_X) \quad \forall z \in X,\ \zeta \in \partial J(z), \tag{3.4}$$

for some $d_J > 0$, which is immediate from the growth condition (2.7) for the integrand j.

Now, we are in the position to state the main result of this section, which extends the stability result of [21] for monotone variational inequalities to extended real-valued HVIs with an unperturbed bifunction φ in the coercive situation.

Theorem 3. *Suppose that the operator $\mathcal{A}$ is continuous and strongly monotone with monotonicity constant $c_{\mathcal{A}} > 0$, the linear operator γ is compact, the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition* (2.21) *and the growth condition* (3.4). *Moreover, suppose the smallness condition* (2.22). *Let $F, F_n : V \to \mathbb{R} \cup \{+\infty\} (n \in \mathbb{N})$ be convex lower semicontinuous proper functions that satisfy the lower estimate* (3.3); *let $F_n \xrightarrow{\mathrm{M}} F$. Then, strong convergence $\mathcal{S}(F_n) \to \mathcal{S}(F)$ holds.*

Proof. We divide the proof into three parts. We first show that the $\hat{u}_n = \mathcal{S}(F_n)$ are bounded, before we can establish the convergence result. In the following, $c_0, c_1, \ldots$ are generic positive constants.

(1) *The sequence $\{\hat{u}_n\} \subset V$ is bounded.*

By definition, $\hat{u}_n$ satisfies for all $v \in V$,

$$\mathcal{A}(\hat{u}_n)(v - \hat{u}_n) + J^0(\gamma\hat{u}_n; \gamma v - \gamma\hat{u}_n) + F_n(v) - F_n(\hat{u}_n) \geq 0. \tag{3.5}$$

Now, let v_0 be an arbitrary element of $\mathrm{dom}F$. Then, by Mosco convergence, (M2), there exist $v_n \in \mathrm{dom}F_n (n \in \mathbb{N})$ such that for $n \to \infty$, the strong convergences hold

$$v_n \to v_0;\ F_n(v_n) \to F(v_0). \tag{3.6}$$

Let $n \in \mathbb{N}$. Then, insert $v = v_n$ in (3.5) and obtain

$$\mathcal{A}(\hat{u}_n)(\hat{u}_n - v_n) \leq J^0(\gamma\hat{u}_n; \gamma v_n - \gamma\hat{u}_n) + F_n(v_n) - F_n(\hat{u}_n).$$

Write $\mathcal{A}(\hat{u}_n) = \mathcal{A}(\hat{u}_n) - \mathcal{A}(v_n) + \mathcal{A}(v_n)$ and use the strong monotonicity of the operator $\mathcal{A}$ and the estimate (3.3) to obtain

$$c_{\mathcal{A}}\|\hat{u}_n - v_n\|_V^2 \leq \|\mathcal{A}(v_n)\|_{V^*}\|\hat{u}_n - v_n\|_V + F_n(v_n) + d_0(1 + \|\hat{u}_n\|) + J^0(\gamma\hat{u}_n; \gamma v_n - \gamma\hat{u}_n). \tag{3.7}$$

On the other hand, write

$$J^0(\gamma\hat{u}_n; \gamma v_n - \gamma\hat{u}_n) = J^0(\gamma\hat{u}_n; \gamma v_n - \gamma\hat{u}_n) + J^0(\gamma v_n; \gamma\hat{u}_n - \gamma v_n) - J^0(\gamma v_n; \gamma\hat{u}_n - \gamma v_n).$$

Hence, by the one-sided Lipschitz condition (2.21),

$$J^0(\gamma\hat{u}_n; \gamma v_n - \gamma\hat{u}_n) \leq c_J\|\gamma\|^2\|v_n - \hat{u}_n\|_V^2 - J^0(\gamma v_n; \gamma\hat{u}_n - \gamma v_n). \tag{3.8}$$

Furthermore, by (3.4),

$$-J^0(\gamma v_n; \gamma\hat{u}_n - \gamma v_n) \leq \max_{\zeta\in\partial J(\gamma v_n)} \|\zeta\|_{X^*}\|\gamma\hat{u}_n - \gamma v_n\|_X \leq d_J\|\gamma\|(1 + \|\gamma\|\|v_n\|_V)\|\hat{u}_n - v_n\|_V. \tag{3.9}$$

By the convergences (3.6), $|F_n(u_n)| \leq c_0$, $\|\mathcal{A}(v_n)\|_{V^*} \leq c_1$, $\|v_n\|_V \leq c_2$. Thus, (3.7)–(3.9) result in

$$(c_{\mathcal{A}} - c_J\|\gamma\|^2)\|\hat{u}_n - v_n\|_V^2 \le c_0 + [c_1 + d_J\|\gamma\|(1 + c_2\|\gamma\|)]\|\hat{u}_n - v_n\|_V + d_0(1 + \|\hat{u}_n\|).$$

Hence, by the smallness condition (2.22), a contradiction argument proves the claimed boundedness of $\{\hat{u}_n\}$.

(2) *$\hat{u}_n = \mathcal{S}(F_n)$ converges weakly to $\hat{u} = \mathcal{S}(F)$ for $n \to \infty$.*

To prove this claim, we employ a "Minty trick" similar to the proof of [23, Prop. 3.2] using the monotonicity of the operator $\mathcal{A}$.

Take $v \in V$ arbitrarily. By $(M2)$, there exist $v_n \in V (n \in \mathbb{N})$ such that

$$\lim_{n\to\infty} v_n = v;\ \lim_{n\to\infty} F_n(v_n) = F(v). \tag{3.10}$$

We test (3.5) with v_n, use the monotonicity of the operator $\mathcal{A}$, and obtain

$$\mathcal{A}(v_n)(v_n - \hat{u}_n) + J^0(\gamma\hat{u}_n;\ \gamma v_n - \gamma\hat{u}_n) \ge F_n(\hat{u}_n) - F_n(v_n). \tag{3.11}$$

On the other hand, by the previous step, there exists a subsequence $\{\hat{u}_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to some $\tilde{u} \in \mathrm{dom}F \subset V$. Furthermore, since γ is completely continuous, $\gamma\hat{u}_{n_k} \to \gamma\tilde{u}$. Thus, the continuity of $\mathcal{A}$, the upper semicontinuity of $(y, z) \in X \times X \mapsto J^0(y;\ z)$, $(M1)$, and (3.10) entail together with (3.11):

$$\begin{aligned}\mathcal{A}(v)(v - \tilde{u}) + J^0(\gamma\tilde{u};\ \gamma v - \gamma\tilde{u}) &\ge \lim_{k\to\infty} \mathcal{A}(v_{n_k})(v_{n_k} - \hat{u}_{n_k}) + \limsup_{k\to\infty} J^0(\gamma\hat{u}_{n_k};\gamma v_{n_k} - \gamma\hat{u}_{n_k})\\ &\ge \liminf_{k\to\infty} F_n(\hat{u}_n) - \lim_{k\to\infty} F_n(v_n)\\ &\ge F(\tilde{u}) - F(v).\end{aligned}$$

Hence, for $v \in \mathrm{dom}F$ fixed, for arbitrary $s \in [0, 1)$ and $w_s := v + s(\tilde{u} - v) \in \mathrm{dom}F$ inserted above, the positive homogeneity of $J^0(\gamma\tilde{u};\ \cdot)$ and the convexity of F imply after division by the factor $(1 - s) > 0$,

$$\mathcal{A}(w_s)(v - \tilde{u}) + J^0(\gamma\tilde{u};\ \gamma v - \gamma\tilde{u}) + F(v) \ge F(\tilde{u}).$$

Letting $s \to 1$, hence $w_s \to \tilde{u}$, $\mathcal{A}(w_s) \to \mathcal{A}(\tilde{u})$ results in:

$$\mathcal{A}(\tilde{u})(v - \tilde{u}) + J^0(\gamma\tilde{u};\ \gamma v - \gamma\tilde{u}) + F(v) \ge F(\tilde{u}), \quad \forall v \in \mathrm{dom}F.$$

This shows by uniqueness that $\tilde{u} = \mathcal{S}(F)$ and the entire sequence $\{\hat{u}_n\}$ converges weakly to $\hat{u} = \mathcal{S}(F)$.

(3) *$\hat{u}_n = \mathcal{S}(F_n)$ converges strongly to $\hat{u} = \mathcal{S}(F)$ for $n \to \infty$.*

By $(M2)$, there exist $u_n \in V (n \in \mathbb{N})$ such that

$$(i)\lim_{n\to\infty} u_n = \hat{u}; \quad (ii)\lim_{n\to\infty} F_n(u_n) = F(\hat{u}). \tag{3.12}$$

Test (3.5) with u_n, use the strong monotonicity of the operator $\mathcal{A}$, and obtain

$$\mathcal{A}(u_n)(u_n - \hat{u}_n) + J^0(\gamma\hat{u}_n;\ \gamma u_n - \gamma\hat{u}_n) + F_n(u_n) - F_n(\hat{u}_n) \ge c_{\mathcal{A}}\|u_n - \hat{u}_n\|^2. \tag{3.13}$$

Analyze the summands in (3.13) separately: by (3.12) (i), $\mathcal{A}(u_n) \to \mathcal{A}(\hat{u})$, hence,

$$\lim_{n\to\infty} \mathcal{A}(u_n)(u_n - \hat{u}_n) = 0.$$

By the upper semicontinuity of $(y, z) \in X \times X \mapsto J^0(y;\ z)$, $(M1)$ and by the complete continuity of γ,

$$\limsup_{n\to\infty} J^0(\gamma\hat{u}_n;\ \gamma u_n - \gamma\hat{u}_n) \le 0.$$

By (3.12) (ii) and by $(M1)$,

$$\limsup_{n\to\infty} [F_n(u_n) - F_n(\hat{u}_n)] \le 0.$$

Thus, from (3.13) finally by the triangle inequality,

$$0 \le \|\hat{u}_n - \hat{u}\| \le \|\hat{u}_n - u_n\| + \|u_n - \hat{u}\| \to 0,$$

and the theorem is proved. □

To conclude this section, let us compare Theorem 3 with a similar stability result of [52, Theorem 6]. There one has the special setting of $F(v) := \chi_C(v) + \lambda(v)$, where $C \subset V$ is closed convex and $\lambda \in V^*$ is given by $\lambda := \kappa^* f$ with $f \in X^*$, κ^* the adjoint to the linear operator $\kappa : V \to X$, which is assumed to be completely continuous or equivalently compact (see [52, (4.3)]).

Likewise, for $n \in \mathbb{N}$, one has convex closed sets $C_n \subset V$ and linear forms $\lambda_n := \kappa^* f_n$ with $f_n \in X^*$ giving rise to $F_n(v) := \chi_{C_n}(v) + \lambda_n(v)$.

Note by the Schauder theorem (see e.g. [46, 3.7.17]), the adjoint $\kappa^* : X^* \to V^*$ is compact. Hence, the assumed weak convergence $f_n \rightharpoonup f$ in X^* entails the strong convergence $\lambda_{n_k} \to \lambda (k \to \infty)$ in V^* for a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$; thus furthermore, the lower estimate (3.3), in view of $\chi_{C_n} \geq 0$. Moreover, the Mosco convergence $F_n \xrightarrow{M} F$ follows at once from the assumed Mosco convergence $C_n \xrightarrow{M} C$, namely from the hypotheses:
(m1) If $v_n \in C_n (n \in \mathbb{N})$ weakly converge to v for $n \to \infty$, then $v \in C$.
(m2) For any $v \in C$, there exist $v_n \in C_n (n \in \mathbb{N})$ strongly converging to v for $n \to \infty$.

On the other hand, an inspection of the above proof of Theorem 3 shows that it is enough to demand for $\tilde{J}^0(v;w) = J^0(\gamma v; \gamma w)$, the generalized directional derivative of the real-valued locally Lipschitz functional $J(\tilde{v}) := J(\gamma v)$,

$$u_n \rightharpoonup u \text{ in } V \text{ and } v_n \to v \text{ in } V \Rightarrow \operatorname{limsup} \tilde{J}^0(u_n; v_n - u_n) \leq \tilde{J}^0(u;\, v - u).$$

This abstract condition [52, (4.2)] is derived in the above proof of Theorem 3 from the compactness of γ and the upper semicontinuity of $(y, z) \in X \times X \mapsto J^0(y;\, z)$.

Concerning the monotone operator $\mathcal{A} : V \to V^*$, we only require its norm continuity, not needing Lipschitz continuity. More importantly, we can also dispense with the condition [52, (4.1)]:

$$u_n \rightharpoonup u \text{ in } V \text{ and } v_n \to v \text{ in } V \Rightarrow \operatorname{limsup} \langle \mathcal{A}u_n, u_n - v_n \rangle \geq \langle \mathcal{A}u, u - v \rangle.$$

It seems that this condition forces an elliptic operator, which stems from an elliptic PDE on the domain, to be linear.

4 Some OCPs governed by the interface problem

In this section, we rely heavily on the stability result of Theorem 3 and present a unified approach to existence results for various OCPs governed by the interface problem, which was described in Section 2. For convenience, let us recall the boundary/domain HVI formulation $(P_{\mathcal{A}})$ of the interface problem: find $(\hat{u}, \hat{v}) \in E$ such that for all $(u, v) \in E$,

$$\mathcal{A}(\hat{u}, \hat{v};\, u - \hat{u}, v - \hat{v}) + J^0(\gamma\hat{v};\, \gamma(v - \hat{v})) \geq \lambda(u - \hat{u}, v - \hat{v}). \tag{4.1}$$

Here, $E = H^1(\Omega) \times \widetilde{H}^{1/2}(\Gamma_s)$ with $\widetilde{H}^{1/2}(\Gamma_s) = \{w \in H^{1/2}(\Gamma) | \operatorname{supp} w \subseteq \overline{\Gamma}_s\}$ on the bounded domain Ω and the boundary part Γ_s. The operator $\mathcal{A}$ is given for all $(u, v), (u', v') \in E$ by:

$$\mathcal{A}(u, v)\,(u', v') = \mathcal{A}(u, v;\, u', v') = DG(u, u') + \langle S(u|_\Gamma + v), u'|_\Gamma + v' \rangle$$

(see (2.12), (2.15)). J^0 denotes the generalized directional derivative of the Lipschitz integral function J (see (2.19), (2.18)) stemming from the generally non-monotone, set-valued transmission condition (2.6). $\gamma : \widetilde{H}^{1/2}(\Gamma_s) \to L^2(\Gamma_s)$ denotes the linear continuous embedding operator, which is compact. The linear functional $\lambda \in E^*$ is defined for $(u, v) \in E$ by:

$$\lambda(u, v) = \int_\Omega f \cdot u \mathrm{d}x + \langle q + Su_0, u|_\Gamma + v \rangle,$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$ extends the L^2 duality on Γ.

Now, for simplicity, we set $u_0 := 0$ and impose for the data f and q that $f \in L^2(\Omega)$ and $q \in L^2(\Gamma)$. Thus, we can write

$$\lambda(u, v) = \langle f, \kappa u\rangle_{L^2(\Omega)\times L^2(\Omega)} + \langle q, \tau u + \iota v\rangle_{L^2(\Gamma)\times L^2(\Gamma)}, \tag{4.2}$$

where $\kappa : H^1(\Omega) \to L^2(\Omega)$ and $\iota : H^{1/2}(\Gamma) \to L^2(\Gamma)$ are the linear compact embedding operators and $\tau : H^1(\Omega) \to L^2(\Gamma)$ is the linear compact trace operator.

In the following OCPs, we stick to a misfit functional with a given target $(u_d, v_d) \in E$ as the simplest case of a cost functional and regularize this functional by the norm of the control with a given regularization parameter $\rho > 0$. The subsequent analysis can be extended to cover more general cost functionals under appropriate lower semicontinuity and coerciveness assumptions (see, e.g., [52, sec. 5]), see also [56, sec. 4.4]), and also to a more general setting of regularization (see, e.g., [33, II Sect. 7.5 (7.51)], [24, (26)]).

4.1 Distributed OCP governed by the interface problem

Here, we control by $f \in L^2(\Omega)$ distributed on the domain Ω. Thus, in the abstract setting of Section 5, we choose the convex functional F as the linear functional:

$$F(u, v) := \langle f, \kappa u\rangle_{L^2(\Omega)\times L^2(\Omega)} = (\kappa^* f)(u), \;\; F = (\kappa^* f, 0) \in E^*.$$

By the abstract existence and uniqueness result of Theorem 2, we have the control-to-state map $f \in L^2(\Omega) \mapsto \mathcal{S}(f) := (\hat{u}, \hat{v})$, the solution of (4.1). Thus, we can pose $(\mathcal{OCP})_1$:

$$\begin{aligned} &\text{minimize} && I_1(f) := \frac{1}{2}\|\mathcal{S}(f) - (u_d, v_d)\|_E^2 + \frac{\rho}{2}\|f\|_{L^2(\Omega)}^2 \\ &\text{subject to} && f \in L^2(\Omega), \end{aligned}$$

for which we can prove the following existence result.

Theorem 4. *Suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition* (2.21) *and the growth condition* (3.4). *Moreover, suppose the smallness condition* (2.22) *with the monotonicity constant $c_{\mathcal{A}}$ of the operator $\mathcal{A}$. Then, there exists an optimal control to $(\mathcal{OCP})_1$.*

Proof. The proof follows the standard pattern of existence proofs in optimal control.

Since the cost function I_1 is bounded below:

$$I_1^* := \inf(\mathcal{OCP})_1 \in \mathbb{R}.$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be a minimizing sequence of $(\mathcal{OCP})_1$, e.g., construct f_n by $I_1(f_n) < I_1^* + \frac{1}{n}$. Since the misfit term in I_1 is nonnegative, $\|f_n\|_{L^2(\Omega)}^2$ is bounded. By reflexivity of $L^2(\Omega)$, we can extract a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ also denoted by $\{f_n\}_{n\in\mathbb{N}}$ that weakly converges to some $\hat{f} \in L^2(\Omega)$. Since κ^* is completely continuous, we have strong convergence $F_n := (\kappa^* f_n, 0) \to \hat{F} := (\kappa^* \hat{f}, 0)$ in E^*. Thus, the linear continuous functionals F_n satisfy the lower estimate (3.3). To show (M1), let $(u_n, v_n) \rightharpoonup (u, v)$ in E for $n \to \infty$. Then clearly, $\hat{F}(u, v) = \lim_{n\to\infty} F_n(u_n, v_n)$. To show (M2), choose for any $(u, v) \in E$, simply $(u_n, v_n) := (u, v) \in E$. Then clearly, $(u_n, v_n) \to (u, v)$ and $\hat{F}(u, v) = \lim_{n\to\infty} F_n(u_n, v_n)$. Hence, $F_n \xrightarrow{\text{M}} \hat{F}$ and Theorem 3 applies; it yields $\mathcal{S}(f_n) \to \mathcal{S}(\hat{f})$. Thus, in view of the weak lower semicontinuity of the norm,

$$I_1^* \le I_1(\hat{f}) \le \liminf_{n\to\infty} I_1(f_n) \le I_1^*,$$

and the theorem is proved. □

4.2 Boundary OCP and a simultaneous distributed-boundary OCP governed by the interface problem

Now, we control by $q \in L^2(\Gamma)$ on the boundary Γ. Thus, in the abstract setting of Section 5, we now choose the convex functional F as the linear functional:

$$F(u, v) := \langle q, \tau u + \iota v\rangle_{L^2(\Gamma)\times L^2(\Gamma)} = (\tau^* q, \iota^* q)(u, v),\ \ F = (\tau^* q, \iota^* q) \in E^*.$$

By the abstract existence and uniqueness result of Theorem 2, we have the control-to-state map $q \in L^2(\Gamma) \mapsto \mathcal{S}(q) := (\hat{u}, \hat{v})$, the solution of (4.1). Thus, we can pose $(\mathcal{OCP})_2$:

$$\begin{aligned} &\text{minimize} && I_2(q) := \frac{1}{2}\|\mathcal{S}(q) - (u_d, v_d)\|_E^2 + \frac{\rho}{2}\|q\|_{L^2(\Gamma)}^2 \\ &\text{subject to} && q \in L^2(\Gamma), \end{aligned}$$

for which we can prove the following existence result.

Theorem 5. *Suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition* (2.21) *and the growth condition* (3.4). *Moreover, suppose the smallness condition* (2.22) *with the monotonicity constant $c_{\mathcal{A}}$ of the operator $\mathcal{A}$. Then, there exists an optimal control to $(\mathcal{OCP})_2$.*

Proof. The proof follows from arguments similar to those that were given in the proof of Theorem 4. So the details are omitted. □

Let us remark that we can also treat the simultaneous distributed-boundary $(\mathcal{OCP})_3$ as in [9], here driven by the interface problem:

$$\begin{aligned} &\text{minimize} && I_3(f, q) := \frac{1}{2}\|\mathcal{S}(f, q) - (u_d, v_d)\|_E^2 + \frac{\rho}{2}[\|f\|_{L^2(\Omega)}^2 + \|q\|_{L^2(\Gamma)}^2] \\ &\text{subject to} && f \in L^2(\Omega),\ \ q \in L^2(\Gamma), \end{aligned}$$

where now we have the control-to-state map $(f, q) \in L^2(\Omega) \times L^2(\Gamma) \mapsto \mathcal{S}(f, q) := (\hat{u}, \hat{v})$, the solution of (4.1). By analogous reasoning, we obtain an optimal solution to $(\mathcal{OCP})_3$. The details are omitted.

4.3 OCP driven by a bilateral obstacle interface problem

To conclude this section, we investigate a related bilateral obstacle interface problem and the associated OCP similar to [34]. First for the strong formulation, we modify in the interior part $\Omega \subset \mathbb{R}^3$ the nonlinear PDE (2.1) to the obstacle problem: find $u = u(x) \in [\underline{u}(x), \overline{u}(x)]$ such that

$$\left.\begin{aligned} -\mathrm{div}(p(|\nabla u|)\cdot\nabla u) &\geq f, \quad \text{if } u = \underline{u}, && \text{a.e. in } \Omega, \\ -\mathrm{div}(p(|\nabla u|)\cdot\nabla u) &= f, \quad \text{if } \underline{u} < u < \overline{u}, && \text{a.e. in } \Omega, \\ -\mathrm{div}(p(|\nabla u|)\cdot\nabla u) &\leq f, \quad \text{if } u = \overline{u}, && \text{a.e. in } \Omega, \end{aligned}\right\} \tag{4.3}$$

where the obstacle functions $\underline{u}, \overline{u} \in H^1(\Omega)$ with $\underline{u} \leq \overline{u}$ a.e. in Ω are given. In the exterior part Ω^c, we consider still the Laplace equation (2.2) with the radiation condition (2.3). The transmission conditions (2.4)–(2.6) remain in force. The variational analysis described in Section 2.2 easily modifies to arrive at the following HVI problem $(P_{\mathcal{A},C})$: find $(\hat{u}, \hat{v}) \in C$ such that for all $(u, v) \in C$,

$$\mathcal{A}(\hat{u}, \hat{v};\ u - \hat{u}, v - \hat{v}) + J^0(\gamma\hat{v};\ \gamma(v - \hat{v})) \geq \lambda(u - \hat{u}, v - \hat{v}), \tag{4.4}$$

where the operator $\mathcal{A}$, the generalized directional derivative J^0, the linear continuous embedding operator γ, the linear functional λ are defined as before and above in this section, and where now the constraint set

$$C := C_{\underline{u},\overline{u}} := \{(u, v) \in E | \underline{u} \le u \le \overline{u} \text{ a.e. in } \Omega\} \tag{4.5}$$

is closed and convex. This gives rise to the closed convex functional:

$$F := F_{\underline{u},\overline{u}} := \chi_C = \chi_{C_{\underline{u},\overline{u}}}.$$

Here, we control by the obstacles $\underline{u}$ and $\overline{u}$ distributed on the domain Ω where (as in [34]), we impose the regularity $\underline{u}, \overline{u} \in H^2(\Omega)$ and introduce the admissible set:

$$U_{\text{ad}} := \{(\underline{u}, \overline{u}) |\ \underline{u} \le \overline{u} \text{ a.e. in } \Omega\}.$$

By the abstract existence and uniqueness result of Theorem 2, we have the control-to-state map $(\underline{u}, \overline{u}) \in U_{\text{ad}} \mapsto \mathcal{S}(\underline{u}, \overline{u}) := (\hat{u}, \hat{v})$, the solution of (4.4). Thus, we can pose $(\mathcal{OCP})_4$:

$$\begin{aligned} &\text{minimize} && I_4(\underline{u}, \overline{u}) := \frac{1}{2}\|\mathcal{S}(\underline{u}, \overline{u}) - (u_d, v_d)\|_E^2 + \frac{\rho}{2}[\|\underline{u}\|_{H^2(\Omega)}^2 + \|\overline{u}\|_{H^2(\Omega)}^2] \\ &\text{subject to} && (\underline{u}, \overline{u}) \in U_{\text{ad}}. \end{aligned}$$

An essential ingredient in the subsequent proof of the existence of an optimal control to $(\mathcal{OCP})_4$ is the Mosco convergence of constraint sets. For that latter result, we exploit the lattice structure of $H^1(\Omega)$. Namely, since Ω is supposed to be a Lipschitz domain, $H^1(\Omega)$ is a *Dirichlet space* ([4, Theorem 5.23], [32, Corollary A.6]) in the following sense: let $\theta : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function such that the derivative θ' exists except at finitely many points and that $\theta(0) = 0$; then, the induced map θ^* on $H^1(\Omega)$ given by $w \in H^1(\Omega) \mapsto \theta \circ w$ is a continuous map into $H^1(\Omega)$. In particular, the map $w \in H^1(\Omega) \mapsto w^+ = \max(0, w) = \frac{1}{2}(w + |w|)$ is a continuous map into $H^1(\Omega)$.

Now, we are in the position to establish the following existence result.

Theorem 6. *Suppose that the generalized directional derivative f^0 satisfies the one-sided Lipschitz condition* (2.21) *and the growth condition* (3.4). *Moreover, suppose the smallness condition* (2.22) *with the monotonicity constant $c_{\mathcal{A}}$ of the operator $\mathcal{A}$. Then, there exists an optimal control to $(\mathcal{OCP})_4$.*

Proof. The proof again follows the standard pattern of existence proofs in optimal control.

Since the cost function I_4 is bounded below,

$$I_4^* := \inf(\mathcal{OCP})_4 \in \mathbb{R}.$$

Let $\{(\underline{u}_n, \overline{u}_n)\}_{n\in\mathbb{N}} \subset U_{\text{ad}}$ be a minimizing sequence of $(\mathcal{OCP})_4$, e.g., construct $(\underline{u}_n, \overline{u}_n) \in U_{\text{ad}}$ by $I_4(\underline{u}_n, \overline{u}_n) < I_4^* + \frac{1}{n}$. Since the misfit term in I_4 is nonnegative, $\|\underline{u}_n\|_{H^2(\Omega)}^2$ and $\|\overline{u}_n\|_{H^2(\Omega)}^2$ are bounded. By reflexivity of $H^2(\Omega)$, closedness of U_{ad}, and the compact embedding $H^2(\Omega) \subset\subset H^1(\Omega)$, we can pass to a subsequence of $\{(\underline{u}_n, \overline{u})\}_{n\in\mathbb{N}}$ also denoted by $\{(\underline{u}_n, \overline{u}_n)\}_{n\in\mathbb{N}}$ such that for some $(\underline{u}_\infty, \overline{u}_\infty) \in U_{\text{ad}}$,

$$\left.\begin{aligned} \underline{u}_n &\rightharpoonup \underline{u}_\infty,\ \ \overline{u}_n \rightharpoonup \overline{u}_\infty, && \text{in } H^2(\Omega), \\ \underline{u}_n &\to \underline{u}_\infty,\ \ \overline{u}_n \to \overline{u}_\infty, && \text{in } H^1(\Omega), \\ \underline{u}_n &\to \underline{u}_\infty,\ \ \overline{u}_n \to \overline{u}_\infty, && \text{a.e. in } \Omega. \end{aligned}\right\} \tag{4.6}$$

We claim the Mosco convergence $C_n \xrightarrow{\text{M}} C_\infty$ for the constraint sets

$$\begin{aligned} C_n &:= C_{\underline{u}_n,\overline{u}_n} := \{(u, v) \in E |\ \underline{u}_n \le u \le \overline{u}_n,\ \text{ a.e. in } \Omega\}, \\ C_\infty &:= C_{\underline{u}_\infty,\overline{u}_\infty} := \{(u, v) \in E |\ \underline{u}_\infty \le u \le \overline{u}_\infty,\ \text{ a.e. in } \Omega\}. \end{aligned}$$

To show (m1), let $(u_n, v_n) \in C_n$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in E for $n \to \infty$. Then, $u_n \rightharpoonup u$ in $H^1(\Omega)$. By compact embedding, $H^1(\Omega) \subset\subset L^2(\Omega)$, for some subsequence $u_n \to u$ in $L^2(\Omega)$ and $u_n \to u$ a.e. in Ω. Thus, by (4.6), $\underline{u}_\infty \le u \le \overline{u}_\infty$ a.e. in Ω. Hence, $(u, v) \in C$ as required and (m1) is proven.

To show (m2), we exploit the aforementioned lattice structure of $H^1(\Omega)$ and employ a cutting technique. Let $(u, v) \in C_\infty$. Then, $\underline{u}_\infty \le u \le \overline{u}_\infty$ a.e. in Ω. This means

$$\max(\underline{u}_\infty, \min(\overline{u}_\infty, u)) = \max(\underline{u}_\infty, u) = u.$$

Then, set

$$u_n := \max(\underline{u}_n, \min(\overline{u}_n, u)), \;\; v_n := v.$$

By construction, $(u_n, v_n) \in C_n$; moreover, by (4.6), $\min(\overline{u}_n, u) \to \min(\overline{u}_\infty, u)$ and $u_n \to u$ in $H^1(\Omega)$. Hence, $(u_n, v_n) \to (u, v)$ in E is required. Thus, (m2) and the claimed Mosco convergence for the constraint sets are proved. This entails $F_n \overset{\mathrm{M}}{\longrightarrow} F_\infty$, where $F_n := \chi_{C_n}, F_\infty := \chi_{C_\infty}$. Therefore, in virtue of Theorem 3, $\mathcal{S}(\underline{u}_n, \overline{u}_n) \to \mathcal{S}(\underline{u}_\infty, \overline{u}_\infty)$. Thus, in view of the weak lower semicontinuity of the norm,

$$I_4^* \le I_4(\underline{u}_\infty, \overline{u}_\infty) \le \liminf_{n\to\infty} I_4(\underline{u}_n, \overline{u}_n) \le I_4^*,$$

and the theorem is proved. □

5 Conclusions and outlook

This article has shown how various techniques from different fields of mathematical analysis can be combined to arrive at well-posedness results for a nonlinear interface problem that models non-monotone frictional contact of elastic infinite media. In particular, we established a stability result for extended real-valued hemivariational inequalities that extends and considerably improves the stability result of [52]. Furthermore, we investigated various OCPs governed by the interface problem and governed by a related obstacle interface problem. Based on our stability result, we could present an unified approach to existence results for optimal controls in these control problems.

In this article, we dealt with the primal HVI formulation of the underlying interface problem in OCP. Here, mixed variational formulations (see, e.g., [22,54]), is another direction of research. Let us also mention the ongoing research on equilibrium problems and related OCPs using generalized monotonicity concepts (see e.g. [11,35]).

The next step toward numerical treatment of such OCPs is the study of relaxation methods (see e.g. [45]), and the derivation of optimality conditions (see e.g. [48]). A major challenge is to arrive at efficient and reliable numerical solution methods, known in optimal control with *linear* elliptic boundary value problems, even on domains with reentrant corners (see e.g. [2]).

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