Research Article

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Existence of normalized peak solutions for a coupled nonlinear Schrödinger system

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Abstract: In this article, we study the following nonlinear Schrödinger system

$$\begin{cases}-\Delta u_1 + V_1(x)u_1 = \alpha u_1u_2 + \mu u_1, & x\in\mathbb{R}^4,\\ -\Delta u_2 + V_2(x)u_2 = \dfrac{\alpha}{2}u_1^2 + \beta u_2^2 + \mu u_2, & x\in\mathbb{R}^4,\end{cases}$$

with the constraint $\int_{\mathbb{R}^4}(u_1^2 + u_2^2)\mathrm{d}x = 1$, where $\alpha > 0$ and $\alpha > \beta$, $\mu \in \mathbb{R}$, $V_1(x)$, and $V_2(x)$ are bounded functions. Under some mild assumptions on $V_1(x)$ and $V_2(x)$, we prove the existence of normalized peak solutions by using the finite dimensional reduction method, combined with the local Pohozaev identities. Because of the inter-species interaction between the components, we aim to obtain some new technical estimates.

Keywords: normalized peak solutions, local Pohozaev identities, reduction method

MSC 2020: 35A10, 35B99, 35J60

1 Introduction and main result

In this article, we consider the following Schrödinger system with coupled quadratic nonlinearities

$$\begin{cases}-\Delta u_1 + V_1(x)u_1 = \alpha u_1u_2 + \mu u_1, & x\in\mathbb{R}^4,\\ -\Delta u_2 + V_2(x)u_2 = \dfrac{\alpha}{2}u_1^2 + \beta u_2^2 + \mu u_2, & x\in\mathbb{R}^4,\end{cases}\tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^4}(u_1^2 + u_2^2)\mathrm{d}x = 1,\tag{1.2}$$

where $\alpha > 0$ and $\alpha > \beta$, $\mu \in \mathbb{R}$ is a chemical potential, and $V_1(x)$ and $V_2(x)$ are bounded functions.

Such type of systems like (1.1) with quadratic interaction have wide applications in physics, such as Bose-Einstein condensates, plasma physics, and nonlinear optics. For example, the following coupled nonlinear Schrödinger equations

$$i\frac{\partial\omega_k}{\partial t} + \Delta\omega_k + \left(\sum_{j=1}^{K}\mu_{j,k}\,|\omega_j|^2\right)\omega_k = 0\tag{1.3}$$

for some $\mu_{j,k} \in \mathbb{R}$ and $j, k = 1, 2, \dots, K$, can describe the propagation of solitons with $\chi^{(3)}$ nonlinear fiber couplers in nonlinear optic theory, where the complex function ω_k denotes the kth component of the light

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beam, and $\sum_{j=1}^{K}|\omega_j|^2$ is the change in refractive index profile created by all the incoherent components in the light beam. This system has gotten a lot of attentions experimentally and theoretically. If we consider the standing wave solutions for (1.3) of the form $\omega_k(x,t) = e^{i\lambda_k t}u_k(x)$ with u_k a function independent of time, then u_k satisfies

$$\Delta u_k - \lambda_k u_k + \left(\sum_{j=1}^{K}\mu_{j,k}u_j^2\right)u_k = 0 \tag{1.4}$$

for $k = 1, 2, \ldots, K$. The existence and multiplicity of standing wave solutions to (1.4) and its relates problem have been explored by many authors in recent years, see previous studies by [2–4,6,9,10,13,22,24,26,30] and the references therein. However, when nonlinear optical effects such as second harmonic generation are investigated in the optical material that has a $\chi^{(2)}$ nonlinear response, it is led to the system (1.1) (c.f. [1,7]). For the existence, multiplicity and asymptotic behavior of solutions for (1.1), we can refer to [27–29,31].

We note that the following Gross-Pitaevskii (GP) equations proposed by Gross [15] and Pitaevskii [25] in the 1960s

$$i\frac{\partial\omega(x,t)}{\partial t} = -\Delta\omega(x,t) + V(x)\omega(x,t) - a\,|\omega(x,t)|^2\omega(x,t), \quad x \in \mathbb{R}^N, \tag{1.5}$$

with the constraint

$$\int_{\mathbb{R}^N}|\omega(x,t)|^2 dx = 1,$$

has been investigated extensively due to some new experimental advances in quantum phenomena (c.f. [5,11,14]). Here, $N \geq 2$, $V(x) \geq 0$ is a real-valued potential and $a \in \mathbb{R}$ is treated as an arbitrary dimensionless parameter. If we want to find a solution for (1.5) of the form $\omega(x,t) = e^{-i\mu t}u(x)$, where μ represents the chemical potential of the condensate and $u(x)$ is a function independent of time, then the unknown pair (μ, u) satisfies the following nonlinear eigenvalue equation

$$-\Delta u + V(x)u = au^3 + \mu u, \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

with the constraint

$$\int_{\mathbb{R}^N}u^2 dx = 1. \tag{1.7}$$

For ground states of equation (1.6), one can refer to the studies by Guo et al. [18–20] and the references therein, where the main tools are the energy comparison under various assumptions of trapping potential $V(x)$, which can be described equivalently by positive L^2 minimizers of the following functional:

$$I_a = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx - \frac{a}{4}\int_{\mathbb{R}^N}u^4 dx.$$

Very recently the existence and the local uniqueness of excites states for a class of degenerated trapping potential with nonisolated critical points were given in the study by Luo et al. [23], where we call any eigenfunction of equations (1.6) and (1.7) whose energy is larger than that of the ground state the excited states in the physics literatures (c.f. [8]). As for two-component Bose-Einstein condensates with the mass constraint, the studies by Guo and Yang [16] showed the existence of the general exited states by using the reduction argument combined with the local Pohozaev identities, which generalize the existence of the ground state with trapping potentials given in the study by Guo et al. [17] to some extent.

To our best knowledge, there seems to be no results on the existence of peak solutions to (1.1) with the L^2 constraint (1.2). So in this article, we want to investigate this problem and suppose the following conditions hold.

(K_1): $V_1(x)$ and $V_2(x)$ have m common critical points $a_1, \ldots, a_m$ for $m \geq 1$ with $a_1, \ldots, a_m$ the nondegenerated critical points of $\Gamma_1^2 V_1(x) + \Gamma_2^2 V_2(x)$, where Γ_1 and Γ_2 are given below.

(K_2): $V_j(x) < V_j(a_1)$ for $x \in B_d(a_1)\backslash\{a_1\}$ and $V_j(x) \in C^{\theta}(\bar{B}_d(a_1))$ for some $\theta \in (0,1)$ with some small $d > 0$ for $j = 1, 2$.

To state our results, we first introduce some notations. Let U be the unique positive solution of the following problem:

$$\begin{cases} -\Delta u + u = u^2, & \text{in } \mathbb{R}^4, \\ u(0) = \max\limits_{x \in \mathbb{R}^4} u(x), & u(x) \in H^1(\mathbb{R}^4), \end{cases}$$

and we denote $b_* = \int_{\mathbb{R}^4} U^2 \mathrm{d}x$. From [21], we know that $U(x) = U(|x|)$ is strictly decreasing and satisfies for $|s| \le 1$

$$|D^s U(x)| e^{|x|} |x|^{-\frac{3}{2}} \le C.$$

Also $(U_1, U_2) \coloneqq (\Gamma_1 U, \Gamma_2 U)$ is the ground state of

$$\begin{cases} -\Delta u_1 + u_1 = \alpha u_1 u_2 & \text{in } \mathbb{R}^4, \\ -\Delta u_2 + u_2 = \dfrac{\alpha}{2} u_1^2 + \beta u_2^2, & \text{in } \mathbb{R}^4, \end{cases} \tag{1.8}$$

provided that $\alpha > \beta$ with

$$\Gamma_1 = \frac{1}{\alpha}\sqrt{\frac{2(\alpha - \beta)}{\alpha}}, \quad \Gamma_2 = \frac{1}{\alpha}.$$

For any $t \in \mathbb{R}^+$ and $y \in \mathbb{R}^4$, we define for $i = 1, 2$, $U_{i,t,y} \coloneqq tU_i(\sqrt{t}(x - y))$, and for any $\delta > 0$, $\|(u, v)\|_\delta^2 \coloneqq \|u\|_{1,\delta}^2 + \|v\|_{2,\delta}^2$ with

$$\|u\|_{i,\delta}^2 \coloneqq \frac{1}{\delta}\int\limits_{\mathbb{R}^4} (|\nabla u|^2 + (\delta + V_i(x))u^2)\mathrm{d}x.$$

The first result of our article is as follows.

Theorem 1.1. *Assume that* (K_1) *holds and* $\alpha > \beta$ *and* $\alpha > 0$, *then there exists a small constant* $\varepsilon > 0$ *such that for any* (α, β) *satisfying* $|\frac{3\alpha - 2\beta}{\alpha^3} - \frac{1}{mb_*}| \le \varepsilon$, *problems* (1.1) *and* (1.2) *have a peak solution* $(u_{1,\mu}, u_{2,\mu}, \mu)$ *depending on* α *and* β *with the form*

$$(u_{1,\mu}(x), u_{2,\mu}(x)) = \left(\sum_{j=1}^{m} U_{1,\mu,y_{\mu,j}}(x) + \varphi_{1,\mu}(x), \sum_{j=1}^{m} U_{2,\mu,y_{\mu,j}}(x) + \varphi_{2,\mu}(x)\right).$$

Also as $\frac{3\alpha - 2\beta}{\alpha^3} \to \frac{1}{mb_*}$, *there hold* $\mu \to -\infty$ *and*

$$|y_{\mu,j} - a_j| = o\left(\frac{1}{\sqrt{-\mu}}\right) \quad \textit{and} \quad \|(\varphi_{1,\mu}, \varphi_{2,\mu})\|_{-\mu} = O\left(-\frac{1}{\mu}\right).$$

Theorem 1.1 tells that we construct a synchronized solution for problems (1.1) and (1.2), where the synchronized solution means that the two components of the solutions for system (1.1) concentrate at the same set of concentrated points. Otherwise, we call it a segregated solution if the components concentrate at two different set of points.

Remark 1.2. If $V_1(x)$ and $V_2(x)$ have no common nondegenerate critical points, that is, $V_1(x)$ has m critical points $a_1, \dots, a_m$, and while $V_2(x)$ has $\bar{m}$ critical points $\bar{a}_1, \dots, \bar{a}_{\bar{m}}$ with $m, \bar{m} \ge 1$, then we also can construct a segregated peak solution $(u_{1,\mu}, u_{2,\mu}, \mu)$ with $u_{1,\mu}$ and $u_{2,\mu}$ concentrating at these points, respectively.

Now we want to consider the clustering peak solutions to problems (1.1) and (1.2), and the another result is as follows:

Theorem 1.3. *Assume that* (K_2) *holds and* $\alpha > \beta$ *and* $\alpha > 0$, *then there exists a small constant* $\varepsilon > 0$ *such that for any* (α, β) *satisfying* $|\frac{3\alpha - 2\beta}{\alpha^3} - \frac{1}{mb_*}| \leq \varepsilon$, *problems* (1.1) *and* (1.2) *have a peak solution* $(u_{1,\mu}, u_{2,\mu}, \mu)$ *depending on* α *and* β *with the form*

$$(u_{1,\mu}(x), u_{2,\mu}(x)) = \left(\sum_{j=1}^{m} U_{1,\mu,y_{\mu,j}}(x) + \varphi_{1,\mu}(x), \sum_{j=1}^{m} U_{2,\mu,y_{\mu,j}}(x) + \varphi_{2,\mu}(x)\right).$$

Also as $\frac{3\alpha - 2\beta}{\alpha^3} \to \frac{1}{mb_*}$, *there hold* $\mu \to -\infty$, $\|(\varphi_{1,\mu}, \varphi_{2,\mu})\|_{-\mu} = O\left(-\frac{1}{\mu}\right)$ *and*

$$y_{\mu,j} \to a_1 \quad \text{and} \quad \sqrt{\mu}|y_{\mu,j} - y_{\mu,l}| \to \infty, \quad \text{for } l \neq j.$$

Remark 1.4. To our best knowledge, this seems to be the first time to consider the existence of the normalized peak (or bubbling) solutions for problems (1.1) and (1.2). Also we note that only one coefficient a in (1.6) need to be discussed with the constraint mass, while there are two parameters α and β in system (1.1) to be considered, corresponding to the interactions within and between the components, respectively, which makes the problems (1.1) and (1.2) more complicated than the single equation, and we have to analyze the mutual influence of the parameters and the constraint condition very carefully.

We will mainly use the finite dimensional reduction to prove our results, which is an effective way to construct solutions for perturbed elliptic problems. Since the singularly perturbed problem has a small parameter naturally, the approximate solutions can be constructed by the standard steps of the reduction argument. However, the constraint problems (1.1) and (1.2) itself do not provide any natural limiting process explicitly. So to apply the reduction method, we have to instigate the relationship between the constraint conditions and the parameters appeared in the system. This process involve some various Pohozaev identities, which play an important role and need some more accurate estimates.

The structure of this article is organized as follows. We carry out the finite dimensional reduction to study the corresponding problem without constraint in Section 2. In Section 3, we prove Theorems 1.1 and 1.3 is proved in Section 4.

2 The finite dimensional reduction

In this section, we consider the following problem without constraint

$$\begin{cases} -\Delta u_1 + (\lambda + V_1(x))u_1 = \alpha u_1 u_2 & \text{in } \mathbb{R}^4, \\ -\Delta u_2 + (\lambda + V_2(x))u_2 = \frac{\alpha}{2}u_1^2 + \beta u_2^2 & \text{in } \mathbb{R}^4, \end{cases} \tag{2.1}$$

where $\lambda > 0$ is a large parameter.

We want to find a peak solution of equation (2.1) with the form

$$u_{i,\lambda}(x) = \sum_{j=1}^{m} U_{i,\lambda,y_{\lambda,j}}(x) + \varphi_{i,\lambda}(x), \; i = 1, 2, \tag{2.2}$$

where $U_{i,\lambda,y} := \lambda U_i(\sqrt{\lambda}(x - y))$ for some $y \in \mathbb{R}^4$. To this aim, we define L_λ be the bounded linear operator from $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ to itself as follows:

$$\begin{aligned} \langle L_\lambda(\varphi_1, \varphi_2), (\psi_1, \psi_2)\rangle_\lambda := & \int_{\mathbb{R}^4}\left[\nabla\varphi_1\nabla\psi_1 + (\lambda + V_1(x))\varphi_1\psi_1 - \alpha\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}\varphi_2\psi_1 - \alpha\sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}\varphi_1\psi_1\right] \\ & + \int_{\mathbb{R}^4}\left[\nabla\varphi_2\nabla\psi_2 + (\lambda + V_2(x))\varphi_2\psi_2 - \alpha\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}\varphi_1\psi_2 - 2\beta\sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}\varphi_2\psi_2\right] \end{aligned}$$

for any $(\psi_1, \psi_2), (\varphi_1, \varphi_2) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$.

Then to obtain the solution $(u_{1,\lambda}, u_{2,\lambda})$ of equation (2.1) with (2.2) is to solve the following problem:

$$L_\lambda(\varphi_{1,\lambda}, \varphi_{2,\lambda}) = l_\lambda + R_\lambda(\varphi_{1,\lambda}, \varphi_{2,\lambda}),$$

where $l_\lambda := (l_{1,\lambda}, l_{2,\lambda})$ with

$$l_{1,\lambda} = -\sum_{j=1}^{m} V_1(x)U_{1,\lambda,y_{\lambda,j}} + \alpha\left(\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}\right)\left(\sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}\right) - \sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}U_{2,\lambda,y_{\lambda,j}},$$

$$l_{2,\lambda} = -\sum_{j=1}^{m} V_2(x)U_{2,\lambda,y_{\lambda,j}} + \frac{\alpha}{2}\left[\left(\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}\right)^2 - \sum_{j=1}^{m} U^2_{1,\lambda,y_{\lambda,j}}\right] + \beta\left[\left(\sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}\right)^2 - \sum_{j=1}^{m} U^2_{2,\lambda,y_{\lambda,j}}\right],$$

and $R_\lambda(\varphi_{1,\lambda}, \varphi_{2,\lambda}) := (R_{1,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}), R_{2,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}))$ with

$$R_{1,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}) = \alpha\varphi_{1,\lambda}\varphi_{2,\lambda},$$
$$R_{2,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}) = \frac{\alpha}{2}\varphi^2_{1,\lambda} + \beta\varphi^2_{2,\lambda}.$$

To carry out the reduction argument, we first introduce the following result, which has been proved in [29].

Proposition 2.1. *For any $\alpha > 0$ and $\alpha > \beta$, (U_1, U_2) is nondegenerate for the system* (1.8) *in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ in the sense that the kernel is given by*

$$\left\{\eta(\alpha,\beta)\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial x_i} \;\middle|\; i = 1, 2, 3, 4\right\}$$

in $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$, where $\eta(\alpha, \beta) \neq 0$.

Now we define the set

$$E_\lambda = \left\{(u_1, u_2) \in H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4) : \left\langle (u_1, u_2), \left(\frac{\partial U_{1,\lambda,y_{\lambda,j}}}{\partial x_k}, \frac{\partial U_{2,\lambda,y_{\lambda,j}}}{\partial x_k}\right)\right\rangle_\lambda = 0, \quad k = 1, \ldots, 4,\; j = 1, \ldots, m\right\},$$

and the norm $\|(u, v)\|^2_\lambda := \|u\|^2_{1,\lambda} + \|v\|^2_{2,\lambda}$ with

$$\|u\|^2_{i,\lambda} := \frac{1}{\lambda}\int_{\mathbb{R}^4}(|\nabla u|^2 + (\lambda + V_i(x))u^2)\mathrm{d}x.$$

Also we set the projection F_λ from $H^1(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ to E_λ as follows:

$$F_\lambda(u_1, u_2) = (u_1, u_2) - \sum_{i=1}^{4}\sum_{j=1}^{m}\Gamma_{\lambda,i,j}\left(\frac{\partial U_{1,\lambda,y_{\lambda,j}}}{\partial x_i}, \frac{\partial U_{2,\lambda,y_{\lambda,j}}}{\partial x_i}\right).$$

Proposition 2.2. *There exist $\lambda_0, \varepsilon_0, \rho > 0$ independent of $a_j (j = 1, \ldots, m)$, such that for any $\lambda \in [\lambda_0, +\infty)$ and $y_{\lambda,j} \in B_{\varepsilon_0}(a_j)$, $F_\lambda L_\lambda$ is bijective in E_λ. Also, for any $(u_1, u_2) \in E_\lambda$, it holds*

$$\|F_\lambda L_\lambda(u_1, u_2)\|_\lambda \geq \rho\lambda\|(u_1, u_2)\|_\lambda.$$

Proof. By contradiction, suppose that there exist $\lambda_n \to +\infty, y_{\lambda_n,j} \to a_j$, $(u_{1,n}, u_{2,n}) \in E_{\lambda_n}$ such that

$$\|F_{\lambda_n}L_{\lambda_n}(u_{1,n}, u_{2,n})\|_{\lambda_n} \leq \frac{\lambda_n}{n}\|(u_{1,n}, u_{2,n})\|_{\lambda_n}, \tag{2.3}$$

with $\|(u_{1,n}, u_{2,n})\|^2_{\lambda_n} = \frac{1}{\lambda_n}$. By (2.3), for any $(\phi_1, \phi_2) \in E_{\lambda_n}$, we have

$$\langle L_{\lambda_n}((u_{1,n},u_{2,n}),(\phi_1,\phi_2))\rangle_{\lambda_n} = o(\lambda_n)\|(u_{1,n},u_{2,n})\|_{\lambda_n}\|(\phi_1,\phi_2)\|_{\lambda_n} = o(\lambda_n^{\frac{1}{2}})\|(\phi_1,\phi_2)\|_{\lambda_n},$$

and from this, it holds

$$\langle L_{\lambda_n}((u_{1,n},u_{2,n}),(u_{1,n},u_{2,n}))\rangle_{\lambda_n} = o(1).$$

Now choosing R sufficiently large such that

$$(\alpha+2\beta)\sum_{j=1}^{m} U_{2,\lambda_n,y_{\lambda_n,j}} < \frac{1}{4}(\lambda_n + V_i(x)), \quad \text{in } \mathbb{R}^4 \setminus \cup_{j=1}^m B_{\frac{R}{\sqrt{\lambda_n}}}(y_{\lambda_n,j}),$$

we find

$$\langle L_{\lambda_n}((u_{1,n},u_{2,n}),(u_{1,n},u_{2,n}))\rangle_{\lambda_n} \geq \frac{\lambda_n}{2}\|(u_{1,n},u_{2,n})\|^2_{\lambda_n} - C\lambda_n \int\limits_{\cup_{j=1}^m B_{\frac{R}{\sqrt{\lambda_n}}}(y_{\lambda_n,j})} (u_{1,n}^2 + u_{2,n}^2),$$

which gives that

$$\int\limits_{\cup_{j=1}^m B_{\frac{R}{\sqrt{\lambda_n}}}(y_{\lambda_n,j})} (u_{1,n}^2 + u_{2,n}^2) \geq \frac{C}{\lambda_n}.$$

To obtain a contradiction, next we want to prove that

$$\int\limits_{\cup_{j=1}^m B_{\frac{R}{\sqrt{\lambda_n}}}(y_{\lambda_n,j})} (u_{1,n}^2 + u_{2,n}^2) = o\left(\frac{1}{\lambda_n}\right). \tag{2.4}$$

To this end, we define $\bar{u}_{k,n,j}(x) = \frac{1}{\sqrt{\lambda_n}} u_{k,n}(\frac{x}{\sqrt{\lambda_n}} + y_{\lambda_n,j})$. Then

$$\sum_{k=1}^{2}\int\limits_{\mathbb{R}^4}(|\nabla \bar{u}_{k,n,j}|^2 + \bar{u}_{k,n,j}^2) = O(\lambda_n)\|(u_{1,n},u_{2,n})\|^2_{\lambda_n} \leq C.$$

Thus, we can assume that as $n\to+\infty$, $\bar{u}_{k,n,j} \rightharpoonup \bar{u}_{k,j}$ weakly in $H^1(\mathbb{R}^4)\times H^1(\mathbb{R}^4)$, and $\bar{u}_{k,n,j}\to\bar{u}_{k,j}$ strongly in $L^2_{loc}(\mathbb{R}^4)\times L^2_{loc}(\mathbb{R}^4)$ for $k=1,2$ and $j=1,\dots,m$. Now we just need to prove that

$$\bar{u}_{k,j} = 0, \;\; k=1,2 \quad \text{and} \quad j=1,\dots,k. \tag{2.5}$$

For any $(\phi_1,\phi_2)\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$, we can decompose (ϕ_1,ϕ_2) as follows:

$$(\phi_1,\phi_2) = F_{\lambda_n}(\phi_1,\phi_2) + \sum_{i=1}^{4}\sum_{j=1}^{m}\Gamma_{\lambda_n,i,j}\left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}, \frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}\right),$$

with $\Gamma_{\lambda_n,i,j} = b_{\lambda_n,i,j}\left\langle (\phi_1,\phi_2), \left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}, \frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}\right)\right\rangle_{\lambda_n}$ for some constants $b_{\lambda_n,i,j}$. Then it holds

$$\langle L_{\lambda_n}(u_{1,n},u_{2,n}),(\phi_1,\phi_2)\rangle_{\lambda_n} = \langle L_{\lambda_n}(u_{1,n},u_{2,n}),F_{\lambda_n}(\phi_1,\phi_2)\rangle_{\lambda_n} + \sum_{i=1}^{4}\sum_{j=1}^{m}\Gamma_{\lambda_n,i,j}\kappa_{\lambda_n,i,j},$$

where $\kappa_{\lambda_n,i,j} = \left\langle L_{\lambda_n}(u_{1,n},u_{2,n}), \left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}, \frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}\right)\right\rangle_{\lambda_n}$. Also, by (2.3),

$$\langle L_{\lambda_n}(u_{1,n},u_{2,n}),F_{\lambda_n}(\phi_1,\phi_2)\rangle_{\lambda_n} = \langle F_{\lambda_n}L_{\lambda_n}(u_{1,n},u_{2,n}),F_{\lambda_n}(\phi_1,\phi_2)\rangle_{\lambda_n} = o\left(\lambda_n^{\frac{1}{2}}\right)\|(\phi_1,\phi_2)\|_{\lambda_n},$$

which gives that

$$\langle L_{\lambda_n}(u_{1,n},u_{2,n}),(\phi_1,\phi_2)\rangle_{\lambda_n} = o\left(\lambda_n^{\frac{1}{2}}\right)\|(\phi_1,\phi_2)\|_{\lambda_n} + \sum_{i=1}^{4}\sum_{j=1}^{m}\vartheta_{\lambda_n,i,j}\left\langle (\phi_1,\phi_2), \left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}, \frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}\right)\right\rangle_{\lambda_n} \tag{2.6}$$

for some constants $\vartheta_{\lambda_n,i,j}$. From this, we estimate $\vartheta_{\lambda_n,i,j}$ as follows:

$$\begin{aligned}
&\sum_{i=1}^{4}\sum_{j=1}^{m}\vartheta_{\lambda_n,i,j}\left\langle\left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h},\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h}\right),\left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_i},\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}\right)\right\rangle_{\lambda_n}\\
&= \left\langle L_{\lambda_n}(u_{1,n},u_{2,n}),\left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h},\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h}\right)\right\rangle_{\lambda_n} + o(\lambda_n)\\
&= \lambda_n\left\langle (u_{1,n},u_{2,n}),\left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h},\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h}\right)\right\rangle_{\lambda_n} + Q_n(u_{1,n},u_{2,n}) + o(\lambda_n)\\
&= Q_n(u_{1,n},u_{2,n}) + o(\lambda_n^{1+\gamma}),
\end{aligned}$$

where $\gamma > 0$ is a small constant and

$$\begin{aligned}
Q_n(u_{1,n},u_{2,n}) = &-\alpha\int_{\mathbb{R}^4} U_{1,\lambda_n,y_{\lambda_n,j}}u_{2,n}\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} - \alpha\int_{\mathbb{R}^4} U_{2,\lambda_n,y_{\lambda_n,j}}u_{1,n}\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h}\\
&-\alpha\int_{\mathbb{R}^4} U_{1,\lambda_n,y_{\lambda_n,j}}u_{1,n}\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} - 2\beta\int_{\mathbb{R}^4} U_{2,\lambda_n,y_{\lambda_n,j}}u_{2,n}\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h}.
\end{aligned}$$

Noting that $(U_{1,\lambda_n,y_{\lambda_n,j}}, U_{2,\lambda_n,y_{\lambda_n,j}})$ satisfies that

$$\begin{cases}
-\Delta U_{1,\lambda_n,y_{\lambda_n,j}} + \lambda_n U_{1,\lambda_n,y_{\lambda_n,j}} = \alpha U_{1,\lambda_n,y_{\lambda_n,j}}U_{2,\lambda_n,y_{\lambda_n,j}} & \text{in } \mathbb{R}^4,\\
-\Delta U_{2,\lambda_n,y_{\lambda_n,j}} + \lambda_n U_{2,\lambda_n,y_{\lambda_n,j}} = \frac{\alpha}{2}U^2_{1,\lambda_n,y_{\lambda_n,j}} + \beta U^2_{2,\lambda_n,y_{\lambda_n,j}} & \text{in } \mathbb{R}^4,
\end{cases}$$

we find

$$\begin{cases}
-\Delta\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} + \lambda_n\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} = \alpha U_{2,\lambda_n,y_{\lambda_n,j}}\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} + \alpha U_{1,\lambda_n,y_{\lambda_n,j}}\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} & \text{in } \mathbb{R}^4,\\
-\Delta\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} + \lambda_n\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} = \alpha U_{1,\lambda_n,j}\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} + 2\beta U_{2,\lambda_n,y_{\lambda_n,j}}\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h} & \text{in } \mathbb{R}^4.
\end{cases}$$

It follows from the definition of E_{λ_n} that

$$Q_n(u_{1,n},u_{2,n}) = o(\lambda_n^{1+\gamma}).$$

Hence, we find

$$\sum_{i=1}^{4}\sum_{j=1}^{m}\vartheta_{\lambda_n,i,j}\left\langle\left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_i},\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_i}\right),\left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_h},\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_h}\right)\right\rangle_{\lambda_n} = o(\lambda_n^{1+\gamma}),$$

which gives $\vartheta_{\lambda_n,i,j} = o(1)$. So (2.6) gives that

$$\langle L_{\lambda_n}(u_{1,n},u_{2,n}),(\phi_1,\phi_2)\rangle_{\lambda_n} = o\left(\lambda_n^{\frac{1}{2}}\right)\|(\phi_1,\phi_2)\|_{\lambda_n}. \tag{2.7}$$

Now setting $\bar{\phi}_{k,j}(x) = \sqrt{\lambda_n}\,\phi_k(\sqrt{\lambda_n}(x - y_{\lambda_n,j}))$ and using (2.7), we obtain

$$\begin{aligned}
\langle L_{\lambda_n}(u_{1,n},u_{2,n}),(\bar{\phi}_{1,j},\bar{\phi}_{2,j})\rangle_{\lambda_n} &= \sum_{i=1}^{2}\int_{\mathbb{R}^4}\nabla\bar{u}_{i,n,j}\nabla\phi_i + \left(1 + \frac{1}{\lambda_n}V_i\left(\frac{z}{\sqrt{\lambda_n}} + y_{\lambda_n,j}\right)\right)\bar{u}_{i,n,j}\phi_i\\
&\quad -2\beta\int_{\mathbb{R}^4} U_2\bar{u}_{2,n,j}\phi_2 - \alpha\int_{\mathbb{R}^4}(U_1\bar{u}_{2,n,j}\phi_1 + U_1\bar{u}_{1,n,j}\phi_2 + U_2\bar{u}_{1,n,j}\phi_1)\\
&= o\left(\lambda_n^{\frac{1}{2}}\right)\|(\bar{\phi}_{1,j},\bar{\phi}_{2,j})\|_{\lambda_n} = o(1).
\end{aligned}$$

Then, $(\bar{u}_{1,j}, \bar{u}_{2,j})$ solves

$$\begin{cases} -\Delta\bar{u}_{1,j} + \bar{u}_{1,j} = \alpha\bar{u}_{1,j}U_2 + \alpha\bar{u}_{2,j}U_1 & \text{in } \mathbb{R}^4, \\ -\Delta\bar{u}_{2,j} + \bar{u}_{2,j} = \alpha\bar{u}_{1,j}U_1 + 2\beta\bar{u}_{2,j}U_2 & \text{in } \mathbb{R}^4. \end{cases}$$

From this and the fact that (U_1, U_2) is nondegenerate, we have

$$(\bar{u}_{1,j}, \bar{u}_{2,j}) = \sum_{k=1}^{4} \tau_{k,j}\left(\frac{\partial U_1}{\partial x_k}, \frac{\partial U_2}{\partial x_k}\right). \tag{2.8}$$

On the other hand, from $(u_{1,n}, u_{2,n}) \in E_{\lambda_n}$ and (2.8), it holds

$$\begin{aligned} 0 &= \left\langle (u_{1,n}, u_{2,n}), \left(\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_k}, \frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_k}\right)\right\rangle_{\lambda_n} \\ &= \int_{\mathbb{R}^4} \frac{1}{\lambda_n}\left(\nabla u_{1,n}\nabla\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_k} + \nabla u_{2,n}\nabla\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_k}\right) \\ &\quad + \int_{\mathbb{R}^4}\left(\left(1 + \frac{1}{\lambda_n}V_1(x)\right)u_{1,n}\frac{\partial U_{1,\lambda_n,y_{\lambda_n,j}}}{\partial x_k} + \left(1 + \frac{1}{\lambda_n}V_2(x)\right)u_{2,n}\frac{\partial U_{2,\lambda_n,y_{\lambda_n,j}}}{\partial x_k}\right) \\ &= \int_{\mathbb{R}^4}\left(\nabla\bar{u}_{1,n,j}\nabla\frac{\partial U_1}{\partial x_k} + \nabla\bar{u}_{2,n,j}\nabla\frac{\partial U_2}{\partial x_k}\right) \\ &\quad + \int_{\mathbb{R}^4}\left(\left(1 + \frac{1}{\lambda_n}V_1(x)\right)\bar{u}_{1,n,j}\frac{\partial U_1}{\partial x_k} + \left(1 + \frac{1}{\lambda_n}V_2(x)\right)\bar{u}_{2,n,j}\frac{\partial U_2}{\partial x_k}\right) \\ &= \int_{\mathbb{R}^4}\left(\nabla\bar{u}_{1,j}\nabla\frac{\partial U_1}{\partial x_k} + \nabla\bar{u}_{2,j}\nabla\frac{\partial U_2}{\partial x_k}\right) + \int_{\mathbb{R}^4}\left(\bar{u}_{1,j}\frac{\partial U_1}{\partial x_k} + \bar{u}_{2,j}\frac{\partial U_2}{\partial x_k}\right) + o(1) \\ &= \alpha\int_{\mathbb{R}^4}\left(U_2\bar{u}_{1,j}\frac{\partial U_1}{\partial x_k} + U_1\bar{u}_{2,j}\frac{\partial U_1}{\partial x_k} + U_1\bar{u}_{1,j}\frac{\partial U_2}{\partial x_k}\right) + 2\beta\int_{\mathbb{R}^4} U_2\bar{u}_{2,j}\frac{\partial U_2}{\partial x_k} + o(1). \end{aligned}$$

By the property of (U_1, U_2), all the constants $\tau_{k,j} = 0$ in (2.8). This gives that (2.5) holds and then (2.4) follows. □

Similar to Proposition 2.2, we have

Proposition 2.3. *There exist $\lambda_1, \varepsilon_1, \rho_1 > 0$ independent of a_1, such that for any $\lambda \in [\lambda_1, +\infty)$ and $y_{\lambda,k}$ close to a_1 with*

$$|y_{\lambda,k} - y_{\lambda,j}| \geq \frac{\varepsilon_1 \ln\lambda}{\sqrt{\lambda}}, \quad \textit{for } k \neq j, \tag{2.9}$$

$F_\lambda L_\lambda$ is bijective in E_λ. Moreover, it holds

$$\|F_\lambda L_\lambda(u_1, u_2)\|_\lambda \geq \rho_1\lambda\|(u_1, u_2)\|_\lambda.$$

Lemma 2.4. *We have*

$$\|l_\lambda\|_\lambda = O(1). \tag{2.10}$$

Proof. Recall that $l_\lambda = (l_{1,\lambda}, l_{2,\lambda})$ and

$$\langle(l_{1,\lambda}, l_{2,\lambda}), (\phi_1, \phi_2)\rangle_\lambda = \langle l_{1,\lambda}, \phi_1\rangle_\lambda + \langle l_{2,\lambda}, \phi_2\rangle_\lambda.$$

Using Hölder inequality, we find

$$\begin{aligned}\langle l_{1,\lambda}, \phi_1\rangle_\lambda &= -\int_{\mathbb{R}^4} \sum_{j=1}^m V_1(x) U_{1,\lambda,y_{\lambda,j}}\phi_1 + \alpha \int_{\mathbb{R}^4}\left[\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}\right)\left(\sum_{j=1}^m U_{2,\lambda,y_{\lambda,j}}\right) - \sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}} U_{2,\lambda,y_{\lambda,j}}\right]\phi_1 \\ &= O(\|\phi_1\|_{1,\lambda}),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\langle l_{2,\lambda}, \phi_2\rangle_\lambda &= -\int_{\mathbb{R}^4} \sum_{j=1}^m V_2(x) U_{2,\lambda,y_{\lambda,j}}\phi_2 + \frac{\alpha}{2} \int_{\mathbb{R}^4}\left[\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}\right)^2 - \sum_{j=1}^m U^2_{1,\lambda,y_{\lambda,j}}\right]\phi_2 \\ &\quad + \beta \int_{\mathbb{R}^4}\left[\left(\sum_{j=1}^m U_{2,\lambda,y_{\lambda,j}}\right)^2 - \sum_{j=1}^m U^2_{2,\lambda,y_{\lambda,j}}\right]\phi_2 \\ &= O(\|\phi_2\|_{2,\lambda}).\end{aligned} \tag{2.12}$$

Hence, (2.12) and (2.11) give (2.10). □

Lemma 2.5. *It holds*

$$\|R_\lambda(\varphi_{1,\lambda}, \varphi_{2,\lambda})\|_\lambda = O\left(\lambda^{\frac{1}{2}}\|(\varphi_{1,\lambda}, \varphi_{2,\lambda})\|^2_\lambda\right). \tag{2.13}$$

Proof. Note that $R_\lambda(\varphi_{1,\lambda}, \varphi_{2,\lambda}) = (R_{1,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}), R_{2,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}))$. By the direct computations, we have

$$\begin{aligned}\langle R_{1,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}), \phi_1\rangle_\lambda &= \alpha \int_{\mathbb{R}^4} \varphi_{1,\lambda}\varphi_{2,\lambda}\phi_1 \le C\|\varphi_{1,\lambda}\|_{L^4}\|\varphi_{2,\lambda}\|_{L^4}\|\phi_1\|_{L^2} \\ &= O\left(\lambda^{\frac{1}{2}}\|\varphi_{1,\lambda}\|_\lambda\|\varphi_{2,\lambda}\|_\lambda\|\phi_1\|_\lambda\right),\end{aligned} \tag{2.14}$$

where we use the following classical Gagliardo-Nirenberg inequality:

$$\left(\int_{\mathbb{R}^2} |\phi|^4\right)^{\frac{1}{4}} = O\left(\|\nabla\phi\|^{\frac{1}{2}}_{L^2}\|\phi\|^{\frac{1}{2}}_{L^2}\right) = O\left(\lambda^{\frac{1}{4}}\|\phi\|_{i,\lambda}\right), \text{ for any } \phi \in H^1(\mathbb{R}^4). \tag{2.15}$$

With the same argument, it holds

$$\langle R_{2,\lambda}(\varphi_{1,\lambda}, \varphi_{2,\lambda}), \phi_2\rangle_\lambda = O\left(\lambda^{\frac{1}{2}}\|\varphi_{1,\lambda}\|_\lambda\|\varphi_{2,\lambda}\|_\lambda\|\phi_2\|_\lambda\right). \tag{2.16}$$

Hence, (2.13) follows by (2.16) and (2.14). □

Proposition 2.6. *Let λ_0 be as in Proposition 2.2. Then for any $\lambda \in [\lambda_0, +\infty)$, and $y_{\lambda,k}$ close to a_k for $k = 1, \dots, m$, there exists unique $\varphi_{i,\lambda} \in E_\lambda$, such that*

$$\begin{aligned}&\int_{\mathbb{R}^4} (\nabla u_{1,\lambda}\nabla\phi_1 + \nabla u_{2,\lambda}\nabla\phi_2 + (\lambda + V_1(x))u_{1,\lambda}\phi_1 + (\lambda + V_2(x))u_{2,\lambda}\phi_2) \\ &= \int_{\mathbb{R}^4}\left(\alpha u_{1,\lambda}u_{2,\lambda}\phi_1 + \frac{\alpha}{2}u^2_{1,\lambda}\phi_2 + \beta u^2_{2,\lambda}\phi_2\right)\end{aligned} \tag{2.17}$$

for any $(\phi_1, \phi_2) \in E_\lambda$, where $u_{i,\lambda}(x) = \sum_{k=1}^m U_{i,\lambda,y_{\lambda,k}}(x) + \varphi_{i,\lambda}(x)$. Moreover, it holds

$$\|(\varphi_{1,\lambda}, \varphi_{2,\lambda})\|_\lambda \le \frac{C}{\lambda}. \tag{2.18}$$

Proof. To obtain (2.17), it is equivalent to consider the following problem:

$$F_\lambda L_\lambda(\varphi_1, \varphi_2) = F_\lambda l_\lambda + F_\lambda R_\lambda(\varphi_1, \varphi_2), \quad \text{for}\,(\varphi_1, \varphi_2) \in E_\lambda. \tag{2.19}$$

By Proposition 2.2, we can rewrite (2.19) as follows:

$$(\varphi_1, \varphi_2) = B(\varphi_1, \varphi_2) := (F_\lambda L_\lambda)^{-1} F_\lambda l_\lambda + (F_\lambda L_\lambda)^{-1} F_\lambda R_\lambda(\varphi_1, \varphi_2). \tag{2.20}$$

Now we define

$$M := \left\{(\varphi_1, \varphi_2) \in E_\lambda : \|(\varphi_1, \varphi_2)\|_\lambda \le \frac{1}{\lambda^{1-\gamma}}\right\},$$

where $\gamma > 0$ is a fixed small constant.

Next we will prove B is a contraction map from M to M. First, from Proposition 2.2 and Lemmas 2.4 and 2.5, we have

$$\|B(\varphi_1, \varphi_2)\|_\lambda \le C\left(\frac{1}{\lambda}\|l_\lambda\|_\lambda + \frac{1}{\lambda}\|R_\lambda(\varphi_1, \varphi_2)\|_\lambda\right) \le \frac{1}{\lambda^{1-\gamma}}.$$

On the other hand, for $(\varphi_1, \varphi_2), (\bar\varphi_1, \bar\varphi_2) \in E_\lambda$, it holds

$$\|B(\varphi_1, \varphi_2) - B(\bar\varphi_1, \bar\varphi_2)\|_\lambda \le \frac{C}{\lambda}(\|R_\lambda(\varphi_1, \varphi_2)\|_\lambda + \|R_\lambda(\bar\varphi_1, \bar\varphi_2)\|_\lambda) \le \frac{1}{\lambda^{1-\gamma}}.$$

By applying the contraction mapping theorem, we find that for any $\lambda \in [\lambda_0, +\infty)$, $y_{\lambda,k}$ close to a_k for $k = 1, \dots, m$, there exists unique $(\varphi_{1,\lambda}, \varphi_{2,\lambda}) \in E_\lambda$ depending on a_k and λ solving (2.20) and then (2.17) follows. Finally, (2.18) follows from (2.20). □

Similar to Proposition 2.6, using Proposition 2.3 and Lemmas 2.4 and 2.5, we have the following result.

Proposition 2.7. *Let λ_1 be as in Proposition 2.3. Then for any $\lambda \in [\lambda_1, +\infty)$, and $y_{\lambda,k}$ close to a_1 for $k = 1, \dots, m$, satisfying (2.9), (2.17) and (2.18) hold with $u_{i,\lambda} = \sum_{k=1}^m U_{i,\lambda,y_{\lambda,k}} + \varphi_{i,\lambda}(x)$.*

3 Proof of Theorem 1.1

From Proposition 2.6, for any $\lambda \in [\lambda_0, +\infty)$, $y_{\lambda,j} \to a_j$ for $j = 1, \dots, m$, there exist $u_{i,\lambda}$ and $\Gamma_{\lambda,i,j}$ such that

$$\begin{cases} -\Delta u_{1,\lambda} + (\lambda + V_1(x))u_{1,\lambda} = \alpha u_{1,\lambda}u_{2,\lambda} + \displaystyle\sum_{i=1}^4\sum_{j=1}^m \Gamma_{\lambda,i,j}\frac{\partial U_{1,\lambda,y_{\lambda,j}}}{\partial x_i}, & \text{in } \mathbb{R}^4, \\ -\Delta u_{2,\lambda} + (\lambda + V_2(x))u_{2,\lambda} = \dfrac{\alpha}{2}u_{1,\lambda}^2 + \beta u_{2,\lambda}^2 + \displaystyle\sum_{i=1}^4\sum_{j=1}^m \Gamma_{\lambda,i,j}\frac{\partial U_{2,\lambda,y_{\lambda,j}}}{\partial x_i}, & \text{in } \mathbb{R}^4, \end{cases}$$

where $u_{i,\lambda}(x) = \sum_{j=1}^m U_{i,\lambda,y_{\lambda,j}}(x) + \varphi_{i,\lambda}(x)$.

Now to obtain a true solution for (2.1), we just need to choose suitable $y_{\lambda,j}$ such that

$$\Gamma_{\lambda,i,j} = 0, \quad \text{for } i = 1, 2, 3, 4 \quad \text{and} \quad j = 1, \dots, m. \tag{3.1}$$

To this end, we want to choose $(y_{\lambda,1}, \dots, y_{\lambda,m})$ solving for any $d > 0$,

$$\begin{aligned} &\int\limits_{B_d(y_{\lambda,j})}\left[-\Delta u_{1,\lambda}\frac{\partial u_{1,\lambda}}{\partial x_i} + (\lambda + V_1(x))u_{1,\lambda}\frac{\partial u_{1,\lambda}}{\partial x_i} - \alpha u_{1,\lambda}u_{2,\lambda}\frac{\partial u_{1,\lambda}}{\partial x_i}\right] \\ &+ \int\limits_{B_d(y_{\lambda,j})}\left[-\Delta u_{2,\lambda}\frac{\partial u_{2,\lambda}}{\partial x_i} + (\lambda + V_2(x))u_{2,\lambda}\frac{\partial u_{2,\lambda}}{\partial x_i} - \frac{\alpha}{2}u_{1,\lambda}^2\frac{\partial u_{2,\lambda}}{\partial x_i} - \beta u_{2,\lambda}^2\frac{\partial u_{2,\lambda}}{\partial x_i}\right] = 0 \end{aligned} \tag{3.2}$$

with $i = 1, 2, 3, 4$ and $j = 1, \dots, m$.

Lemma 3.1. *Assume that* (K_1) *holds. Then there exists some large* λ_0 *such that for any* $\lambda \in [\lambda_0, +\infty)$, *problem* (2.1) *has a peak solution* $(u_{1,\lambda}, u_{2,\lambda})$ *with the form*

$$u_{i,\lambda}(x) = \sum_{j=1}^{m} U_{i,\lambda,y_{\lambda,j}}(x) + \varphi_{i,\lambda}(x), \tag{3.3}$$

satisfying for $j = 1, \dots, m$,

$$|y_{\lambda,j} - a_j| = o\left(\frac{1}{\sqrt{\lambda}}\right) \quad \text{and} \quad \|(\varphi_{1,\lambda}, \varphi_{2,\lambda})\|_\lambda = O\left(\frac{1}{\lambda}\right). \tag{3.4}$$

Proof. From the aforementioned discussions, we have to choose suitable $y_{\lambda,j} \in \mathbb{R}^4$ for $j = 1, \dots, m$ satisfying (3.2). Integrating on $B_d(y_{\lambda,j})$, we find that (3.2) is equivalent to

$$\begin{aligned}
&\int_{B_d(y_{\lambda,j})} \left[\frac{\partial V_1(x)}{\partial x_i} u_{1,\lambda}^2 + \frac{\partial V_2(x)}{\partial x_i} u_{2,\lambda}^2\right] \\
&= \int_{\partial B_d(y_{\lambda,j})} |\nabla u_{1,\lambda}|^2 \nu_i - 2 \int_{\partial B_d(y_{\lambda,j})} \frac{\partial u_{1,\lambda}}{\partial x_i} \frac{\partial u_{1,\lambda}}{\partial \nu} - \alpha \int_{B_d(y_{\lambda,j})} u_{2,\lambda} \frac{\partial u_{1,\lambda}^2}{\partial x_i} + \int_{\partial B_d(y_{\lambda,j})} (\lambda + V_1(x)) u_{1,\lambda}^2 \nu_i \\
&\quad + \int_{\partial B_d(y_{\lambda,j})} |\nabla u_{2,\lambda}|^2 \nu_i - 2 \int_{\partial B_d(y_{\lambda,j})} \frac{\partial u_{2,\lambda}}{\partial x_i} \frac{\partial u_{1,\lambda}}{\partial \nu} + \int_{\partial B_d(y_{\lambda,j})} (\lambda + V_2(x)) u_{2,\lambda}^2 \nu_i \\
&\quad - \alpha \int_{B_d(y_{\lambda,j})} u_{1,\lambda}^2 \frac{\partial u_{2,\lambda}}{\partial x_i} - \frac{2}{3}\beta \int_{\partial B_d(y_{\lambda,j})} u_{2,\lambda}^3 \nu_i,
\end{aligned}$$

where $\nu = (\nu_1, \dots, \nu_4)$ is the outward unit normal of $\partial B_d(y_{\lambda,j})$. Also it holds

$$\int_{B_d(y_{\lambda,j})} \left(u_{1,\lambda}^2 \frac{\partial u_{2,\lambda}}{\partial x_i} + u_{2,\lambda} \frac{\partial u_{1,\lambda}^2}{\partial x_i}\right) = \int_{\partial B_d(y_{\lambda,j})} u_{1,\lambda}^2 u_{2,\lambda} \nu_i = O(e^{-\theta\sqrt{\lambda}})$$

for some $\theta > 0$. Then we have

$$\int_{B_d(y_{\lambda,j})} \left[\frac{\partial V_1(x)}{\partial x_i} u_{1,\lambda}^2 + \frac{\partial V_2(x)}{\partial x_i} u_{2,\lambda}^2\right] = O(e^{-\theta\sqrt{\lambda}}). \tag{3.5}$$

On the other hand,

$$\begin{aligned}
\int_{B_d(y_{\lambda,j})} \frac{\partial V_1(x)}{\partial x_i} u_{1,\lambda}^2 &= \int_{B_d(y_{\lambda,j})} \sum_{l=1}^{4} \frac{\partial^2 V_1(a_j)}{\partial x_i x_l} (x_l - a_{j,l}) u_{1,\lambda}^2 + O\left(\int_{B_d(y_{\lambda,j})} |x - a_j|^2 u_{1,\lambda}^2\right) \\
&= m \sum_{l=1}^{4} \frac{\partial^2 V_1(a_j)}{\partial x_i x_l} (y_{\lambda,j,l} - a_{j,l}) \int_{\mathbb{R}^4} U_1^2(x) + O(e^{-\theta\sqrt{\lambda}} + \|\varphi_{1,\lambda}\|_{1,\lambda}^2) + o(|y_{\lambda,j} - a_j|) \\
&= m \sum_{l=1}^{4} \frac{\partial^2 V_1(a_j)}{\partial x_i x_l} (y_{\lambda,j,l} - a_{j,l}) \int_{\mathbb{R}^4} U_1^2(x) + o\left(\frac{1}{\sqrt{\lambda}} + |y_{\lambda,j} - a_j|\right).
\end{aligned} \tag{3.6}$$

Similarly, we have

$$\int_{B_d(y_{\lambda,j})} \frac{\partial V_2(x)}{\partial x_i} u_{2,\lambda}^2 = m \sum_{l=1}^{4} \frac{\partial^2 V_1(a_j)}{\partial x_i x_l} (y_{\lambda,j,l} - a_{j,l}) \int_{\mathbb{R}^4} U_2^2(x) + o\left(\frac{1}{\sqrt{\lambda}} + |y_{\lambda,j} - a_j|\right). \tag{3.7}$$

Then (3.5), (3.6), and (3.7) give us that

$$\sum_{l=1}^{4}\left(\frac{\partial^2 V_1(a_j)}{\partial x_i x_l}\Gamma_1^2+\frac{\partial^2 V_2(a_j)}{\partial x_i x_l}\Gamma_2^2\right)(y_{\lambda,j,l}-a_{j,l})=o\left(\frac{1}{\sqrt{\lambda}}+|y_{\lambda,j}-a_j|\right). \tag{3.8}$$

By the assumption (K_1), we find

$$|y_{\lambda,j}-a_j|=o\left(\frac{1}{\sqrt{\lambda}}\right)\quad \text{for } j=1,\dots,m. \qquad \square$$

Lemma 3.2. *It holds*

$$\int_{\mathbb{R}^4}U^3=3\int_{\mathbb{R}^4}U^2=\frac{3}{2}\int_{\mathbb{R}^4}|\nabla U|^2. \tag{3.9}$$

Proof. Note that U satisfies

$$-\Delta U+U=U^2,\quad \text{in } \mathbb{R}^4. \tag{3.10}$$

By direct computations, we have

$$\int_{\mathbb{R}^4}(|\nabla U|^2+U^2)=\int_{\mathbb{R}^4}U^3.$$

Also multiplying $(x\cdot\nabla U)$ on both sides of equation (3.10) and integrating on $\mathbb{R}^4$, it holds

$$\int_{\mathbb{R}^4}|\nabla U|^2+2\int_{\mathbb{R}^4}U^2=\frac{4}{3}\int_{\mathbb{R}^4}U^3.$$

So the aforementioned two equalities give (3.9). $\square$

Proposition 3.3. *If* $(u_{1,\lambda},u_{2,\lambda})$ *is a solution of* (2.1) *with the form* (3.3) *satisfying* (3.4), *then it holds*

$$\lambda\left(mb_*\left(\frac{3}{2}\alpha\Gamma_2\Gamma_1^2+\beta\Gamma_2^3\right)-\int_{\mathbb{R}^4}(u_{1,\lambda}^2+u_{2,\lambda}^2)\right)=b_*\sum_{i=1}^{2}\sum_{j=1}^{m}\Gamma_i^2V_i(a_j)+O\left(\frac{1}{\lambda}\right). \tag{3.11}$$

Proof. Let $(u_{1,\lambda},u_{2,\lambda})$ be a solution of equation (2.1), multiplying $\langle x-y_{\lambda,j},\nabla u_{i,\lambda}\rangle$ on both sides of the ith equation of equation (2.1) and integrating on $B_d(y_{\lambda,j})$, we find

$$\int_{B_d(y_{\lambda,j})}\left[\sum_{i=1}^{2}((2V_i(x)+\langle\nabla V_i(x),x-y_{\lambda,j}\rangle)u_{i,\lambda}^2+2\lambda u_{i,\lambda}^2)-\alpha u_{2,\lambda}u_{1,\lambda}^2-\frac{2}{3}\beta u_{2,\lambda}^3\right]=\int_{\partial B_d(y_{\lambda,j})}W(x)\mathrm{d}\sigma, \tag{3.12}$$

where

$$W(x):=\sum_{i=1}^{2}\left(-2\frac{\partial u_{i,\lambda}}{\partial\nu}\langle x-y_{\lambda,j},\nabla u_{i,\lambda}\rangle+\langle x-y_{\lambda,j},\nu\rangle[(\lambda+V_i(x))u_{i,\lambda}^2]\right)-\langle x-y_{\lambda,j},\nu\rangle\left(\alpha u_{2,\lambda}u_{1,\lambda}^2+\frac{2}{3}\beta u_{2,\lambda}^3\right).$$

Now, being $u_{i,\lambda}(x)=\sum_{j=1}^{m}U_{i,\lambda,y_{\lambda,j}}(x)+\varphi_{i,\lambda}(x)$ with $\|(\varphi_{1,\lambda},\varphi_{2,\lambda})\|_\lambda=O\left(\frac{1}{\lambda}\right)$, it holds

$$\begin{aligned}
&\int_{B_d(y_{\lambda,j})}(2V_i(x)+\langle\nabla V_i(x),x-y_{\lambda,j}\rangle)u_{i,\lambda}^2\\
&=\int_{B_d(y_{\lambda,j})}(\langle\nabla V_i(x),x-x_{\lambda,j}\rangle+2(V_i(x)-V_i(a_j)))U_{i,\lambda,y_{\lambda,j}}^2(x)+2V_i(a_j)\int_{B_d(y_{\lambda,j})}u_{i,\lambda}^2\\
&\quad+O\left(\int_{B_d(y_{\lambda,j})}|x-a_j|^2U_{i,\lambda,y_{\lambda,j}}\varphi_{i,\lambda}+\|\varphi_{i,\lambda}\|_\lambda^2\right)\\
&=2V_i(a_j)\int_{B_d(y_{\lambda,j})}u_{i,\lambda}^2+O\left(\frac{1}{\lambda}\right)\\
&=2\Gamma_i^2b_*V_i(a_j)+O\left(\frac{1}{\lambda}\right),
\end{aligned}$$

where $b_* = \int_{\mathbb{R}^4} U^2$. On the other hand, using (3.9), we find

$$\int_{B_d(y_{\lambda,j})} \left(\alpha u_{2,\lambda} u_{1,\lambda}^2 + \frac{2}{3}\beta u_{2,\lambda}^3\right) = \lambda \int_{\mathbb{R}^4} \left(\alpha U_1^2 U_2 + \frac{2}{3}\beta U_2^3\right) + O\left(\frac{1}{\lambda}\right) = 2\lambda\left(\frac{3}{2}\alpha\Gamma_2\Gamma_1^2 + \beta\Gamma_2^3\right) b_* + O\left(\frac{1}{\lambda}\right).$$

So summing (3.12) from $j = 1$ to $j = m$, we find (3.11). □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Now we set $b_\lambda = \int_{\mathbb{R}^4}(u_{1,\lambda}^2 + u_{2,\lambda}^2)$ and $v_{i,\lambda} = u_{i,\lambda}/\sqrt{b_\lambda}$, then $(v_{1,\lambda}, v_{2,\lambda})$ satisfies that

$$\begin{cases} -\Delta v_{1,\lambda} + (\lambda + V_1(x)) v_{1,\lambda} = \alpha b_\lambda^{\frac{1}{2}} v_{1,\lambda} v_{2,\lambda}, & \text{in } \mathbb{R}^4, \\ -\Delta v_{2,\lambda} + (\lambda + V_2(x)) v_{2,\lambda} = \frac{\alpha}{2} b_\lambda^{\frac{1}{2}} v_{1,\lambda}^2 + \beta b_\lambda^{\frac{1}{2}} v_{2,\lambda}^2, & \text{in } \mathbb{R}^4, \end{cases}$$

with

$$\int_{\mathbb{R}^4} (v_{1,\lambda}^2 + v_{2,\lambda}^2) = 1.$$

Therefore, from Proposition 3.3, we obtain that

$$\lambda(mD(\alpha,\beta)b_* - b_\lambda) = b_* \sum_{i=1}^{2}\sum_{j=1}^{m} \Gamma_i^2 V_i(a_j) + o(1), \tag{3.13}$$

where $D(\alpha,\beta) = \frac{3}{2}\alpha\Gamma_2\Gamma_1^2 + \beta\Gamma_2^3 = \frac{3\alpha - 2\beta}{\alpha^3}$. Take (α^*, β^*) is a root of $D(\alpha,\beta) = \frac{1}{mb_*}$.

If $D(\alpha,\beta)\downarrow \frac{1}{mb_*}$ as $(\alpha,\beta) \to (\alpha^*,\beta^*)$, letting $N_1 \in \mathbb{R}$ such that $N_1 + V_i(a_j) > 0$ and $\bar\lambda = \lambda - N_1$, we have

$$\begin{cases} -\Delta u_{1,\lambda} + (\bar\lambda + (N_1 + V_1(x))) u_{1,\lambda} = \alpha u_{1,\lambda} u_{2,\lambda}, & \text{in } \mathbb{R}^4, \\ -\Delta u_{2,\lambda} + (\bar\lambda + (N_1 + V_2(x))) u_{2,\lambda} = \frac{\alpha}{2} u_{1,\lambda}^2 + \beta u_{2,\lambda}^2, & \text{in } \mathbb{R}^4. \end{cases}$$

Similar to (3.13), it holds

$$\bar\lambda(mD(\alpha,\beta)b_* - b_{\bar\lambda}) = b_* \sum_{i=1}^{2}\sum_{j=1}^{m} \Gamma_i^2 (N_1 + V_i(a_j)) + o(1) > 0, \tag{3.14}$$

which gives that there exists some $\bar\lambda_0$ large enough such that $b_{\bar\lambda_0} \le 1$. In fact, if $b_{\bar\lambda_0} > 1$, then from the fact that $D(\alpha,\beta)\downarrow \frac{1}{mb_*}$ as $(\alpha,\beta) \to (\alpha^*,\beta^*)$, we can take (α,β) such that $b_{\bar\lambda_0} > mD(\alpha,\beta)b_*$, which contradicts to (3.14).

Now we define $T(\lambda) := (1 - b_\lambda)\lambda^2$. If $b_{\bar\lambda_0} = 1$, then we have got the solution. Now we consider $b_{\bar\lambda_0} < 1$ and $T(\bar\lambda_0) > 0$ for $(\alpha,\beta) \to (\alpha^*,\beta^*)$. On the other hand, by (3.13),

$$\begin{aligned} \lim_{\lambda\to+\infty} T(\lambda) &= \lim_{\lambda\to+\infty} ((mD(\alpha,\beta)b_* - b_\lambda)\lambda + (1 - mD(\alpha,\beta)b_*)\lambda)\lambda \\ &= \lim_{\lambda\to+\infty} (1 - mD(\alpha,\beta)b_* + o(1))\lambda^2 = -\infty. \end{aligned}$$

Thus, there exists some $\lambda_{\alpha,\beta} \in [\bar\lambda_0, +\infty)$ such that $T(\lambda_{\alpha,\beta}) = 0$, which means that $b_{\lambda_{\alpha,\beta}} = 1$. So we have derived the existence for (1.1) with $\mu = -\lambda_{\alpha,\beta}$ and $\int_{\mathbb{R}^4}(u_1^2 + u_2^2) = 1$ if $D(\alpha,\beta) \in (\frac{1}{mb_*}, \frac{1}{mb_*} + \varepsilon]$ for small ε.

When $D(\alpha,\beta)\uparrow \frac{1}{mb_*}$ as $(\alpha,\beta) \to (\alpha^*,\beta^*)$, letting $N_2 \in \mathbb{R}$ such that $N_2 + V_i(a_j) < 0$ and $\tilde\lambda = \lambda - N_2$, we have

$$\begin{cases} -\Delta u_{1,\lambda} + (\tilde\lambda + (N_2 + V_1(x))) u_{1,\lambda} = \alpha u_{1,\lambda} u_{2,\lambda}, & \text{in } \mathbb{R}^4, \\ -\Delta u_{2,\lambda} + (\tilde\lambda + (N_2 + V_2(x))) u_{2,\lambda} = \frac{\alpha}{2} u_{1,\lambda}^2 + \beta u_{2,\lambda}^2, & \text{in } \mathbb{R}^4. \end{cases}$$

With the same argument as mentioned earlier, we can also obtain the existence of a concentrated solution to (1.1)–(1.2) if $D(\alpha,\beta) \in [\frac{1}{mb_*} - \varepsilon, \frac{1}{mb_*})$. □

4 Proof of Theorem 1.3

In this section, we come to prove Theorem 1.3. First from Proposition 2.7, for any $\lambda \in [\lambda_1, +\infty)$, $y_{\lambda,j} \to a_1$, $\sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}| \to \infty$, there exist $u_{i,\lambda}$ and $\Gamma_{\lambda,i,j}$ such that

$$\begin{cases} -\Delta u_{1,\lambda} + (\lambda + V_1(x))u_{1,\lambda} = \alpha u_{1,\lambda}u_{2,\lambda} + \sum_{i=1}^{4}\sum_{j=1}^{m}\Gamma_{\lambda,i,j}\frac{\partial U_{1,\lambda,y_{\lambda,j}}}{\partial x_i}, & \text{in } \mathbb{R}^4,\\ -\Delta u_{2,\lambda} + (\lambda + V_2(x))u_{2,\lambda} = \frac{\alpha}{2}u_{1,\lambda}^2 + \beta u_{2,\lambda}^2 + \sum_{i=1}^{4}\sum_{j=1}^{m}\Gamma_{\lambda,i,j}\frac{\partial U_{2,\lambda,y_{\lambda,j}}}{\partial x_i}, & \text{in } \mathbb{R}^4 \end{cases}$$

with $u_{i,\lambda}(x) = \sum_{j=1}^{m} U_{i,\lambda,y_{\lambda,j}}(x) + \varphi_{i,\lambda}(x)$ satisfying (2.18).

Now we define

$$I_\lambda(u_1, u_2) = \frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^4}(|\nabla u_i|^2 + (\lambda + V_1(x))u_i^2) - \frac{\alpha}{2}\int_{\mathbb{R}^4} u_1^2 u_2 - \frac{\beta}{3}\int_{\mathbb{R}^4} u_2^3,$$

and $Q(\mathbf{y}) := I_\lambda\left(\sum_{j=1}^{m} U_{1,\lambda,y_j} + \varphi_{1,\lambda}, \sum_{j=1}^{m} U_{2,\lambda,y_j} + \varphi_{2,\lambda}\right)$ with $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^{4m}$.

Then to obtain a true solution of (2.1), we need to choose suitable $y_{\lambda,j}$ such that

$$\Gamma_{\lambda,i,j} = 0, \quad \text{for } i = 1, 2, 3, 4 \quad \text{and} \quad j = 1, \dots, m. \tag{4.1}$$

To this aim, similar to Lemma 2.2.13 in [12], we will find the critical points of $Q(\mathbf{y})$ because that if $\mathbf{y}_\lambda = (y_{\lambda,1}, \dots, y_{\lambda,m})$ is the critical point of $Q(\mathbf{y})$, then (4.1) holds.

Lemma 4.1. *It holds*

$$\begin{aligned} I_\lambda\left(\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}, \sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}\right) &= \frac{mb_*\lambda}{2}\sum_{i=1}^{2}\Gamma_i^2 + \frac{b_*}{2}\sum_{i=1}^{2}\sum_{j=1}^{m}\Gamma_i^2 V_i(y_{\lambda,j})\\ &\quad - \sum_{l\neq j}\frac{(\Gamma_0 + o(1))\lambda}{2}e^{-\sqrt{\lambda}|y_{\lambda,j}-y_{\lambda,l}|}\left(\frac{1}{\sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}|}\right)^{\frac{3}{2}}\\ &\quad + O\left(\frac{1}{\lambda^{\frac{\theta}{2}}} + \lambda\sum_{l\neq j} e^{-\frac{3}{2}\sqrt{\lambda}|y_{\lambda,j}-y_{\lambda,l}|}\left(\frac{1}{\sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}|}\right)^{\frac{3}{2}}\right), \end{aligned} \tag{4.2}$$

for some constant $\Gamma_0 > 0$.

Proof. First, by the definition of I_λ, we have

$$\begin{aligned} I_\lambda\left(\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}, \sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}\right) &= \frac{\lambda^2}{2}\sum_{i=1}^{2}\Gamma_i^2\int_{\mathbb{R}^4}\left|\sum_{j=1}^{m}\nabla U(\sqrt{\lambda}(x - y_{\lambda,j}))\right|^2\\ &\quad + \frac{\lambda^2}{2}\sum_{i=1}^{2}\Gamma_i^2\int_{\mathbb{R}^4}(\lambda + V_i(x))\left[\sum_{j=1}^{m} U(\sqrt{\lambda}(x - y_{\lambda,j}))\right]^2\\ &\quad - \frac{\lambda^3}{3}\sum_{i=1}^{2}\Gamma_i^2\int_{\mathbb{R}^4}\left[\sum_{j=1}^{m} U(\sqrt{\lambda}(x - y_{\lambda,j}))\right]^3, \end{aligned}$$

where we use the fact that $\frac{3\alpha - 2\beta}{\alpha^3} = \Gamma_1^2 + \Gamma_2^2$. Now, we find

$$\int_{\mathbb{R}^4}\left|\sum_{j=1}^{m}\nabla U(\sqrt{\lambda}(x - y_{\lambda,j}))\right|^2 = \frac{2m}{\lambda}b_* + \underbrace{\sum_{l\neq j}\int_{\mathbb{R}^4}\nabla U(\sqrt{\lambda}(x - y_{\lambda,j}))\cdot\nabla U(\sqrt{\lambda}(x - y_{\lambda,l}))}_{:=A_1}.$$

And from the assumption (K_2) and $|y_{\lambda,j}-y_{\lambda,l}|\to 0$, we compute that

$$\begin{aligned}
&\int_{\mathbb{R}^4}(\lambda+V_i(x))\left[\sum_{j=1}^m U(\sqrt{\lambda}(x-y_{\lambda,j}))\right]^2\\
&=\sum_{j=1}^m(\lambda+V_i(y_{\lambda,j}))\int_{\mathbb{R}^4}U^2(\sqrt{\lambda}(x-y_{\lambda,j}))+\sum_{j=1}^m\int_{\mathbb{R}^4}(V_i(x)-V_i(y_{\lambda,j}))U^2(\sqrt{\lambda}(x-y_{\lambda,j}))\\
&\quad+\sum_{l\neq j}^m\int_{\mathbb{R}^4}(\lambda+V_i(x))U(\sqrt{\lambda}(x-y_{\lambda,j}))U(\sqrt{\lambda}(x-y_{\lambda,l}))\\
&=\frac{1}{\lambda}mb_*+\frac{1}{\lambda^2}\sum_{j=1}^m V_i(y_{\lambda,j})b_*+O\left(\frac{1}{\lambda^{2+\frac{1}{2}\theta}}\right)+\underbrace{\sum_{l\neq j}^m\int_{\mathbb{R}^4}\lambda U(\sqrt{\lambda}(x-y_{\lambda,j}))U(\sqrt{\lambda}(x-y_{\lambda,l}))}_{:=A_2}\\
&\quad+o\left(\frac{1}{\lambda}e^{-\sqrt{\lambda}|y_{\lambda,j}-y_{\lambda,l}|}\left(\frac{1}{\sqrt{\lambda}|y_{\lambda,j}-y_{\lambda,l}|}\right)^{\frac{3}{2}}\right).
\end{aligned}$$

On the other hand, it holds

$$\begin{aligned}
\int_{\mathbb{R}^4}\left[\sum_{j=1}^m U(\sqrt{\lambda}(x-y_{\lambda,j}))\right]^3&=\frac{3kb_*}{\lambda^2}+3\underbrace{\int_{\mathbb{R}^4}\sum_{l\neq j}U^2(\sqrt{\lambda}(x-y_{\lambda,j}))U(\sqrt{\lambda}(x-y_{\lambda,l}))}_{:=A_3}\\
&\quad+O\left(\sum_{l\neq j}\int_{\mathbb{R}^4}U^{\frac{3}{2}}(\sqrt{\lambda}(x-y_{\lambda,j}))U^{\frac{3}{2}}(\sqrt{\lambda}(x-y_{\lambda,l}))\right)\\
&=\frac{3kb_*}{\lambda^2}+3A_3+O\left(\frac{1}{\lambda^2}e^{-\frac{3}{2}\sqrt{\lambda}|y_{\lambda,j}-y_{\lambda,l}|}\left(\frac{1}{\sqrt{\lambda}|y_{\lambda,j}-y_{\lambda,l}|}\right)^{\frac{3}{2}}\right).
\end{aligned}$$

Finally, by using the equation satisfied by U, $A_1+A_2-\lambda A_3=0$. So the aforementioned estimates give (4.2). □

Now we have the following energy expansion.

Lemma 4.2. *It holds*

$$I_\lambda(u_{1,\lambda},u_{2,\lambda})=I_\lambda\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}},\sum_{j=1}^m U_{2,\lambda,y_{\lambda,j}}\right)+O\left(\frac{1}{\lambda}\right).\tag{4.3}$$

Proof. First, by definition, we have

$$\begin{aligned}
&I_\lambda\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}(x)+\varphi_{1,\lambda},\sum_{j=1}^m U_{2,\lambda,y_{\lambda,j}}(x)+\varphi_{2,\lambda}\right)\\
&=I_\lambda\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}(x),\sum_{j=1}^m U_{2,\lambda,y_{\lambda,j}}(x)\right)+\sum_{j=1}^m\sum_{i=1}^2\int_{\mathbb{R}^4}(\nabla U_{i,\lambda,y_{\lambda,j}}\nabla\varphi_{i,\lambda}+(\lambda+V_i(x))U_{i,\lambda,y_{\lambda,j}}\varphi_{i,\lambda})\\
&\quad-\frac{\alpha}{2}\int_{\mathbb{R}^4}\left[\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}\right)^2\varphi_{2,\lambda}+2\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}\right)\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}\right)\varphi_{1,\lambda}+2\left(\sum_{j=1}^m U_{1,\lambda,y_{\lambda,j}}\right)\varphi_{1,\lambda}\varphi_{2,\lambda}+\sum_{j=1}^m(U_{2,\lambda,y_{\lambda,j}})\varphi_{1,\lambda}^2+\varphi_{1,\lambda}^2\varphi_{2,\lambda}\right]\\
&\quad-\frac{\beta}{3}\int_{\mathbb{R}^4}\left[3\left(\sum_{j=1}^m U_{2,\lambda,y_{\lambda,j}}\right)^2\varphi_{2,\lambda}+3\left(\sum_{j=1}^m U_{2,\lambda,y_{\lambda,j}}\right)\varphi_{2,\lambda}+\varphi_{2,\lambda}^3\right]+O(\lambda\|(\varphi_{1,\lambda},\varphi_{2,\lambda})\|_\lambda^2).
\end{aligned}\tag{4.4}$$

By using the classical Gagliardo-Nirenberg inequality (2.15), similar to (2.14), we have

$$\int_{\mathbb{R}^4}\left[\left(\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}\right)\varphi_{1,\lambda}\varphi_{2,\lambda} + \sum_{j=1}^{m}(U_{2,\lambda,y_{\lambda,j}})\varphi_{1,\lambda}^2 + \varphi_{1,\lambda}^2\varphi_{2,\lambda} + \varphi_{2,\lambda}^3\right] = O\left(\lambda^{\frac{1}{2}}\|(\varphi_{1,\lambda},\varphi_{2,\lambda})\|_\lambda^2\right).$$

On the other hand, from $(\varphi_{1,\lambda}, \varphi_{2,\lambda}) \in E_\lambda$,

$$\int_{\mathbb{R}^4}(\nabla U_{i,\lambda,y_{\lambda,j}}\nabla\varphi_{i,\lambda} + (\lambda + V_i(x))U_{i,\lambda,y_{\lambda,j}}\varphi_{i,\lambda}) = 0.$$

So, (4.4) gives that

$$\begin{aligned} & I_\lambda\left(\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}(x) + \varphi_{1,\lambda}, \sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}(x) + \varphi_{2,\lambda}\right) \\ &= I_\lambda\left(\sum_{j=1}^{m} U_{1,\lambda,y_{\lambda,j}}(x), \sum_{j=1}^{m} U_{2,\lambda,y_{\lambda,j}}(x)\right) - \lambda^2\int_{\mathbb{R}^4}\left[\left(\frac{\alpha}{2}\Gamma_1^2 + \beta\Gamma_2^2\right)\left(\sum_{j=1}^{m} U(\sqrt{\lambda}(x - y_{\lambda,j}))\right)^2\varphi_{2,\lambda}\right. \\ &\quad \left. + \alpha\Gamma_1\Gamma_2\left(\sum_{j=1}^{m} U(\sqrt{\lambda}(x - y_{\lambda,j}))\right)^2\varphi_{1,\lambda}\right] + O(\lambda\|(\varphi_{1,\lambda},\varphi_{2,\lambda})\|_\lambda^2). \end{aligned} \tag{4.5}$$

Now using the classical Gagliardo-Nirenberg inequality (2.15) again, we have

$$\begin{aligned} & \int_{\mathbb{R}^4}\left(\sum_{j=1}^{m} U(\sqrt{\lambda}(x - y_{\lambda,j}))\right)^2\varphi_{i,\lambda} \\ &= \sum_{j=1}^{m}\int_{\mathbb{R}^4} U^2(\sqrt{\lambda}(x - y_{\lambda,j}))\varphi_{i,\lambda} + \sum_{l\neq j}\int_{\mathbb{R}^4} U(\sqrt{\lambda}(x - y_{\lambda,j}))U(\sqrt{\lambda}(x - y_{\lambda,l}))\varphi_{i,\lambda} \\ &= O\left(\frac{1}{\lambda^{\frac{5}{4}}}\|(\varphi_{1,\lambda},\varphi_{2,\lambda})\|_\lambda\right). \end{aligned}$$

Then (4.3) follows by the aforementioned estimates. □

Proposition 4.3. *Assume that* (K_2) *holds. Then there exist some large* λ_1 *such that for any* $\lambda \in [\lambda_1, +\infty)$, *problem* (2.1) *has a peak solution* $(u_{1,\lambda}, u_{2,\lambda})$ *with the form*

$$u_{i,\lambda}(x) = \sum_{j=1}^{m} U_{i,\lambda,y_{\lambda,j}}(x) + \varphi_{i,\lambda} \tag{4.6}$$

satisfying

$$y_{\lambda,j} \to a_1,\ \sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}| \to \infty, \quad \text{for } l \neq j \quad \text{and} \quad \|(\varphi_{1,\lambda},\varphi_{2,\lambda})\|_\lambda = O\left(\frac{1}{\lambda}\right). \tag{4.7}$$

Proof. First, it follows from Lemmas 4.1 and 4.2 that

$$\begin{aligned} Q(\mathbf{y}) &= \frac{mb_*\lambda}{2}\sum_{i=1}^{2}\Gamma_i^2 + \frac{b_*}{2}\sum_{i=1}^{2}\sum_{j=1}^{m}\Gamma_i^2 V_i(y_{\lambda,j}) \\ &\quad - \sum_{l\neq j}\frac{(\Gamma_0 + o(1))\lambda}{2}e^{-\sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}|}\left(\frac{1}{\sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}|}\right)^{\frac{3}{2}} \\ &\quad + O\left(\frac{1}{\lambda^{\min\{\frac{\theta}{2},1\}}} + \lambda\sum_{l\neq j}e^{-\frac{3}{2}\sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}|}\left(\frac{1}{\sqrt{\lambda}|y_{\lambda,j} - y_{\lambda,l}|}\right)^{\frac{3}{2}}\right). \end{aligned} \tag{4.8}$$

Now we define

$$\mathcal{H} = \left\{ \mathbf{y} = (y_1, \dots, y_m) : y_j \in \bar{B}_\varepsilon(a_1), |y_l - y_j| \geq \frac{\varepsilon \ln\lambda}{\sqrt{\lambda}}, j = 1, \dots, m, l \neq j \right\},$$

where ε is a small fixed constant. Consider the following problem

$$\max_{\mathbf{y}\in\mathcal{H}} Q(y). \tag{4.9}$$

Suppose that it is achieved by $\mathbf{y}_\lambda \in \mathcal{H}$. To prove this $\mathbf{y}_\lambda$ is a critical point of $Q(\mathbf{y})$, we just need to prove that $\mathbf{y}_\lambda$ is the interior point of $\mathcal{H}$.

Let $\bar{y}_{\lambda,j}$, $j = 1, \dots, m$, satisfy

$$|\bar{y}_{\lambda,j} - a_1| \leq \left(\frac{1}{\sqrt{\lambda}}\right)^{\gamma} \quad \text{and} \quad |\bar{y}_{\lambda,j} - \bar{y}_{\lambda,l}| \geq \left(\frac{1}{\sqrt{\lambda}}\right)^{1-\gamma} \quad \text{if } l \neq j,$$

where $0 < \varepsilon < \gamma < <1$ is a small fixed constant. Then for $\bar{\mathbf{y}}_\lambda = (\bar{y}_{\lambda,1}, \dots, \bar{y}_{\lambda,m}) \in \mathcal{H}$, it holds

$$Q(\bar{\mathbf{y}}_\lambda) = \frac{\lambda m b_*}{2} \sum_{i=1}^{2} \Gamma_i^2 + \frac{m b_*}{2} \sum_{i=1}^{2} \Gamma_i^2 V_i(a_1) + O\left(\frac{1}{\lambda^{\frac{\gamma\theta}{2}}}\right).$$

Suppose that there exists y_{λ,j_0} such that $|y_{\lambda,j_0} - a_1| = \varepsilon$. Then by (4.8) and the assumption (K_2), we have

$$Q(\mathbf{y}_\lambda) \leq \frac{\lambda m b_*}{2} \sum_{i=1}^{2} \Gamma_i^2 + \frac{(m-1) b_*}{2} \sum_{i=1}^{2} \Gamma_i^2 V_i(a_1) + \frac{b_*}{2} \sum_{i=1}^{2} \Gamma_i^2 V_i(y_{\lambda,j_0}) + o(1) < Q(\bar{\mathbf{y}}_\lambda),$$

which contradicts the fact that $\mathbf{y}_\lambda$ is a maximum point of $Q(\mathbf{y})$ in $\mathcal{H}$.

Now suppose that there exist y_{λ,j_0} and y_{λ,m_0} such that $|y_{\lambda,j_0} - y_{\lambda,m_0}| = \frac{\varepsilon \ln\lambda}{\sqrt{\lambda}}$ for $m_0 \neq j_0$. Then by using (4.8), we find

$$Q(\mathbf{y}_\lambda) \leq \frac{\lambda m b_*}{2} \sum_{i=1}^{2} \Gamma_i^2 - \tilde{\kappa}_0 \lambda^{1-\varepsilon} (\ln\lambda)^{-\frac{3}{2}} < Q(\bar{\mathbf{y}}_\lambda)$$

for some $\tilde{\kappa}_0 > 0$, which gives a contradiction again. Thus, we have proved that $\mathbf{y}_\lambda$ is the interior point of $\mathcal{H}$, and thus, it is a critical point of $Q(\mathbf{y})$. □

Finally, we conclude our second main result.

Proof of Theorem 1.3. Now we set $b_\lambda = \int_{\mathbb{R}^4} (u_{1,\lambda}^2 + u_{2,\lambda}^2)$ and $v_{i,\lambda} = u_{i,\lambda}/\sqrt{b_\lambda}$, then $(v_{1,\lambda}, v_{2,\lambda})$ satisfies that

$$\begin{cases} -\Delta v_{1,\lambda} + (\lambda + V_1(x)) v_{1,\lambda} = \alpha b_\lambda^{\frac{1}{2}} v_{1,\lambda} v_{2,\lambda}, & \text{in } \mathbb{R}^4, \\ -\Delta v_{2,\lambda} + (\lambda + V_2(x)) v_{2,\lambda} = \frac{\alpha}{2} b_\lambda^{\frac{1}{2}} v_{1,\lambda}^2 + \beta b_\lambda^{\frac{1}{2}} v_{2,\lambda}^2, & \text{in } \mathbb{R}^4, \end{cases}$$

with

$$\int_{\mathbb{R}^4} (v_{1,\lambda}^2 + v_{2,\lambda}^2) = 1.$$

Therefore, similar to Proposition 3.3, we obtain that

$$\lambda(m D(\alpha,\beta) b_* - b_\lambda) = b_* \sum_{i=1}^{2} \sum_{j=1}^{m} \Gamma_i^2 V_i(a_1) + o(1).$$

Next, similar to the proofs of Theorem 1.1, we can finish the proof. □

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References

[1] B. Alexander and K. Yuri, *Solitons due to second harmonic generation*, Phys. Lett. A **197** (1995), 407–412.

[2] A. Ambrosetti and E. Colorado, *Bound and ground states of coupled nonlinear Schrödinger equations*, C. R. Math. Acad. Sci. Paris **342** (2006), 453–458.

[3] A. Ambrosetti and E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations*, J. London Math. Soc. **75** (2007), no. 2, 67–82.

[4] A. Ambrosetti, E. Colorado, and D. Ruiz, *Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations*, Calc. Var. Partial Differential Equations **30** (2007), 85–112.

[5] M. Anderson, J. Ensher, M. Matthews, C. Wieman, and E. Cornell, *Observation of Bose-Einstein condensation in a dilute atomic vapor*, Science **269** (1995), 198–201.

[6] W. Ao and J. Wei, *Infinitely many positive solutions for nonlinear equations with non-symmetric potential*, Calc. Var. Partial Differential Equations **51** (2014), 761–798.

[7] O. Bang, Y.-S. Kivshar, A.-V. Buryak, A.-D. Rossi, and S. Trillo, *Two-dimensional solitary waves in media with quadratic and cubic nonlinearity*, Phys. Rev. E **58** (1998), 5057–5069.

[8] W. Bao and Y. Cai, *Mathmatical theory and numerical methods for Bose-Einstein condensation*, Kinetic Related Models **6** (2013), 1–135.

[9] T. Bartsch, N. Dancer, and Z.-Q. Wang, *A Liouville theorem, a priori bounds, and bifurcating branches of positive solutions for a linear elliptic system*, Calc. Var. Partial Differential Equations **37** (2010), 345–361.

[10] T. Bartsch and Z.-Q. Wang, *Note on ground states of nonlinear Schrödinger systems*, J. Partial Differ. Equations **19** (2006), 200–207.

[11] I. Bloch, J. Dalibard, and W. Zwerger, *Many-body physics with ultracold gases*, Rev. Mod. Phys. **80** (2008), 885.

[12] D. Cao, S. Peng, and S. Yan, *Singularly Perturbed Methods for Nonlinear Elliptic Problems*, Cambridge University Press, Cambridge, 2021

[13] N. Dancer, J. Wei, and T. Weth, *A priori bounds versus multiple existence of positive solutions for a linear Schrödinger system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 953–969.

[14] K. Davis, M. Mewes, M. Andrews, N. van Druten, D. Durfee, D. Kurn, et al., *Bose-Einstein condensation in a gas of sodium atoms*, Phys. Rev. Lett. **75** (1995), 3969–3973.

[15] E. Gross, *Structure of a quantized vortex in boson systems*. Nuovo Cimento **20** (1961), 454–466.

[16] Q. Guo and J. Yang, *Excited states for two-component Bose-Estein condenstates in dimension two*, J. Differential Equations **343** (2023), 658–686.

[17] Y. Guo, S. Li, J. Wei, and X. Zeng, *Ground states of two-component attractive Bose-Einstein condensates I: Existence and uniqueness*, J. Funct. Anal. **276** (2019), no. 1, 183–230.

[18] Y. Guo and R. Seiringer, *On the mass concentration for Bose-Einstein condensates with attractive interactions*. Lett. Math. Phys. **104** (2014), 141–156.

[19] Y. Guo, Z. Wang, X. Zeng, and H. Zhou, *Properties of ground states of attractive Gross-Pitaevskii equations with multi-well potentials*. Nonlinearity **31** (2018), 957–979.

[20] Y. Guo, X. Zeng, and H. Zhou, *Energy estimates and symmetry breaking in attractive Bose-Einsteincondensates with ring-shaped potentials*, Ann. Inst. H. Poincaré Anal. Non Linéeaire **33** (2016), 809–828.

[21] M. Kwong, *Uniqueness of positive solutions of* $\Delta u \quad u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **105** (1989), no. 3, 243–266.

[22] T.C. Lin and J. Wei, *Ground state of N coupled nonlinear Schrödinger equations in* $\mathbb{R}^N$, $N \leq 3$, Commun. Math. Phys. **255** (2005), 629–653.

[23] P. Luo, S. Peng, J. Wei, and S. Yan, *Excited states on Bose-Einstein condesates with attactive interactions*, Calc. Var. Partial Differential Equations **60** (2021), 155.

[24] S. Peng and Z.-Q. Wang, *Segregated and synchronized vector solutions for nonlinear Schrödinger systems*, Arch. Ration. Mech. Anal. **208** (2013), 305–339.

[25] L. Pitaevskii, *Vortex lines in an imperfect Bose gas*, Sov. Phys. JETP **13** (1961), 451–454.

[26] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equation in* $\mathbb{R}^N$, Comm. Math. Phys. **271** (2007), no. 1, 199–221.

[27] Z. Tang and H. Xie, *Muti-spikes solutions for a system of coupled elliptic equations with quadratic nonlinearity*, Commun. Pure Appl. Anal **19** (2020), 311–328.

[28] R. Tian, Z.-Q. Wang, and L. Zhao, *Schrödinger systems with quadratic interactions*, Commun. Contemp. Math. **21** (2019), no. 8, **1850077**, 20pp.
[29] C. Wang and J. Zhou, *Infinitely many solitary waves due to the second-harmonic generation in quadratic media*, Acta Math. Sci. Ser. B (Engl. Ed.) **40** (2020), no. 1, 16–34.
[30] J. Wei and T. Weth, *Radial solutions and phase separation in a system of two coupled Schrödinger equations*, Arch. Ration. Mech. Anal. **190** (2008), no. 1, 83–106.
[31] J. Yang and T. Zhou, *Existence of single peak solutions for a nonlinear Schróodinger system with coupled quadratic nonlinearity*, Adv. Nonlinear Anal. **11** (2022), 417–431.